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On sufficient conditions for the total positivity and for the multiple positivity of matrices

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Abstract

The following theorem is proved.

Theorem. *Suppose* $M = (a_{i,j})$ *be a* $k \times k$ *matrix with positive entries and* $a_{i,j}a_{i+1,j+1} > 4\cos^2 \frac{\pi}{k+1}\, a_{i,j+1}a_{i+1,j}\quad (1 \leqslant i \leqslant k-1,\ 1 \leqslant j \leqslant k-1)$. *Then* $\det M > 0$.

The constant $4\cos^2 \frac{\pi}{k+1}$ in this theorem is sharp. A few other results concerning totally positive and multiply positive matrices are obtained.

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1. Introduction and statement of results

This paper is inspired by the work [5] in which some useful and easily verified conditions of strict total positivity of a matrix are obtained. We recall that a matrix A is said to be k-times positive, if all minors of A of order not greater than k are non-negative. A matrix A is said to be multiply positive if it is k-times positive for some $k \in \mathbf{N}$. A matrix A is said to be totally positive, if

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all minors of A are non-negative. For more information about these notions and their applications we refer the reader to [3,12]. According to [12] we will denote the class of all k-times positive matrices by TP_k and the class of all totally positive matrices by TP. By STP we will denote the class of matrices with all minors being strictly positive and by STP_k the class of matrices with all minors of order not greater than k being strictly positive.

In [5] the following theorem was proved

Theorem A. *Denote by* $\tilde{c}$ *the unique real root of* $x^3 - 5x^2 + 4x - 1 = 0$ ($\tilde{c} \approx 4.0796$). *Let* $M = (a_{i,j})$ *be an* $n \times n$ *matrix with the property that*

(a) $a_{i,j} > 0$ $(1 \leqslant i, j \leqslant n)$ *and*
(b) $a_{i,j}a_{i+1,j+1} \geqslant \tilde{c}\, a_{i,j+1}a_{i+1,j}$ $(1 \leqslant i, j \leqslant n-1)$.

Then M *is strictly totally positive.*

Note that the verification of total positivity is, in general, a very difficult problem. Surely, it is not difficult to calculate the determinant of a given matrix with numerical entries. But if the order of a matrix or the entries of a matrix depend on some parameters then the testing of multiple positivity is complicated. Theorem A provides a convenient sufficient condition for total positivity of a matrix.

For $c \geqslant 1$ we will denote by $TP_2(c)$ the class of all matrices $M = (a_{i,j})$ with positive entries which satisfy the condition

$$a_{i,j}a_{i+1,j+1} \geqslant c\, a_{i,j+1}a_{i+1,j} \quad \text{for all } i, j. \tag{1}$$

For $c \geqslant 1$ we will denote by $STP_2(c)$ the class of all matrices $M = (a_{i,j})$ with positive entries which satisfy the condition

$$a_{i,j}a_{i+1,j+1} > c\, a_{i,j+1}a_{i+1,j} \quad \text{for all } i, j. \tag{2}$$

It is easy to verify that $STP_2 = STP_2(1)$. Theorem A states that $TP_2(\tilde{c}) \subset STP$. Denote by

$$c_k := 4\cos^2 \frac{\pi}{k+1}, \quad k = 2, 3, 4, \ldots$$

The main result of this paper is the following:

Theorem 1. *Suppose* $M = (a_{i,j})$ *be a* $k \times k$ *matrix with positive entries.*

(i) *if* $M \in TP_2(c_k)$ *then* $\det M \geqslant 0$;
(ii) *if* $M \in STP_2(c_k)$ *then* $\det M > 0$.

In the proof of Theorem 1 we will show that if $M \in TP_2(c)$ then every submatrix of M belongs to $TP_2(c)$. Therefore the following theorem is the simple consequence of Theorem 1.

Theorem 2. *For every* $c \geqslant c_k$ *we have*

(i) *if* $M \in TP_2(c)$ *then* $M \in TP_k$;
(ii) *if* $M \in STP_2(c)$ *then* $M \in STP_k$.

The following fact is a simple consequence of this theorem.

Theorem 3. *For every* $c \geqslant 4$ *we have if* $M \in TP_2(c)$ *then* $M \in STP$.

The following statement demonstrates that the constants in Theorems 1 and 3 are unimprovable not only in the class of matrices with positive entries but in the classes of Toeplitz matrices and of Hankel matrices. We recall that a matrix M is a Toeplitz matrix if it is of the form $M = (a_{j-i})$ and a matrix M is a Hankel matrix if it is of the form $M = (a_{j+i})$.

Theorem 4

(i) *For every* $1 \leqslant c < c_k$ *there exists a* $k \times k$ *Toeplitz matrix* $M \in TP_2(c)$ *with* $\det M < 0$;
(ii) *for every* $1 \leqslant c < c_k$ *there exists a* $k \times k$ *Hankel matrix* $M \in TP_2(c)$ *with* $\det M < 0$.

A simple consequence of Theorem 4 is the following fact.

Corollary of Theorem 4

(i) *For every* $1 \leqslant c < 4$ *there exists a Toeplitz matrix* $M \in TP_2(c)$ *but* $M \notin TP$;
(ii) *for every* $1 \leqslant c < 4$ *there exists a Hankel matrix* $M \in TP_2(c)$ *but* $M \notin TP$.

The following theorem shows that Theorem 1 remains valid for some special classes of matrices with non-negative elements.

Theorem 5. *Let* $M = (a_{i,j})$ *be a* $k \times k$ *matrix. Suppose that* $\exists s, l \in \mathbf{Z} : -(k-1) \leqslant s < l \leqslant k-1$ *such that* $a_{i,j} > 0$ *for* $s \leqslant j-i \leqslant l$ *and* $a_{i,j} = 0$ *for* $j-i < s$ *or* $j-i > l$. *If* $a_{i,j}a_{i+1,j+1} \geqslant c_k\, a_{i,j+1}a_{i+1,j}$ $(1 \leqslant i < m,\ 1 \leqslant j < n)$ *then* $\det M \geqslant 0$.

We will show how to prove Theorem 5 in the section "Proof of Theorem 4".

A variation of Theorem 3 for the class of Toeplitz matrices was proved by Hutchinson in [11]. To formulate his result we need some notions.

The class of m-times positive sequences consists of the sequences $\{a_k\}_{k=0}^{\infty}$ such that all minors of the infinite matrix

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{Vmatrix} \tag{3}$$

of order not greater than m are non-negative. The class of m-times positive sequences is denoted by PF_m. A sequence is called a multiply positive sequence if it is m-times positive for some $m \in \mathbf{N}$ (see [16]). A sequence $\{a_k\}_{k=0}^{\infty}$ such that all minors of the infinite matrix (3) are non-negative is called a totally positive sequence. The class of totally positive sequences is denoted by PF_∞. The corresponding classes of generating functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

are also denoted by PF_m and PF_∞.

The multiply positive sequences (also called Pólya frequency sequences) were introduced by Fekete and Pólya in 1912 see [7] in connection with the problem of exact calculation of the number of positive zeros of a real polynomial.

The class PF_∞ was completely described by Aissen et al. in [1] (see also [12, p. 412]):

Theorem ASWE. *A function $f \in PF_\infty$ iff*

$$f(z) = Cz^n e^{\gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z)/(1 - \beta_k z),$$

where $C \geqslant 0$, $n \in \mathbf{Z}$, $\gamma \geqslant 0$, $\alpha_k \geqslant 0$, $\beta_k \geqslant 0$, $\sum(\alpha_k + \beta_k) < \infty$.

By Theorem ASWE a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \geqslant 0$, has only real zeros if and only if the sequence $(a_0, a_1, \ldots, a_n, 0, 0, \ldots) \in PF_\infty$.

In 1926, Hutchinson [11, p. 327] extended the work of Petrovitch [15] and Hardy [9] or [10, pp. 95–100] and proved the following theorem.

Theorem B. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, $\forall k$. Inequality*

$$a_n^2 \geqslant 4a_{n-1}a_{n+1}, \quad \forall n \geqslant 1 \tag{4}$$

holds if and only if the following two properties hold:

(i) *the zeros of f(x) are all real, simple and negative and*
(ii) *the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, formed by taking any number of consecutive terms of $f(x)$, are all real and non-positive.*

It is easy to see that (4) implies

$$a_n \leqslant \frac{a_1}{4^{n(n-1)/2}} \left(\frac{a_1}{a_0}\right)^{n-1}, \quad n \geqslant 2,$$

that is f is an entire function of the order 0. So by the Hadamard theorem (see, for example, [14, p. 24])

$$f(z) = Cz^n \prod_{k=1}^{\infty} (1 + \alpha_k z),$$

where $C \geqslant 0$, $n \in \mathbf{N} \cup \{\mathbf{0}\}$, $\alpha_k \geqslant 0$, $\sum(\alpha_k) < \infty$.

Using ASWE Theorem we obtain from Theorem B that

$$a_n^2 \geqslant 4\, a_{n-1}a_{n+1}, \ \forall n \geqslant 1 \Rightarrow \{a_n\}_{n=0}^{\infty} \in PF_\infty. \tag{5}$$

In [13] it was proved that the constant 4 in (5) is sharp.

Thus, Theorem B provides a simple sufficient condition for deducing when a sequence is a totally positive sequence. Theorem 5 provides the following simple sufficient condition of multiple positivity for a sequence.

Corollary of Theorem 5. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of non-negative numbers. Then*

$$a_n^2 \geqslant c_m a_{n-1}a_{n+1}, \quad \forall n \geqslant 1 \ \Rightarrow \ \{a_n\}_{n=0}^{\infty} \in PF_m.$$

Our results are applicable also to the moment problem. Recall that a sequence of positive numbers $\{s_k\}_{k=0}^{\infty}$ is said to be the moment sequence of a non-decreasing function $F : \mathbf{R} \to \mathbf{R}$ if

$$s_k = \int_{-\infty}^{\infty} t^k \,\mathrm{d}F(t).$$

A sequence of positive numbers is called a Hamburger moment sequence if it is a moment sequence of a function F having infinitely many points of growth. The following famous theorem gives the description of Hamburger moment sequences.

Theorem C ([8], see also [2, Chapter 2]). *A sequence of positive numbers $\{s_k\}_{k=0}^{\infty}$ is a Hamburger moment sequence if and only if*

$$\det \begin{pmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k} \end{pmatrix} > 0, \quad k = 0, 1, 2, \ldots \tag{6}$$

The following statement is proved in [4].

Theorem D. *Let d be the positive solution of $\sum_{n=1}^{\infty} d^{-n^2} = 1/4$ $(d \approx 4.06)$. Then any positive sequence $\{s_k\}_{k=0}^{\infty}$ satisfying*

$$s_{n-1}s_{n+1} \geqslant d s_n^2, \quad n = 0, 1, 2, \ldots$$

is a Hamburger moment sequence.

Theorem 3 implies the following statement.

Corollary of Theorem 3. *Any positive sequence $\{s_k\}_{k=0}^{\infty}$ satisfying*

$$s_{n-1}s_{n+1} \geqslant 4 s_n^2, \quad n = 0, 1, 2, \ldots$$

is a Hamburger moment sequence.

The constant 4 in the corollary above cannot be improved.

2. Proof of Theorem 1

We need the following sequence of functions:

$$F_m(c) = \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m-j}{j} (-1)^j \frac{1}{c^j}, \quad m = 0, 1, 2, \ldots, \; c \geqslant 1, \tag{7}$$

where by $\lfloor x \rfloor$ we denote the integral part of x.

The following lemma provides some properties for this sequence of functions.

Lemma 1

(i) *The following identities hold*

$$F_0(c) = F_1(c) = 1,$$
$$F_m(c) = F_{m-1}(c) - \frac{1}{c}F_{m-2}(c), \quad m = 2, 3, 4, \dots \tag{8}$$

(ii) *For $c = 4\cos^2\phi$ we have*

$$F_m(c) = \frac{\sin(m+1)\phi}{c^{m/2}\sin\phi}. \tag{9}$$

(iii) *For $c_k = 4\cos^2\frac{\pi}{k+1}$ we have*

$$F_{j-1}(c_k) - \frac{1}{c_k^2}F_{j-2}(c_k) - \frac{1}{c_k^j} \geqslant F_j(c_k), \quad k \geqslant 3, \quad j = 2, 3, \dots, k-1. \tag{10}$$

Proof. Formula (8) follows directly from (7). Formula (9) is a simple consequence of the well-known trigonometric identity (see, for example, [17, p. 696])

$$\frac{\sin(m+1)\phi}{\sin\phi} = \sum_{j=0}^{\lfloor m/2\rfloor}\binom{m-j}{j}(-1)^j(2\cos\phi)^{m-2j}.$$

Using the identity $4\cos^2\phi - 1 = \frac{\sin(3\phi)}{\sin\phi}$ we have

$$\begin{aligned}
&F_{j-1}(c_k) - \frac{1}{c_k^2}F_{j-2}(c_k) - \frac{1}{c_k^j} - F_j(c_k)\\
&\quad= \left(\frac{1}{c_k} - \frac{1}{c_k^2}\right)F_{j-2}(c_k) - \frac{1}{c_k^j}\\
&\quad= \frac{1}{c_k^{(j+2)/2}}\left(\frac{\sin\left(3\frac{\pi}{k+1}\right)}{\sin\frac{\pi}{k+1}}\cdot\frac{\sin\left((j-1)\frac{\pi}{k+1}\right)}{\sin\frac{\pi}{k+1}} - \frac{1}{\left(2\cos\frac{\pi}{k+1}\right)^{j-2}}\right)\\
&\quad\geqslant \frac{1}{c_k^{(j+2)/2}}\left(\frac{\sin\left(3\frac{\pi}{k+1}\right)}{\sin\frac{\pi}{k+1}}\cdot\frac{\sin\left((j-1)\frac{\pi}{k+1}\right)}{\sin\frac{\pi}{k+1}} - 1\right) \geqslant 0,
\end{aligned}$$

for $k \geqslant 3$ and $j = 2, 3, \dots, k-1$. Inequality (10) is proved.

Lemma 1 is proved. □

The following lemma was proved in [5].

Lemma A. *Let $M = (a_{i,j})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and $M \in TP_2(c)$, $c \geqslant 1$. Then*

$$a_{i,j}a_{k,l} \geqslant c^{(l-j)(k-i)}a_{i,l}a_{k,j} \quad \textit{for all } i < k,\ j < l.$$

A simple consequence of Lemma A is the fact that if $M \in TP_2(c)$ then any submatrix of M also belongs to $TP_2(c)$. Analogously if $M \in STP_2(c)$ then any submatrix of M also belongs to $STP_2(c)$.

For a matrix $M = (a_{i,j})$ we will denote by $M\binom{i_1,i_2,\ldots,i_k}{j_1,j_2,\ldots,j_k}$ the following submatrix of M:

$$M\binom{i_1,i_2,\ldots,i_k}{j_1,j_2,\ldots,j_k} = \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_k} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_k,j_1} & a_{i_k,j_2} & \cdots & a_{i_k,j_k} \end{pmatrix}.$$

We now prove the following claim (which consists of three parts) by induction on n. Let $M = (a_{i,j})$ be an $n \times n$ matrix and $M \in TP_2(c)$, where $c \geqslant 4\cos^2 \frac{\pi}{n+1}$. Then the following inequalities hold:

$$\det M \geqslant 0, \tag{11}$$

$$\det M \geqslant a_{1,1} \det M\binom{2,3,\ldots,n}{2,3,\ldots,n} - a_{1,2}a_{2,1} \det M\binom{3,4,\ldots,n}{3,4,\ldots,n}, \tag{12}$$

$$\det M \leqslant a_{1,1} \det M\binom{2,3,\ldots,n}{2,3,\ldots,n}. \tag{13}$$

Since $M \in TP_2(c)$ then hypothesis (11)–(13) are true for $n = 2$. The proof below is based on the following lemma.

Lemma 2. *Let $c_0 \geqslant 1$, $M = (a_{i,j}) \in TP_2(c_0)$ be an $n \times n$ matrix satisfying the following conditions*:

(i) $\forall i = 2, 3, \ldots, n \quad \det M\binom{i,i+1,\ldots,n}{i,i+1,\ldots,n} \geqslant 0$;

(ii) $\forall i = 1, 2, \ldots, n-2$

$$\det M\binom{i,i+1,\ldots,n}{i,i+1,\ldots,n} \geqslant a_{i,i} \det M\binom{i+1,i+2,\ldots,n}{i+1,i+2,\ldots,n} - a_{i,i+1}a_{i+1,i} \det M\binom{i+2,i+3,\ldots,n}{i+2,i+3,\ldots,n}.$$

Then for all c, $1 \leqslant c \leqslant c_0$ the following inequalities are valid:

$$\begin{aligned} &\det M\binom{m+1,m+2,\ldots,n}{m+1,m+2,\ldots,n} \\ &\quad \geqslant a_{m+1,m+1}\left(\det M\binom{m+2,m+3,\ldots,n}{m+2,m+3,\ldots,n} - \frac{1}{c}a_{m+2,m+2} \det M\binom{m+3,m+4,\ldots,n}{m+3,m+4,\ldots,n}\right), \\ &\quad m = 0, 1, \ldots, n-3. \end{aligned} \tag{14}$$

$$\begin{aligned} \det M \geqslant a_{1,1}a_{2,2}\cdots a_{m,m}\Bigg(& F_m(c) \det M\binom{m+1,m+2,\ldots,n}{m+1,m+2,\ldots,n} \\ & -\frac{1}{c}F_{m-1}(c)a_{m+1,m+1} \det M\binom{m+2,m+3,\ldots,n}{m+2,m+3,\ldots,n}\Bigg), \quad m = 1, 2, \ldots, n-2. \end{aligned} \tag{15}$$

$$F_m(c) \det M\binom{m+1,m+2,\ldots,n}{m+1,m+2,\ldots,n} - \frac{1}{c}F_{m-1}(c)a_{m+1,m+1} \det M\binom{m+2,m+3,\ldots,n}{m+2,m+3,\ldots,n}$$

$$\geqslant a_{m+1,m+1}\Bigg(F_{m+1}(c)\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}$$

$$-\frac{1}{c}F_m(c)a_{m+2,m+2}\det M\binom{m+3,m+4,\dots,n}{m+3,m+4,\dots,n}\Bigg),\quad m=1,2,\dots,n-3. \tag{16}$$

$$F_m(c)\det M\binom{m+1,m+2,\dots,n}{m+1,m+2,\dots,n}-\frac{1}{c}F_{m-1}(c)a_{m+1,m+1}\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}$$

$$\geqslant a_{m+1,m+1}a_{m+2,m+2}\cdots a_{n,n}F_n(c),\quad m=1,2,\dots,n-2. \tag{17}$$

Proof. First we prove (14). Since $M\in TP_2(c)$ and by (ii) we have

$$\det M\binom{m+1,m+2,\dots,n}{m+1,m+2,\dots,n}$$

$$\geqslant a_{m+1,m+1}\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}-a_{m+1,m+2}a_{m+2,m+1}\det M\binom{m+3,m+4,\dots,n}{m+3,m+4,\dots,n}$$

$$\geqslant a_{m+1,m+1}\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}-\frac{1}{c}a_{m+1,m+1}a_{m+2,m+2}\det M\binom{m+3,m+4,\dots,n}{m+3,m+4,\dots,n},$$

$$m=0,1,\dots,n-3.$$

Inequality (14) is proved.

Let us prove (16). Multiplying (14) by $F_m(c)$ we have

$$F_m(c)\det M\binom{m+1,m+2,\dots,n}{m+1,m+2,\dots,n}-\frac{1}{c}F_{m-1}(c)a_{m+1,m+1}\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}$$

$$\geqslant a_{m+1,m+1}\Bigg(\bigg(F_m(c)-\frac{1}{c}F_{m-1}(c)\bigg)\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}$$

$$-\frac{1}{c}F_m(c)a_{m+2,m+2}\det M\binom{m+3,m+4,\dots,n}{m+3,m+4,\dots,n}\Bigg),\quad m=1,2,\dots,n-3,$$

and, using (8) we obtain (16).

To prove (17) we apply (16) $(n-2-m)$ times. We derive

$$F_m(c)\det M\binom{m+1,m+2,\dots,n}{m+1,m+2,\dots,n}-\frac{1}{c}F_{m-1}(c)a_{m+1,m+1}\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}$$

$$\geqslant a_{m+1,m+1}a_{m+2,m+2}\cdots a_{n-2,n-2}\left(F_{n-2}(c)\det M\binom{n-1,n}{n-1,n}-\frac{1}{c}F_{n-3}(c)a_{n-1,n-1}a_{n,n}\right).$$

Since $M\in TP_2(c_0)$ the following inequality holds for all c, $1\leqslant c\leqslant c_0$,

$$\det M\binom{n-1,n}{n-1,n}\geqslant\left(1-\frac{1}{c}\right)a_{n-1,n-1}a_{n,n}, \tag{18}$$

so by (8) we obtain

$$F_m(c)\det M\binom{m+1,m+2,\dots,n}{m+1,m+2,\dots,n}-\frac{1}{c}F_{m-1}(c)a_{m+1,m+1}\det M\binom{m+2,m+3,\dots,n}{m+2,m+3,\dots,n}$$

$$\geqslant a_{m+1,m+1}a_{m+2,m+2}\cdots a_{n,n}\left(\left(F_{n-2}(c)-\frac{1}{c}F_{n-3}(c)\right)-\frac{1}{c}F_{n-2}(c)\right)$$

$$= a_{m+1,m+1}a_{m+2,m+2}\cdots a_{n,n}\left(F_{n-1}(c)-\frac{1}{c}F_{n-2}(c)\right)$$
$$= a_{m+1,m+1}a_{m+2,m+2\cdots a_{n,n}}F_n(c).$$

Inequality (17) is proved.

By (8) we rewrite inequality (14) for $m = 0$ in the following form:

$$\det M \geqslant a_{1,1}\left(F_1(c)\det M\begin{pmatrix}2,3,\ldots,n\\2,3,\ldots,n\end{pmatrix}-\frac{1}{c}F_0(c)a_{2,2}\det M\begin{pmatrix}3,4,\ldots,n\\3,4,\ldots,n\end{pmatrix}\right).$$

To prove (15) we apply (16) $(m-1)$ times.

Lemma 2 is proved. □

Remark. If a matrix M satisfies the conditions of Lemma 2 and, moreover, $a_{n-1,n-1}a_{n,n} > c_0 a_{n-1,n}a_{n,n-1}$, then inequality (18) is strict, hence (17) is strict, i.e.,

$$F_m(c)\det M\begin{pmatrix}m+1,m+2,\ldots,n\\m+1,m+2,\ldots,n\end{pmatrix}-\frac{1}{c}F_{m-1}(c)a_{m+1,m+1}\det M\begin{pmatrix}m+2,m+3,\ldots,n\\m+2,m+3,\ldots,n\end{pmatrix}$$
$$> a_{m+1,m+1}a_{m+2,m+2}\cdots a_{n,n}F_n(c), \quad m = 1,2,\ldots,n-2. \tag{19}$$

In particular, for all matrices $M \in STP(c_0)$ inequality (19) is valid for all c, $1 \leqslant c \leqslant c_0$.

Assume that conditions (11)–(13) hold for all matrices of sizes smaller than k. Let us prove these conditions for $n = k$.

Lemma 3. *Let $M = (a_{i,j})$ be a $k \times k$ matrix, $M \in TP_2(c)$, $c \geqslant c_k := 4\cos^2\frac{\pi}{k+1}$. For all $j = 2,3,\ldots,k-1$ the following inequality holds:*

$$a_{1,j}\det M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j-1,j+1,\ldots,k\end{pmatrix}-a_{1,j+1}\det M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j,j+2,\ldots,k\end{pmatrix}\geqslant 0.$$

Proof. Since $m \in TP_2(c)$, $M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j-1,j+1,\ldots,k\end{pmatrix} \in TP_2(c)$ and $M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j,j+2,\ldots,k\end{pmatrix} \in TP_2(c)$. Since $4\cos^2\frac{\pi}{n+1} \leqslant 4\cos^2\frac{\pi}{k+1}$ for $n = 2,3,\ldots,k-1$ we can apply the induction hypothesis to the matrices $M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j-1,j+1,\ldots,k\end{pmatrix}$, $M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j,j+2,\ldots,k\end{pmatrix}$ and to all their square submatrices. We apply inequality (13) j times and obtain

$$\det M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j,j+2,\ldots,k\end{pmatrix}\leqslant a_{2,1}a_{3,2}\cdots a_{j+1,j}\det M\begin{pmatrix}j+2,j+3,\ldots,k\\j+2,j+3,\ldots,k\end{pmatrix}.$$

From Lemma A and from the fact $a_{1,j+1}a_{j+1,j} \leqslant \frac{1}{c_k^j}a_{1,j}a_{j+1,j+1}$ now we conclude

$$a_{1,j+1}\det M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j,j+2,\ldots,k\end{pmatrix}\leqslant\frac{1}{c_k^j}a_{1,j}a_{2,1}a_{3,2}\cdots a_{j,j-1}a_{j+1,j+1}\det M\begin{pmatrix}j+2,j+3,\ldots,k\\j+2,j+3,\ldots,k\end{pmatrix}. \tag{20}$$

By the induction hypothesis the matrix $M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j-1,j+1,\ldots,k\end{pmatrix}$ satisfies the assumptions of Lemma 2. Applying to this matrix (15) with $m = j-2$ we obtain

$$\det M\begin{pmatrix}2,3,\ldots,k\\1,2,\ldots,j-1,j+1,\ldots,k\end{pmatrix}$$

$$\geqslant a_{2,1}a_{3,2}\cdots a_{j-1,j-2}\Bigg(F_{j-2}(c_k)\det M\binom{j,j+1,j+2,\ldots,k}{j-1,j+1,j+2,\ldots,k}$$
$$-\frac{1}{c_k}F_{j-3}(c_k)a_{j,j-1}\det M\binom{j+1,j+2,\ldots,k}{j+1,j+2,\ldots,k}\Bigg).$$

Applying (12) to the matrix $M\binom{j,j+1,j+2,\ldots,k}{j-1,j+1,j+2,\ldots,k}$ and plugging the result into the last formula we have

$$\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k}$$
$$\geqslant a_{2,1}a_{3,2}\cdots a_{j-1,j-2}\Bigg(a_{j,j-1}\left(F_{j-2}(c_k)-\frac{1}{c_k}F_{j-3}(c_k)\right)\det M\binom{j+1,j+2,\ldots,k}{j+1,j+2,\ldots,k}$$
$$-a_{j,j+1}a_{j+1,j-1}F_{j-2}(c_k)\det M\binom{j+2,j+3,\ldots,k}{j+2,j+3,\ldots,k}\Bigg),$$

whence, by Lemma A and (8) we obtain

$$\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k}$$
$$\geqslant a_{2,1}a_{3,2}\cdots a_{j-1,j-2}a_{j,j-1}\Bigg(F_{j-1}(c_k)\det M\binom{j+1,j+2,\ldots,k}{j+1,j+2,\ldots,k}$$
$$-\frac{1}{c_k^2}a_{j+1,j+1}F_{j-2}(c_k)\det M\binom{j+2,j+3,\ldots,k}{j+2,j+3,\ldots,k}\Bigg).$$

Further applying (14) to the matrix $M\binom{j+1,j+2,\ldots,k}{j+1,j+2,\ldots,k}$ we have

$$\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k}$$
$$\geqslant a_{2,1}a_{3,2}\cdots a_{j,j-1}a_{j+1,j+1}\Bigg(\det M\binom{j+2,j+3,\ldots,k}{j+2,j+3,\ldots,k}\Bigg(F_{j-1}(c_k)$$
$$-\frac{1}{c_k^2}F_{j-2}(c_k)\Bigg)-\frac{1}{c_k}a_{j+2,j+2}F_{j-1}(c_k)\det M\binom{j+3,j+4,\ldots,k}{j+3,j+4,\ldots,k}\Bigg). \tag{21}$$

By (20) and (21) we derive

$$a_{1,j}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k}-a_{1,j+1}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j,j+2,\ldots,k}$$
$$\geqslant a_{1,j}a_{2,1}a_{3,2}\cdots a_{j,j-1}a_{j+1,j+1}\Bigg(\Bigg(F_{j-1}(c_k)-\frac{1}{c_k^2}F_{j-2}(c_k)-\frac{1}{c_k^j}\Bigg)$$
$$\times\det M\binom{j+2,j+3,\ldots,k}{j+2,j+3,\ldots,k}-\frac{1}{c_k}a_{j+2,j+2}F_{j-1}(c_k)\det M\binom{j+3,j+4,\ldots,k}{j+3,j+4,\ldots,k}\Bigg). \tag{22}$$

It follows from (22), (10) and (17) that

$$a_{1,j}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k}-a_{1,j+1}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j,j+2,\ldots,k}$$

$$\geqslant a_{1,j}a_{2,1}a_{3,2}\cdots a_{j,j-1}a_{j+1,j+1}\Bigg(F_j(c_k)\det M\binom{j+2,j+3,\ldots,k}{j+2,j+3\ldots,k}$$
$$-\frac{1}{c_k}a_{j+2,j+2}F_{j-1}(c_k)\det M\binom{j+3,j+4,\ldots,k}{j+3,j+4,\ldots,k}\Bigg)$$
$$\geqslant a_{1,j}a_{2,1}a_{3,2}\cdots a_{j,j-1}a_{j+1,j+1}a_{j+2,j+2}\cdots a_{k,k}F_{k-1}(c_k).$$

Hence by Lemma 1 and (9) with $m = k-1$ we conclude that

$$a_{1,j}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k} - a_{1,j+1}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j,j+2,\ldots,k}$$
$$\geqslant a_{1,j}a_{2,1}a_{3,2}\cdots a_{j,j-1}a_{j+1,j+1}a_{j+2,j+2}\cdots a_{k,k}\frac{\sin(k\frac{\pi}{k+1})}{c_k^{(k-1)/2}\sin\frac{\pi}{k+1}} \geqslant 0.$$

Lemma 3 is proved. □

Now we will prove (12). Using Lemma 3 we have

$$\det M\binom{1,2,\ldots,k}{1,2,\ldots,k} = \sum_{j=1}^{k}(-1)^{j+1}a_{1,j}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k}$$
$$\geqslant a_{1,1}\det M\binom{2,3,\ldots,k}{2,3,\ldots,k} - a_{1,2}\det M\binom{2,3,\ldots,k}{1,3,4,\ldots,k}.$$

We apply the induction hypothesis (13) to the matrix $M\binom{2,3,\ldots,k}{1,3,4,\ldots,k}$. We have

$$\det M\binom{1,2,\ldots,k}{1,2,\ldots,k} \geqslant a_{1,1}\det M\binom{2,3,\ldots,k}{2,3,\ldots,k} - a_{1,2}a_{2,1}\det M\binom{3,4,\ldots,k}{3,4,\ldots,k}.$$

The inequality (12) is proved.

By Lemma 3

$$\det M\binom{1,2,\ldots,k}{1,2,\ldots,k} = \sum_{j=1}^{k}(-1)^{j+1}a_{1,j}\det M\binom{2,3,\ldots,k}{1,2,\ldots,j-1,j+1,\ldots,k} \leqslant a_{1,1}\det M\binom{2,3,\ldots,k}{2,3,\ldots,k}.$$

The inequality (13) is proved.

To prove (11) we note that by (12) and induction hypothesis the matrix M satisfies the assumptions of Lemma 2. It follows from (15), (17) and Lemma 1 that

$$\det M \geqslant a_{1,1}a_{2,2}\cdots a_{k,k}F_k(c_k) = a_{1,1}a_{2,2}\cdots a_{k,k}\frac{\sin\pi}{c_k^{k/2}\sin\frac{\pi}{k+1}} = 0.$$

Hence the statement (i) in Theorem 1 is proved.

Now we will prove the statement (ii) in Theorem 4. If $M \in STP_k(c_k)$ then by (19) we can rewrite the last inequality in the following form:

$$\det M > a_{1,1}a_{2,2}\cdots a_{k,k}F_k(c_k) = a_{1,1}a_{2,2}\cdots a_{k,k}\frac{\sin\pi}{c_k^{k/2}\sin\frac{\pi}{k+1}} = 0.$$

Hence the statement (ii) in Theorem 1 is proved, which completes the proof of Theorem 1. □

In fact, we have proved a slightly stronger theorem, which may be of independent interest.

Theorem 6. *Suppose* $c \geqslant 4\cos^2 \frac{\pi}{k+1}$. *Let* $M = (a_{i,j}) \in TP_2(c)$ *be a* $k \times k$ *matrix. Then*

$$\det M \geqslant a_{1,1}a_{2,2}\cdots a_{k,k}F_k(c).$$

3. Proof of Theorem 4

Note that $TP_2(c_1) \subset TP_2(c_2)$ for $c_1 \geqslant c_2$. Thus it is sufficient to prove Theorem 4 with $c \in (c_k - \varepsilon, c_k)$ for $\varepsilon > 0$ being small enough.

Consider the following $n \times n$ symmetrical Toeplitz matrix.

$$M_n(\phi) := \begin{Vmatrix} 2\cos\phi & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\phi & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos\phi & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 2\cos\phi & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2\cos\phi \end{Vmatrix}, \tag{23}$$

where $0 \leqslant \phi < \pi/2$. Obviously, $M_n(\phi) \in TP_2(4\cos^2\phi)$. The matrix $M_n(\phi)$ satisfies the following recursion relation $\det M_n(\phi) = 2\cos\phi \det M_{n-1}(\phi) - \det M_{n-2}(\phi)$ and $M_1(\phi) = 2\cos\phi$, $M_2(\phi) = 4\cos^2\phi - 1$. It is easy to verify that $\det M_n(\phi) = \frac{\sin(n+1)\phi}{\sin\phi}$. So for all $\phi \in \left(\frac{\pi}{n+1}, \frac{2\pi}{n+1}\right)$ we have $\det M_n(\phi) < 0$. For $\phi \in \left(\frac{\pi}{n+1}, \frac{2\pi}{n+1}\right)$ consider the following $n \times n$ symmetrical Toeplitz matrix

$$T_n(\phi, \varepsilon_1, \ldots, \varepsilon_{n-2}) := \begin{Vmatrix} 2\cos\phi & 1 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{n-3} & \varepsilon_{n-2} \\ 1 & 2\cos\phi & 1 & \varepsilon_1 & \cdots & \varepsilon_{n-4} & \varepsilon_{n-3} \\ \varepsilon_1 & 1 & 2\cos\phi & 1 & \varepsilon_1 & \cdots & \varepsilon_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \varepsilon_{n-3} & \varepsilon_{n-4} & \cdots & \varepsilon_1 & 1 & 2\cos\phi & 1 \\ \varepsilon_{n-2} & \varepsilon_{n-3} & \varepsilon_{n-4} & \cdots & \varepsilon_1 & 1 & 2\cos\phi \end{Vmatrix}, \tag{24}$$

where $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{n-2} > 0$ and ε_1 is chosen to satisfy the inequality $1 \geqslant 4\cos^2\phi \cdot 2\cos\phi \cdot \varepsilon_1$, then ε_2 is chosen to satisfy the inequality $\varepsilon_1^2 \geqslant 4\cos^2\phi \cdot \varepsilon_2$, then ε_3 is chosen to satisfy the inequality $\varepsilon_2^2 \geqslant 4\cos^2\phi \cdot \varepsilon_1 \cdot \varepsilon_3, \ldots$ and then ε_{n-2} is chosen to satisfy the inequality $\varepsilon_{n-3}^2 \geqslant 4\cos^2\phi \cdot \varepsilon_{n-4} \cdot \varepsilon_{n-2}$. Under these conditions we have $T_n(\phi, \varepsilon_1, \ldots, \varepsilon_{n-2}) \in TP_2(4\cos^2\phi)$. Since $T_n(\phi, 0, 0, \ldots, 0) = M_n(\phi)$ we obtain $\det T_n(\phi, 0, 0, \ldots, 0) < 0$ for $\phi \in \left(\frac{\pi}{n+1}, \frac{2\pi}{n+1}\right)$. Therefore we have $\det T_n(\phi, \varepsilon_1, \ldots, \varepsilon_{n-2}) < 0$ for $\phi \in \left(\frac{\pi}{n+1}, \frac{2\pi}{n+1}\right)$ if ε_1 is small enough.

Thus, for every $c \in (4\cos^2\frac{2\pi}{n+1}, c_n)$ the statement (i) of Theorem 4 is proved. Since $TP_2(c_1) \subset TP_2(c_2)$ for $c_1 \geqslant c_2$ the statement (i) of Theorem 4 follows.

We use the same method to obtain the proof of Theorem 5.

To prove the statement (ii) we consider the following Hankel matrix $D_n(p,q)$ with $p \geqslant 1$, $q \geqslant 1$.

$$D_n(p,q) := \left(p^{\lfloor (i+j-2)/2\rfloor\lfloor (i+j-1)/2\rfloor} q^{\lfloor (i+j-3)/2\rfloor\lfloor (i+j-2)/2\rfloor}\right), \quad 1 \leqslant i, j \leqslant n, \tag{25}$$

or,

$$D_n(p,q) = \begin{Vmatrix} 1 & 1 & p & p^2q & \cdots & * & * \\ 1 & p & p^2q & p^4q^2 & \cdots & * & * \\ p & p^2q & p^4q^2 & p^6q^4 & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ * & * & * & * & \cdots & p^{(n-2)^2}q^{(n-2)(n-3)} & p^{(n-1)(n-2)}q^{(n-2)^2} \\ * & * & * & * & \cdots & p^{(n-1)(n-2)}q^{(n-2)^2} & p^{(n-1)^2}q^{(n-1)(n-2)} \end{Vmatrix}. \tag{26}$$

By direct calculation we obtain $D_n(p,q) \in TP_2(\min(p,q))$.

Lemma 4. *For all $n \geqslant 3$ we have*

$$\det D_n(p,q) = p^{\beta_n} q^{\alpha_n} F_n(p) + Q_{\alpha_n - 1}(p,q), \tag{27}$$

where $\alpha_n = \frac{n(n-1)(n-2)}{3}$, $\beta_n = \frac{n(n-1)(2n-1)}{6}$ and $Q_{\alpha_n-1}(p,q)$ is a polynomial in p, q such that $\deg_q Q_{\alpha_n-1}(p,q) \leqslant \alpha_n - 1$. (*Here and further by* $\deg_q Q(p,q)$ *we will denote the degree of $Q(p,q)$ with respect to q.*)

Proof. We will prove this lemma by induction in n. For $n = 3$ the statement is true as can be verified directly. The expansion of $\det D_n(p,q)$ along column n gives

$$\begin{aligned} &\det D_n(p,q) \\ &\quad = R_{\alpha_n-1}(p,q) \\ &\qquad + \det \begin{Vmatrix} 1 & 1 & p & p^2q & \cdots & * & 0 \\ 1 & p & p^2q & p^4q^2 & \cdots & * & 0 \\ p & p^2q & p^4q^2 & p^6q^4 & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ * & * & * & * & \cdots & * & 0 \\ * & * & * & * & \cdots & p^{(n-2)^2}q^{(n-2)(n-3)} & p^{(n-1)(n-2)}q^{(n-2)^2} \\ * & * & * & * & \cdots & p^{(n-1)(n-2)}q^{(n-2)^2} & p^{(n-1)^2}q^{(n-1)(n-2)} \end{Vmatrix}, \end{aligned} \tag{28}$$

where $R_{\alpha_n-1}(p,q)$ is a polynomial in p, q and $\deg_q R_{\alpha_n-1}(p,q) \leqslant \alpha_n - 1$.

The expansion of the determinant on the right-hand side of the last equation along row n gives

$$\begin{aligned} &\det D_n(p,q) \\ &\quad = S_{\alpha_n-1}(p,q) \end{aligned}$$

$$+\det\begin{Vmatrix}1 & 1 & p & p^2q & \cdots & * & 0\\ 1 & p & p^2q & p^4q^2 & \cdots & * & 0\\ p & p^2q & p^4q^2 & p^6q^4 & \cdots & * & 0\\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots\\ * & * & * & * & \cdots & * & 0\\ * & * & * & * & \cdots & p^{(n-2)^2}q^{(n-2)(n-3)} & p^{(n-1)(n-2)}q^{(n-2)^2}\\ 0 & 0 & 0 & \cdots & 0 & p^{(n-1)(n-2)}q^{(n-2)^2} & p^{(n-1)^2}q^{(n-1)(n-2)}\end{Vmatrix}, \tag{29}$$

where $S_{\alpha_n-1}(p,q)$ is a polynomial in p,q and $\deg_q S_{\alpha_n-1}(p,q)\leqslant\alpha_n-1$.

The last equation provides the following recursion relation:

$$D_n(p,q)=p^{(n-1)^2}q^{(n-1)(n-2)}D_{n-1}(p,q)-p^{2(n-1)(n-2)}q^{2(n-2)^2}D_{n-2}(p,q)$$
$$+T_{\alpha_n-1}(p,q),$$

where $T_{\alpha_n-1}(p,q)$ is a polynomial in p,q and $\deg_q T_{\alpha_n-1}(p,q)\leqslant\alpha_n-1$.

Using the induction hypothesis and formula (8) we obtain the statement of Lemma 4.

Lemma 4 is proved. □

Note that $p^{\lfloor n/2\rfloor}F_n(p)$ is a polynomial in p of degree $\lfloor n/2\rfloor$. By (9) it has the following $\lfloor n/2\rfloor$ roots:

$$4\cos^2\frac{\pi}{n+1},4\cos^2\frac{2\pi}{n+1},\ldots,4\cos^2\frac{\lfloor n/2\rfloor\pi}{n+1}.$$

Obviously, $4\cos^2\frac{\pi}{n+1}$ is the largest root of this polynomial. Hence for $p\in\left(4\cos^2\frac{2\pi}{n+1},4\cos^2\frac{\pi}{n+1}\right)$ we have $F_n(p)<0$.

Let us fix an arbitrary $p_0\in(4\cos^2\frac{2\pi}{n+1},4\cos^2\frac{\pi}{n+1})$. Since

$$\det D_n(p_0,q)=q^{\alpha_n}(p_0^{\beta_n}F_n(p_0)+q^{-\alpha_n}Q_{\alpha_n-1}(p_0,q)),$$

where $Q_{\alpha_n-1}(p_0,q)$ is a polynomial in q and $\deg Q_{\alpha_n-1}(p_0,q)\leqslant\alpha_n-1$, for q being large enough (and $q>p_0$) we obtain $D_n(p_0,q)\in TP_2(p_0)$ but $\det D_n(p_0,q)<0$.

Thus, for every $p\in\left(4\cos^2\frac{2\pi}{n+1},c_n\right)$ the statement (ii) of Theorem 4 is proved. Since $TP_2(c_1)\subset TP_2(c_2)$ for $c_1\geqslant c_2$ the statement (ii) of Theorem 4 follows.

Theorem 4 is proved. □

Remark. This is a revised version of the paper originally submitted to the journal "Linear Algebra and its Applications" in summer 2004. Recently in the paper [6] the authors formulated a conjecture which coincides with the statement proved in our Theorem 1.

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