## Note

# A sufficient condition guaranteeing large cycles in graphs 

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#### Abstract

We generalize Bedrossian-Chen-Schelp's condition (1993) for the existence of large cycles in graphs, and give infinitely many examples of graphs which fulfill the new condition for hamiltonicity, while the related condition by Bedrossian, Chen, and Schelp is not fulfilled.


## 1. Introduction

Througout, the graphs $K_{1,3}$ and $K_{1,3}+e, e$ is an edge, are called a claw and a modified claw. Let $C(G)$ denote the set of all pairs of non-adjacent vertices of each induced claw or induced modified claw of $G$. Let $G$ be a graph of order $n$ and let $k$ be an integer; $0 \leqslant k \leqslant n$. The graph $G$ is said to satisfy the property $P C(k)$ if $\max \{d(x)$, $d(y)\} \geqslant k / 2$ for each pair of vertices $[x, y] \in C(G)$, where $d(u)$ denotes the degree of the vertex $u$. Considering this property, Bedrossian, Chen and Schelp proved:

Theorem 1 ([1]). Let $G$ be a 2-connected graph of order $n \geqslant 3$ and let $3 \leqslant k \leqslant n$. If $G$ satisfies $P C(k)$ then $G$ contains a cycle of length $\geqslant k$.

For any subset $S$ of $V(G)$ the subgraph induced by the set $S$ is denoted by $\langle S\rangle$. Let $\psi_{G}(u, v)$ be the number of components of $\langle N(u)\rangle$ containing no neighbour of $v$ in $G$. Let $a b s(k)=\max \{k, 0\}$, and let $\alpha_{G}(u, v)$ denote the number of vertices $x \in V(G)$ such that $u x, v x \in E(G)$. Similarly, let $\beta_{G}(u, v)$ denote the number of vertices $x \in V(G)$ such that $x \neq v, u x, v x \notin E(G)$ and $\operatorname{dist}_{G}(u, x)=2$, where $\operatorname{dist}_{G}(a, b)$ denotes the distance

[^0]between the vertices $a$ and $b$ in $G$. Finally, let us define $\chi_{G}(u, v)=\psi_{G}(u, v)+$ $\operatorname{abs}\left(\alpha_{G}(u, v)-\beta_{G}(u, v)-1\right)$. In [3] it was proved that if for any two non-adjacent vertices $x, y$ of a graph $G$ of order $n$ it holds that $d(x)+d(y)+\max \left\{\psi_{G}(x, y)\right.$, $\left.\psi_{G}(y, x)\right\} \geqslant n$, then $G$ is hamiltonian, thus generalizing the well known Ore's result. The invariant $\chi$ was introduced in [4], where Bondy's [2] and Policky's [3] conditions for hamiltonicity were generalized. One can see that if two vertices $x$ and $y$ of a graph $G$ have "a small number" of neighbours in common, then $\psi_{G}(x, y)$ can be non-zero. If the vertices $x$ and $y$ have "many" neighbours in common, then $\psi_{G}(x, y)$ is "small" (often zero), but $\chi_{G}(x, y)$ can be non-zero. The complete bipartite graphs $G=K_{n, n}, n \geqslant 2$, provide examples of graphs in which $\psi_{G}(x, y)<\chi_{G}(x, y)$ for any pair of non-adjacent vertices $x, y$.

The aim of this note is to improve Theorem 1 by considering the invariant $\chi$ again. More precisely, we describe a parameter (related to $\chi$ ) which gives an adjustment to the degree requirement for non-adjacent vertices which are part of an induced claw or modified claw in the graph. The parameter, say $\omega(G)$, is defined as $\omega(G)=\min _{x y \notin E(G)}$ $\max \left\{\chi_{G}(x, y), \chi_{G}(y, x)\right\}$. Note that there exist graphs with non-zero $\omega$. The cycles $C_{n}$ can serve as simple examples. For other examples we refer the reader to Remark 1.

We define, involving the new parameter $\omega$, a property, say $P(k)$, (related to the property $P C(k)$ ) as follows. Let $G$ be a graph of order $n$ and let $k$ be an integer; $0 \leqslant k \leqslant n$. The graph $G$ is said to satisfy the property $P(k)$ if $\max \{d(x), d(y)\} \geqslant(k-\omega(G)) / 2$ for each pair of vertices $[x, y] \in C(G)$. Following this notation, our main result is then

Theorem 2. Let $G$ be a 2 -connected graph of order $n \geqslant 3$ and let $3 \leqslant k \leqslant n$. If $G$ satisfies $P(k)$ then $G$ contains a cycle of length $\geqslant k$.

The proof of Theorem 2 is based on Corollary 1 of Lemma 4 which is a generalization of the result of Bondy [2]. With Corollary 1, a proof of Theorem 2 can easily be obtained from [1, proof of Theorem 1] by replacing $k$ with $k-\omega(G)$ and $P C(k)$ with $P(k)$. Therefore the proof of Theorem 2 is omitted.

## 2. Results

We will need some definitions and auxiliary results. Let $P_{i}$ be a path. For simplicity, we will refer to the first vertex of $P_{i}$, as $f_{i}$ and to the last vertex of $P_{i}$ as $l_{i}$. If $P=\left(f, x_{1}, x_{2}, \ldots, x_{k}, l\right)$ is a path, then the reverse path to $P$ is the path $\bar{P}=\left(\bar{f}, x_{k}, x_{k-1}, \ldots, x_{1}, \bar{l}\right)$, where $\bar{f}=l$ and $\bar{l}=f$. When $u, v \in V(P)$ and $u$ precedes $v$ on P we write $u<_{P} v$. The subpath of $P$ starting at $u$ and ending at $v$ will be denoted by $[u, v]$; similarly, $[u, v]_{i}$ will denote the section of $P_{i}$. We write $p(v)$ and $s(v)$ for the predecessor and successor of $v$ on $P$, respectively. If $P_{i}$ and $P_{j}$ are two paths for which $l_{i}=f_{j}$, then the composition $P_{i} \cdot P_{j}$ is the path $\left[f_{i}, p\left(l_{i}\right)\right]_{i}$ followed by $P_{j}$. A path $P$ has
length $L(P)=|V(P)|-1$; a cycle $C$ has length $L(C)=|V(C)|$. Let $P, P_{i}$ and $P_{j}$ be paths such that $V(P) \cap V\left(P_{i}\right)=\left\{f_{i}, l_{i}\right\}, V(P) \cap V\left(P_{j}\right)=\left\{f_{j}, l_{j}\right\}$ and $V\left(P_{i}\right) \cap V\left(P_{j}\right)=\emptyset$. Then $P_{i}$ overlaps with $P_{j}$ on $P$ if $f_{i} \prec_{P} f_{j}<_{P} l_{i} \prec_{P} l_{j}$.

Lemma 1 ([2]). Let $G$ be a 2-connected graph and let $P$ be any path in $G$. Then for some $m \geqslant 1$, there is a sequence of $m$ pairwise edge-disjoint paths $P_{1}, \ldots, P_{m}$, satisfying

$$
f_{1}=f, \quad l_{m}=l, \quad V(P) \cap V\left(P_{i}\right)=\left\{f_{i}, l_{i}\right\}, \quad 1 \leqslant i \leqslant m
$$

and such that, for $1 \leqslant i<m-1, P_{i}$ overlaps with $P_{i+1}$ on $P$.

Lemma 2 ([4]). Let $u$, v be a pair of vertices of a graph $G$. Let $H$ be the graph induced by a set of vertices $S$ satisfying

$$
\{u\} \cup N(u) \cup\{v\} \cup N(v) \subseteq S \subseteq V(G)
$$

Then $\chi_{G}(u, v) \leqslant \chi_{H}(u, v)$ and $\chi_{G}(v, u) \leqslant \chi_{H}(v, u)$.
The proof of the following result can be found in [4]. Because of its importance in the proof of Lemma 4, we outline the proof.

Lemma 3 ([4]). Let $G$ be a graph with a hamiltonian path $P=\left(f, x_{1}, \ldots, x_{n-2}, l\right)$, where $f$ and $l$ are non-adjacent vertices with

$$
d(f)+d(l)+\max \left\{\chi_{G}(f, l), \chi_{G}(l, f)\right\} \geqslant n .
$$

Then there is an integer $i, 1 \leqslant i \leqslant n-3$, such that $\left(f, x_{i+1}\right),\left(x_{i}, l\right) \in E(G)$ and $G$ is hamiltonian.

Sketch of Proof. Suppose $\max \left\{\chi_{G}(f, l), \chi_{G}(l, f)\right\}=\chi_{G}(f, l)$. We proceed by way of contradiction. Then $l$ is not adjacent to any vertex from the sets $A, B$ and $C$, where the set $A=\left\{x_{m} \mid f x_{m+1} \in E(G)\right\}, B=\left\{x_{m} \mid f x_{m} \in E(G), f x_{m+1} \notin E(G), x_{m} l \notin E(G)\right\}$ and the set $C=\left\{x_{m} \mid f x_{m} \notin E(G), f x_{m+1} \notin E(G), x_{m} l \notin E(G)\right\}$. These sets are obviously disjoint. We determine their cardinalities to obtain an upper bound of $d(l)$. One can prove that $|A|=d(f), \quad|B| \geqslant \psi_{G}(f, l) \quad$ and $\quad|C| \geqslant a b s\left(\alpha_{G}(f, l)\right.$ $\left.-\beta_{G}(f, l)-1\right)$. Then $\quad d(l) \leqslant|V(G)|-|\{l\}|-|A|-|B|-|C| \leqslant n-1-d(f)-$ $\chi_{G}(f, l)=n-1-d(f)-\max \left\{\chi_{G}(f, l), \chi_{G}(l, f)\right\}$, which is a contradiction.

Lemma 4. Let $G$ be a 2-connected graph of order $n \geqslant 3$ and let $P$ be a longest path in G. If

$$
d(f)+d(l)+\max \left\{\chi_{G}(f, l), \chi_{G}(l, f)\right\} \geqslant k
$$

then $G$ has a cycle of length $\geqslant \min \{k, n\}$.

Proof. We prove the case $\max \left\{\chi_{G}(f, l), \chi_{G}(l, f)\right\}=\chi_{G}(f, l)$; the second case is analogous. Let us distinguish two cases according to $k$.
(a) $k \geqslant n$; let $H=\langle V(P)\rangle$. If the vertices $f$ and $l$ are adjacent, then $H$ is hamiltonian. Thus, let $f$ and $l$ are non-adjacent. Since $P$ is a longest path, it holds that $\{f\} \cup N(f) \cup\{l\} \cup N(l) \subseteq V(P) \subseteq V(G)$. It follows from Lemma 2 that $\chi_{G}(f, l) \leqslant \chi_{H}(f, l)$ and $\chi_{G}(l, f) \leqslant \chi_{H}(l, f)$. Further, from the same argument, it follows that $d_{G}(f)=d_{H}(f), d_{G}(l)=d_{H}(l)$. Thus we can apply Lemma 3 to the graph $H$ with its hamiltonian path $P$ and deduce that $H$ is hamiltonian (because $d_{H}(f)+d_{H}(l)+\chi_{H}(f, l) \geqslant d_{G}(f)+d_{G}(l)+\chi_{G}(f, l)=d_{G}(f)+d_{G}(l)+\max \left\{\chi_{G}(f, l)\right.$, $\left.\chi_{G}(l, f)\right\} \geqslant n$ ). Now, $G$ is either hamiltonian (if $V(G)=V(P)$ ) or there is a contradiction with the maximality of $P$ (because $G$ is 2 -connected and we can prolong the hamiltonian cycle in $H$ to a path of length at least $L(P)+1$ ).
(b) $k<n$; assume that $L(P)=p$. We claim that $p \geqslant k$. If this is not the case, then $p \leqslant k-1$. Considering the graph $H=\langle V(P)\rangle$ again, one can obtain a contradiction by a way similar to the above case (because $|\boldsymbol{V}(H)| \leqslant k$ ). Thus, it holds that $\mathrm{p} \geqslant k$.

By Lemma 1 there are paths $P_{1}, \ldots, P_{m}$ satisfying the conditions of that Lemma. Since $P$ is maximal, it follows that $P_{1}$ and $P_{m}$ both have length 1 . Choose the minimum such $m$.
(i) $m=1$. Then $(f, l) \in E(G)$ and the cycle $P \cdot(l, f)$ has length $p \geqslant k$.
(ii) $m=2$. Choose the paths $P_{1}, P_{2}$ so that the length of the path $\left[f_{2}, l_{1}\right]$ is as small as possible. First suppose that $L\left(\left[f_{2}, l_{1}\right]\right) \geqslant p-k+3 \geqslant 3$. Let $H^{\prime}$ be the graph induced by the set of vertices $V\left(\left[f_{1}, s\left(f_{2}\right)\right]\right) \cup V\left(\left[l_{1}, l_{2}\right]\right)$ and $H=H^{\prime}+\left(s\left(f_{2}\right), l_{1}\right)$. The order of $H$ is at most $|V(P)|-\left|V\left(\left[s\left(s\left(f_{2}\right)\right), p\left(l_{1}\right)\right]\right)\right| \leqslant p+1-p+k-1=k$. From the maximality of $P$ and from Lemma 2 it follows that $\chi_{G}\left(f_{1}, l_{2}\right) \leqslant \chi_{H}\left(f_{1}, l_{2}\right)$ and $\chi_{G}\left(l_{2}, f_{1}\right) \leqslant \chi_{H}\left(l_{2}, f_{1}\right)$. Similarly, by the minimality of $L\left(\left[f_{2}, l_{1}\right]\right)$ both $f$ and $l$ have no neighbour in $V\left(\left[s\left(f_{2}\right), p\left(l_{1}\right)\right]\right)$ and thus $d_{H}\left(f_{1}\right)=d_{G}\left(f_{1}\right)$ and $d_{H}\left(l_{2}\right)=d_{G}\left(l_{2}\right)$. Since the vertices $f_{1}$ and $l_{2}$ are non-adjacent, by Lemma 3, the graph $H$ with the hamiltonian path $P^{\prime}=\left[f_{1}, s\left(f_{2}\right)\right] \cdot\left(s\left(f_{2}\right), l_{1}\right) \cdot\left[l_{1}, l_{2}\right]$ contains vertices $x$ and $s(x)$ and edges $\left(f_{1}, s(x)\right)$ and $\left(x, l_{2}\right)$. It holds that $x \neq s\left(f_{2}\right)$ since $s\left(f_{2}\right)$ is non-adjacent to $l_{2}=l$ in $G$. Therefore the successor of $x$ in $P$ is the same as the successor of $x$ in $P^{\prime}$. Moreover, $\left(f_{1}, s(x)\right.$ ), $\left(x, l_{2}\right) \in E(G)$ as well. But then $P_{1}^{\prime}=(f, s(x))$ and $P_{2}^{\prime}=(x, l)$ are paths satisfying the conditions of Lemma with $L\left(\left[f_{2}^{\prime}, l_{1}^{\prime}\right]\right)=1$, contradicting the choice of $P_{1}$ and $P_{2}$. Therefore it must hold that $L\left(\left[f_{2}, l_{1}\right]\right) \leqslant p-k+2$. Then the cycle $P_{1} \cdot\left[l_{1}, l_{2}\right] \cdot \bar{P}_{2} \cdot \overline{\left[f_{1}, f_{2}\right]}$ has length at least $\left|V\left(\left[l_{1}, l_{2}\right]\right)\right|+\left|V\left(\left[f_{1}, f_{2}\right]\right)\right| \geqslant k$.
(iii) $m \geqslant 3$. Let $J, K$ be the sets of vertices adjacent to $f, l$, respectively. From the minimality of $m$ it holds that $u \in J \Rightarrow u<_{P} s\left(f_{3}\right)$ and $v \in K \Rightarrow p\left(l_{m-2}\right)<_{P} v$. Choose $P_{1}, P_{m}$ so that $L\left(\left[f_{1}, l_{1}\right]\right)$ and $L\left(\left[f_{m}, l_{m}\right]\right)$ are as small as possible. If $m$ is odd, then the cycle $\left.C=P_{1} \cdot\left[l_{1}, f_{3}\right] \cdot \mathrm{P}_{3} \cdot\left[l_{3}, f_{5}\right] \cdot \cdots \cdot\left[l_{m-2}, f_{m}\right] \cdot P_{m} \cdot \overline{\left[l_{m-1}, l_{m}\right]} \cdot \overline{P_{m-1}} \cdot \overline{\left[l_{m-3},\right.}, f_{m-1}\right)$. $\overline{P_{m-3}} \cdot \cdots \cdot \overline{P_{2}} \cdot \overline{\left[f_{1}, f_{2}\right]}$ has length at least $k$. If $m$ is even, then the cycle $C=P_{1} \cdot\left[l_{1}, f_{3}\right] \cdot P_{3} \cdot\left[l_{3}, f_{5}\right] \cdot \cdots \cdot P_{m-1} \cdot\left[l_{m-1}, l_{m}\right] \cdot \overline{P_{m}} \cdot \overline{\left[l_{m-2}, f_{m}\right]} \cdot \overline{P_{m-2}} \cdot \cdots \cdot \overline{P_{2}}$. [ $\overline{\left.f_{1}, f_{2}\right]}$ has length at least $k$. Indeed, in both cases these cycles contain all vertices of $X=V\left(\left[f_{1}, f_{2}\right]\right) \cup V\left(\left[l_{1}, f_{3}\right]\right)$ and $Y=V\left(\left[l_{m-2}, f_{m}\right]\right) \cup V\left(\left[l_{m-1}, l_{m}\right]\right)$. Moreover, by
minimality of $L\left(\left[f_{1}, l_{1}\right]\right), L\left(\left[f_{m}, l_{m}\right]\right)$, and minimality of $m$ it holds that $J \subseteq X$ and $K \subseteq Y$. Since $m \geqslant 3,|J \cap K| \leqslant 1$, and thus $a b s\left(\alpha_{G}(f, l)-\beta_{G}(f, l)-1\right)=0$. One can observe that if $|J \cap K|=1(|J \cap K|=0)$ and $\psi_{G}(f, l)=r$, then there are at least $r-1$ $(r-2)$ vertices from $\mathrm{X}-\{f, l\}$ which are adjacent neither to $f$ nor $l$. From all these facts we can claim that $|V(C)|$ is at least

1 if $|J \cap K|=1:|\{f\}|+|J|+|\{l\}|+|K|-1+\psi_{G}(f, l)-1 \geqslant d(f)+d(l)+$ $\max \left\{\chi_{G}(f, l), \chi_{G}(l, f)\right\} \geqslant k$;

2 if $|J \cap K|=0:|\{f\}|+|J|+|\{l\}|+|K|+\psi_{G}(f, l)-2 \geqslant d(f)+d(l)+\max \left\{\chi_{G}(f, l)\right.$, $\left.\chi_{G}(l, f)\right\} \geqslant k$.

This proves the Lemma.

Corollary 1. Let $G$ be a 2-connected graph of order $n \geqslant 3$ and let $P$ be a longest path in G. If

$$
d(f)+d(l) \geqslant k-\omega(G)
$$

then $G$ has a cycle of length $\geqslant \min \{k, n\}$.
Proof. If the vertices $f$ and $l$ were adjacent, then either $G$ would be hamiltonian or it would follow from 2 -connectivity of $G$ that $P$ is not the longest path. Thus we can assume that $f$ and $l$ are non-adjacent and use Lemma 4.

Remark 1. Finally, it should be noted that the condition $P$ is a new condition in the sense that there are hamiltonian graphs which satisfy $P(n)$, but fail to satisfy $P C(n)$. In order to show this, we define for each $p \geqslant 4$ a class $\mathfrak{F}_{p}$ of graphs of order $2 p^{2}+2 p+1$. Let the graph $H_{1}$ consist of $p+1$ copies of $K_{p+1}$ with the vertex sets $\left\{u_{1, i}, u_{2, i}, \ldots, u_{p+1, i}\right\}$ for $i=1,2, \ldots, p+1$, and of $p^{2}$ vertices $v_{1}, v_{2}, \ldots, v_{p^{2}}$, and all edges $v_{j} u_{k, l}$, where $j=1,2, \ldots, p^{2}, k=2,3, \ldots, p+1$ and $l=1,2, \ldots, p+1$. Let $S$ be the set of all graphs with minimum degree $\geqslant p^{2}-p+1$ on the vertex set $\left\{v_{1}, v_{2}, \ldots, v_{p^{2}}\right\}$. Finally, for $p \geqslant 4$ let us define $\mathfrak{F}_{p}=\left\{G \mid G=H_{1} \cup H_{2}, H_{2} \in S\right\}$.

For a graph $G$ from $\mathfrak{F}_{p}$ it is an easy but time consuming exercise to observe that $\omega(G)=1$ and that $G$ satisfies the condition $P(n)$. But $P C(n)$ is not satisfied because $u_{2,1}$ and $u_{2,2}$ lie in an induced $K_{1,3}$, but both are of degree $p^{2}+p$.

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