Criteria for the Nuclearity of Spaces
of Functions of Infinitely Many Variables

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The present paper evolves from Berezanskii and Gali (Ukrainian Math. J. 24 (4) (1973), 435-464) and Berezanskii, Gali, and Zuk (Soviet. Math. Dokl. 13 (2) (1972)), in which it was shown how one can construct a weighted infinite tensor product $H_{e,\delta} = \bigotimes_{n=1}^{\infty} H_n$ of Hilbert spaces $H_n$ with a given stabilizing sequence $\delta = (\delta_n)_{n=1}^{\infty}$ ($\delta_n > 0$). Here a weighted infinite tensor product $\phi_e = \bigotimes_{n=1}^{\infty} \phi_n$ of nuclear spaces $\phi_n$ is established first. Criteria for nuclearity of the constructed spaces are also given. Some examples of nuclear spaces of functions of infinite many variables $K(T^\infty)$ and $A(T^\infty)$ are obtained.

1. The Spaces under Consideration

The following construction of a weighted infinite tensor product of Hilbert spaces with a given stabilizing sequence [1-3] will be used further. Suppose $(H_n)_{n=1}^{\infty}$ is a sequence of Hilbert spaces, $e = (e^{(n)})_{n=1}^{\infty}$ ($e^{(n)} \in H_n$) be a fixed sequence of unit vectors and $\delta = (\delta_n)_{n=1}^{\infty}$ ($\delta_n > 0$) is a fixed numerical sequence (a weight). In each $H_n$ we consider an orthonormalized basis $(e_j^{(n)})_{j=1}^{\infty}$ such that $e^{(n)} = e^{(n)}$.

We form a formal product

$$e_\alpha = e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\alpha_2} \otimes \cdots (\alpha = (\alpha_j)_{j=1}^{\infty}),$$

where $\alpha_1, \alpha_2, \ldots = 1, 2, \ldots$, and, moreover, $\alpha_{n+1} = \alpha_{n+2} = \cdots = 1$ beginning from some number $n$ depending on $\alpha$; let $\nu(a)$ be the minimal $n = 1, 2, \ldots$, possessing this property. Let $A$ be the countable set of all vectors indices $\alpha$ of this kind.

We define the weighted infinite tensor product $H_{e,\delta} = \bigotimes_{n=1}^{\infty} e^{(n)} \otimes \phi_n$ of the Hilbert spaces $H_n$ with stabilizing sequence $e$ and weight $\delta$ as the Hilbert space spanned by the basis $(\delta_{\nu(a)}^{-1/2} e_\alpha)_{\alpha \in A}$, which is assumed orthonormalized.
by definition. Thus, the element of $H_{e, \delta}$ has the form $f = \sum_{\alpha \in A} f_{\alpha} a_{\alpha}$, where $\sum_{\alpha \in A} |f_{\alpha}|^2 \delta_{\lambda(\alpha)} = \|f\|_{H_{e, \delta}} < \infty$; $(f, g)_{H_{e, \delta}} = \sum_{\alpha \in A} f_{\alpha} \bar{g}_{\alpha} \delta_{\lambda(\alpha)}$.

Of course, this definition does not depend on the choice of the basis $(e_{\lambda})_{\lambda=1}^\infty$ (such that $e_{1}^{(n)} = e(\lambda)$). We set $A_n = \{ \alpha \in A \mid \nu(\alpha) = n \}$ $(n = 1, 2, \ldots)$. These sets are pairwise nonintersecting and their union is $A$.

If $\delta = 1$, i.e., $\delta_{\lambda} = 1$ $(n = 1, 2, \ldots)$, then $H_{e, 1}$ coincides with a separable subspace of the complete Neumann product of the spaces $H_n$. In particular, if $H_n - L_2(R, du_n(x_n))$ $(\mu_n(R) = 1)$ and $e = (e^{(n)})_{n=1}^\infty$, where $e_{\lambda}^{(n)}(x_{\lambda}) = 1$, then $H_{e, 1} = L_2(R^\infty, du_1(x_1) \otimes du_2(x_2) \otimes \cdots)$ ($R^\infty = R^1 \times R^1 \times \cdots$).

We recall the construction of a nuclear space $\Phi$ as the projective limit of Hilbert spaces. Let $(H_n)_{n \in T}$ be a family of Hilbert spaces and $T$ an arbitrary set of indices. We shall assume that each $H_n$ is a subset of a certain enveloping Hilbert space $H_0$ $(0 \in T)$, the imbedding $H_n \to H_0$ being continuous with a norm not exceeding 1. It is assumed that $\Phi = \bigcap_{\tau \in T} H_{\tau}$ is dense in every $H_\tau$ and that for all $\tau', \tau'' \in T$ there exists $\tau''' \in T$ such that the imbedding $H_{\tau'''} \to H_{\tau'}$, and $H_{\tau'''} \to H_{\tau''}$ being quasinuclear (i.e., the inclusion operators) are of the Hilbert–Schmidt type. Then $\Phi$ can be transformed into a nuclear space if it is equipped with the projective limit topology (i.e., if one takes as a basis of the neighborhood of the origin the set $\{ f \in \Phi \mid \| f \|_{H_{\tau}} < \epsilon \}$ $(\tau \in T, \epsilon > 0)$.

Note that any nuclear space $\Phi$ can be obtained from Hilbert spaces by means of a construction that slightly generalizes the one described in [5, 6].

We introduce the concept of a weighted infinite tensor product $\phi_{n} = \otimes_{n=1}^{\infty} \Phi_n$ of nuclear spaces $\Phi_n$. Let $\Phi_n = \bigcap_{\tau_n \in T_n} H_{\tau_n}$ be the representation of $\Phi_n$ described above as a projective limit of Hilbert spaces $H_{\tau_n}$; $H_0$ is the corresponding space $H_0$. We shall assume that there exists a sequence $e = (e^{(n)})_{n=1}^{\infty}$, where $e^{(n)} \subset \bigcap_{\tau_n \in T_n} H_{\tau_n}$ and $\| e^{(n)} \|_{H_{\tau_n}} = 1$ for any $\tau_n \in T$. We construct the weighted tensor product of the Hilbert spaces $\otimes_{n=1}^{\infty} H_{\tau_n}$, where the role of the stabilizing sequence is now played by the sequence $e$ that we have introduced and the weight $\delta$ is arbitrary, of the type indicated in the above section.

By $\otimes_{n=1}^{\infty} \Phi_n$, we shall understand the projective limit $\bigcap_{\delta, (\tau_n)} H_{\tau_n}$, where the intersection is taken with respect to all weights $\delta = (\delta_n)_{n=1}^{\infty}$ and with respect to all sequences $\tau = (\tau_n)_{n=1}^{\infty}$ $(\tau_n \in T_n)$. Thus, we have

$$T = \{ \tau = (\delta, \tau) = ((\delta_n)_{n=1}^{\infty}, (\tau_n)_{n=1}^{\infty}) \mid \delta_n \geq 1, \tau_n \in T_n \};$$

we set $0 = (1, (O_n)_{n=1}^{\infty})$.

From that we have said and the arguments of [1, 3, 4], it is easy to obtain the following:
Lemma. The weighted infinite tensor product $\Phi_\mathbf{e} = \bigotimes_{n=1}^{\infty} \Phi_n$ of nuclear spaces $\mathbf{Q}_n$ is nuclear if and only if for every weight $m = (m_i)_{i=1}^{\infty}$ there exists $l = (l_i)_{i=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \delta_n^{-1} \prod_{a=1}^{\infty} (|O^I_{n_a}| - 1) < \infty,$$

where $|O^I_{n_a}|$ is the Hilbert norm of the imbedding operator $O^I_m: H_l \rightarrow H_m (l \geq m)$.

2. Criteria for Nuclearity of Nuclear Spaces

Wolk [7] has given an interesting criterion for the nuclearity of projective (inductive) limit of Hilbert spaces. In this section two criteria are given, in terms of a given family of weights $P$, where

$$P = \{P^\tau = (P^\tau_n)_{n=0}^{\infty} : P \in R^\infty; P^0_0 = 1; P^\tau_n \geq 1 \text{ for every } n, \tau \in \{0, 1, \ldots\} = T\}.$$

We consider the Hilbert spaces $H_\tau$ which are the completions of the class $C^\infty (R^\tau)$ of infinitely differentiable functions of compact support with respect to the norm

$$\|u\|^2 = \sum_{n=0}^{\infty} |C_n|^2 P^\tau_n < \infty; \quad (C_n)_{n=0}^{\infty} \in C^\infty = C^1 \times C^1 \ldots.$$

Then, we can construct the space $\Phi_p$ as the projective limit of $H_\tau$ in exactly the same way as in [2].

$$\Phi_p = \bigcap_{\tau \in T} H_\tau.$$

We shall say that the family $P$ has the property $(N)$ if for any weight $\tau' \in T$ there exists a weight $\tau'' \in T$ such that

$$\sum_{n=0}^{\infty} P^\tau_n / P^{\tau''}_n < \infty.$$

Then we can prove

Theorem 1. The space $\Phi_p$ is nuclear if and only if the set $P$ has the property $(N)$.
Proof. Indeed, if the property (N) is satisfied, and \((P_n^{-\frac{1}{2}} e_n)^{\infty}_{n=0}\) be an orthonormal basis in \(H_{\tau}^\prime\). Then,
\[
|O_{\tau}^{\prime\prime}| \sum_{n=0}^{\infty} ||(P_n^{-\frac{1}{2}} e_n)||^2 = \sum_{n=0}^{\infty} P_n^{\tau}/P_n^{\tau}\prime < \infty,
\]
where \(O_{\tau}^{\prime\prime}: H_{\tau}^\prime \rightarrow H_{\tau}\) is the inclusion operator. Thus, the space \(\Phi_{\rho}\) is nuclear.

The necessity of the condition (N) follows from the nuclearity of \(\Phi_{\rho}\). This completes the proof.

The space \(\Phi_{\rho}^\prime\) is the inductive limit of spaces \(H_{-\tau}\)
\[
\Phi_{\rho}^\prime = \bigcup_{\tau \in T} H_{-\tau},
\]
where
\[
H_{-\tau} = \left\{ w = \sum_{n=0}^{\infty} w_n e_n, (w_n)^{\infty}_{n=0} \in C^\infty : ||w||^2_{-\tau}\right\}
\]

is the conjugate to \(H_{\tau}\).

We assume that in every \(H_{\tau}\), the unit vector \(e_0\) is chosen such that \(\|e_0\|_\tau = 1\); let \(P_0^\prime = 1\). Then, with the aid of a well-known procedure [1, 3], it is possible to construct the following chain of spaces:

\[
\mathcal{K}_{\tau} \subseteq \mathcal{K}_0 \subseteq \mathcal{K}_{-\tau}; \quad \Phi_{\rho} \subseteq \mathcal{K}_0 \subseteq \Phi_{\rho}^\prime
\]

with
\[
\|\cdot\|_\tau \gg \|\cdot\|_0 \gg \|\cdot\|_{-\tau}; \quad \|\cdot\|_{\rho} \gg \|\cdot\|_0 \gg \|\cdot\|_{-\rho},
\]
where \(\mathcal{K}_0\) is the infinite tensor product of \(H_0\) constructed by the stabilizing sequence \(e = (e_0, e_0, \ldots)\) and \(e_{\alpha} = e_{\alpha} \otimes e_{\alpha} \otimes e_{\alpha} \ldots\) is considered as an orthonormal basis in \(\mathcal{K}_0\); \(e_{\alpha} = (P_{\alpha_1}^{\tau})^{-\frac{1}{2}} \ldots (P_{\alpha_p}^{\tau})^{-\frac{1}{2}} e_{\alpha}\) be an orthonormal basis in \(\mathcal{K}_{\tau}\).

We shall construct the space \(\Phi_{\rho}\) exactly as the projective limit of the Hilbert spaces \(\mathcal{K}_{\tau}\). The corresponding conjugate \(\Phi_{\rho}^\prime\) is defined as the inductive limit of the negative space \(\mathcal{K}_{-\tau}\), where \(\mathcal{K}_{-\tau} = \bigotimes_{i=1}^{\infty} H_{-\tau_i}\), for every \(\tau \in T\) has been constructed by stabilizing sequence \(e = (e_0, e_0, \ldots)\) with basis \(e_{\alpha}^{-\tau} = (P_{\alpha_1}^{\tau})^{1/2} \ldots (P_{\alpha_p}^{\tau})^{1/2} e_{\alpha}\).
Then we shall say that the family $p$ has the property $(N*)$ if for every \( \tau' \in T \) there exists \( \tau'' \in T \) such that
\[
\sum_{n=0}^{\infty} \frac{P_n^{\tau'}(p^{\tau''})}{P_n^{\tau''}} \leq 1 + \varepsilon, \quad \varepsilon > 0
\]

**Theorem 2.** The space $\Phi$ is nuclear if and only if the family $P$ has the property $(N*)$.

**Proof.** The proof is similar to Theorem 1. For the sufficient conditions let $P$ have the property $(N*)$, i.e., for \( \tau' = (\tau_i')_i \), we can choose \( \tau'' = (\tau_i'')_i \) such that
\[
\sum_{n=0}^{\infty} \frac{P_n^{\tau_i'}(p^{\tau_i''})}{P_n^{\tau_i''}} \leq 1 + \frac{1}{2^i} \quad (i = 1, 2, \ldots),
\]
since the Hilbert norm for the inclusion operator is \( |O^{\tau_i'}| = \sum_{n=-\infty}^{\infty} P_n^{\tau_i'}/P_n^{\tau_i''} \), which is convergent. Then the condition for nuclearity of $\Phi_p$ are satisfied.

Conversely, suppose that $\Phi_p$ is nuclear. This means that the inclusion operator is quasinuclear. Hence the necessity of the condition $(N*)$ follows directly.

3. **Examples of Nuclear Spaces of Functions of Infinitely Many Variables**

In this section, we shall construct two illustrative examples for such spaces. We introduce the locally convex linear topological space $K(S^1)$ as the projective limit of Sobolev spaces $W_{\alpha}^{\infty}(S^1)$. The last space is equal to the completion of trigonometric polynomials
\[
T_n(t) = \sum_{|\alpha| \leq n} C_\alpha e^{i\alpha t} \quad (n = 0, 1, \ldots)
\]
by the norm
\[
\|T_n\|_{W_{\alpha}^{\infty}(S^1)}^2 = \sum_{\ell=0}^{m} \|T_{n_{\ell}}\|_{W_{\alpha}^{\infty}(S^1)}^2 = \sum_{\alpha} |C_\alpha|^2 (1 + \alpha^2 + \cdots + \alpha^{2m}).
\]

Consider the space $K(T^\infty) = \bigotimes_{\lambda=1}^{\infty} K(S^1)$, \( T^\infty = S^1 \times S^1 \times \cdots \) to be regarded as the infinite tensor product of spaces constructed by the stabilizing sequence $e = (e^a)_{a=1}^{\infty}$. 
On the other hand, let us consider the spaces
\[ K^m = \left\{ u(t) = \sum_{\alpha=0}^{\infty} a(t^{\alpha}) e^{i\alpha t} \|u\|^2 = \sum_{\alpha=0}^{\infty} u^\alpha (1 + |\alpha|)^m < \infty \right\}, \]
in which
\[ \{P^m = (P^m_{\alpha=0})_{\infty=0}, P^m_{\alpha} = (1 + |\alpha|)^m \}_{m=0}^{\infty} \]
be a family of weights satisfies the property \( N^* \).

We form the projective limit \( K(S^1) = \sum_{m=(m_i)_{i=1}} K^m(S^1) \), and then introduce \( \hat{K}(T^\infty) \) as the weighted infinite tensor product of the spaces \( K(S^1) \) with the stabilizing sequence \( e^\alpha = e^{i\alpha t} (\alpha = 0, 1, 2, \ldots) \). The two spaces \( K(T^\infty) \) and \( \hat{K}(T^\infty) \) are identical in the topological sense. By Theorem 2 we then obtain a nuclear space.

For another example, we fix a family of weights
\[ B = \{P^m = (P^m_{\alpha=0})_{\infty=0}, P^m_{\alpha} = m^{\alpha+1} \}_{m=0}^{\infty}, \]
which satisfies the property \( N^* \).

We construct the corresponding spaces \( A^m(S^1) \) to form the projective limit \( A(S^1) = \sum_{m=(m_i)_{i=1}} A^m(S^1) \). Then the weighted infinite tensor product of these spaces are denoted by \( A(T^\infty) \), which are nuclear.

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