ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 254 (2008) 109–153
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients

Alessio Figalli

Scuola Normale Superiore of Pisa, Piazza dei Cavalieri 7, Pisa, Italy

Received 1 December 2006; accepted 27 September 2007

Available online 13 November 2007

Communicated by C. Villani

Abstract

In this paper we extend recent results on the existence and uniqueness of solutions of ODEs with non-smooth vector fields to the case of martingale solutions, in the Stroock–Varadhan sense, of SDEs with non-smooth coefficients. In the first part we develop a general theory, which roughly speaking allows to deduce existence, uniqueness and stability of martingale solutions for $\mathcal{L}^d$-almost every initial condition x whenever existence and uniqueness is known at the PDE level in the L^∞-setting (and, conversely, if existence and uniqueness of martingale solutions is known for $\mathcal{L}^d$-a.e. initial condition, then existence and uniqueness for the PDE holds). In the second part of the paper we consider situations where, on the one hand, no pointwise uniqueness result for the martingale problem is known and, on the other hand, well-posedness for the Fokker–Planck equation can be proved. Thus, the theory developed in the first part of the paper is applicable. In particular, we will study the Fokker–Planck equation in two somehow extreme situations: in the first one, assuming uniform ellipticity of the diffusion coefficients and Lipschitz regularity in time, we are able to prove existence and uniqueness in the L^2-setting; in the second one we consider an additive noise and, assuming the drift b to have BV regularity and allowing the diffusion matrix a to be degenerate (also identically 0), we prove existence and uniqueness in the L^∞-setting. Therefore, in these two situations, our theory yields existence, uniqueness and stability results for martingale solutions.

Keywords: Martingale solutions; Existence and uniqueness almost everywhere; Fokker–Planck equation; Absolutely continuous solutions

E-mail address: a.figalli@sns.it.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.09.020

1. Introduction and preliminary results

Recent research activity has been devoted to study transport equations with rough coefficients, showing that a well-posedness result for the transport equation in a certain subclass of functions allows to prove existence and uniqueness of a flow for the associated ODE. The first result in this direction is due to DiPerna and P.-L. Lions [10], where the authors study the connection between the transport equation and the associated ODE $\dot\gamma = b(t,\gamma)$, showing that existence and uniqueness for the transport equation is equivalent to a sort of well-posedness of the ODE which says, roughly speaking, that the ODE has a unique solution for $\mathcal{L}^d$-almost every initial condition (here and in the sequel, $\mathcal{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$). In that paper they also show that the transport equation $\partial_t u + \sum_i b_i \partial_i u = c$ is well-posed in L^∞ if $b = (b_1, \dots, b_n)$ is Sobolev and satisfies suitable global conditions (including L^∞-bounds on the spatial divergence), which yields the well-posedness of the ODE.

In [1] (see also [2]), using a slightly different philosophy, Ambrosio studied the connection between the continuity equations $\partial_t u + \sum_i \partial_i(b_i u) = c$ and the ODE $\dot\gamma = b(t,\gamma)$. This different approach allows him to develop the general theory of the so-called Regular Lagrangian Flows (see [2, Remark 31] for a detailed comparison with the DiPerna–Lions axiomatization), which relates existence and uniqueness for the continuity equation with well-posedness of the ODE, without assuming any regularity on the vector field b. Indeed, since the transport equation is in a conservative form, it has a meaning in the sense of distributions even when b is only L^∞_{loc} and u is L^1_{loc}. Thus, a general theory is developed in [1] under very general hypotheses, showing as in [10] that existence and uniqueness for the continuity equation is equivalent to a sort of well-posedness of the ODE. After having proved this, in [1] the well-posedness of the continuity equations in L^∞ is proved in the case of vector fields with BV regularity whose distributional divergence belongs to L^∞ (for other similar results on the well-posedness of the transport/continuity equation, see also [6,7,11,13,17]).

Our aim is to develop a stochastic counterpart of this theory: in our setting the continuity equation becomes the Fokker–Planck equation, while the ODE becomes an SDE.

Let us consider the following SDE

$$\begin{cases} dX = b(t,X)\,dt + \sigma(t,X)\,dB(t), \\ X(0) = x, \end{cases} \tag{1}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^r, \mathbb{R}^d)$ are bounded (here $\mathcal{L}(\mathbb{R}^r, \mathbb{R}^d)$ denotes the vector space of linear maps from $\mathbb{R}^r$ to $\mathbb{R}^d$) and B is an r-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We want to study the existence and uniqueness of martingale solutions for this equation. Let us define $a(t,x) := \sigma(t,x)\sigma^*(t,x)$ (that is $a_{ij} := \sum_k \sigma_{ik}\sigma_{jk}$). We consider the so called Fokker–Planck equation

$$\begin{cases} \partial_t \mu_t + \sum_i \partial_i(b_i \mu_t) - \frac{1}{2}\sum_{ij} \partial_{ij}(a_{ij}\mu_t) = 0 & \text{in } [0,T] \times \mathbb{R}^d, \\ \mu_0 = \bar\mu & \text{in } \mathbb{R}^d. \end{cases} \tag{2}$$

We recall that, for a (possibly signed) measure $\mu = \mu(t,x) = \mu_t(x)$, being a solution of (2) simply means that

$$\frac{d}{dt}\int_{\mathbb{R}^d}\varphi(x)\,d\mu_t(x) = \int_{\mathbb{R}^d}\left[\sum_i b_i(t,x)\partial_i\varphi(x)+\frac{1}{2}\sum_{ij}a_{ij}(t,x)\partial_{ij}\varphi(x)\right]d\mu_t(x)\quad \forall\varphi\in C_c^\infty(\mathbb{R}^d) \tag{3}$$

in the distributional sense on $[0,T]$, and the initial condition means that μ_t w^*-converges to $\bar\mu$ (i.e. converges in the duality with $C_c(\mathbb{R}^d)$) as $t\to 0$. We observe that, since Eq. (2) is in divergence form, it makes sense without any regularity assumption on a and b, provided that

$$\int_0^T\int_A\left(\left|b(t,x)\right|+\left|a(t,x)\right|\right)d|\mu_t|(x)\,dt<+\infty\quad \forall A\Subset\mathbb{R}^d$$

(here and in the sequel, $|\mu_t|$ denotes the total variation of μ_t). Since b and a will always be assumed to be bounded, in the definition of measure-valued solution of the PDE we assume that

$$\int_0^T|\mu_t|(A)\,dt<+\infty\quad \forall A\Subset\mathbb{R}^d, \tag{4}$$

so that (2) surely makes sense. However, if μ_t is singular with respect to the Lebesgue measure $\mathcal{L}^d$, then the products $b(t,\cdot)\mu_t$ and $a(t,\cdot)\mu_t$ are sensitive to modification of $b(t,\cdot)$ and $a(t,\cdot)$ in $\mathcal{L}^d$-negligible sets. Since in the case of singular measures the coefficients a and b will be assumed to be continuous, while in the case of coefficients in L^∞ the measures will be assumed to be absolutely continuous, (2) will always make sense.

Recall also that it is not restrictive to consider only solutions $t\mapsto\mu_t$ of the Fokker–Planck equation that are w^*-continuous on $[0,T]$, i.e. continuous in the duality with $C_c(\mathbb{R}^d)$ (see Lemma 2.1). Thus, we can assume that μ_t is defined for all t and even at the endpoints of $[0,T]$.

For simplicity of notation, we define

$$L_t:=\sum_i b_i(t,\cdot)\partial_i+\frac{1}{2}\sum_{ij}a_{ij}(t,\cdot)\partial_{ij}.$$

In this way the PDE can be written as

$$\partial_t\mu_t=L_t^*\mu_t\quad \text{in } [0,T]\times\mathbb{R}^d,$$

where L_t^* denotes the (formal) adjoint of L_t in $L^2(\mathbb{R}^d)$. Using Itô's formula it is simple to check that, if $X(t,x,\omega)\in L^2(\Omega,C([0,T],\mathbb{R}^d))$ is a family of solutions of (1), measurable in (t,x,ω), then the measure μ_t defined by

$$\int_{\mathbb{R}^d}f(x)\,d\mu_t(x):=\int_{\mathbb{R}^d}\mathbb{E}\left[f\left(X(t,x,\omega)\right)\right]d\bar\mu(x)\quad \forall f\in C_c\left(\mathbb{R}^d\right)$$

is a solution of (2) with $\mu_0=\bar\mu$ (see also Lemma 2.4).

We define $\Gamma_T := C([0,T],\mathbb{R}^d)$, and $e_t : \Gamma_T \to \mathbb{R}^d$, $e_t(\gamma) := \gamma(t)$. Let us recall the Stroock–Varadhan definition of martingale solutions.

Definition 1.1. A measure $\nu_{x,s}$ on Γ_T is a martingale solution of (1) starting from x at time s if:

(i) $\nu_{x,s}(\{\gamma \mid \gamma(s) = x\}) = 1$;
(ii) for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, the stochastic process on Γ_T

$$\varphi\big(\gamma(t)\big) - \int_s^t L_u\varphi\big(\gamma(u)\big)\,du$$

is a $\nu_{x,s}$-martingale after time s with respect to the canonical filtration.

We will say that the martingale problem is well-posed if, for any $(s,x) \in \mathbb{R}^d$, we have existence and uniqueness of martingale solutions.

In the sequel, we will deal with families $\{\nu_x\}_{x\in\mathbb{R}^d}$ of probability measures that are measurable with respect to x according to the following standard definition.

Definition 1.2. We say that a family of probability measures on a probability space $(\Omega,\mathcal{A})$ $\{\nu_x\}_{x\in\mathbb{R}^d}$ is measurable if, for any $A \in \mathcal{A}$, the real-valued map $x \mapsto \nu_x(A)$ is measurable.

1.1. Plan of the paper

1.1.1. The theory of Stochastic Lagrangian Flows

In the first part of the paper, we develop a general theory (independent of specific regularity or ellipticity assumptions), which roughly speaking allows to deduce existence, uniqueness and stability of martingale solutions for $\mathcal{L}^d$-almost every initial condition x whenever existence and uniqueness is known at the PDE level in the L^∞-setting (and, conversely, if existence and uniqueness of martingale solutions is known for $\mathcal{L}^d$-a.e. initial condition, then existence and uniqueness for the PDE in the L^∞-setting holds).

More precisely, in Section 2 we study how uniqueness of the SDE is related to that of the PDE. In Section 2.1 we prove a representation formula for solutions of the PDE, which shows that they can always be seen as a superposition of solutions of the SDE also when standard existence results for martingale solutions of SDE do not apply. In particular, assuming only the boundedness of the coefficients, we will show that, whenever we have existence of a solution of the PDE starting from μ_0, there exists at least one martingale solution of the SDE for μ_0-a.e. initial condition x.

In Section 3 we introduce the main object of our study, what we call Stochastic Lagrangian Flow. In Section 3.1 we state and prove our main result regarding the existence and uniqueness of Stochastic Lagrangian Flows, showing that these flows exist and are unique whenever the PDE is well-posed in the L^∞-setting. We also prove a stability result, and we show that Stochastic Lagrangian Flows satisfy the Chapman–Kolmogorov equation. Moreover, in Section 3.2 we investigate the relation between our result and its deterministic counterpart and, applying our stability result, we deduce a vanishing viscosity theorem for Ambrosio's Regular Lagrangian Flows.

1.1.2. The Fokker–Planck equation

In the second part of the paper we study by purely PDE methods the well-posedness of the Fokker–Planck equation in two extreme (with respect to the regularity imposed in time, or in space) situations: in the first one, assuming uniform ellipticity of the coefficients and Lipschitz regularity in time, we are able to prove existence and uniqueness in the L^2-settings assuming no regularity in space, but only suitable divergence bounds (see Theorem 4.3). This result, together with Proposition 4.4, directly implies the following theorem (here and in the sequel, $\mathcal{S}_+(\mathbb{R}^d)$ denotes the set of symmetric and non-negative definite $d \times d$ matrices).

Theorem 1.3. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i = 1, \dots, d$;
(ii) $\partial_t a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i, j = 1, \dots, d$;
(iii) $(\sum_i \partial_i b_i - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij})^- \in L^\infty([0,T] \times \mathbb{R}^d)$;
(iv) $\langle \xi, a(t,x)\xi \rangle \geqslant \alpha |\xi|^2$ $\forall (t,x) \in [0,T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(v) $\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d)$, $\frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d)$.

Then there exists a unique solution of (2) *in* $\mathscr{L}_+$, *where*

$$\mathscr{L}_+ := \left\{ u \in L^\infty\big([0,T], L^1_+\big(\mathbb{R}^d\big)\big) \cap L^\infty\big([0,T], L^\infty_+\big(\mathbb{R}^d\big)\big) \;\big|\; u \in C\big([0,T], w^*\text{-}L^\infty\big(\mathbb{R}^d\big)\big) \right\},$$

and L^1_+ *and* L^∞_+ *denote the convex subsets of* L^1 *and* L^∞ *consisting of non-negative functions.*

In the second case, a does not depend on the space variables, but it can be degenerate and it is allowed to depend on t even in a measurable way. Since a can also be identically 0, we need to assume BV regularity on the vector field b, and so we can prove:

Theorem 1.4. *Let us assume that* $a : [0,T] \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $(\sum_i \partial_i b_i)^- \in L^1([0,T], L^\infty(\mathbb{R}^d))$.

Then there exists a unique solution of (2) *in* $\mathscr{L}_+$.

This theorem is a direct consequence of Theorem 4.12. Other existence and uniqueness results for the Fokker–Planck equation, which are in some sense intermediate with respect the two extreme ones stated above, have been proved in a recent paper of LeBris and P.-L. Lions [14]. As in our case, in that paper the authors are interested in the well-posedness of the Fokker–Planck equation as a tool to deduce existence and uniqueness results at the SDE level (see also [15]). In particular, in [14, Section 4] the authors give a list of interesting situations in the modelization of polymeric fluids when SDEs with irregular drift b and dispersion matrix σ arise (see also [12] and the references therein for other existence and uniqueness results for non-smooth SDEs).

1.1.3. Conclusions and appendix

In Section 5 we apply the theory developed in Section 3.1 to obtain, in the cases considered above, the generic well-posedness of the associated SDE.

Finally, in Appendix A we generalize an important uniqueness result of Stroock and Varadhan (see Theorem 2.2 and the remarks at the end of Theorem 5.4).

2. SDE–PDE uniqueness

In this section we study the main relations between the SDE and the PDE. The main result is a general representation formula for solutions of the PDE (Theorem 2.6) which allows to relate uniqueness of the SDE to that of the PDE (Lemma 2.3).

As we already said in the introduction, here and in the sequel b and a are always assumed to be bounded. Let us recall the following result on the time regularity of $t \mapsto \mu_t$ (see for example [2, Remark 3] or [4, Lemma 8.1.2]):

Lemma 2.1. *Up to modification of μ_t in a negligible set of times, $t \mapsto \mu_t$ is w^*-continuous on $[0, T]$. Moreover, if $|\mu_t|(\mathbb{R}^d) \leqslant C$ for any $t \in [0, T]$, then $t \mapsto \mu_t$ is narrowly continuous.*

We also recall the following important theorem of Stroock and Varadhan (for a proof, see [18, Theorem 6.2.3]).

Theorem 2.2. *Assume that for any $(s, x) \in [0, T] \times \mathbb{R}^d$, for any $\nu_{x,s}$ and $\tilde{\nu}_{x,s}$ martingale solutions of* (1) *starting from x at time s, one has*

$$(e_t)_\# \nu_{x,s} = (e_t)_\# \tilde{\nu}_{x,s} \quad \forall t \in [s, T].$$

Then the martingale solution of (1) *starting from any $(s, x) \in [0, T] \times \mathbb{R}^d$ is unique.*

We start studying how the uniqueness of (1) is related to that of (2).

Lemma 2.3. *Let $A \subset \mathbb{R}^d$ be a Borel set. The following two properties are equivalent*:

(a) *Time-marginals of martingale solutions of the SDE are unique for any $x \in A$.*
(b) *Finite non-negative measure-valued solutions of the PDE are unique for any non-negative Radon measure μ_0 concentrated in A.*

Proof. (b) $\Rightarrow$ (a). Let us choose $\mu_0 = \delta_x$, with $x \in A$. Then, if ν_x and $\tilde{\nu}_x$ are two martingale solutions of the SDE, we get that $\mu_t := (e_t)_\# \nu_x$ and $\tilde{\mu}_t := (e_t)_\# \tilde{\nu}_x$ are two solutions of the PDE with $\mu_0 = \delta_x$ (see Lemma 2.4). This implies that $\mu_t = \tilde{\mu}_t$, that is

$$\langle \mu_t, \varphi \rangle = \int_{\Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma) = \int_{\Gamma_T} \varphi\big(\gamma(t)\big)\, d\tilde{\nu}_x(\gamma) = \langle \tilde{\mu}_t, \varphi \rangle \quad \forall \varphi \in C_c^\infty\big(\mathbb{R}^d\big),$$

that is $(e_t)_\# \nu_x = (e_t)_\# \tilde{\nu}_x$ (observe in particular that, if $A = \mathbb{R}^d$ and we have uniqueness for the PDE for any initial time $s \geqslant 0$, by Theorem 2.2 we get that $\nu_x = \tilde{\nu}_x$ for any $x \in \mathbb{R}^d$).

(a) $\Rightarrow$ (b). This implication follows by Theorem 2.6, which provides, for every finite non-negative measure-valued solutions of the PDE, the representation

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x), \tag{5}$$

where, for μ_0-a.e. x, ν_x is a martingale solution of SDE starting from x (at time 0). Therefore, by the uniqueness of $(e_t)_\#\nu_x$, we obtain that solutions of the PDE are unique. □

We now prove that, if ν_x is a martingale solution of the SDE starting from x (at time 0) for μ_0-a.e. x, the right-hand side of (5) always defines a non-negative solution of the PDE. We recall that a locally finite measure is a possibly signed measure with locally finite total variation.

Lemma 2.4. *Let μ_0 be a locally finite measure on $\mathbb{R}^d$, and let $\{\nu_x\}_{x\in\mathbb{R}^d}$ be a measurable family of probability measures on Γ_T such that ν_x is a martingale solution of the SDE starting from x (at time 0) for $|\mu_0|$-a.e. x. Define on Γ_T the measure $\nu := \int_{\mathbb{R}^d} \nu_x\, d\mu_0(x)$, and assume that*

$$\int_0^T \int_{\mathbb{R}^d \times \Gamma_T} \chi_{B_R}\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d|\mu_0|(x)\, dt < +\infty \quad \forall R > 0 \tag{6}$$

(this property is trivially true if, for example, $|\mu_0|(\mathbb{R}^d) < +\infty$). Then the measure μ_t^ν on $\mathbb{R}^d$ defined by

$$\big\langle \mu_t^\nu, \varphi\big\rangle := \big\langle (e_t)_\#\nu, \varphi\big\rangle = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x) \quad \forall \varphi \in C_c^\infty\big(\mathbb{R}^d\big)$$

is a solution of the PDE.

Proof. Let us first show that the map $t \mapsto \langle \mu_t^\nu, \varphi\rangle$ is absolutely continuous for any $\varphi \in C_c^\infty(\mathbb{R}^d)$. We recall that a real-valued map $t \mapsto f(t)$ is said absolutely continuous if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, given any family of disjoint intervals $(s_k, t_k) \subset [0, T]$, the following implication holds:

$$\sum_k |t_k - s_k| \leqslant \delta \quad \Rightarrow \quad \sum_k \big|f(t_k) - f(s_k)\big| \leqslant \varepsilon.$$

Take $R > 0$ such that $\operatorname{supp}(\varphi) \subset B_R$, and let $I = \bigcup_{k=1}^n (s_k, t_k)$ be a subset of $[0, T]$ with (s_k, t_k) disjoint and such that $|t_k - s_k| \leqslant 1$. For μ_0-a.e. x, by the definition of martingale solution we have

$$\int_{\Gamma_T} \varphi\big(\gamma(t_k)\big)\, d\nu_x(\gamma) - \int_{\Gamma_T} \varphi\big(\gamma(s_k)\big)\, d\nu_x(\gamma)$$

$$= \int_{s_k}^{t_k} \int_{\Gamma_T} L_t\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, dt$$

$$= \int_{s_k}^{t_k} \int_{\Gamma_T} \sum_i b_i\big(t,\gamma(t)\big)\partial_i\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, dt + \frac{1}{2}\int_{s_k}^{t_k} \int_{\Gamma_T} \sum_{ij} a_{ij}\big(t,\gamma(t)\big)\partial_{ij}\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, dt$$

and so, integrating with respect to μ_0, we obtain

$$\big|\langle\mu^\nu_{t_k},\varphi\rangle - \langle\mu^\nu_{s_k},\varphi\rangle\big| \leqslant \|\varphi\|_{C^2}\bigg[\|b\|_\infty + \frac{1}{2}\|a\|_\infty\bigg]\int_{s_k}^{t_k} \int_{\mathbb{R}^d\times\Gamma_T} \chi_{B_R}\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d|\mu_0|(x)\, dt.$$

Thus

$$\sum_{k=1}^n \big|\langle\mu^\nu_{t_k},\varphi\rangle - \langle\mu^\nu_{s_k},\varphi\rangle\big| \leqslant \|\varphi\|_{C^2}\bigg[\|b\|_\infty + \frac{1}{2}\|a\|_\infty\bigg]\sum_{k=1}^n\int_{s_k}^{t_k} \int_{\mathbb{R}^d\times\Gamma_T} \chi_{B_R}\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d|\mu_0|(x)\, dt,$$

which shows that the map $t \mapsto \langle\mu^\nu_t,\varphi\rangle$ is absolutely continuous thanks to (6) and the absolute continuity property of the integral. So, in order to conclude that μ^ν_t solves the PDE, it suffices to compute the time derivative of $t \mapsto \langle\mu^\nu_t,\varphi\rangle$, and, by the computation we made above, one simply gets

$$\frac{d}{dt}\langle\mu^\nu_t,\varphi\rangle = \int_{\mathbb{R}^d} \frac{d}{dt}\bigg(\int_{\Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\bigg)\, d\mu_0(x)$$

$$= \int_{\mathbb{R}^d}\int_{\Gamma_T} L_t\varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x) = \langle\mu^\nu_t, L_t\varphi\rangle. \qquad \square$$

Remark 2.5. We observe that, by the definition of μ^ν_t, the following implications hold:

(i) $\mu_0 \geqslant 0 \Rightarrow \forall t \geqslant 0$, $\mu^\nu_t \geqslant 0$ and $\mu^\nu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$ (the total mass can also be infinite);
(ii) μ_0 signed $\Rightarrow \forall t \geqslant 0$, $|\mu^\nu_t|(\mathbb{R}^d) \leqslant |\mu_0|(\mathbb{R}^d)$ (the total variation can also be infinite).

2.1. A representation formula for solutions of the PDE

We denote by $\mathcal{M}_+(\mathbb{R}^d)$ the set of non-negative finite measures on $\mathbb{R}^d$.

Theorem 2.6. *Let μ_t be a solution of the PDE such that $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ for any $t \in [0,T]$, with $\mu_t(\mathbb{R}^d) \leqslant C$ for any $t \in [0,T]$. Then there exists a measurable family of probability measures $\{\nu_x\}_{x\in\mathbb{R}^d}$ such that ν_x is a martingale solution of* (1) *starting from x* (*at time* 0) *for μ_0-a.e. x, and the following representation formula holds*:

$$\int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x(\gamma)\, d\mu_0(x). \tag{7}$$

By this theorem it follows that, whenever we have existence of a solution of the PDE starting from μ_0, there exists a martingale solution of the SDE for μ_0-a.e. initial condition x.

Proof. Up to a renormalization of μ_0, we can assume that $\mu_0(\mathbb{R}^d) = 1$.

Step 1 (*Smoothing*). Let $\rho : \mathbb{R}^d \to (0, +\infty)$ be a convolution kernel such that $|D^k\rho(x)| \leqslant C_k|\rho(x)|$ for any $k \geqslant 1$ ($\rho(x) = Ce^{-\sqrt{1+|x|^2}}$, for instance). We consider the measures $\mu_t^\varepsilon := \mu_t * \rho_\varepsilon$. They are smooth solutions of the PDE

$$\partial_t \mu_t^\varepsilon + \sum_i \partial_i\big(b_i^\varepsilon \mu_t^\varepsilon\big) - \frac{1}{2}\sum_{ij}\partial_{ij}\big(a_{ij}^\varepsilon \mu_t^\varepsilon\big) = 0, \tag{8}$$

where

$$b_t^\varepsilon = b^\varepsilon(t,\cdot) := \frac{(b(t,\cdot)\mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}, \qquad a_t^\varepsilon = a^\varepsilon(t,\cdot) := \frac{(a(t,\cdot)\mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}.$$

Then it is immediate to see that

$$\big\|b_t^\varepsilon\big\|_\infty \leqslant \|b_t\|_\infty, \qquad \big\|a_t^\varepsilon\big\|_\infty \leqslant \|a_t\|_\infty. \tag{9}$$

Since $|D^k\rho(x)| \leqslant C_k|\rho(x)|$, it is simple to check that b^ε and a^ε are smooth and bounded together with all their spatial derivatives. By [18, Corollary 6.3.3], the martingale problem for a^ε and b^ε is well-posed (see Definition 1.1) and the family $\{\nu_x^\varepsilon\}_{x\in\mathbb{R}^d}$ of martingale solutions (starting at time 0) is measurable (see Definition 1.2). By (9) we can apply Lemma 2.4, which tells us that $\tilde\mu_t^\varepsilon := (e_t)_\# \int_{\mathbb{R}^d} \nu_x^\varepsilon \, d\mu_0^\varepsilon(x)$ is a finite measure which solves the smoothed PDE (8) with initial datum μ_0^ε. Then, since the solution of (8) is unique (Proposition 4.1), we obtain $\tilde\mu_t^\varepsilon = \mu_t^\varepsilon$, that is

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t^\varepsilon = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big)\, d\nu_x^\varepsilon(\gamma)\, d\mu_0^\varepsilon(x). \tag{10}$$

Step 2 (*Tightness*). It is clear that the measures $\mu_0^\varepsilon = \mu_0 * \rho_\varepsilon$ are tight. So, if we define $\nu^\varepsilon := \int_{\mathbb{R}^d} \nu_x^\varepsilon \, d\mu_0^\varepsilon$, we have

$$\lim_{R\to\infty} \sup_{0<\varepsilon<1} \nu^\varepsilon\big(\big\{\big|\gamma(0)\big| > R\big\}\big) = 0.$$

For any $\varphi \in C_c^\infty(\mathbb{R}^d)$, let us define $A_\varphi := \|\varphi\|_{C^2}[\|b\|_\infty + \frac{1}{2}\|a\|_\infty]$. Since for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and any $0 < \varepsilon < 1$

$$\varphi\big(\gamma(t)\big) - \int_0^t \bigg(\sum_i b_i^\varepsilon\big(u,\gamma(u)\big)\partial_i\varphi\big(\gamma(u)\big) + \frac{1}{2}\sum_{ij} a_{ij}^\varepsilon\big(u,\gamma(u)\big)\partial_{ij}\varphi\big(\gamma(u)\big)\bigg)\, du$$

is a ν^ε-martingale with respect to the canonical filtration, by (9) we obtain that $\varphi(\gamma(t)) + A_\varphi t$ is a ν^ε-submartingale with respect to the canonical filtration. Thus [18, Theorem 1.4.6] can be applied, and the tightness of ν^ε follows.

Let ν be any limit point of ν^ε, and consider the disintegration of ν with respect to $\mu_0 = (e_0)_\# \nu$, i.e. $\nu = \int_{\mathbb{R}^d} \nu_x \, d\mu_0(x)$. Passing to the limit in (10), we get

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t(x) = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(\gamma(t)\big) \, d\nu_x(\gamma) \, d\mu_0(x).$$

Step 3 (ν_x *is a martingale solution of the SDE for* μ_0*-a.e.* x). Let $\varepsilon_n \to 0$ be a sequence such that ν is the weak limit of ν^{ε_n}. Let us fix a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ with $0 \leqslant f \leqslant 1$, $s \in [0, T]$, and an $\mathcal{F}_s$-measurable continuous function $\Phi^s : \Gamma_T \to \mathbb{R}$ with $0 \leqslant \Phi^s \leqslant 1$, where $(\mathcal{F}_s)_{0 \leqslant s \leqslant T}$ denotes the canonical filtration on Γ_T. We define

$$L_t^n := \sum_i b_i^{\varepsilon_n}(t, \cdot)\partial_i + \frac{1}{2} \sum_{ij} a_{ij}^{\varepsilon_n}(t, \cdot)\partial_{ij}.$$

Since each $\nu_x^{\varepsilon_n}$ is a martingale solution, we know that for any $t \in [s, T]$ and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \int_0^t L_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x) \\
&\quad = \int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(s)\big) - \int_0^s L_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x)
\end{aligned}$$

(see Definition 1.1), or equivalently

$$\int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \varphi\big(\gamma(s)\big) - \int_s^t L_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x) = 0.$$

Let us take $\tilde{b} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{a} : [0, T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ bounded and continuous, and define

$$\tilde{L}_t := \sum_i \tilde{b}_i(t, \cdot)\partial_i + \frac{1}{2} \sum_{ij} \tilde{a}_{ij}(t, \cdot)\partial_{ij},$$

$$\tilde{L}_t^n := \sum_i \tilde{b}_i^{\varepsilon_n}(t, \cdot)\partial_i + \frac{1}{2} \sum_{ij} \tilde{a}_{ij}^{\varepsilon_n}(t, \cdot)\partial_{ij},$$

where $\tilde{b}_i^{\varepsilon_n}$ and $\tilde{a}_{ij}^{\varepsilon_n}$ are defined analogously to $b_i^{\varepsilon_n}$ and $a_{ij}^{\varepsilon_n}$. Thus we can write

$$\int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \varphi\big(\gamma(s)\big) - \int_s^t \tilde{L}_u^n \varphi\big(\gamma(u)\big) \, du \right] \Phi^s(\gamma) \, d\nu_x^{\varepsilon_n}(\gamma) f(x) \, d\mu_0^{\varepsilon_n}(x)$$

$$= \int_{\mathbb{R}^d \times \Gamma_T} \left[\int_s^t \big(L_u^n - \tilde{L}_u^n\big)\varphi\big(\gamma(u)\big)\,du \right] \Phi^s(\gamma)\,d\nu_x^{\varepsilon_n}(\gamma)\,f(x)\,d\mu_0^{\varepsilon_n}(x).$$

Then, recalling that $0 \leqslant f \leqslant 1$ and $0 \leqslant \Phi^s \leqslant 1$, we get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \varphi\big(\gamma(s)\big) - \int_s^t \tilde{L}_u^n \varphi\big(\gamma(u)\big)\,du \right] \Phi^s(\gamma)\,d\nu_x^{\varepsilon_n}(\gamma)\,f(x)\,d\mu_0^{\varepsilon_n}(x) \right| \\
&\quad \leqslant \int_{\mathbb{R}^d \times \Gamma_T} \left[\int_s^t \big|\big(L_u^n - \tilde{L}_u^n\big)\varphi\big(\gamma(u)\big)\big|\,du \right] \Phi^s(\gamma)\,d\nu_x^{\varepsilon_n}(\gamma)\,f(x)\,d\mu_0^{\varepsilon_n}(x) \\
&\quad \leqslant \int_{\mathbb{R}^d \times \Gamma_T} \left[\int_s^t \big|\big(L_u^n - \tilde{L}_u^n\big)\varphi\big(\gamma(u)\big)\big|\,du \right] d\nu_x^{\varepsilon_n}(\gamma)\,d\mu_0^{\varepsilon_n}(x) \\
&\quad = \int_s^t \int_{\mathbb{R}^d} \big|\big(L_u^n - \tilde{L}_u^n\big)\varphi(x)\big|\,d\mu_u^{\varepsilon_n}(x)\,du \\
&\quad \leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \left| \left(\frac{(b_i(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} - \frac{(\tilde{b}_i(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} \right) \partial_i \varphi \right|(x)\,d\mu_u^{\varepsilon_n}(x)\,du \\
&\qquad + \frac{1}{2} \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \left| \left(\frac{(a_{ij}(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} - \frac{(\tilde{a}_{ij}(u,\cdot)\mu_u) * \rho_{\varepsilon_n}}{\mu_u^{\varepsilon_n}} \right) \partial_{ij} \varphi \right|(x)\,d\mu_u^{\varepsilon_n}(x)\,du \\
&\quad \leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \big|b_i(u,\cdot) - \tilde{b}_i(u,\cdot)\big|(x)\partial_i\varphi * \rho_{\varepsilon_n}(x)\,d\mu_u(x)\,du \\
&\qquad + \frac{1}{2} \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \big|a_{ij}(u,\cdot) - \tilde{a}_{ij}(u,\cdot)\big|(x)\partial_{ij}\varphi * \rho_{\varepsilon_n}(x)\,d\mu_u(x)\,du.
\end{aligned}$$

Since $\tilde{a}$ and $\tilde{b}$ are continuous, $\tilde{a}^{\varepsilon_n}$ and $\tilde{b}^{\varepsilon_n}$ converge to $\tilde{a}$ and $\tilde{b}$ locally uniformly. So we can pass to the limit in the above equation as $n \to \infty$, obtaining

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d \times \Gamma_T} \left[\varphi\big(\gamma(t)\big) - \varphi\big(\gamma(s)\big) - \int_s^t \tilde{L}_u \varphi\big(\gamma(u)\big)\,du \right] \Phi^s(\gamma)\,d\nu_x(\gamma)\,f(x)\,d\mu_0(x) \right| \\
&\quad \leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \big|b_i(u,x) - \tilde{b}_i(u,x)\big|\partial_i\varphi(x)\,d\mu_u(x)\,du
\end{aligned}$$

$$+\frac{1}{2}\sum_{ij}\int_s^t\int_{\mathbb{R}^d}\left|a_{ij}(u,x)-\tilde{a}_{ij}(u,x)\right|\partial_{ij}\varphi(x)\,d\mu_u(x)\,du.$$

Choosing two sequences of continuous functions $(\tilde{b}^k)_{k\in\mathbb{N}}$ and $(\tilde{a}^k)_{k\in\mathbb{N}}$ converging respectively to b and a in $L^1([0,T]\times\mathbb{R}^d,\eta)$, with $\eta := \int_0^T \mu_t\,dt$, we finally obtain

$$\int_{\mathbb{R}^d\times\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\varphi\big(\gamma(s)\big)-\int_s^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\,f(x)\,d\mu_0(x)=0,$$

that is

$$\int_{\mathbb{R}^d\times\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\int_0^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\,f(x)\,d\mu_0(x)$$
$$=\int_{\mathbb{R}^d\times\Gamma_T}\left[\varphi\big(\gamma(s)\big)-\int_0^s L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\,f(x)\,d\mu_0(x).$$

By the arbitrariness of f we get that, for any $0\leqslant s\leqslant t\leqslant T$, and for any $\mathcal{F}_s$-measurable function Φ^s, we have

$$\int_{\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\int_0^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$
$$=\int_{\Gamma_T}\left[\varphi\big(\gamma(s)\big)-\int_0^s L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)\quad\text{for } \mu_0\text{-a.e. } x.$$

Letting Φ^s vary in a dense countable subset of $\mathcal{F}_s$-measurable functions, by approximations we deduce that, for any $0\leqslant s\leqslant t\leqslant T$, for μ_0-a.e. x,

$$\int_{\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\int_0^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$
$$=\int_{\Gamma_T}\left[\varphi\big(\gamma(s)\big)-\int_0^s L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$

for any $\mathcal{F}_s$-measurable function Φ^s (here the μ_0-a.e. depends on s and t but not on Φ^s). Taking now $s,t\in[0,T]\cap\mathbb{Q}$, we deduce that, for μ_0-a.e. x,

$$\int_{\Gamma_T}\left[\varphi\big(\gamma(t)\big)-\int_0^t L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$
$$=\int_{\Gamma_T}\left[\varphi\big(\gamma(s)\big)-\int_0^s L_u\varphi\big(\gamma(u)\big)\,du\right]\Phi^s(\gamma)\,d\nu_x(\gamma)$$

for any $s, t \in [0, T] \cap \mathbb{Q}$, for any $\mathcal{F}_s$-measurable function Φ^s. By the continuity of the above equality with respect to both s and t, and the continuity in time of the filtration $\mathcal{F}_s$, we conclude that ν_x is a martingale solution for μ_0-a.e. x. □

Remark 2.7. We observe that by (7) it follows that

$$\mu_t\big(\mathbb{R}^d\big) \leqslant C \quad \forall t \quad \Rightarrow \quad \mu_t\big(\mathbb{R}^d\big) = \mu_0\big(\mathbb{R}^d\big)$$

(this result can also be proved more directly using as test functions in (2) a suitable sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$, with $0 \leqslant \varphi_n \leqslant 1$ and $\varphi_n \nearrow 1$, and, even in the case when the measures μ_t are signed, under the assumption $|\mu_t|(\mathbb{R}^d) \leqslant C$ one obtains the constancy of the map $t \mapsto \mu_t(\mathbb{R}^d)$).

3. Stochastic Lagrangian Flows

In this section we want to prove an existence and uniqueness result for martingale solutions which satisfy certain properties, in the spirit of the Regular Lagrangian Flows (RLF) introduced in [1].

Definition 3.1. Given a measure $\mu_0 = \rho_0\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^\infty(\mathbb{R}^d)$, we say that a measurable family of probability measures $\{\nu_x\}_{x\in\mathbb{R}^d}$ on Γ_T is a μ_0-*Stochastic Lagrangian Flow* (μ_0-SLF) (starting at time 0), if:

(i) for μ_0-a.e. x, ν_x is a martingale solution of the SDE starting from x (at time 0);
(ii) for any $t \in [0, T]$

$$\mu_t := (e_t)_\#\left(\int \nu_x\,d\mu_0(x)\right) \ll \mathcal{L}^d,$$

and, denoting $\mu_t = \rho_t\mathcal{L}^d$, we have $\rho_t \in L^\infty(\mathbb{R}^d)$ uniformly in t.

More generally, one can analogously define a μ_0-SLF starting at time s with $s \in (0, T)$ requiring that ν_x is a martingale solution of the SDE starting from x at time s.

Remark 3.2. If $\{\nu_x\}_{x\in\mathbb{R}^d}$ is a μ_0-SLF, then it is also a μ_0'-SLF for any $\mu_0' \in \mathcal{M}_+(\mathbb{R}^d)$ with $\mu_0' \leqslant C\mu_0$. Indeed, this easily follows by the inequality

$$0 \leqslant (e_t)_\# \int_{\mathbb{R}^d} \tilde{\nu}_x\,d\mu_0'(x) \leqslant C(e_t)_\# \int_{\mathbb{R}^d} \tilde{\nu}_x\,d\mu_0(x).$$

3.1. Existence, uniqueness and stability of SLF

We denote by L^1_+ and L^∞_+ the convex subsets of L^1 and L^∞ consisting of non-negative functions, and, following [1], we define

$$\mathscr{L} := \{u \in L^\infty([0,T], L^1(\mathbb{R}^d)) \cap L^\infty([0,T], L^\infty(\mathbb{R}^d)) \mid u \in C([0,T], w^*\text{-}L^\infty(\mathbb{R}^d))\},$$

and

$$\mathscr{L}_+ := \{u \in L^\infty([0,T], L^1_+(\mathbb{R}^d)) \cap L^\infty([0,T], L^\infty_+(\mathbb{R}^d)) \mid u \in C([0,T], w^*\text{-}L^\infty(\mathbb{R}^d))\}.$$

Under an existence and uniqueness result for the PDE in the class $\mathscr{L}_+$, we prove existence and uniqueness of SLF.

Theorem 3.3 *(Existence of SLF starting from a fixed measure). Let us suppose that, for some initial datum $\mu_0 = \rho_0 \mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^\infty(\mathbb{R}^d)$, there exists a solution of the PDE in $\mathscr{L}_+$. Then there exists a μ_0-SLF.*

Proof. It suffices to apply Theorem 2.6 to the solution of the PDE in $\mathscr{L}_+$. □

Let us assume now that forward uniqueness for the PDE holds in the class $\mathscr{L}_+$ for any initial time, that is, for any $s \in [0,T]$, for any $\rho_s \in L^1_+(\mathbb{R}^d) \cap L^\infty_+(\mathbb{R}^d)$, if we denote by $\rho_t \mathcal{L}^d$ and $\tilde{\rho}_t \mathcal{L}^d$ two solutions of the PDE in the class $\mathscr{L}_+$ starting from $\rho_s \mathcal{L}^d$ at time s, then

$$\rho_t = \tilde{\rho}_t \quad \text{for any } t \in [s,T].$$

Before stating and proving our main theorem, we first introduce some notation that will be used also in Appendix A.

Let $\mathcal{B}$ be the Borel σ-algebra on $\Gamma_T = C([0,T], \mathbb{R}^d)$, and define the filtrations $\mathcal{F}_t := \sigma[e_s \mid 0 \leqslant s \leqslant t]$ and $\mathcal{F}^t := \sigma[e_s \mid t \leqslant s \leqslant T]$. Set $\mathcal{P}(\Gamma_T)$ the set of probability measures on Γ_T. Now, given $\nu \in \mathcal{P}(\Gamma_T)$, we denote by

$$\Gamma_T \ni \gamma \mapsto \nu^\gamma_{\mathcal{F}_t} \in \mathcal{P}(\Gamma_T)$$

a regular conditional probability distribution of ν given $\mathcal{F}_t$, that is a family of probability measures on $(\Gamma_T, \mathcal{B})$ indexed by γ such that:

- for each $B \in \mathcal{B}$, $\gamma \mapsto \nu^\gamma_{\mathcal{F}_t}(B)$ is $\mathcal{F}_t$-measurable;
- $$\nu(A \cap B) = \int_A \nu^\gamma_{\mathcal{F}_t}(B)\, d\nu(\gamma) \quad \forall A \in \mathcal{F}_t,\ \forall B \in \mathcal{B}. \tag{11}$$

Since Γ_T is a Polish space and every σ-algebra $\mathcal{F}_t$ is finitely generated, such a function exists and is unique, up to ν-null sets. In particular, up to changing this function in a ν-null set, the following fact holds:

$$\nu^\gamma_{\mathcal{F}_t}(\{\tilde{\gamma} \mid \tilde{\gamma}(s) = \gamma(s)\ \forall s \in [0,t]\}) = 1 \quad \forall \gamma \in \Gamma_T. \tag{12}$$

Finally, given $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T$, we set $M^{t_1,\dots,t_n} := \sigma[e_{t_1},\dots,e_{t_n}]$, and one can analogously define $\nu^{\gamma}_{M^{t_1,\dots,t_n}}$. For $\nu^{\gamma}_{M^{t_1,\dots,t_n}}$ an analogous of (12) holds:

$$\nu^{\gamma}_{M^{t_1,\dots,t_n}}\big(\big\{\tilde\gamma \;\big|\; \tilde\gamma(t_i)=\gamma(t_i)\ \forall i=1,\dots,n\big\}\big)=1 \quad \forall \gamma\in\Gamma_T. \tag{13}$$

If $\gamma(t_i)=x_i$ for $i=1,\dots,n$, then we will also use the notation $\nu^{\gamma}_{M^{t_1,\dots,t_n}} = \nu^{x_1,\dots,x_n}_{M^{t_1,\dots,t_n}}$.

By (11) one can check that $\int_{\Gamma_T} \nu^{\tilde\gamma}_{\mathcal{F}_{t_n}}\, d\nu^{\gamma}_{M^{t_1,\dots,t_n}}(\tilde\gamma)$ is a regular conditional probability distribution of ν given $M^{t_1,\dots,t_n}$, which implies by uniqueness that

$$\nu^{\gamma}_{M^{t_1,\dots,t_n}} = \int_{\Gamma_T} \nu^{\tilde\gamma}_{\mathcal{F}_{t_n}}\, d\nu^{\gamma}_{M^{t_1,\dots,t_n}}(\tilde\gamma) \quad \text{for } \nu\text{-a.e. } \gamma. \tag{14}$$

Theorem 3.4 *(Uniqueness of SLF starting from a fixed measure). Let us assume that forward uniqueness for the PDE holds in the class $\mathscr{L}_+$ for any initial time. Then, for any $\mu_0 = \rho_0\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^\infty(\mathbb{R}^d)$, the μ_0-SLF is uniquely determined μ_0-a.e. (in the sense that, if $\{\nu_x\}$ and $\{\tilde\nu_x\}$ are two μ_0-SLF, then $\nu_x = \tilde\nu_x$ for μ_0-a.e. x).*

Proof. Let $\{\nu_x\}$ and $\{\tilde\nu_x\}$ be two μ_0-SLF. Take now a function $\psi \in C_c(\mathbb{R}^d)$, with $\psi \geqslant 0$. By Remark 3.2, $\{\nu_x\}$ and $\{\tilde\nu_x\}$ are two $\psi\mu_0$-SLF. Thus, by Lemma 2.4 and the uniqueness of the PDE in $\mathscr{L}_+$, for any $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\begin{aligned}&\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_t(\gamma)\big)\, d\nu_x(\gamma)\psi(x)\, d\mu_0(x)\\ &\quad = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_t(\gamma)\big)\, d\tilde\nu_x(\gamma)\psi(x)\, d\mu_0(x) \quad \forall t\in[0,T].\end{aligned} \tag{15}$$

This clearly implies that, for any $t\in[0,T]$,

$$(e_t)_\#\nu_x = (e_t)_\#\tilde\nu_x \quad \text{for } \mu_0\text{-a.e. } x.$$

We now want to use an analogous argument to deduce that, for any $0 < t_1 < t_2 < \cdots < t_n \leqslant T$,

$$(e_{t_1},\dots,e_{t_n})_\#\nu_x = (e_{t_1},\dots,e_{t_n})_\#\tilde\nu_x \quad \text{for } \mu_0\text{-a.e. } x. \tag{16}$$

The idea is that, given a measure $\tilde\mu_s = \tilde\rho_s\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\tilde\rho_s \in L^\infty$, once we have a $\tilde\mu_s$-SLF starting at time s we can multiply $\tilde\mu_s$ by a function $\psi_s \in C_c(\mathbb{R}^d)$ with $\psi_s \geqslant 0$, and by Remark 3.2 our $\tilde\mu_s$-SLF is also a $\psi_s\tilde\mu_s$-SLF starting at time s. Using this argument n times at different times and the time marginals uniqueness, we will obtain (16).

Fix $0 < t_1 < \cdots < t_n \leqslant T$. Take $\psi_0 \geqslant 0$ with $\psi_0 \in C_c(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}\psi_0\, d\mu_0 = 1$, and denote by $\mu^{\psi_0}_{t_1}$ the value at time t_1 of the (unique) solution in $\mathscr{L}_+$ of the PDE starting from $\psi_0\mu_0$ (which is induced both by $\{\nu_x\}$ and $\{\tilde\nu_x\}$ by uniqueness, see Eq. (15)). Let $\{\nu_{x,t_1}\}_{x\in\mathbb{R}^d}$ and $\{\tilde\nu_{x,t_1}\}_{x\in\mathbb{R}^d}$ be the families of probability measures on Γ_T given by the disintegration of

$$\nu^{\psi_0} := \int_{\mathbb{R}^d} \nu_x\psi_0(x)\, d\mu_0(x) \quad \text{and} \quad \tilde\nu^{\psi_0} := \int_{\mathbb{R}^d} \tilde\nu_x\psi_0(x)\, d\mu_0(x)$$

with respect to $\mu_{t_1}^{\psi_0} = (e_{t_1})_\# \nu^{\psi_0} = (e_{t_1})_\# \tilde{\nu}^{\psi_0}$, that is

$$\nu^{\psi_0} = \int_{\mathbb{R}^d} \nu_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x), \qquad \tilde{\nu}^{\psi_0} = \int_{\mathbb{R}^d} \tilde{\nu}_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x). \tag{17}$$

It is easily seen that $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are regular conditional probability distributions, given $M^{t_1} = \sigma[e_{t_1}]$, of ν^{ψ_0} and $\tilde{\nu}^{\psi_0}$, respectively (that is, with the notation introduced before, $\nu_{x,t_1} = (\nu^{\psi_0})^x_{M_{t_1}}$ and $\tilde{\nu}_{x,t_1} = (\tilde{\nu}^{\psi_0})^x_{M_{t_1}}$). Thus, looking at $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ as their restriction to $C([t_1,T],\mathbb{R}^d)$, $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are $\mu_{t_1}^{\psi_0}$-SLF starting at time t_1. Indeed, by the stability of martingale solutions with respect to regular conditional probability (see [18, Chapter 6]), $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are martingale solutions of the SDE starting from x at time t_1 for $\mu_{t_1}^{\psi_0}$-a.e. x (see also the remarks at the end of the proof of Proposition A.1), while (ii) of Definition 3.1 is trivially true since $\{\nu_x\}$ and $\{\tilde{\nu}_x\}$ are $\psi_0\mu_0$-SLF. As before, since $\{\nu_{x,t_1}\}$ and $\{\tilde{\nu}_{x,t_1}\}$ are also $\psi_1\mu_{t_1}^{\psi_0}$-SLF for any $\psi_1 \in C_c(\mathbb{R}^d)$ with $\psi_1 \geqslant 0$, using again the uniqueness of the PDE in $\mathscr{L}_+$ we get

$$\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\, d\nu_{x,t_1}(\gamma)\psi_1(x)\, d\mu_{t_1}^{\psi_0}(x) = \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\, d\tilde{\nu}_{x,t_1}(\gamma)\psi_1(x)\, d\mu_{t_1}^{\psi_0}(x)$$

for any $\varphi \in C_c(\mathbb{R}^d)$, which can also be written as

$$\begin{aligned}&\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\nu_{x,t_1}(\gamma)\, d\mu_{t_1}^{\psi_0}(x)\\ &\quad= \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\tilde{\nu}_{x,t_1}(\gamma)\, d\mu_{t_1}^{\psi_0}(x).\end{aligned} \tag{18}$$

Recalling that by (17)

$$\int_{\mathbb{R}^d} \nu_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x) = \int_{\mathbb{R}^d} \nu_x \psi_0(x)\, d\mu_0(x), \qquad \int_{\mathbb{R}^d} \tilde{\nu}_{x,t_1}\, d\mu_{t_1}^{\psi_0}(x) = \int_{\mathbb{R}^d} \tilde{\nu}_x \psi_0(x)\, d\mu_0(x),$$

by (18) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\nu_x(\gamma)\psi_0(x)\, d\mu_0(x)\\ &\quad= \int_{\mathbb{R}^d\times\Gamma_T} \varphi\big(e_{t_2}(\gamma)\big)\psi_1\big(e_{t_1}(\gamma)\big)\, d\tilde{\nu}_x(\gamma)\psi_0(x)\, d\mu_0(x)\end{aligned}$$

for any non-negative $\psi_0, \psi_1, \varphi \in C_c(\mathbb{R}^d)$ (the constraint $\int_{\mathbb{R}^d} \psi_0\, d\mu_0 = 1$ can be easily removed multiplying the above equality by a positive constant). Iterating this argument, we finally get

$$\int\limits_{\mathbb{R}^d\times\Gamma_T} \psi_n\big(e_{t_n}(\gamma)\big)\cdots\psi_1\big(e_{t_1}(\gamma)\big)\,d\nu_x(\gamma)\psi_0(x)\,d\mu_0(x)$$
$$= \int\limits_{\mathbb{R}^d\times\Gamma_T} \psi_n\big(e_{t_n}(\gamma)\big)\cdots\psi_1\big(e_{t_1}(\gamma)\big)\,d\tilde\nu_x(\gamma)\psi_0(x)\,d\mu_0(x),$$

for any non-negative $\psi_0,\dots,\psi_n \in C_c(\mathbb{R}^d)$, and thus (16) follows.

Considering now only rational times, we get that there exists a subset $A \subset \mathbb{R}^d$, with $\mu_0(A^c)=0$, such that, for any $x\in A$,

$$(e_{t_1},\dots,e_{t_n})_\#\nu_x = (e_{t_1},\dots,e_{t_n})_\#\tilde\nu_x \quad \text{for any } t_1,\dots,t_n\in[0,T]\cap\mathbb{Q}.$$

By continuity, this implies that, for any $x\in A$, $\nu_x=\tilde\nu_x$, as wanted. □

Remark 3.5. Suppose that forward uniqueness for the PDE holds in the class $\mathscr{L}_+$, and take $\mu_0=\rho_0\mathcal{L}^d$ and $\tilde\mu_0=\tilde\rho_0\mathcal{L}^d$, with $\rho_0,\tilde\rho_0\in L^1_+(\mathbb{R}^d)\cap L^\infty_+(\mathbb{R}^d)$. If $\{\nu_x\}$ is a μ_0-SLF and $\{\tilde\nu_x\}$ is a $\tilde\mu_0$-SLF, then

$$\nu_x=\tilde\nu_x \quad \text{for } \mu_0\wedge\tilde\mu_0\text{-a.e. } x.$$

In fact, by Remark 3.2 $\{\nu_x\}$ and $\{\tilde\nu_x\}$ are both $\mu_0\wedge\tilde\mu_0$-SLF, and thus we conclude by the uniqueness result proved above.

By Theorems 3.3 and 3.4, and by the remark above, we obtain the following:

Corollary 3.6 *(Existence and uniqueness of SLF). Let us assume that we have forward existence and uniqueness for the PDE in $\mathscr{L}_+$. Then there exists a measurable selection of martingale solution $\{\nu_x\}_{x\in\mathbb{R}^d}$ which is a μ_0-SLF for any $\mu_0=\rho_0\mathcal{L}^d$ with $\rho_0\in L^1_+(\mathbb{R}^d)\cap L^\infty_+(\mathbb{R}^d)$, and if $\{\tilde\nu_x\}_{x\in\mathbb{R}^d}$ is a $\tilde\mu_0$-SLF for a fixed $\tilde\mu_0=\tilde\rho_0\mathcal{L}^d$ with $\tilde\rho_0\in L^1_+(\mathbb{R}^d)\cap L^\infty_+(\mathbb{R}^d)$, then $\nu_x=\tilde\nu_x$ for $\mathcal{L}^d$-a.e. $x\in\operatorname{supp}(\tilde\mu_0)$.*

Proof. It suffices to consider a SLF starting from a Gaussian measure (which exists by Theorem 3.3), and to apply Remark 3.5. □

By now, the above selection of martingale solutions $\{\nu_x\}$, which is uniquely determined $\mathcal{L}^d$-a.e., will be called the SLF (starting at time 0 and relative to (b,a)).

We finally prove a stability result for SLF.

Theorem 3.7 *(Stability of SLF starting from a fixed measure). Let us suppose that $b^n, b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $a^n, a:[0,T]\times\mathbb{R}^d\to\mathcal{S}_+(\mathbb{R}^d)$ are uniformly bounded functions, and that we have forward existence and uniqueness for the PDE in $\mathscr{L}_+$ with coefficients (b,a). Let $\mu_0=\rho_0\mathcal{L}^d\in\mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0\in L^\infty(\mathbb{R}^d)$, and let $\{\nu^n_x\}_{x\in\mathbb{R}^d}$ and $\{\nu_x\}_{x\in\mathbb{R}^d}$ be μ_0-SLF for (b^n,a^n) and (b,a), respectively. Define $\nu^n:=\int_{\mathbb{R}^d}\nu^n_x\,d\mu_0(x)$, $\nu:=\int_{\mathbb{R}^d}\nu_x\,d\mu_0(x)$. Assume that*:

(i) $(b^n,a^n)\to(b,a)$ *in* $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$;

(ii) *setting* $\rho_t^n \mathcal{L}^d = \mu_t^n := (e_t)_\# \nu^n$, *for any* $t \in [0,T]$

$$\left\| \rho_t^n \right\|_{L^\infty(\mathbb{R}^d)} \leqslant C \quad \text{for a certain constant } C = C(T).$$

Then $\nu^n \rightharpoonup^* \nu$ *in* $\mathcal{M}(\Gamma_T)$.

Proof. Since (b^n, a^n) are uniformly bounded in L^∞, as in Step 2 of the proof of Theorem 2.6 one proves that the sequence of probability measures (ν^n) on $\mathbb{R}^d \times \Gamma_T$ is tight. In order to conclude, we must show that any limit point of (ν^n) is ν.

Let $\tilde{\nu}$ be any limit point of (ν^n). We claim that $\tilde{\nu}$ is concentrated on martingale solutions of the SDE with coefficients (b,a). Indeed, let us define $\tilde{\mu}_t := (e_t)_\# \tilde{\nu}$. Since $\mu_t^n \to \tilde{\mu}_t$ narrowly and ρ_t^n are non-negative functions bounded in $L^\infty(\mathbb{R}^d)$, we get $\tilde{\mu}_t = \rho_t \mathcal{L}^d$ for a certain non-negative function $\rho_t \in L^\infty(\mathbb{R}^d)$. We now observe that the argument used in Step 3 of the proof of Theorem 2.6 was using only the property that, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\limsup_{n\to+\infty} \sum_i \int_s^t \int_{\mathbb{R}^d} \left| \left(b_i^n(u,x) - \tilde{b}_i(u,x) \right) \partial_i \varphi(x) \right| \rho_u^n(x)\, dx\, du$$

$$\leqslant \sum_i \int_s^t \int_{\mathbb{R}^d} \left| \left(b_i(u,x) - \tilde{b}_i(u,x) \right) \partial_i \varphi(x) \right| \rho_u(x)\, dx\, du,$$

$$\limsup_{n\to+\infty} \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \left| \left(a_{ij}^n(u,x) - \tilde{a}_{ij}(u,x) \right) \partial_{ij} \varphi(x) \right| \rho_u^n(x)\, dx\, du$$

$$\leqslant \sum_{ij} \int_s^t \int_{\mathbb{R}^d} \left| \left(a_{ij}(u,x) - \tilde{a}_{ij}(u,x) \right) \partial_{ij} \varphi(x) \right| \rho_u(x)\, dx\, du$$

for any $\tilde{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\tilde{a} : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ bounded and continuous. This property simply follows by (i) and the w^*-convergence of ρ_t^n to ρ_t in $L^\infty([0,T] \times \mathbb{R}^d)$.

Since $t \mapsto \rho_t \mathcal{L}^d$ is w^*-continuous in the sense of measures, the w^*-continuity of $t \mapsto \rho_t$ in $L^\infty(\mathbb{R}^d)$ follows. Thus, if we write $\tilde{\nu} := \int_{\mathbb{R}^d} \tilde{\nu}_x \, d\mu_0(x)$ (considering the disintegration of $\tilde{\nu}$ with respect to $\mu_0 = (e_0)_\# \tilde{\nu}$), we have proved that $\{\tilde{\nu}_x\}$ is a μ_0-SLF for (b,a). Therefore, by Theorem 3.4, we conclude that $\nu = \tilde{\nu}$. □

We remark that the theory just developed could be generalized to more general situations. Indeed the key property of the convex class $\mathscr{L}_+$ is the following monotonicity property:

$$0 \leqslant \tilde{\mu}_t \leqslant \mu_t \in \mathscr{L}_+ \quad \Rightarrow \quad \tilde{\mu}_t \in \mathscr{L}_+$$

(see also [2, Section 3]).

3.2. SLF versus RLF

We remark that, in the special case $a = 0$, our SLF coincides with a sort of superposition of the RLF introduced in [1]:

Lemma 3.8. *Let us assume $a = 0$. Then $\nu_{x,s}$ is a martingale solution of the SDE (which, in this case, is just an ODE) starting from x at time s if and only if it is concentrated on integral curves of the ODE, that is, for $\nu_{x,s}$-a.e. γ,*

$$\gamma(t) - \gamma(s) = \int_s^t b\big(\tau, \gamma(\tau)\big)\,d\tau \quad \forall t \in [s, T].$$

Proof. It is clear from the definition of martingale solution that, if $\nu_{x,s}$ is concentrated on integral curves on the ODE, then it is a martingale solution. Let us prove the converse implication. By the definition of martingale solution and the fact that $a = 0$, it is a known fact that

$$M_t := \gamma(t) - \gamma(s) - \int_s^t b\big(\tau, \gamma(\tau)\big)\,d\tau, \quad t \in [s, T],$$

is a $\nu_{x,s}$-martingale with zero quadratic variation. This implies that also M_t^2 is a martingale, and since $M_s = 0$ we get

$$0 = \mathbb{E}^{\nu_{x,s}}\big[M_t^2\big] = \int_{\Gamma_T} \left(\gamma(t) - \gamma(s) - \int_s^t b\big(\tau, \gamma(\tau)\big)\,d\tau\right)^2 d\nu_{x,s}(\gamma) \quad \forall t \in [s, T],$$

which gives the thesis. □

Thus, in the case $a = 0$, a martingale solution of the SDE starting from x is simply a measure on Γ_T concentrated on integral curves of b. By the results in [1] we know that, if we have forward uniqueness for the PDE in $\mathscr{L}_+$, then any measure ν on Γ_T concentrated on integral curves of b such that its time marginals induces a solution of the PDE in $\mathscr{L}_+$ is concentrated on a graph, i.e. there exists a function $x \mapsto X(\cdot, x) \in \Gamma_T$ such that

$$\nu = X(\cdot, x)_\# \mu_0, \quad \text{with } \mu_0 := (e_0)_\# \nu$$

(see for instance [3, Theorem 18]). Then, if we assume forward uniqueness for the PDE in $\mathscr{L}_+$, our SLF coincides exactly with the RLF in [1]. Applying the stability result proved in the above paragraph, we obtain that, as the noise tends to 0, our SLF converges to the RLF associated to the ODE $\dot{\gamma} = b(\gamma)$. So we have a vanishing viscosity result for RLF.

Corollary 3.9. *Let us suppose that $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is uniformly bounded, and that we have forward existence and uniqueness for the PDE in $\mathscr{L}_+$ with coefficients $(b, 0)$. Let $\{\nu_x^\varepsilon\}_{x \in \mathbb{R}^d}$ and $\{\nu_x\}_{x \in \mathbb{R}^d}$ be the SLF relative to $(b, \varepsilon I)$ and $(b, 0)$, respectively (existence and uniqueness of martingale solutions for the SDE with coefficients $(b, \varepsilon I)$, together with the measurability* *of the family* $\{\nu_x^\varepsilon\}_{x\in\mathbb{R}^d}$, *follows by* [18, *Theorem* 7.2.1]). *Let* $\mu_0 = \rho_0\mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, *with* $\rho_0 \in L^\infty(\mathbb{R}^d)$, *and define* $\nu^\varepsilon := \int_{\mathbb{R}^d} \nu_x^\varepsilon \, d\mu_0(x)$, $\nu := \int_{\mathbb{R}^d} \nu_x \, d\mu_0(x)$.

Set $\rho_t^\varepsilon \mathcal{L}^d = \mu_t^\varepsilon := (e_t)_\# \nu^\varepsilon$, *and assume that for any* $t \in [0, T]$

$$\left\|\rho_t^\varepsilon\right\|_{L^\infty(\mathbb{R}^d)} \leqslant C \quad \text{for a certain constant } C = C(T).$$

Then $\nu^\varepsilon \rightharpoonup^* \nu$ *in* $\mathcal{M}(\Gamma_T)$.

In [1], the uniqueness of RLF implies the semigroup law (see [1,2] for more details). In our case, by the uniqueness of SLF, we have as a consequence that the Chapman–Kolmogorov equation holds:

Proposition 3.10. *For any* $s \geqslant 0$, *let* $\{\nu_{x,s}\}_{x\in\mathbb{R}^d}$ *denotes the unique SLF starting at time* s. *Let us denote by* $\nu_{s,x}(t, dy)$ *the probability measure on* $\mathbb{R}^d$ *given by* $\nu_{s,x}(t,\cdot) := (e_t)_\#\nu_{s,x}$. *Then, for any* $0 \leqslant s < t < u \leqslant T$,

$$\int_{\mathbb{R}^d} \nu_{t,y}(u,\cdot)\nu_{s,x}(t,dy) = \nu_{s,x}(u,\cdot) \quad \text{for } \mathcal{L}^d\text{-a.e. } x.$$

Proof. Let us define

$$\tilde\nu_{s,x} := \begin{cases} \nu_{s,x} & \text{on } C([s,t],\mathbb{R}^d), \\ \int_{\mathbb{R}^d} \nu_{t,y}\nu_{s,x}(t,dy) & \text{on } C([t,T],\mathbb{R}^d). \end{cases}$$

This gives a family of martingale solution starting from x at time s (see [18]), and, using that $\{\nu_{x,s}\}$ and $\{\nu_{x,t}\}$ are SLF starting at time s and t, respectively, it is simple to check that $\{\tilde\nu_{s,x}\}_{x\in\mathbb{R}^d}$ is a SLF starting at time s. Thus, by Theorem 3.4, we have the thesis. □

4. Fokker–Planck equation

We now want to study the Fokker–Planck equation

$$\partial_t\mu_t + \sum_i \partial_i(b_i\mu_t) - \frac{1}{2}\sum_{ij}\partial_{ij}(a_{ij}\mu_t) = 0 \quad \text{in } [0,T]\times\mathbb{R}^d, \tag{19}$$

where $a = (a_{ij})$ is symmetric and non-negative definite (that is, $a : [0,T]\times\mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$).

4.1. Existence and uniqueness of measure-valued solutions

Proposition 4.1. *Let us assume that* $a : [0,T]\times\mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions, having two bounded continuous spatial derivatives. Then, for any finite measure* μ_0 *there exists a unique finite measure-valued solution of* (19) *starting from* μ_0 *such that* $|\mu_t|(\mathbb{R}^d) \leqslant C$ *for any* $t \in [0,T]$.

Proof. *Existence*. Let $\{\nu_x\}_{x\in\mathbb{R}^d}$ be the measurable family of martingale solutions of the SDE

$$\begin{cases} dX = b(t,X)\,dt + \sqrt{a(t,X)}\,dB(t), \\ X(0) = x \end{cases}$$

(which exists and is unique by [18, Corollary 6.3.3]). Then, by Lemma 2.4 and Remark 2.5, the measure $\mu_t := (e_t)_\# \int_{\mathbb{R}^d} \nu_x\, d\mu_0(x)$ solves (19) and $|\mu_t|(\mathbb{R}^d) \leqslant |\mu_0|(\mathbb{R}^d)$.

Uniqueness. By linearity, it suffices to prove that, if $\mu_0 = 0$, then $\mu_t = 0$ for all $t \in [0,T]$. Fix $\psi \in C_c^\infty(\mathbb{R}^d)$, $\bar t \in [0,T]$, and let $f(t,x)$ be the (unique) solution of

$$\begin{cases} \partial_t f + \sum_i b_i \partial_i f + \frac{1}{2}\sum_{ij} a_{ij}\partial_{ij} f = 0 & \text{in } [0,\bar t] \times \mathbb{R}^d, \\ f(\bar t) = \psi & \text{on } \mathbb{R}^d \end{cases}$$

(which exists and is unique by [18, Theorem 3.2.6]). By [18, Theorems 3.1.1 and 3.2.4], we know that $f \in C_b^{1,2}$, i.e. it is uniformly bounded with one bounded continuous time derivative and two bounded continuous spatial derivatives. Since μ_t is a finite measure by assumption, and $t \mapsto \mu_t$ is narrowly continuous (Lemma 2.1), we can use $f(t,\cdot)$ as test functions in (3), and we get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)\,d\mu_t(x) \\ &\quad = \int_{\mathbb{R}^d} \left[\partial_t f(t,x) + \sum_i b_i(t,x)\partial_i f(t,x) + \frac{1}{2}\sum_{ij} a_{ij}(t,x)\partial_{ij} f(t,x)\right] d\mu_t(x) = 0 \end{aligned}$$

(the above computation is admissible since $f \in C_b^{1,2}$). This implies in particular that

$$0 = \int_{\mathbb{R}^d} f(0,x)\,d\mu_0(x) = \int_{\mathbb{R}^d} f(\bar t,x)\,d\mu_{\bar t}(x) = \int_{\mathbb{R}^d} \psi(x)\,d\mu_{\bar t}(x).$$

By the arbitrariness of ψ and $\bar t$ we obtain $\mu_t = 0$ for all $t \in [0,T]$. □

We remark that, in the uniformly parabolic case, the above proof still works under weaker regularity assumptions. Indeed, in that case, one has existence of a measurable family of martingale solutions of the SDE and of a solution $f \in C_b^{1,2}([0,\bar t] \times \mathbb{R}^d)$ of the adjoint equation if a and b are just Hölder continuous (see [18, Theorem 3.2.1]). So we get:

Proposition 4.2. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\langle \xi, a(t,x)\xi\rangle \geqslant \alpha|\xi|^2$ $\forall (t,x) \in [0,T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(ii) $|b(t,x) - b(s,y)| + \|a(t,x) - a(s,y)\| \leqslant C(|x-y|^\delta + |t-s|^\delta)$ $\forall (t,x), (s,y) \in [0,T] \times \mathbb{R}^d$, *for some* $\delta \in (0,1]$, $C \geqslant 0$.

Then, for any finite measure μ_0 *there exists a unique finite measure-valued solution of* (19) *starting from* μ_0.

4.2. Existence and uniqueness of absolutely continuous solutions in the uniformly parabolic case

We are now interested in absolutely continuous solutions of (2). Therefore, we consider the following equation

$$\begin{cases} \partial_t u + \sum_i \partial_i (b_i u) - \frac{1}{2} \sum_{ij} \partial_{ij} (a_{ij} u) = 0 & \text{in } [0,T] \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{20}$$

which must be understood in the distributional sense on $[0,T] \times \mathbb{R}^d$. We now first prove an existence and uniqueness result in the L^2-setting under a regularity assumption on the divergence of a, which enables us to write (20) in a variational form, and thus to apply classical existence results (the uniqueness part in L^2 is much more involved). After, we will give a maximum principle result.

Let us make the following assumptions on the coefficients:

$$\begin{aligned} &\sum_j \partial_j a_{ij} \in L^\infty\big([0,T] \times \mathbb{R}^d\big) \quad \text{for } i = 1, \dots, d, \\ &\bigg(\sum_i \partial_i b_i - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}\bigg)^- \in L^\infty\big([0,T] \times \mathbb{R}^d\big), \\ &\big\langle \xi, a(t,x)\xi \big\rangle \geqslant \alpha |\xi|^2 \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \text{ for a certain } \alpha > 0. \end{aligned} \tag{21}$$

Theorem 4.3. *Let us assume that $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are bounded functions such that* (21) *is fulfilled. Then, for any $u_0 \in L^2(\mathbb{R}^d)$,* (20) *has a unique solution $u \in Y$, where*

$$Y := \big\{ u \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big) \ \big|\ \partial_t u \in L^2\big([0,T], H^{-1}\big(\mathbb{R}^d\big)\big)\big\}.$$

If moreover $\partial_t a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ for $i, j = 1, \dots, d$, then existence and uniqueness holds in $L^2([0,T] \times \mathbb{R}^d)$, and so in particular any solution $u \in L^2([0,T] \times \mathbb{R}^d)$ of (20) *belongs to Y.*

The proof the above theorem is quite standard, except for the uniqueness result in the large space L^2, which is indeed quite technical and involved. The motivation for this more general result is that $L^1_+(\mathbb{R}^d) \cap L^\infty_+(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, and $L^1_+(R^d) \cap L^\infty_+(R^d)$ is the space where we need well-posedness of the PDE if we want to apply the theory on martingale solutions developed in the last section (see Theorems 1.3 and 5.1).

We now give some properties of the family of solutions of (20):

Proposition 4.4. *We assume that $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are bounded functions, and that* (21) *is fulfilled. Then the solution $u \in Y$ provided by Theorem* 4.3 *satisfies*:

(a) $u_0 \geqslant 0 \Rightarrow u \geqslant 0$;

(b) $u_0 \in L^\infty(\mathbb{R}^d) \Rightarrow u \in L^\infty([0,T] \times \mathbb{R}^d)$ *and we have*

$$\left\|u(t)\right\|_{L^\infty(\mathbb{R}^d)} \leqslant \|u_0\|_{L^\infty(\mathbb{R}^d)} e^{t\|(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij})^-\|_\infty};$$

(c) *if moreover*

$$\frac{a}{1+|x|^2} \in L^2\big([0,T] \times \mathbb{R}^d\big), \qquad \frac{b}{1+|x|} \in L^2\big([0,T] \times \mathbb{R}^d\big),$$

then $u_0 \in L^1 \Rightarrow \|u(t)\|_{L^1(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d)}$ $\forall t \in [0,T]$.

We observe that, by the above results together with Proposition 4.2, we obtain:

Corollary 4.5. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $$\big\langle \xi, a(t,x)\xi \big\rangle \geqslant \alpha|\xi|^2 \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d, \text{ for a certain } \alpha > 0;$$

(ii) $$\big|b(t,x) - b(s,y)\big| + \big\|a(t,x) - a(s,y)\big\| \leqslant C\big(|x-y|^\gamma + |t-s|^\gamma\big)$$
$$\forall (t,x), (s,y) \in [0,T] \times \mathbb{R}^d, \text{ for some } \gamma \in (0,1],\ C \geqslant 0;$$

(iii) $$\sum_j \partial_j a_{ij} \in L^\infty\big([0,T] \times \mathbb{R}^d\big) \quad \text{for } i = 1, \dots, d,$$
$$\Big(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij} \partial_{ij} a_{ij}\Big)^- \in L^\infty\big([0,T] \times \mathbb{R}^d\big);$$

(iv) $$\frac{a}{1+|x|^2} \in L^2\big([0,T] \times \mathbb{R}^d\big), \qquad \frac{b}{1+|x|} \in L^2\big([0,T] \times \mathbb{R}^d\big).$$

Then, for any $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$ *there exists a unique finite measure-valued solution* $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ *of* (2) *starting from* μ_0. *Moreover, if such that* $\mu_0 = \rho_0 \mathcal{L}^d$ *with* $\rho_0 \in L^2(\mathbb{R}^d)$, *then* $\mu_t \ll \mathcal{L}^d$ *for all* $t \in [0,T]$.

Proof. Existence and uniqueness of finite measure-valued solutions follows by Proposition 4.2. So the only thing to prove is that, if $\rho_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is non-negative, then $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ and $\mu_t \ll \mathcal{L}^d$ for all $t \in [0,T]$. This simply follows by the fact that the solution $u \in Y$ provided by Theorem 4.3 belongs to $L^1_+(\mathbb{R}^d)$ by Proposition 4.4, and thus coincides with μ_t by uniqueness in the set of finite measure-valued solutions. □

In order to prove the results stated before, we need the following theorem of J.-L. Lions (see [16]).

Theorem 4.6. *Let* H *be an Hilbert space, provided with a norm* $|\cdot|$, *and inner product* $(\cdot,\cdot)$. *Let* $\Phi \subset H$ *be a subspace endowed with a prehilbertian norm* $\|\cdot\|$, *such that the injection* $\Phi \hookrightarrow H$ *is continuous. We consider a bilinear form* $B : H \times \Phi \to \mathbb{R}$ *such that*:

- *$H \ni u \mapsto B(u, \varphi)$ is continuous on H for any fixed $\varphi \in \Phi$;*
- *there exists $\alpha > 0$ such that $B(\varphi, \varphi) \geqslant \alpha \|\varphi\|^2$ for any $\varphi \in \Phi$.*

Then, for any linear continuous form L on Φ there exists $v \in H$ such that

$$B(v, \varphi) = L(\varphi) \quad \forall \varphi \in \Phi.$$

Proof of Theorem 4.3. We will first prove existence and uniqueness of a solution in the space Y. Once this will be done, we will show that, if u is a weak solution of (20) belonging to $L^2([0,T] \times \mathbb{R}^d)$ and $\partial_t a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ for $i, j = 1, \dots, d$, then u belongs to Y, and so it coincides with the unique solution provided before.

The change of unknown

$$v(t,x) = e^{-\lambda t} u(t,x)$$

leads to the equation

$$\begin{cases} \partial_t v + \sum_i \partial_i(\tilde{b}_i v) - \frac{1}{2} \sum_{ij} \partial_i(a_{ij} \partial_j v) + \lambda v = 0 & \text{in } [0,T] \times \mathbb{R}^d, \\ v_0 = u_0, \end{cases} \tag{22}$$

where $\tilde{b}_i := b_i - \frac{1}{2} \sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$. Assuming that λ satisfies $\lambda > \frac{1}{2} \|(\sum_i \partial_i \tilde{b}_i)^-\|_\infty$, we will prove existence and uniqueness for u.

Step 1 (*Existence in Y*). We want to apply Theorem 4.6.

Let us take $H := L^2([0,T], H^1(\mathbb{R}^d))$, $\Phi := \{\varphi \in C^\infty([0,T] \times \mathbb{R}^d) \mid \operatorname{supp} \varphi \Subset [0,T) \times \mathbb{R}^d\}$. Φ is endowed with the norm

$$\|\varphi\|_\Phi^2 := \|\varphi\|_H^2 + \frac{1}{2} \int_{\mathbb{R}^d} \left|\varphi(0,x)\right|^2 dx.$$

The bilinear form B and the linear form L are defined as

$$B(u, \varphi) := \int_0^T \int_{\mathbb{R}^d} \left[u \left(-\partial_t \varphi - \sum_i \tilde{b}_i \partial_i \varphi + \lambda \varphi \right) + \frac{1}{2} \sum_{ij} a_{ij} \partial_j u \partial_i \varphi \right] dx\, dt,$$

$$L(\varphi) := \int_{\mathbb{R}^d} u_0(x) \varphi(0,x)\, dx.$$

Thanks to these definitions and our assumptions, Lions' theorem applies, and we find a distributional solution v of (22). In particular,

$$\partial_t v = -\sum_i \partial_i(\tilde{b}_i v) + \frac{1}{2} \sum_{ij} \partial_i(a_{ij} \partial_j v) - \lambda v \in H^* = L^2\big([0,T], H^{-1}\big(\mathbb{R}^d\big)\big),$$

and thus $v \in Y$. In order to give a meaning to the initial condition and to show the uniqueness, we recall that for functions in Y there exists a well-defined notion of trace at 0 in $L^2(\mathbb{R}^d)$, and the following Gauss–Green formula holds:

$$\int_0^T \int_{\mathbb{R}^d} \partial_t u\tilde{u} + \partial_t \tilde{u} u \, dx \, dt = \int_{\mathbb{R}^d} u(T,x)\tilde{u}(T,x)\, dx - \int_{\mathbb{R}^d} u(0,x)\tilde{u}(0,x)\, dx \quad \forall u, \tilde{u} \in Y \tag{23}$$

(both facts follow by a standard approximation with smooth functions and by the fact that, if u is smooth and compactly supported in $[0,T) \times \mathbb{R}^d$, $\int_{\mathbb{R}^d} u^2(0,x)\, dx \leqslant 2\|\partial_t u\|_{H^*}\|u\|_H$). Thus, by (22) and (23), we obtain that v satisfies

$$\int_{\mathbb{R}^d} \big(v(0,x) - u_0(x)\big)\varphi(0,x)\, dx = 0 \quad \forall \varphi \in \Phi,$$

and therefore the initial condition is satisfied in $L^2(\mathbb{R}^d)$.

Step 2 (*Uniqueness in Y*). For the uniqueness, if $v \in Y$ is a solution of (22) with $u_0 = 0$, again by (23) we get

$$\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^d} \left(\partial_t v + \sum_i \partial_i(\tilde{b}_i v) - \frac{1}{2}\sum_{ij} \partial_i(a_{ij}\partial_j v) + \lambda v\right) v \, dx \, dt \\
&= \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \left[\frac{d}{dt}v^2 - \sum_i \tilde{b}_i \partial_i\big(v^2\big) + \sum_{ij} a_{ij}\partial_i v \partial_j v + 2\lambda v^2\right] dx \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^d} v^2(T,x)\, dx + \left(\lambda - \frac{1}{2}\left\|\left(\sum_i \partial_i \tilde{b}_i\right)^-\right\|_\infty\right) \int_0^T \int_{\mathbb{R}^d} v^2 \, dx \, dt \\
&\geqslant \left(\lambda - \frac{1}{2}\left\|\left(\sum_i \partial_i \tilde{b}_i\right)^-\right\|_\infty\right) \int_0^T \int_{\mathbb{R}^d} v^2 \, dx \, dt.
\end{aligned}$$

Since $\lambda > \frac{1}{2}\|(\sum_i \partial_i \tilde{b}_i)^-\|_\infty$, we get $v = 0$.

Remark 4.7. We observe that the above proof still works for the PDE

$$\begin{cases} \partial_t u + \sum_i \partial_i(b_i u) - \frac{1}{2}\sum_{ij}\partial_{ij}(a_{ij}u) = U & \text{in } [0,T] \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases}$$

with $U \in H^* = L^2([0,T], H^{-1}(\mathbb{R}^d))$. Indeed, it suffices to define L as

$$L(\varphi) := \langle U, \varphi\rangle_{H^*,H} + \int_{\mathbb{R}^d} u_0(x)\varphi(x)\, dx,$$

and all the rest of the proof works without any changes.

Thanks to this remark, we can now prove uniqueness in the larger space $L^2([0,T]\times\mathbb{R}^d)$ under the assumption $\partial_t a_{ij}\in L^\infty([0,T]\times\mathbb{R}^d)$ for $i,j=1,\dots,d$.

Step 3 (*Uniqueness in L^2*). If $u\in L^2([0,T]\times\mathbb{R}^d)$ is a (distributional) solution of (19), then

$$\partial_t u-\frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j u)=-\sum_i\partial_i(\tilde{b}_i u)\in L^2\big([0,T],H^{-1}\big(\mathbb{R}^d\big)\big).$$

By Remark 4.7, there exists $\tilde{u}\in Y$ solution of the above equation, with the same initial condition. Let us define $w:=u-\tilde{u}\in L^2([0,T]\times\mathbb{R}^d)$. Then w is a distributional solution of

$$\begin{cases}\partial_t w-A(\partial_x)w:=\partial_t w-\frac{1}{2}\sum_{ij}\partial_i(a_{ij}\partial_j w)=0 & \text{in } [0,T]\times\mathbb{R}^d,\\ w(0)=0.\end{cases}$$

In order to conclude the proof, it suffices to prove that $w=0$.

Step 3.1 (*Regularization*). Let us consider the PDE

$$w_\varepsilon-\varepsilon A(\partial_x)w_\varepsilon=w\quad\text{in } [0,T]\times\mathbb{R}^d \tag{24}$$

(this is an elliptic problem degenerate in the time variable). Applying Theorem 4.6, with $H=\Phi:=L^2([0,T],H^1(\mathbb{R}^d))$,

$$B(u,\varphi):=\int_0^T\int_{\mathbb{R}^d}\bigg(u\varphi+\frac{\varepsilon}{2}\sum_{ij}a_{ij}\partial_j u\partial_i\varphi\bigg)\,dx\,dt,$$

$$L(\varphi):=\int_0^T\int_{\mathbb{R}^d}w\varphi\,dx\,dt,$$

we find a unique solution w_ε of (24) in $L^2([0,T],H^1(\mathbb{R}^d))$, that is $w_\varepsilon=(I-\varepsilon A(\partial_x))^{-1}w$, with $(I-\varepsilon A(\partial_x)):L^2([0,T],H^1(\mathbb{R}^d))\to L^2([0,T],H^{-1}(\mathbb{R}^d))$ isomorphism. Now we want to find the equation solved by w_ε. We observe that, since $(I-\varepsilon A(\partial_x))^{-1}$ commutes with $A(\partial_x)$ and $\partial_t w=A(\partial_x)w$, the parabolic equation solved by w_ε formally looks

$$\partial_t w_\varepsilon-A(\partial_x)w_\varepsilon=\big[\partial_t,\big(I-\varepsilon A(\partial_x)\big)^{-1}\big]w.$$

Formally computing the commutator between ∂_t and $(I-\varepsilon A(\partial_x))^{-1}$, one obtains

$$\partial_t w_\varepsilon-A(\partial_x)w_\varepsilon=\varepsilon\big(I-\varepsilon A(\partial_x)\big)^{-1}\sum_{ij}\partial_j\big(\partial_t a_{ij}\partial_i w^\varepsilon\big) \tag{25}$$

in the distributional sense (see (27) below). Let us assume for a moment that (25) has been rigorously justified, and let us see how we can conclude.

Step 3.2 (*Gronwall argument*). By (25) it follows that $\partial_t w_\varepsilon \in L^2([0,T], H^{-1}(\mathbb{R}^d))$. Thus, recalling that $w_\varepsilon \in L^2([0,T], H^1(\mathbb{R}^d))$, we can multiply (25) by w_ε and integrate on $\mathbb{R}^d$, obtaining

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |w_\varepsilon|^2\,dx + \alpha \int_{\mathbb{R}^d} |\nabla_x w_\varepsilon|^2\,dx \leqslant -\varepsilon \int_{\mathbb{R}^d} \sum_{ij} (\partial_t a_{ij}) \partial_i w_\varepsilon \partial_j \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} w_\varepsilon\big)\,dx.$$

We observe that $w_\varepsilon(t) \to 0$ in L^2 as $t \searrow 0$. Indeed, since $w_\varepsilon \in Y$ there is a well-defined notion of trace at 0 in L^2 (see (23)), and it is not difficult to see that this trace is 0 since $w(0) = 0$ in the sense of distributions. Thus, integrating in time the above inequality, we get

$$\begin{aligned} &\big\|w_\varepsilon(t)\big\|^2_{L^2(\mathbb{R}^d)} + 2\alpha \|\nabla_x w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)} \\ &\quad \leqslant 2C\varepsilon \|\nabla_x w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)} \big\|\nabla_x \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} w_\varepsilon\big)\big\|_{L^2([0,T]\times\mathbb{R}^d)} \quad \forall t \in [0,T]. \end{aligned} \tag{26}$$

Let us consider, for a general $v \in L^2$, the function $v_\varepsilon := (I - \varepsilon A(\partial_x))^{-1} v$. Multiplying the identity $v_\varepsilon - \varepsilon A(\partial_x) v_\varepsilon = v$ by v_ε and integrating on $[0,T] \times \mathbb{R}^d$, we get

$$\|v_\varepsilon\|^2_{L^2} + \alpha\varepsilon \|\nabla_x v_\varepsilon\|^2_{L^2} \leqslant \|v_\varepsilon\|_{L^2} \|v\|_{L^2},$$

which implies $\|v_\varepsilon\|_{L^2} \leqslant \|v\|_{L^2}$, and therefore $\alpha\varepsilon \|\nabla_x v_\varepsilon\|^2_{L^2} \leqslant \|v\|^2_{L^2}$. Applying this last inequality with $v = w_\varepsilon$, we obtain

$$\big\|\nabla_x \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} w_\varepsilon\big)\big\|_{L^2([0,T]\times\mathbb{R}^d)} \leqslant \frac{1}{\sqrt{\alpha\varepsilon}} \|w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)}.$$

Substituting the above inequality in (26), we have

$$\begin{aligned} \big\|w_\varepsilon(t)\big\|^2_{L^2(\mathbb{R}^d)} + 2\alpha \|\nabla_x w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)} &\leqslant 2C\sqrt{\frac{\varepsilon}{\alpha}} \|\nabla_x w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)} \|w_\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d)} \\ &\leqslant C\sqrt{\frac{\varepsilon}{\alpha}} \|\nabla_x w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)} + C\sqrt{\frac{\varepsilon}{\alpha}} \|w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)}, \end{aligned}$$

which implies, for ε small enough (say $\varepsilon \leqslant 4\frac{\alpha^3}{C^2}$),

$$\big\|w_\varepsilon(t)\big\|^2_{L^2(\mathbb{R}^d)} \leqslant C\sqrt{\frac{\varepsilon}{\alpha}} \|w_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d)} \quad \forall t \in [0,T].$$

By Gronwall inequality $w_\varepsilon = 0$, and thus by (24) $w = 0$.

Step 3.3 (*Rigorous justification of (25)*). In order to conclude the proof of the theorem, we only need to rigorously justify (25).

Let $(a^n_{ij})_{n\in\mathbb{N}}$ be a sequence of smooth functions bounded in L^∞, such that $\langle a^n \xi, \xi\rangle \geqslant \frac{\alpha}{2}|\xi|^2$, $\sum_j \partial_j a^n_{ij}$ and $\partial_t a^n_{ij}$ are uniformly bounded, and $a^n_{ij} \to a_{ij}$, $\sum_j \partial_j a^n_{ij} \to \sum_j \partial_j a_{ij}$, $\partial_t a^n_{ij} \to \partial_t a_{ij}$ a.e. We now compute $[\partial_t, (I - \varepsilon A^n(\partial_x))^{-1}]$, where $A^n(\partial_x) := \sum_{ij} \partial_i (a^n_{ij} \partial_j \cdot)$:

$$\begin{aligned}
\bigl[\partial_t,\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\bigr] &= \Bigl[\partial_t,\sum_{k\geqslant 0}\varepsilon^k A^n(\partial_x)^k\Bigr] = \sum_{n\geqslant 0}\varepsilon^k\bigl[\partial_t, A^n(\partial_x)^k\bigr] \\
&= \varepsilon\sum_{k=0}^{\infty}\sum_{i=0}^{k-1}\bigl(\varepsilon A^n(\partial_x)\bigr)^i\bigl[\partial_t, A^n(\partial_x)\bigr]\bigl(\varepsilon A^n(\partial_x)\bigr)^{k-i-1} \\
&= \varepsilon\sum_{i=0}^{\infty}\bigl(\varepsilon A^n(\partial_x)\bigr)^i\bigl[\partial_t, A^n(\partial_x)\bigr]\sum_{k>i}\bigl(\varepsilon A^n(\partial_x)\bigr)^{k-i-1} \\
&= \varepsilon\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\bigl[\partial_t, A^n(\partial_x)\bigr]\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}, \qquad (27)
\end{aligned}$$

where at the second equality we used the algebraic identity $[A, B^k] = \sum_{i=0}^{k-1} B^i[A,B]B^{k-i-1}$. Thus, for any $\varphi, \psi \in C_c^\infty([0,T]\times\mathbb{R}^d)$, we have

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\psi\,\partial_t\Bigl(\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\varphi\Bigr)\,dx\,dt \\
&\quad= \int_0^T\int_{\mathbb{R}^d}\psi\bigl[\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\partial_t\varphi\bigr]\,dx\,dt \\
&\qquad+\varepsilon\int_0^T\int_{\mathbb{R}^d}\psi\bigl[\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\bigl[\partial_t, A^n(\partial_x)\bigr]\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\varphi\bigr]\,dx\,dt. \qquad (28)
\end{aligned}$$

We now want to pass to the limit in the above identity as $n\to\infty$. Since $(I-\varepsilon A^n(\partial_x))^{-1}$ is selfadjoint in $L^2([0,T]\times\mathbb{R}^d)$ and it commutes with $A^n(\partial_x)$, we get

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\psi\bigl[\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\bigl[\partial_t, A^n(\partial_x)\bigr]\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\varphi\bigr]\,dx\,dt \\
&\quad= \int_0^T\int_{\mathbb{R}^d}\bigl[\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\psi\bigr]\bigl[\bigl[\partial_t, A^n(\partial_x)\bigr]\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\varphi\bigr]\,dx\,dt \\
&\quad= -\int_0^T\int_{\mathbb{R}^d}\bigl[\partial_t\bigl(\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}\psi\bigr)\bigr]\bigl[\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}A^n(\partial_x)\varphi\bigr]\,dx\,dt \\
&\qquad-\int_0^T\int_{\mathbb{R}^d}\bigl[\bigl(I-\varepsilon A^n(\partial_x)\bigr)^{-1}A^n(\partial_x)\psi\bigr]\bigl[\partial_t\bigl(\bigl(I-\varepsilon A^n(\partial_x)^{-1}\bigr)\varphi\bigr)\bigr]\,dx\,dt.
\end{aligned}$$

By (27) we have

$$\begin{aligned}\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big) &= \big(I-\varepsilon A^n(\partial_x)\big)^{-1}\partial_t\varphi\\ &\quad+\varepsilon\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t, A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi,\end{aligned}$$

and, observing that $[\partial_t, A^n(\partial_x)] = \sum_{ij}\partial_i(\partial_t a^n_{ij}\partial_j\cdot)$, we deduce that the right-hand side is uniformly bounded in $L^2([0,T], H^1(\mathbb{R}^d))$. In the same way one obtains

$$\begin{aligned}\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi\big) &= \big(I-\varepsilon A^n(\partial_x)\big)^{-1}\partial_t\big(A^n(\partial_x)\varphi\big)\\ &\quad+\varepsilon\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t, A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi\\ &= \big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t, A^n(\partial_x)\big]\varphi\\ &\quad+\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\partial_t\varphi\\ &\quad+\varepsilon\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\big[\partial_t, A^n(\partial_x)\big]\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi,\end{aligned}$$

and, as above, the right-hand side is uniformly bounded in $L^2([0,T], H^1(\mathbb{R}^d))$. Thus $\partial_t(I-\varepsilon A^n(\partial_x))^{-1}\varphi$ is uniformly bounded in $L^2([0,T], H^1(\mathbb{R}^d)) \subset L^2([0,T]\times\mathbb{R}^d)$ (the same obviously holds for ψ in place of φ), while $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$ is uniformly bounded in $H^1([0,T]\times\mathbb{R}^d)$ (again the same fact holds for ψ in place of φ). Therefore, since $H^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d) \hookrightarrow L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ compactly, all we have to check is that

$$\partial_t\big(\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi\big) \to \partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)$$

and

$$\big(I-\varepsilon A^n(\partial_x)\big)^{-1}A^n(\partial_x)\varphi \to \big(I-\varepsilon A(\partial_x)\big)^{-1}A(\partial_x)\varphi$$

in the sense of distribution (indeed, by what we have shown above, $\partial_t((I-\varepsilon A^n(\partial_x))^{-1}\varphi)$ will converge weakly in L^2 while $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$ will converge strongly in L^2_{loc}, and therefore it is not difficult to see that the product converges to the product of the limits). We observe that, since the solution of

$$\varphi_\varepsilon - \varepsilon A(\partial_x)\varphi_\varepsilon = \varphi \quad \text{in } [0,T]\times\mathbb{R}^d \tag{29}$$

belonging to $L^2([0,T], H^1(\mathbb{R}^d))$ is unique, and any limit point of $(I-\varepsilon A^n(\partial_x))^{-1}\varphi$ belongs to $L^2([0,T], H^1(\mathbb{R}^d))$ and is a distributional solution of (29), one obtains that

$$\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\varphi \to \big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi$$

in the distributional sense, which implies the convergence of $\partial_t(I-\varepsilon A^n(\partial_x))^{-1}\varphi$ to $\partial_t(I-\varepsilon A(\partial_x))^{-1}\varphi$. Regarding $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$, let us take $\chi \in C^\infty_c([0,T]\times\mathbb{R}^d)$. Then we consider

$$\int_0^T\int_{\mathbb{R}^d} A^n(\partial_x)\varphi\big[\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\chi\big]\,dx\,dt = -\int_0^T\int_{\mathbb{R}^d}\sum_{ij} a^n_{ij}\partial_j\varphi\big(\partial_i\big(I-\varepsilon A^n(\partial_x)\big)^{-1}\chi\big)\,dx\,dt.$$

Recalling that $(I-\varepsilon A^n(\partial_x))^{-1}\chi$ is uniformly bounded in $L^2([0,T],H^1(\mathbb{R}^d))$, we get that $\partial_j(I-\varepsilon A^n(\partial_x))^{-1}\chi$ converges to $\partial_j(I-\varepsilon A(\partial_x))^{-1}\chi$ weakly in $L^2([0,T]\times\mathbb{R}^d)$ while $a_{ij}^n\to a_{ij}$ a.e., and so the convergence of $(I-\varepsilon A^n(\partial_x))^{-1}A^n(\partial_x)\varphi$ to $(I-\varepsilon A(\partial_x))^{-1}A(\partial_x)\varphi$ follows.

Thus we are able to pass to the limit in (28), and we get

$$\partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\in L^2\big([0,T],H^1\big(\mathbb{R}^d\big)\big)$$

and

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\psi\,\partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\,dx\,dt\\
&\quad=\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\partial_t\varphi\big]\,dx\,dt\\
&\qquad+\varepsilon\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\big[\partial_t,A(\partial_x)\big]\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big]\,dx\,dt.
\end{aligned}$$

Observing that $(I-\varepsilon A(\partial_x))^{-1}$ is selfadjoint in $L^2([0,T]\times\mathbb{R}^d)$ (for instance, this can be easily proved by approximation), we have that the second integral in the right-hand side can be written as

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\psi\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\big[\partial_t,A(\partial_x)\big]\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big]\,dx\,dt\\
&\quad=\int_0^T\int_{\mathbb{R}^d}\big[\big(I-\varepsilon A(\partial_x)\big)^{-1}\psi\big]\big[\big[\partial_t,A(\partial_x)\big]\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\big]\,dx\,dt.
\end{aligned}$$

Using now that $[\partial_t,A(\partial_x)]=\sum_{ij}\partial_i(\partial_t a_{ij}\partial_j\cdot)$ in the sense of distributions, it can be easily proved by approximation that the right-hand side above coincides with

$$-\int_0^T\int_{\mathbb{R}^d}\sum_{ij}(\partial_t a_{ij})\partial_i\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\psi\big)\partial_j\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\,dx\,dt.$$

Therefore we finally obtain

$$\int_0^T\int_{\mathbb{R}^d}\psi\,\partial_t\big(\big(I-\varepsilon A(\partial_x)\big)^{-1}\varphi\big)\,dx\,dt$$

$$= \int_0^T \int_{\mathbb{R}^d} \psi\big[\big(I - \varepsilon A(\partial_x)\big)^{-1} \partial_t \varphi\big]\, dx\, dt$$
$$- \varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_{ij} (\partial_t a_{ij}) \partial_i \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \psi\big) \partial_j \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big)\, dx\, dt. \tag{30}$$

By what we have proved above, it follows that

$$\partial_t \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big) \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big),$$
$$A(\partial_x)\big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big) = \big(I - \varepsilon A(\partial_x)\big)^{-1} A(\partial_x)\varphi \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big). \tag{31}$$

This implies that (30) holds also for $\psi \in L^2([0,T] \times \mathbb{R}^d)$, and that $(I - \varepsilon A(\partial_x))^{-1}\varphi$ is an admissible test function in the equation $\partial_t w - A(\partial_x) w = 0$. By these two facts we obtain

$$0 = \int_0^T \int_{\mathbb{R}^d} w\big[\big(\partial_t + A(\partial_x)\big)\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big]\, dx\, dt$$
$$= \int_0^T \int_{\mathbb{R}^d} w\big[\big(I - \varepsilon A(\partial_x)\big)^{-1} \big(\partial_t + A(\partial_x)\big)\varphi\big]\, dx\, dt$$
$$- \varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_{ij} (\partial_t a_{ij}) \partial_i \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} w\big) \partial_j \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big)\, dx\, dt$$
$$= \int_0^T \int_{\mathbb{R}^d} w_\varepsilon \big[\big(\partial_t + A(\partial_x)\big)\varphi\big]\, dx\, dt - \varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_{ij} (\partial_t a_{ij}) \partial_i w_\varepsilon \partial_j \big(\big(I - \varepsilon A(\partial_x)\big)^{-1} \varphi\big)\, dx\, dt,$$

which exactly means that

$$\partial_t w_\varepsilon - A(\partial_x) w_\varepsilon = \varepsilon \big(I - \varepsilon A(\partial_x)\big)^{-1} \sum_{ij} \partial_j \big(\partial_t a_{ij} \partial_i w^\varepsilon\big)$$

in the distributional sense. □

Proof of Proposition 4.4. (a) Arguing as in the first part of the proof of Theorem 4.3, with the same notation we have

$$0 = \int_0^T \int_{\mathbb{R}^d} \bigg(\partial_t v + \sum_i \partial_i (\tilde{b}_i v) - \frac{1}{2} \sum_{ij} \partial_i (a_{ij} \partial_j v) + \lambda v\bigg) v^-\, dx\, dt$$

$$= \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \left[-\frac{d}{dt}(v^-)^2 - \sum_i \tilde{b}_i \partial_i \big((v^-)^2\big) - \sum_{ij} a_{ij} \partial_i v^- \partial_j v^- - 2\lambda (v^-)^2 \right] dx$$

$$\leqslant -\frac{1}{2}\int_{\mathbb{R}^d} (v^-)^2(T,x)\,dx - \left(\lambda - \frac{1}{2}\left\| \left(\sum_i \partial_i \tilde{b}_i\right)^- \right\|_\infty\right) \int_0^T \int_{\mathbb{R}^d} (v^-)^2\,dx\,dt$$

$$\leqslant -\left(\lambda - \frac{1}{2}\left\| \left(\sum_i \partial_i \tilde{b}_i\right)^- \right\|_\infty\right) \int_0^T \int_{\mathbb{R}^d} (v^-)^2\,dx\,dt,$$

and then $v^- = 0$.

(b) It suffices to observe that the above argument works for every $v \in Y$ such that $v(0) \geqslant 0$ and

$$\partial_t v + \sum_i \partial_i (\tilde{b}_i v) - \frac{1}{2}\sum_{ij} \partial_i (a_{ij} \partial_j v) \geqslant 0.$$

Applying this remark to the function $v := \|u_0\|_{L^\infty(\mathbb{R}^d)} - ue^{-\lambda t}$ with $\lambda > \|(\sum_i \partial_i \tilde{b}_i)^-\|_\infty$, and then letting $\lambda \to \|(\sum_i \partial_i \tilde{b}_i)^-\|_\infty$, the thesis follows.

(c) The argument we use here is reminiscent of the one that we will use in the next subsection for renormalized solutions. Indeed, in order to prove the thesis, we will implicitly prove that, if $u \in L^2([0,T], H^1(\mathbb{R}^d))$ is a solution of (20), then it is also a renormalized solution (see Definition 4.9).

Let us define

$$\beta_\varepsilon(s) := \left(\sqrt{s^2 + \varepsilon^2} - \varepsilon\right) \in C^2(\mathbb{R}).$$

Notice that β_ε is convex and

$$\beta_\varepsilon(s) \to |s| \quad \text{as } \varepsilon \to 0, \qquad \beta_\varepsilon(s) - s\beta_\varepsilon'(s) \in [-\varepsilon, 0].$$

Moreover, since $\beta_\varepsilon', \beta_\varepsilon'' \in W^{1,\infty}(\mathbb{R})$, it is easily seen that

$$u \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big) \quad \Rightarrow \quad \beta_\varepsilon(u), \beta_\varepsilon'(u) \in L^2\big([0,T], H^1\big(\mathbb{R}^d\big)\big).$$

Fix now a non-negative cut-off function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\varphi) \subset B_2(0)$, and $\varphi = 1$ in $B_1(0)$, and consider the functions $\varphi_R(x) := \varphi(x/R)$ for $R \geqslant 1$.

Thus, since $\beta_\varepsilon'' \geqslant 0$ and a_{ij} is positive definite, recalling that $\tilde{b}_i = b_i - \frac{1}{2}\sum_j \partial_j a_{ij}$, for any $t \in [0,T]$ we have

$$0 = \int_0^t \int_{\mathbb{R}^d} \left(\partial_t u + \sum_i \partial_i (\tilde{b}_i u) - \frac{1}{2}\sum_{ij} \partial_i (a_{ij} \partial_j u) \right) \beta_\varepsilon'(u) \varphi_R \,dx\,ds$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \biggl(\frac{d}{dt}\bigl(\varphi_R \beta_\varepsilon(u)\bigr) - 2\sum_i \tilde{b}_i \partial_i \bigl(u\beta_\varepsilon'(u)\varphi_R\bigr) + 2\sum_i \tilde{b}_i \partial_i \bigl(\beta_\varepsilon(u)\bigr)\varphi_R \\
&\quad + \sum_{ij} a_{ij}\partial_i u \partial_j u \beta_\varepsilon''(u)\varphi_R + \sum_{ij} a_{ij}\partial_i\bigl(\beta_\varepsilon(u)\bigr)\partial_j\varphi_R \biggr)\,dx\,ds \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(t)\bigr)\,dx - \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(0)\bigr)\,dx \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \sum_i \tilde{b}_i \Bigl(\partial_i\Bigl(\bigl(u\beta_\varepsilon'(u) - \beta_\varepsilon(u)\bigr)\varphi_R\Bigr) + \beta_\varepsilon(u)\partial_i\varphi_R\Bigr)\,dx\,ds \\
&\quad - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \sum_{ij} \bigl((\partial_j a_{ij})\partial_i\varphi_R + a_{ij}\partial_{ij}\varphi_R\bigr)\beta_\varepsilon(u)\,dx\,ds \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(t)\bigr)\,dx - \frac{1}{2}\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(0)\bigr)\,dx - \int_0^t \int_{\mathbb{R}^d} \biggl(\sum_i \partial_i\tilde{b}_i\biggr)^- \bigl(u\beta_\varepsilon'(u) - \beta_\varepsilon(u)\bigr)\varphi_R\,dx\,ds \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \biggl(\sum_i b_i\partial_i\varphi_R + \frac{1}{2}\sum_{ij} a_{ij}\partial_{ij}\varphi_R\biggr)\beta_\varepsilon(u)\,dx\,ds.
\end{aligned}
$$

Observing that $|\beta_\varepsilon(u)| \leqslant |u|$, and using Hölder inequality and the inequalities

$$
\frac{1}{R}\chi_{\{R\leqslant|x|\leqslant 2R\}} \leqslant \frac{3}{1+|x|}\chi_{\{|x|\geqslant R\}}, \qquad \frac{1}{R^2}\chi_{\{R\leqslant|x|\leqslant 2R\}} \leqslant \frac{5}{1+|x|^2}\chi_{\{|x|\geqslant R\}}, \tag{32}
$$

we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(t)\bigr)\,dx \\
&\quad \leqslant \int_{\mathbb{R}^d} \varphi_R\beta_\varepsilon\bigl(u(0)\bigr)\,dx + 2\varepsilon\int_0^t \int_{|x|\leqslant 2R} \biggl(\sum_i \partial_i\tilde{b}_i\biggr)^-\,dx\,ds \\
&\qquad + \|\varphi\|_{C^2}\biggl(6\biggl\|\frac{b}{1+|x|}\biggr\|_{L^2([0,T]\times\{|x|\geqslant R\})} + 5\biggl\|\frac{a}{1+|x|^2}\biggr\|_{L^2([0,T]\times\{|x|\geqslant R\})}\biggr)\|u\|_{L^2([0,t]\times\mathbb{R}^d)}.
\end{aligned}
$$

Letting first $\varepsilon \to 0$ and then $R \to \infty$, we obtain

$$
\bigl\|u(t)\bigr\|_{L^1(\mathbb{R}^d)} \leqslant \bigl\|u(0)\bigr\|_{L^1(\mathbb{R}^d)} \quad \forall t \in [0,T]. \qquad \square
$$

4.3. Existence and uniqueness in the degenerate parabolic case

We now want to drop the uniform ellipticity assumption on a. In this case, to prove existence and uniqueness in $\mathscr{L}_+$, we will need to assume a independent of the space variables.

4.3.1. Uniqueness in $\mathscr{L}$

The uniqueness result is a consequence of the following comparison principle in $\mathscr{L}$ (recall that the comparison principle in said to hold if the inequality between two solutions at time 0 is preserved at later times).

Theorem 4.8 *(Comparison principle in $\mathscr{L}$). Let us assume that $a : [0,T] \to \mathcal{S}_+(\mathbb{R}^d)$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $a \in L^\infty([0,T], \mathcal{S}_+(\mathbb{R}^d))$.

Then (19) *satisfies the comparison principle in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In particular solutions of the PDE in $\mathscr{L}$, if they exist, are unique.*

Since we do not assume any ellipticity of the PDE, in order to prove the above result we use the technique of renormalized solutions, which was first introduced in the study of the Boltzmann equation by DiPerna and P.-L. Lions [8,9], and then applied in the context of transport equations by many authors (see for example [1,5–7,10]).

Definition 4.9. Let $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be such that:

(i) $b, \sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $a, \sum_j \partial_j a_{ij}, \sum_{ij} \partial_{ij} a_{ij} \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$.

Let $u \in L^\infty_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ and assume that

$$c := \partial_t u + \sum_i b_i \partial_i u - \frac{1}{2} \sum_{ij} a_{ij} \partial_{ij} u \in L^1_{\mathrm{loc}}\big([0,T] \times \mathbb{R}^d\big). \tag{33}$$

We say that u is a renormalized solution of (33) if, for any convex function $\beta : \mathbb{R} \to \mathbb{R}$ of class C^2, we have

$$\partial_t \beta(u) + \sum_i b_i \partial_i \beta(u) - \frac{1}{2} \sum_{ij} a_{ij} \partial_{ij} \beta(u) \leqslant c\beta'(u).$$

Equivalently the definition could be given in a partially conservative form:

$$\partial_t \beta(u) + \sum_i \partial_i \big(b_i \beta(u)\big) - \frac{1}{2} \sum_{ij} a_{ij} \partial_{ij} \beta(u) \leqslant c\beta'(u) + \Big(\sum_i \partial_i b_i\Big)\beta(u).$$

Recalling that a is non-negative definite and β is convex, it is simple to check that, if everything is smooth so that one can apply the standard chain rule, every solution of (33) is a renormalized solution. Indeed, in that case, one gets

$$\partial_t\beta(u)+\sum_i b_i\partial_i\beta(u)-\frac{1}{2}\sum_{ij}a_{ij}\partial_{ij}\beta(u)=c\beta'(u)-\frac{1}{2}\beta''(u)\sum_{ij}a_{ij}\partial_i u\partial_j u\leqslant c\beta'(u).$$

In our case, a solution of the Fokker–Planck equation is renormalized if

$$\partial_t\beta(u)+\sum_i\Big(b_i-\sum_j\partial_j a_{ij}\Big)\partial_i\beta(u)-\frac{1}{2}\sum_{ij}a_{ij}\partial_{ij}\beta(u)\leqslant\Big(\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}-\sum_i\partial_i b_i\Big)u\beta'(u),$$

or equivalently, writing everything in the partially conservative form,

$$\begin{aligned}\partial_t\beta(u)&+\sum_i\partial_i\Big(\Big(b_i-\sum_j\partial_j a_{ij}\Big)\beta(u)\Big)-\frac{1}{2}\sum_{ij}a_{ij}\partial_{ij}\beta(u)\\&\leqslant\Big(\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}-\sum_i\partial_i b_i\Big)u\beta'(u)+\sum_i\partial_i\Big(b_i-\sum_j\partial_j a_{ij}\Big)\beta(u)\\&=\Big(\sum_i\partial_i b_i-\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Big)\big(\beta(u)-u\beta'(u)\big)-\frac{1}{2}\Big(\sum_{ij}\partial_{ij}a_{ij}\Big)\beta(u).\end{aligned}$$

Now, since

$$\begin{aligned}\sum_{ij}a_{ij}\partial_{ij}\beta(u)&=\sum_{ij}\partial_j\big(a_{ij}\partial_i\beta(u)\big)-\sum_{ij}\partial_j a_{ij}\partial_i\beta(u)\\&=\sum_{ij}\partial_{ij}\big(a_{ij}\beta(u)\big)-2\sum_{ij}\partial_i\big((\partial_j a_{ij})\beta(u)\big)+\Big(\sum_{ij}\partial_{ij}a_{ij}\Big)\beta(u),\end{aligned}$$

the above expression can be simplified, and we obtain that a solution of the Fokker–Planck equation is renormalized if and only if

$$\begin{aligned}\partial_t\beta(u)&+\sum_i\partial_i\big(b_i\beta(u)\big)-\frac{1}{2}\sum_{ij}\partial_{ij}\big(a_{ij}\beta(u)\big)\\&\leqslant\Big(\sum_i\partial_i b_i-\frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Big)\big(\beta(u)-u\beta'(u)\big).\end{aligned}\tag{34}$$

It is not difficult to prove the following lemma.

Lemma 4.10. *Assume that there exist $p,q\in[1,\infty]$ such that*

$$\frac{a}{1+|x|^2}\in L^1\big([0,T],L^p\big(\mathbb{R}^d\big)\big),\qquad\frac{b}{1+|x|}\in L^1\big([0,T],L^q\big(\mathbb{R}^d\big)\big),$$

and that

$$\Big(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Big)^- \in L^1_{\mathrm{loc}}\big([0,T]\times\mathbb{R}^d\big).$$

Setting $a, b = 0$ for $t < 0$, assume moreover that any solution $u \in \mathscr{L}$ of the Fokker–Planck equation in $(-\infty, T)\times\mathbb{R}^d$ is renormalized. Then the comparison principle holds in $\mathscr{L}$.

Proof. By the linearity of the equation, it suffices to prove that

$$u_0 \leqslant 0 \quad \Rightarrow \quad u(t) \leqslant 0 \quad \forall t \in [0,T].$$

Fix a non-negative cut-off function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\varphi) \subset B_2(0)$, and $\varphi = 1$ in $B_1(0)$, and take as renormalization function

$$\beta_\varepsilon(s) := \frac{1}{2}\big(\sqrt{s^2+\varepsilon^2} + s - \varepsilon\big) \in C^2(\mathbb{R}).$$

Notice that β_ε is convex and

$$\beta_\varepsilon(s) \to s^+ \quad \text{as } \varepsilon \to 0, \qquad \beta_\varepsilon(s) - s\beta'_\varepsilon(s) \in [-\varepsilon, 0].$$

By (34), we know that

$$\partial_t\beta_\varepsilon(u) + \sum_i \partial_i\big(b_i\beta_\varepsilon(u)\big) - \frac{1}{2}\sum_{ij}\partial_{ij}\big(a_{ij}\beta_\varepsilon(u)\big) \leqslant \Big(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}\Big)\big(\beta_\varepsilon(u) - u\beta'_\varepsilon(u)\big)$$

in the sense of distributions in $(-\infty, T)\times\mathbb{R}^d$. Using as test function $\varphi_R(x) := \varphi(\frac{x}{R})$ for $R \geqslant 1$, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\varphi_R\beta_\varepsilon(u)\,dx \leqslant{}& \int_{\mathbb{R}^d}\Big(\sum_i b_i(t)\partial_i\varphi_R + \frac{1}{2}\sum_{ij}a_{ij}(t)\partial_{ij}\varphi_R\Big)\beta_\varepsilon(u)\,dx\\ &+ \int_{\mathbb{R}^d}\varphi_R\Big(\sum_i \partial_i b_i(t) - \frac{1}{2}\sum_{ij}\partial_{ij}a_{ij}(t)\Big)\big(\beta_\varepsilon(u) - u\beta'_\varepsilon(u)\big)\,dx.\end{aligned}$$

Observing that $|\beta_\varepsilon(u)| \leqslant |u|$, by Hölder inequality and the inequalities (32) we can bound the first integral in the right-hand side, uniformly with respect to ε, with

$$\begin{aligned}&\|\varphi\|_{C^2}\int_{\{|x|\geqslant R\}}\Big(3\frac{|b(t,x)|}{1+|x|} + \frac{5}{2}\frac{|a(t,x)|}{(1+|x|^2)}\Big)\big|u(t,x)\big|\,dx\\ &\leqslant \|\varphi\|_{C^2}\bigg(3\bigg\|\frac{b(t)}{1+|x|}\bigg\|_{L^p(\{|x|\geqslant R\})}\big\|u(t)\big\|_{L^{p'}(\mathbb{R}^d)} + \frac{5}{2}\bigg\|\frac{a(t)}{1+|x|^2}\bigg\|_{L^q(\{|x|\geqslant R\})}\big\|u(t)\big\|_{L^{q'}(\mathbb{R}^d)}\bigg)\end{aligned}$$

(recall that $u \in \mathscr{L}$, and thus $u \in L^\infty([0,T], L^r(\mathbb{R}^d))$ for any $r \in [1,\infty]$), while the second integral is bounded by

$$\varepsilon \int\limits_{\{|x|\leqslant 2R\}} \Big(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij} \partial_{ij} a_{ij}\Big)^- \, dx.$$

Letting first $\varepsilon \to 0$ and then $R \to \infty$, we get

$$\frac{d}{dt}\int\limits_{\mathbb{R}^d} u^+ \, dx \leqslant 0$$

in the sense of distribution in $(-\infty, T)$. Since the function vanishes for negative times, we conclude $u^+ = 0$. □

Now Theorem 4.8 is a direct consequence of the following proposition.

Proposition 4.11. *Let us assume that* $a : [0,T] \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $a \in L^\infty([0,T], \mathcal{S}_+(\mathbb{R}^d))$.

Then any distributional solution $u \in L^\infty_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ *of* (33) *is renormalized.*

Proof. We take η, a smooth convolution kernel in $\mathbb{R}^d$, and we mollify the equation with respect to the spatial variable obtaining

$$\partial_t u^\varepsilon + \sum_i b_i \partial_i u^\varepsilon - \frac{1}{2}\sum_{ij} a_{ij} \partial_{ij} u^\varepsilon = c * \eta_\varepsilon - r^\varepsilon, \tag{35}$$

where

$$r^\varepsilon := \sum_i (b_i \partial_i u) * \eta_\varepsilon - \sum_i b_i \partial_i (u * \eta_\varepsilon), \quad u^\varepsilon := u * \eta_\varepsilon.$$

By the smoothness of u^ε with respect to x, by (35) we have that $\partial_t u^\varepsilon \in L^1_{\mathrm{loc}}$. Thus by the standard chain rule in Sobolev spaces we get that u^ε is a renormalized solution, that is

$$\partial_t \beta\big(u^\varepsilon\big) + \sum_i b_i \partial_i \beta\big(u^\varepsilon\big) - \frac{1}{2}\sum_{ij} a_{ij} \partial_{ij} \beta\big(u^\varepsilon\big) \leqslant \big(c * \eta_\varepsilon - r^\varepsilon\big)\beta'\big(u^\varepsilon\big)$$

for any $\beta \in C^2(\mathbb{R})$ convex. Passing to the limit in the distributional sense as $\varepsilon \to 0$ in the above identity, the convergence of all the terms is trivial except for $r^\varepsilon \beta'(u^\varepsilon)$.

Let σ_η be any weak limit point of $r^\varepsilon \beta'(u^\varepsilon)$ in the sense of measures (such a cluster point exists since $r^\varepsilon \beta'(u^\varepsilon)$ is bounded in L^1_{loc}). Thus we get

$$\partial_t \beta(u) + \sum_i b_i \partial_i \beta(u) - \frac{1}{2}\sum_{ij} a_{ij}\partial_{ij}\beta(u) - c\beta'(u) \leqslant -\sigma_\eta \leqslant |\sigma_\eta|.$$

Since the left-hand side is independent of η, in order to conclude the proof it suffices to prove that $\bigwedge_\eta |\sigma_\eta| = 0$, where η varies in a dense countable set of convolution kernels. This fact is implicitly proved in [2, Theorem 34], see in particular Step 3 therein. □

4.3.2. Existence in $\mathscr{L}_+$

We can now prove an existence and uniqueness result in the class $\mathscr{L}_+$.

Theorem 4.12. *Let us assume that $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}(\mathbb{R}^d)$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are bounded functions such that*

$$\left(\sum_i \partial_i b_i - \frac{1}{2}\sum_{ij} \partial_{ij} a_{ij}\right)^- \in L^1\big([0,T], L^\infty\big(\mathbb{R}^d\big)\big).$$

Then, for any $\mu_0 = \rho_0 \mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, with $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exists a solution of (2) *in $\mathscr{L}_+$. If moreover $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$, and a is independent of x, then this solution turns out to be unique.*

Proof. *Existence*. It suffices to approximate the coefficients a and b locally uniformly with smooth uniformly bounded coefficients a^n and b^n such that $(\sum_i \partial_i b_i^n - \frac{1}{2}\sum_{ij}\partial_{ij} a_{ij}^n)^-$ is uniformly bounded in $L^1([0,T], L^\infty(\mathbb{R}^d))$. Indeed, if we now consider the approximate solutions $\mu_t^n = \rho_t^n \mathcal{L}^d \in \mathcal{M}_+(\mathbb{R}^d)$, we know that

$$\partial_t \rho_t^n + \sum_i \partial_i\big(b_i^n \rho_t^n\big) - \frac{1}{2}\sum_{ij}\partial_{ij}\big(a_{ij}^n \rho_t^n\big) = 0,$$

that is

$$\partial_t \rho_t^n - \frac{1}{2} a_{ij}^n \partial_{ij}\rho_t^n + \sum_i \left(b_i^n - \sum_j \partial_j a_{ij}^n\right)\partial_i \rho_t^n + \left(\sum_i \partial_i b_i^n - \frac{1}{2}\sum_{ij}\partial_{ij} a_{ij}^n\right)\rho_t^n = 0.$$

Using the Feynman–Kac's formula, we obtain the bound

$$\big\|\rho_t^n\big\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\rho_0\|_{L^\infty(\mathbb{R}^d)} e^{\int_0^t \|(\sum_i \partial_i b_i^n(s) - \frac{1}{2}\sum_{ij}\partial_{ij} a_{ij}^n(s))^-\|_{L^\infty(\mathbb{R}^d)}\, ds}.$$

So we see that the approximate solutions are non-negative and uniformly bounded in $L^1 \cap L^\infty$ (the bound in L^1 follows by the constancy of the map $t \mapsto \|\rho_t^n\|_{L^1}$ (observe that $\rho_t^n \geqslant 0$ and recall Remark 2.7)). Therefore, any weak limit is a solution of the PDE in $\mathscr{L}_+$.

Uniqueness. It follows by Theorem 4.8. □

5. Conclusions

Let us now combine the results proved in Sections 2 and 4 in order to get existence and uniqueness of SLF. The first theorem follows directly by Corollary 3.6 and Theorem 1.3, while the second is a consequence of Corollary 3.6 and Theorem 1.4.

Theorem 5.1. *Let us assume that* $a : [0,T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\sum_j \partial_j a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i = 1, \dots, d$;
(ii) $\partial_t a_{ij} \in L^\infty([0,T] \times \mathbb{R}^d)$ *for* $i, j = 1, \dots, d$;
(iii) $(\sum_i \partial_i b_i - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij})^- \in L^\infty([0,T] \times \mathbb{R}^d)$;
(iv) $\langle \xi, a(t,x)\xi \rangle \geqslant \alpha |\xi|^2$ $\forall (t,x) \in [0,T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(v) $\frac{a}{1+|x|^2} \in L^2([0,T] \times \mathbb{R}^d)$, $\frac{b}{1+|x|} \in L^2([0,T] \times \mathbb{R}^d)$.

Then there exists a unique SLF (*in the sense of Corollary* 3.6).

If moreover $(b^n, a^n) \to (b, a)$ *in* $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ *and* $(\sum_i \partial_i b_i^n - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}^n)^-$ *are uniformly bounded in* $L^1([0,T], L^\infty(\mathbb{R}^d))$, *then the Feynman–Kac formula implies* (ii) *of Theorem* 3.7 (*see the proof of Theorem* 4.12). *Thus we have stability of SLF.*

Theorem 5.2. *Let us assume that* $a : [0,T] \to \mathcal{S}(\mathbb{R}^d)$ *and* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $(\sum_i \partial_i b_i)^- \in L^1([0,T], L^\infty(\mathbb{R}^d))$.

Then there exists a unique SLF (*in the sense of Corollary* 3.6).

If moreover $(b^n, a^n) \to (b, a)$ *in* $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ *and* $(\sum_i \partial_i b_i^n - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij}^n)^-$ *are uniformly bounded in* $L^1([0,T], L^\infty(\mathbb{R}^d))$, *then the Feynman–Kac formula implies* (ii) *of Theorem* 3.7 (*see the proof of Theorem* 4.12). *Thus we have stability of SLF.*

In particular, by Corollary 3.9 and the Feynman–Kac formula (see the proof of Theorem 4.12), the following vanishing viscosity result for RLF holds:

Theorem 5.3. *Let us assume that* $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *is bounded and*:

(i) $b \in L^1([0,T], BV_{\mathrm{loc}}(\mathbb{R}^d))$, $\sum_i \partial_i b_i \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$;
(ii) $(\sum_i \partial_i b_i)^- \in L^1([0,T], L^\infty(\mathbb{R}^d))$.

Let $\{\nu_x^\varepsilon\}_{x \in \mathbb{R}^d}$ *be the unique SLF relative to* $(b, \varepsilon I)$, *with* $\varepsilon > 0$, *and* $\{\nu_x\}_{x \in \mathbb{R}^d}$ *be the RLF relative to* $(b, 0)$ (*which is uniquely determined* $\mathcal{L}^d$*-a.e. by the results in* [1]). *Then, as* $\varepsilon \to 0$,

$$\int_{\mathbb{R}^d} \nu_x^\varepsilon f(x)\, dx \rightharpoonup^* \int_{\mathbb{R}^d} \nu_x f(x)\, dx \quad \text{in } \mathcal{M}(\Gamma_T) \text{ for any } f \in C_c\big(\mathbb{R}^d\big).$$

We finally combine an important uniqueness result of Stroock and Varadhan (see Theorem 2.2) with the well-posedness results on Fokker–Planck of the previous section. By Theorem 2.2, Lemma 2.3 applied with $A = \mathbb{R}^d$ and Corollary 4.5, we have:

Theorem 5.4. *Let us assume that* $a : [0, T] \times \mathbb{R}^d \to \mathcal{S}_+(\mathbb{R}^d)$ *and* $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions such that*:

(i) $\langle \xi, a(t, x)\xi \rangle \geqslant \alpha|\xi|^2$ $\forall (t, x) \in [0, T] \times \mathbb{R}^d$, *for a certain* $\alpha > 0$;
(ii) $|b(t, x) - b(s, y)| + \|a(t, x) - a(s, y)\| \leqslant C(|x - y|^\gamma + |t - s|^\gamma)$ $\forall (t, x), (s, y) \in [0, T] \times \mathbb{R}^d$, *for some* $\gamma \in (0, 1]$, $C \geqslant 0$;
(iii) $\sum_j \partial_j a_{ij} \in L^\infty([0, T] \times \mathbb{R}^d)$ *for* $i = 1, \ldots, d$, $(\sum_i \partial_i b_i - \frac{1}{2} \sum_{ij} \partial_{ij} a_{ij})^- \in L^\infty([0, T] \times \mathbb{R}^d)$;
(iv) $\frac{a}{1+|x|^2} \in L^2([0, T] \times \mathbb{R}^d)$, $\frac{b}{1+|x|} \in L^2([0, T] \times \mathbb{R}^d)$.

Then, there exists a unique martingale solution starting from x (*at time* 0) *for any* $x \in \mathbb{R}^d$.

We remark that this result is not interesting by itself, since it can be proved that the martingale problem starting from any $x \in \mathbb{R}^d$ at any initial time $s \in [0, T]$ is well-posed also under weaker regularity assumptions (see [18, Chapters 6 and 7]). We stated it just because we believe that it is an interesting example of how existence and uniqueness at the PDE level can be combined with a refined analysis at the level of the uniqueness of martingale solutions. It is indeed in this spirit that we generalize Theorem 2.2 in Appendix A, hoping that it could be useful for further analogous applications.

Acknowledgments

I wish to thank Luigi Ambrosio and Franco Flandoli for many useful suggestions and improvements, and for their continuous support during the preparation of this paper. I also thank the referees of the paper for their helpful comments.

Appendix A. A generalized uniqueness result for martingale solutions

Here we generalize Theorem 2.2, using the notation introduced in Section 3.1.

Proposition A.1. *For any* $(s, x) \in [0, T] \times \mathbb{R}^d$, *let* $C_{x,s}$ *be a subset of martingale solutions of the SDE starting from* x *at time* s, *and let us make the following assumptions*: *there exists a measure* $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$ *such that*:

(i) $\forall s \in [0, T]$, $C_{x,s}$ *is convex for* μ_0-*a.e.* x;
(ii) $\forall s \in [0, T]$, $\forall t \in [s, T]$,

$$\text{for } \mu_0\text{-a.e. } x, \quad (e_t)_\# \nu^1_{x,s} = (e_t)_\# \nu^2_{x,s} \quad \forall \nu^1_{x,s}, \nu^2_{x,s} \in C_{x,s};$$

(iii) *for* μ_0-*a.e.* x, *for any* $\nu_x \in C_x := C_{x,0}$, *for* ν_x-*a.e.* γ,

$$\forall t \in [0, T], \quad \nu^{i,\gamma}_{x,\mathcal{F}_t} := \left(\nu^i_x\right)^\gamma_{\mathcal{F}_t} \in C_{\gamma(t),t},$$

where, with the above notation, we mean that the restriction of $\nu^{i,\gamma}_{x,\mathcal{F}_t}$ *to* $\Gamma^t_T := C([t,T],\mathbb{R}^d)$ *is a martingale solution starting from* $\gamma(t)$ *at time* t;

(iv) *the solution of* (2) *starting from* μ_0 *given by* $\mu_t := (e_t)_\# \int_{\mathbb{R}^d} \nu^1_x \, d\mu_0(x)$ *for a measurable selections* $\{\nu_x\}_{x\in\mathbb{R}^d}$ *with* $\nu_x \in C_x$ *(observe that* μ_t *does not depends on the choice of* $\nu_x \in C_x$ *by* (ii)*), satisfies* $\mu_t \ll \mu_0$ *for any* $t \in [0,T]$.

Then, given two measurable families of probability measures $\{\nu^1_x\}_{x\in\mathbb{R}^d}$ *and* $\{\nu^2_x\}_{x\in\mathbb{R}^d}$ *with* $\nu^1_x, \nu^2_x \in C_x$, $\nu^1_x = \nu^2_x$ *for* μ_0*-a.e.* x. *In particular, by standard measurable selection theorems* (*see for instance* [18, *Chapter* 12]), C_x *is a singleton for* μ_0*-a.e.* x.

Proof. Let $\{\nu^1_x\}_{x\in\mathbb{R}^d}$ and $\{\nu^2_x\}_{x\in\mathbb{R}^d}$ be two measurable families of probability measures with $\nu^1_x, \nu^2_x \in C_x$, and fix $0 < t_1 < \cdots < t_n \leqslant T$.

Claim. *For* μ_0*-a.e.* x, *for* ν^i_x*-a.e.* γ $(i = 1,2)$,

$$\nu^{i,\tilde\gamma}_{x,\mathcal{F}_{t_n}} \in C_{\tilde\gamma(t_n),t_n} \quad \textit{for } \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}\textit{-a.e. } \tilde\gamma$$

where $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}} := (\nu^i_x)^\gamma_{M^{t_1,\dots,t_n}}$.

This claim follows observing that, by assumption (iii), for μ_0-a.e. x there exists a subset $\Gamma_x \subset \Gamma_T$ such that $\nu^i_x(\Gamma_x) = 1$ and $\nu^{i,\gamma}_{x,\mathcal{F}_{t_n}} \in C_{\gamma(t_n),t_n}$ for any $\gamma \in \Gamma_x$. Thus, by (11) applied with $\nu := \nu^i_x$, $A := \Gamma_T$, $B := \Gamma_x$, and with $M^{t_1,\dots,t_n}$ in place of $\mathcal{F}_{t_n}$, one obtains

$$0 = \nu^i_x\big(\Gamma^c_x\big) = \int_{\Gamma_T} \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}\big(\Gamma^c_x\big)\, d\nu^i_x(\gamma),$$

that is,

$$\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}(\Gamma_x) = 1 \quad \text{for } \nu^i_x\text{-a.e. } \gamma.$$

This, together with assumption (iii), implies the claim.

By (13), $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}$ is concentrated on the set $\{\tilde\gamma \mid \tilde\gamma(t_n) = \gamma(t_n)\}$, and so, by the claim above, we get

$$\nu^{i,\tilde\gamma}_{x,\mathcal{F}_{t_n}} \in C_{\gamma(t_n),t_n} \quad \text{for } \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}\text{-a.e. } \tilde\gamma.$$

Let $A \subset \mathbb{R}^d$ be such that $\mu_0(A^c) = 0$ and assumption (i) is true for any $x \in A$. By assumption (iv), we have $\mu_{t_n}(A^c) = 0 = \int_{\mathbb{R}^d\times\Gamma_T} 1_{A^c}(\gamma(t_n))\, d\nu^i_x(\gamma)\, d\mu_0(x)$, that is

$$\text{for } \mu_0\text{-a.e. } x, \quad \gamma(t_n) \in A \quad \text{for } \nu^i_x\text{-a.e } \gamma. \tag{A.1}$$

Thus, for μ_0-a.e. x, $C_{\gamma(t_n),t_n}$ is convex for ν^i_x-a.e γ, and so, by (14) applied with ν^i_x, we obtain that

$$\text{for } \mu_0\text{-a.e. } x, \quad \nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}} \in C_{\gamma(t_n),t_n} \quad \text{for } \nu^i_x\text{-a.e. } \gamma \tag{A.2}$$

(where, with the above notation, we again mean that the restriction of $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_n}}$ to $\Gamma^{t_n}_T$ is a martingale solution starting from $\gamma(t_n)$ at time t_n). We now want to prove that, for all $n \geqslant 1$, $0 < t_1 < \cdots < t_n \leqslant T$, we have that, for μ_0-a.e. x,

$$\int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^1_x(\gamma) = \int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^2_x(\gamma) \tag{A.3}$$

for any $f_i \in C_c(\mathbb{R}^d)$. We observe that (A.3) is true for $n = 1$ by assumption (ii). We want to prove it for any n by induction. Let us assume (A.3) true for $n - 1$, and let us prove it for n. We want to show that

$$\int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^1_x(\gamma) = \int_{\Gamma_T} f_1\big(e_{t_1}(\gamma)\big)\dots f_n\big(e_{t_n}(\gamma)\big)\,d\nu^2_x(\gamma),$$

which can be written also as

$$\mathbb{E}^{\nu^1_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n})\big] = \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n})\big],$$

where $\mathbb{E}^\nu := \int_{\Gamma_T} d\nu$. Now we observe that, for $i = 1, 2$,

$$\begin{aligned}\mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n})\big] &= \mathbb{E}^{\nu^i_x}\big[\mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_n(e_{t_n}) \bigm| M^{t_1,\dots,t_{n-1}}\big]\big]\\ &= \mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\mathbb{E}^{\nu^i_x}\big[f_n(e_{t_n}) \mid M^{t_1,\dots,t_{n-1}}\big]\big]\\ &= \mathbb{E}^{\nu^i_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^i_x(e_{t_1},\dots,e_{t_{n-1}})\big],\end{aligned}$$

where $\psi^i_x(e_{t_1},\dots,e_{t_{n-1}}) := \mathbb{E}^{\nu^i_x}[f_n(e_{t_n}) \mid M^{t_1,\dots,t_{n-1}}]$. Let $\phi \in C_c(\mathbb{R}^d)$, and let us prove that

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbb{E}^{\nu^1_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^1_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x)\\ &\quad= \int_{\mathbb{R}^d} \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^2_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x).\end{aligned} \tag{A.4}$$

Let $B \subset \mathbb{R}^d$ be such that $\mu_0(B^c) = 0$ and assumption (ii′) is true for any $x \in B$. By assumption (iv), we also have $\mu_{t_{n-1}}(B^c) = 0 = \int_{\mathbb{R}^d\times\Gamma_T} 1_{B^c}(e_{t_{n-1}}(\gamma))\,d\nu^i_x(\gamma)\,d\mu_0(x)$, that is

$$\text{for } \mu_0\text{-a.e. } x, \quad \gamma(t_{n-1}) \in B \quad \text{for } \nu^i_x\text{-a.e. } \gamma. \tag{A.5}$$

Let us consider $\nu^{i,\gamma}_{x,M^{t_1,\dots,t_{n-1}}}$. By (A.2),

$$\text{for } \mu_0\text{-a.e. } x, \quad \nu^{i,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} \in C_{\gamma(t_{n-1}),t_{n-1}} \quad \text{for } \nu^i_x\text{-a.e. } \gamma,$$

and, combining this with (A.5), we obtain

$$\text{for } \mu_0\text{-a.e. } x, \quad \nu^{i,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} \in C_{\gamma(t_{n-1}),t_{n-1}} \quad \text{and} \quad \gamma(t_{n-1}) \in B \quad \text{for } \nu^i_x\text{-a.e. } \gamma.$$

By assumption (ii) applied with $t = t_n$, this implies that

$$\text{for } \mu_0\text{-a.e. } x, \quad (e_{t_n})_\# \nu^{1,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} = (e_{t_n})_\# \nu^{2,\gamma}_{x,M^{t_1,\dots,t_{n-1}}} \quad \text{for } \nu^i_x\text{-a.e. } \gamma,$$

which give us that

$$\text{for } \mu_0\text{-a.e. } x, \quad \psi^1_x(e_{t_1},\dots,e_{t_{n-1}}) = \psi^2_x(e_{t_1},\dots,e_{t_{n-1}}) \quad \text{for } \nu^i_x\text{-a.e. } \gamma. \tag{A.6}$$

Thus we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}^{\nu^1_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^1_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x) \\
&\quad = \int_{\mathbb{R}^d} \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^1_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x) \\
&\quad \overset{(A.6)}{=} \int_{\mathbb{R}^d} \mathbb{E}^{\nu^2_x}\big[f_1(e_{t_1})\dots f_{n-1}(e_{t_{n-1}})\psi^2_x(e_{t_1},\dots,e_{t_{n-1}})\big]\phi(x)\,d\mu_0(x),
\end{aligned}$$

where the first equality in the above equation follows by the inductive hypothesis. Now, by (A.4) and the arbitrariness of ϕ and of f_j, with $j = 1,\dots,n$, we obtain that, for all $n \geqslant 1$, $0 < t_1 < \dots < t_n \leqslant T$, we have

$$\text{for } \mu_0\text{-a.e. } x, \quad (e_{t_1},\dots,e_{t_n})_\# \nu_x = (e_{t_1},\dots,e_{t_n})_\# \tilde{\nu}_x \quad \forall t_1,\dots,t_n \in [0,T].$$

Considering only rational times, we get that there exists a subset $D \subset \mathbb{R}^d$, with $\mu_0(D^c) = 0$, such that, for any $x \in D$,

$$(e_{t_1},\dots,e_{t_n})_\# \nu_x = (e_{t_1},\dots,e_{t_n})_\# \tilde{\nu}_x \quad \text{for any } t_1,\dots,t_n \in [0,T] \cap \mathbb{Q}.$$

By continuity, this implies that, for any $x \in D$, $\nu_x = \tilde{\nu}_x$, as wanted. □

The above result apply, for example, in the case when $C_{x,s}$ denotes the set of all martingale solutions starting from x. In particular, we remark that, by the above proof, one obtains the well-known fact that, if ν_x is a martingale solution starting from x (at time 0), then, for any $0 \leqslant t_1 \leqslant \dots \leqslant t_n \leqslant T$, $\nu^\gamma_{x,M^{t_1,\dots,t_n}}$ is a martingale solution starting from $\gamma(t_n)$ at time t_n. More generally, since martingale solutions are closed by convex combination, is μ is a probability measure on $\mathbb{R}^d$, the average $\int_{\mathbb{R}^d} \nu^\gamma_{x,M^{t_1,\dots,t_n}}\,d\mu(x)$ is a martingale solution starting from $\gamma(t_n)$ at time t_n.

Observe that assumption (iv) in the above theorem was necessary only to deduce, from a μ_0-a.e. assumption, a μ_t-a.e. property. Thus, the above proof give us the following result:

Proposition A.2. *For any $(s, x) \in [0, T] \times \mathbb{R}^d$, let $C_{x,s}$ be a convex subset of martingale solutions of the SDE starting from x at time s, and let us make the following assumption: there exists a measure $\mu_0 \in \mathcal{M}_+(\mathbb{R}^d)$ such that:*

(i) $\forall t \in [0, T]$, *for μ_0-a.e. x,*

$$(e_t)_\# \nu_x^1 = (e_t)_\# \nu_x^2 \quad \forall \nu_x^1, \nu_x^2 \in C_x := C_{x,0}.$$

If (i) *holds, we can define $\mu_t := (e_t)_\# \int_{\mathbb{R}^d} \nu_x \, d\mu_0(x)$ for a measurable selections $\{\nu_x\}_{x \in \mathbb{R}^d}$ with $\nu_x \in C_x$, and this definition does not depends on the choice of $\nu_x \in C_x$. We now assume that:*

(i′) $\forall s \in [0, T]$, $\forall t \in [s, T]$, *for μ_s-a.e. x,*

$$(e_t)_\# \nu_{x,s}^1 = (e_t)_\# \nu_{x,s}^2 \quad \forall \nu_{x,s}^1, \nu_{x,s}^2 \in C_{x,s};$$

(ii) $\forall s \in [0, T]$, *$C_{x,s}$ is convex for μ_s-a.e. x;*
(iii) *for μ_0-a.e. x, for any $\nu_x \in C_x$, for ν_x-a.e. γ,*

$$\forall t \in [0, T], \quad \nu_{x,\mathcal{F}_t}^{i,\gamma} := \left(\nu_x^i\right)_{\mathcal{F}_t}^{\gamma} \in C_{\gamma(t),t},$$

where, with the above notation, we mean that the restriction of $\nu_{x,\mathcal{F}_t}^{i,\gamma}$ to Γ_T^t is a martingale solution starting from $\gamma(t)$ at time t.

Then, given two measurable families of probability measures $\{\nu_x^1\}_{x \in \mathbb{R}^d}$ and $\{\nu_x^2\}_{x \in \mathbb{R}^d}$ with $\nu_x^1, \nu_x^2 \in C_x$, $\nu_x^1 = \nu_x^2$ for μ_0-a.e. x. In particular, by standard measurable selection theorems (*see for instance* [18, *Chapter* 12]), *C_x is a singleton for μ_0-a.e. x.*

References

[1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Invent. Math. 158 (2) (2004) 227–260.
[2] L. Ambrosio, Lecture notes on transport equation and Cauchy problem for non-smooth vector fields, preprint, 2005, available at http://cvgmt.sns.it.
[3] L. Ambrosio, G. Crippa, Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields, preprint, 2006.
[4] L. Ambrosio, N. Gigli, G. Savaré, Gradient Flows in Metric Spaces and in the Wasserstein Space of Probability Measures, Lectures Math. ETH Zurich, Birkhäuser, Basel, 2005.
[5] F. Bouchut, Renormalized solutions to the Vlasov equations with coefficient of bounded variations, Arch. Ration. Mech. Anal. 157 (2001) 75–90.
[6] F. Colombini, N. Lerner, Uniqueness of continuous solutions for BV vector fields, Duke Math. J. 111 (2002) 357–384.
[7] F. Colombini, N. Lerner, Uniqueness of L^∞ solutions for a class of conormal BV vector fields, in: Contemp. Math., vol. 368, Amer. Math. Soc., Providence, RI, 2005, pp. 133–156.
[8] R.J. DiPerna, P.-L. Lions, On the Fokker–Planck–Boltzmann equation, Comm. Math. Phys. 120 (1988) 1–23.
[9] R.J. DiPerna, P.-L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability, Ann. of Math. 130 (2) (1989) 321–366.
[10] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (3) (1989) 511–547.
[11] M. Hauray, C. LeBris, P.-L. Lions, Deux remarques sur les flots généralisées d'équations différentielles ordinaires (Two remarks generalized flows for ordinary differential equations), C. R. Math. Acad. Sci. Paris 344 (12) (2007) 759–764.

[12] N.V. Krylov, M. Röckner, Strong solutions of stochastic equations with singular time dependent drift, Probab. Theory Related Fields 131 (2) (2005) 154–196.
[13] C. LeBris, P.-L. Lions, Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications, Ann. Mat. Pura Appl. (4) 183 (2004) 97–130.
[14] C. LeBris, P.-L. Lions, Existence and uniqueness of solutions to Fokker–Planck type equations with irregular coefficients, Comm. Partial Differential Equations, in press.
[15] C. LeBris, P.-L. Lions, Generalized flows for stochastic differential equations with irregular coefficients, in preparation.
[16] J.-L. Lions, Équations différentielles opérationnelles et problèmes aux limites, Grundlehren Math. Wiss., vol. 111, Springer-Verlag, Berlin, 1961 (in French).
[17] P.-L. Lions, Sur les équations différentielles ordinaires et les équations de transport (On ordinary differential equations and transport equations), C. R. Math. Acad. Sci. Paris 326 (7) (1998) 833–838.
[18] D.W. Stroock, S.R.S. Varadhan, Multidimensional Diffusion Processes, Grundlehren Math. Wiss., vol. 233, Springer-Verlag, Berlin, 1979.