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LIMIT CYCLES IN A PREY-PREDATOR SYSTEM

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Abstract—We consider a generalised Gause-type prey-predator system, where both the prey and the predator species have independent specific growth rate. We proved that the system has limit cycles globally.

1. INTRODUCTION

We consider a generalised Gause-type model of two interacting species which are in prey-predator relationship and where both species have independent specific growth rate in the absence of other, by the following set of autonomous differential equations:

$$\dot{x} = xg(x) - yp(x) \tag{1.1a}$$

$$\dot{y} = yG(x,y) \tag{1.1b}$$

$$x(0) = x_0 > 0, \quad y(0) = y_0 > 0 \quad \text{and } \cdot \equiv \tfrac{d}{dt}.$$

Here, $x(t)$ and $y(t)$ denote the biomass of the prey species x and the predator species y respectively at time t. The specific growth rate of the prey species x is $g(x)$,and $p(x)$ is the rate at which the species x are consumed by the predator y. $G(0,y)$ is the specific growth rate of the predator y in the absence of the prey species x. It is considered in (1.1b) that the predator species has an alternative resource so that in the absence of prey, it can survive. Thus, this prey-predator model is quite different from the usual prey-predator model (see [1–3]). Such pre-predator model (1.1) can be obtained as a subsystem of a generalised model of three-species cyclic loops [4,5]. Three-species cyclic loops are found in some aquatic ecosystems [6]. Cyclic loops can occur in ecological communities [7] and also in bio-geo-chemical food webs [8].

We consider the following assumptions:

(H1)

(a) g, p and G are continuously differentiable functions.

(b) $g(0) = \alpha > 0$. There exists a unique $k > 0$ such that $g(k) = 0$. $g'(x) < 0$ for $x \geq 0$.

(c) $G(x, 0) = \alpha_1 > 0$, $\frac{\partial G(x,y)}{\partial y} < 0$, x, $y \geq 0$. This implies that the predator population growth rate is density dependent and slows down as the population increases. Since species x and y are in prey-predator relation,

$$\frac{\partial G(x,y)}{\partial x} > 0 \text{ holds.}$$

(d) There exists a $L(x)$ such that $G(x,\ L(x)) = 0$ where $L : R_+ \to R_+$ and $L'(x) \geq 0$. This implies the existence of a monotonically increasing density dependent carrying capacity for the predator and also we assume

$$\lim_{x\to\infty} L(x) = \bar{L} < +\infty.$$

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(e) $p(0) = 0 \cdot p'(x) > 0$ for $x \geq o$.
$\lim_{x\to\infty} p(x) = p_\infty$ where $0 < p_\infty < +\infty$.
$\lim_{x\to 0+} p(x)/x = p'(0) = \beta > 0$.
This means that the predator functional response $p(x)$ is zero when x is sufficiently small and it is the increasing function of x, but approaches asymptotically to a finite positive value when x is large enough. The specific loss rate $p(x)/x$ is a non-zero quantity when x is sufficiently small.

(H2) All the feasible equilibria are hyperbolic.

2. DISSIPATIVITY OF THE SYSTEM (1.1)

The system (1.1) describing the evolution of $X(t) = (x,(t), y(t))$, is said to be dissipative if all the trajectories of (1.1) are uniformly asymptotically bounded for $t \geq 0$. In other words, there exists a constant M such that

$$\lim_{t\to\infty} \sup \|X(t)\| \leq M$$

For our system (1.1) the result regarding dissipativity is given in the following lemma.

LEMMA 2.1. *All solutions of (1.1) that initiate in R^2_+ are uniformly asymptotically bounded for some $\bar{x}$, $\bar{y} > 0$ to be specified later and they are ultimately in the region $B \subseteq R^2_+$, where*

$$B = \{(x,y) : 0 \leq x \leq \bar{x}; \quad 0 \leq y \leq \bar{y}\}.$$

PROOF. By (H1b) $g'(x) < 0 \Rightarrow x \leq xg(x) \Rightarrow x(t) \leq \max(k, x_0) = \bar{x}$ (say).

Again, $\partial G(x,y)/\partial y < 0 \Rightarrow$ if $y_0 < \bar{L}$, $y(t) \leq L(x) + \in \leq \bar{L} + \in$ for any $\in > 0$. If $y_0 \geq \bar{L}$ then $y(t) \leq \max(\bar{L} + \in, y_0) = \bar{y}$ (say) (see [9]).

2.1. Equilibria and Their Stability

By (H1), the trivial equilibrium $E_0(0,0)$ and the two axial equilibria $E_1(k,0)$, $E_2(0, L(0))$, always exist.

The interior positive equilibrium of the system (1.1) is the intersection of the two isoclines $\dot{x} = \dot{y} = 0$ in the positive quadrant of the xy plane.

From (1.1a), the prey isocline: $\dot{x} = 0$ is equivalent to

$$y = \frac{xg(x)}{p(x)} = F(x) \text{ say} \tag{2.1a}$$

and from (1.1b), the predator isocline: $\dot{y} = 0$ gives $G(x,y) = 0$. This is equivalent to the curve

$$y = L(x). \tag{2.1b}$$

The predator isocline (2.1b) meets the y-axis at $(0, L(0))$ and it is a monotonically increasing function. The prey isocline $F(x)$ intersects the x-axis at E_1 and the y-axis at $(0, \alpha/\beta)$. For mathematical convenience we impose the following assumption on $F(x)$.

(H3) The prey isocline $F(x)$ possesses a unique global maximum at $x_M \geq 0$ satisfying

$$\begin{aligned} \frac{dF}{dx} &> 0 \quad \text{for } 0 \leq x < x_M \\ &< 0 \quad \text{for } x_M < x \leq k. \end{aligned}$$

Together with (H1)–(H3), we get the following result which ensured the existence of the interior positive equilibrium of the system (1.1).

PROPOSITION 2.2. *Let (H1)–(H3) hold. Further let $L(0) < \alpha/\beta$ hold. Then there is a positive equilibrium $E(\hat{x}, \hat{y})$ in the interior of B.*

PROOF. By Lemma (2.1), the system (1.1) is dissipative. The variational matrix $V(x,y)$ of the system (1.1) is

$$V(x,y) = \begin{bmatrix} p(x)F'(x) & -p(x) \\ yG_x(x,y) & -yG_y(x,y) \end{bmatrix} \tag{2.2}$$

From (2.2), it follows that E_0 is a repeller along both the x- and y-directions. E_1 is locally stable along the x-direction but unstable along the y-direction. E_2 is locally unstable along the x-direction since $L(0) < \alpha/\beta$. So there exists an equilibrium E in the interior of the positive xy plane. This follows from an application of the Poincare-Bendixson theorem.

Now if the condition $L(0) < \alpha/\beta$ is satisfied there may be more than one equilibria in the interior of B and hence, we further assume.

(H4) If $L(0) < \alpha/\beta$, then the positive equilibrium E is unique in the interior of B.
Moreover, the equilibrium E in the interior of B may lie on the increasing part of $F(x)$ or may lie on the decreasing part of $F(x)$.

REMARK 2.3. If $L(0) \geq \alpha/\beta$, then there does not exist or exists multiple interior equilibria (see [5]) in the positive xy plane. We shall not consider this case in this paper.

PROPOSITION 2.4. *Let (H1)–(H4) and $L(0) < \alpha/\beta$ hold. In addition,*

(a) *let E lie on the decreasing part of $F(x)$, then E is a sink, and*
(b) *let E lie on the increasing part of $F(x)$, then E is a sink provided $\bar{A} < -G_y(\hat{x},\hat{y}) < \bar{B}$ and E is a source if $-G_y(\hat{x},\hat{y}) < \bar{A}$, where*

$$\bar{A} = \frac{p(\hat{x})F'(\hat{x})}{\hat{y}} \text{ and } \bar{B} = \frac{G_x(\hat{x},\hat{y})}{F'(\hat{x})} \tag{2.3}$$

PROOF. The characteristic equation for $E(\hat{x},\hat{y})$ is obtained from (2.2):

$$\mu^2 + p_1\mu + q_1 = 0 \tag{2.4}$$

where

$$p_1 = -(p(\hat{x})F'(\hat{x}) + \hat{y}G_y(\hat{x},\hat{y})) \tag{2.5a}$$

and

$$q_1 = \hat{y}p(\hat{x})\left\{G_y(\hat{x},\hat{y})F'(\hat{x}) + G_x(\hat{x},\hat{y})\right\} \tag{2.5b}$$

(a) From (2.5), it is clear that if E lies on the decreasing part of $F(x)$, that is, $F'(x) < 0$, then $p_1 > 0$ and $q_1 > 0$. This implies that E is a sink. This completes the proof of part (a).
(b) Let E lie on the increasing part of $F(x)$. We shall prove first that E cannot be a saddle point. E must be either a sink or a source. It follows from the Index Theorem [3] that the sum of the indices of all the saturated regular equilibria of a dissipative two-dimensional system is $(-1)^2 = +1$. By Lemma (2.1), the system (1.1) is dissipative. In the positive xy plane, the axial equilibria E_1, E_2 are non-saturated and the only saturated fixed point is E. So the index of E must be $(+1)$. Then E must be a sink with two-dimensional (local) stable manifolds or a source with two dimensional (local) unstable manifolds. Let E be a saddle point, then its index is (-1) which contradicts the Index Theorem [3]. Hence, E can never be a saddle point. E can only be either a sink or a source.

From (2.5),

$$p_1 > 0 \Rightarrow -G_y(\hat{x},\hat{y}) > \bar{A} \quad \text{and}$$
$$q_1 > 0 \Rightarrow -G_y(\hat{x},\hat{y}) < \bar{B}$$

Thus, whenever E lies on the increasing part of $F(x)$, E is a sink provided $\bar{A} < -G_y < \bar{B}$ and E is a source if $-G_y < \bar{A}$.

2.2. The Existence of Limit Cycles

Next, we investigate the existence of limit cycles globally. For this purpose, we consider that the equilibrium E lies on the increasing part of $F(x)$ and $-G_y(\hat{x},\hat{y}) < \tilde{A}$, that is, the equilibrium E is a source. Then by boundedness of solutions, there is an attracting limit cycle around E. This follows from our last result.

THEOREM 2.6. *Let (H1)–(H4) and $L(o) < \alpha/\beta$ hold. Further, let E lie on the increasing part of $F(x)$ and $-G_y(\hat{x},\hat{y}) < \tilde{A}$. Then there exists a limit cycle enclosing E, which is globally stable from the outside.*

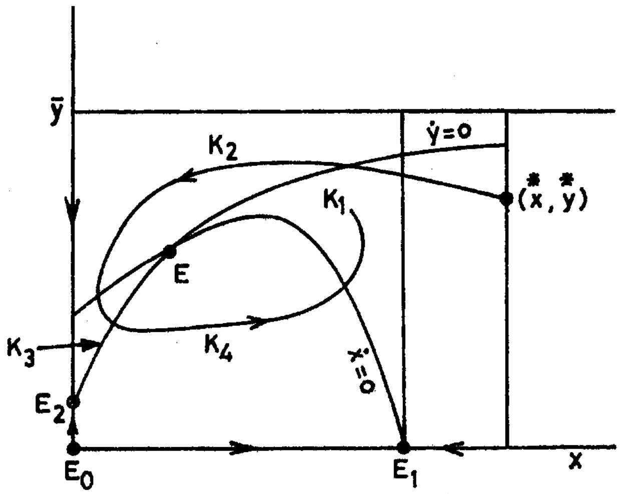

Figure 1. Illustrating the proof of Theorem 2.6.

PROOF. Let R be the rectangle with $0 < x < k$ and $0 < y < \bar{y}$. The closure $\bar{R}$ of R is divided by the isoclines $\dot{x} = \dot{y} = 0$ into four compact regions k_1, k_2, k_3 and k_4 (see Figure 1). Let $\overset{*}{R}$ be the compact rectangle: $0 \le x \le \max(k, \overset{*}{x})$ and $0 \le y \le \max(\bar{y}, \overset{*}{y})$ for arbitrary fixed $\overset{*}{x}$, $\overset{*}{y} > 0$ and $(\overset{*}{x}, \overset{*}{y}) \notin R$. Also let γ^+ be the positive semiorbit of (1.1) with initial value $(\overset{*}{x}, \overset{*}{y})$ and $\omega(\gamma^+)$ be its omega limit set. Since $\overset{*}{R}$ is positively invariant, $\gamma^+ \in \mathrm{Int}\overset{*}{R}$ (interior of $\overset{*}{R}$) and $\omega(\gamma^+) \subseteq \overset{*}{R}$. Again, while γ^+ is contained in a compact region where $\dot{x}$ and $\dot{y}$ of (1.1) do not change sign, γ^+ tends to a limit as $t \to \infty$ and this limit is an equilibrium. But we get the following:

(1) As the equilibrium E_1 is locally unstable along the direction into the interior of positive xy plane, γ^+ cannot tend to E_1 for $t \to \infty$. Rather γ^+ reaches k_1 (or k_2 if $(\overset{*}{x}, \overset{*}{y})$ lies above the $\dot{y} = o$ isocline).
(2) Entering region k_1, after a finite time, as stated earlier γ^+ cannot approach to E_1. E is a source. So γ^+ cannot also converge E. γ^+ must leave k_1 and enter k_2.
(3) From k_2, after finite times, γ^+ moves to k_3. In region k_3, convergence of γ^+ to E_2 is not possible, as E_2 is locally unstable along its orthogonal direction. So after some times on γ^+ leaves k_3 and moves to k_4.
(4) In k_4, E_0 is a repellor. So γ^+ cannot converge E_0 or E_1. γ^+ must enter k_1.

Following the same process described in (1)–(4), for $t \to \infty$. γ^+ spirals inwards R. (γ^+ cannot leave R, as R is positively invariant). Further, γ^+ starts outside of R and γ^+ cannot intersect itself. Hence γ^+ is not a closed trajectory.

From (1)–(4) it is also clear that E_0, E_1, E_2, $E \notin \omega(\gamma^+)$. By Poincare-Bendixson theorem, $\omega(\gamma^+)$ is a closed orbit and since $\gamma^+ \neq \omega(\gamma^+)$, $\omega(\gamma^+)$ is a limit cycle. Again by Poincare-Bendixson theorem, any closed orbit of (1.1) enclosing E must be in a positively invariant rectangle R. Thus, any closed orbit in $R \neq \omega(\gamma^+)$ must ben encircled by $\omega(\gamma^+)$. Hence, $\omega(\gamma^+)$ is the ω-limit set for each initial value $(\overset{*}{x}, \overset{*}{y})$ lying outside domain of $\omega(\gamma^+)$.

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