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Hypersurfaces and variational formulas in sub-Riemannian Carnot groups

Francescopaolo Montefalcone [1]

Dipartimento di Matematica, Università degli Studi di Bologna, P.zza di P.ta S. Donato 5, 40126 Bologna, Italy

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Abstract

In this paper we study smooth immersed non-characteristic submanifolds (with or without boundary) of k-step sub-Riemannian Carnot groups, from a differential-geometric point of view. The methods of exterior differential forms and moving frames are extensively used. Particular emphasis is given to the case of hypersurfaces. We state divergence-type theorems and integration by parts formulas with respect to the intrinsic measure σ_H^{n-1} on hypersurfaces. General formulas for the first and the second variation of the measure σ_H^{n-1} are proved.
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Résumé

Dans cet article nous étudions les sous-variétés non caractéristiques (avec ou sans bord) immergées dans un groupe de Carnot sous-riemannien, selon le point de vue de la géométrie différentielle classique, en utilisant la méthode du repère mobile et le formalisme des formes différentielles. En particulier, nous étudions le cas des variétés de codimension 1 en établissant des formules de type divergence et d'intégration par parties par rapport à la mesure intrinsèque σ_H^{n-1}. Enfin, nous établissons des formules générales pour les variations première et seconde de la mesure σ_H^{n-1}.
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1. Introduction

Over the last years considerable efforts have been devoted to extending the methods of Analysis, Calculus of Variations and Geometric Measure Theory to general metric spaces. This type of study, in some sense already contained in the classical Federer book's [15], has received new stimuli, among the others, by the works of Ambrosio and Kirchheim [2,3], Cheeger [8], De Giorgi [14], Gromov [22,23], David and Semmes [13], Pansu [39,40].

E-mail address: montefal@dm.unibo.it.

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In this respect, the so-called *sub-Riemannian* or *Carnot–Carathéodory* geometries have become of great interest. The setting of sub-Riemannian geometry is that of a smooth manifold N endowed with a smooth non-integrable distribution $H \subset TN$ of h-planes, or *horizontal subbundle* (h is a constant less than $\dim N$). Such a distribution is endowed with a positive definite metric g_H defined only on the subbundle H. The manifold N is said to be a *Carnot–Carathéodory space* (abbreviated *CC-space*) when one considers the so-called *CC-metric* d_H (see Definition 2.2). With respect to this metric the only paths on the manifold which have finite length are tangent to the distribution H and therefore called *horizontal*. Roughly speaking, for connecting two points we are only allowed to follow horizontal paths joining them.

We would stress that sub-Riemannian geometry has many connections with many different areas of Mathematics and Physics: Analysis, PDEs, Calculus of Variations, Control Theory, Mechanics, etc. For references, comments and perspectives, we refer the reader to Montgomery's book [38] and the surveys by Gromov [23], and Vershik and Gershkovich [49]. We also mention, specifically for sub-Riemannian geometry [47], and the recent [42].

The geometric setting of this paper is that of *Carnot groups*. Roughly speaking, a Carnot group $\mathbb{G}$ is a nilpotent and stratified Lie group endowed with a one-parameter family of dilations adapted to the stratification.

In sub-Riemannian geometry, Carnot groups are of special interest and one of the main reasons is that they constitute a wide class of concrete examples of sub-Riemannian geometries.

Another reason comes from the fact that, by virtue of a theorem due to Mitchell (see [35,38]), the *Gromov–Hausdorff tangent cone* at regular points of a sub-Riemannian manifold is a suitable Carnot group. This further justifies the interest towards the study of Carnot groups, which play, for sub-Riemannian geometries, a similar role to that of Euclidean spaces for Riemannian geometry.

The initial interest in developing Analysis and Geometric Measure Theory in this setting was the proof of the existence of intrinsic isoperimetric inequalities, first proved in Pansu's Thesis [39], for the case of the *Heisenberg group* $\mathbb{H}^1$. For a survey of results about isoperimetric inequalities on Lie groups, see [48]. More recently, a new impulse in this direction has come from a *Rectifiability Theorem* for sets of *finite H-perimeter*, obtained by Franchi, Serapioni and Serra Cassano in [16], first in the case of Heisenberg groups and then generalized to the case of 2-step Carnot groups; see [18].

For recent results on these topics and for more detailed bibliographic references, we shall refer the reader, for instance, to [1,5,16–19,30,31,36,37].

Object of the present paper is the differential geometry of immersed hypersurfaces in Carnot groups. In particular, we shall prove some variational formulas concerning the "intrinsic volume" of hypersurfaces.

The point of view adopted here is that of the classical differential geometry. In this respect, we stress that we will extensively use moving frames and differential forms as a tool. For a somewhat different, but still differential-geometric, approach to sub-Riemannian geometry, we refer to the articles [23,44], and [41,42].

As is common in differential geometry, we will study smooth submanifolds. We would remark that, since Carnot groups are naturally equipped with a left-invariant Riemannian metric, they can also be naturally equipped with the Levi-Civita connection related to such a metric. We will also introduce a notion of *partial connection* or *horizontal connection* (see Definition 2.8), to bring to light some typically sub-Riemannian features.

In Section 2.2 we introduce some basic notions about hypersurfaces and submanifolds.

We stress that the submanifolds we consider are supposed to be *geometrically H-regular* (see Definition 2.23) with respect to the horizontal distribution H, and equipped with *homogeneous measures* with respect to the intrinsic Carnot dilations. In the case of the hypersurfaces, such measure coincides with the *H-perimeter measure*, extensively studied in recent literature; see [1,5,16,17,19,30]. The idea here is to look at the H-perimeter measure of sets having regular boundary, like a measure associated to a suitable $(n-1)$-differential form σ_H^{n-1}. In such a manner we can use the formalism of differential forms to make computations. We then give some more general definitions for higher codimensional submanifolds.

In Section 3, we introduce some geometrical basic notions aiming at studying *non-characteristic* hypersurfaces, like for example the notion of *sub-Riemannian horizontal II^a fundamental form* and that of *horizontal mean curvature* (see Definition 3.2).

In Section 3.2, we then illustrate and prove some integration by parts formulas on non-characteristic hypersurfaces equipped with the measure σ_H^{n-1}.

Section 4 is entirely devoted to prove the formula for the *1st variation* and that of the *2nd variation of* σ_H^{n-1}. The last one is, of course, the main result of this paper. For precise statements, we refer to Section 4.3. These results have many consequences. As an example, we will show in Corollary 4.5 that smooth isoperimetric sets in Carnot groups must have constant horizontal mean curvature. Actually, these formulae are basic tools in many problems, as for instance, in studying the sub-Riemannian minimal surfaces equation, that is the object of a great deal of recent study; see [20,12,28,43,9].

We would like to stress that the methods used in this paper are general enough to be used also in at least two different ways. Indeed we could use them not only to generalize our results to the case of higher codimensional submanifolds of Carnot groups but also to study hypersurfaces and, more generally, submanifolds in the setting of *equiregular CC-spaces* in the sense of Gromov's definition; see [22] and [42].

2. Carnot groups, submanifolds and measures

2.1. Sub-Riemannian geometry of Carnot groups

In this section, we will introduce the definitions and the main features concerning the sub-Riemannian geometry of Carnot groups. References for this subject are, for instance, [5,19,21–23,30,35,38–42,47].

First, let us consider a $\mathbf{C}^\infty$-smooth connected n-dimensional manifold N and let $H \subset TN$ be a h_1-dimensional smooth subbundle of TN. For any $p \in N$, let T_p^k denote the vector subspace of T_pN spanned by a local basis of smooth vector fields $X_1(p), \dots, X_{h_1}(p)$ for the subbundle H around p, together with all commutators of these vector fields of order $\leqslant k$. The subbundle H is called *generic* if for all $p \in N$ $\dim T_p^k$ is independent of the point p and *horizontal* if $T_p^k = TN$ for some $k \in \mathbb{N}$. The pair (N, H) is a *k-step CC-space* if is generic and horizontal and if $k := \inf\{r\colon\ T_p^r = TN\}$. In this case, we have that

$$0 = T^0 \subset H = T^1 \subset T^2 \subset \cdots \subset T^k = TN \tag{1}$$

is a strictly increasing filtration of *subbundles* of constant dimensions $n_i := \dim T^i$ $(i = 1, \dots, k)$. Setting $(H_i)_p := T_p^i \setminus T_p^{i-1}$, then $\mathtt{gr}(T_pN) := \bigoplus_{i=1}^k (H_k)_p$ is the associated *graded Lie algebra*, at the point $p \in N$, with Lie product induced by $[\cdot,\cdot]$. Moreover, we shall set $h_i := \dim H_i = n_i - n_{i-1}$ $(n_0 = h_0 = 0)$. The k-vector $h = (h_1, \dots, h_k)$ is called the *growth vector* of H. Notice that every H_i is a smooth subbundle of the tangent bundle $\pi : TN \to N$, i.e. $\pi_{H_i} : H_i \to N$, where $\pi_{H_i} = \pi|_{H_i}$ $(i = 1, \dots, k)$.

Definition 2.1. We will call *graded frame* $\underline{X} = \{X_1, \dots, X_n\}$ for N, any frame for N such that, for any $p \in N$ we have that $\{X_{i_j}(p)\colon\ n_{j-1} < i_j \leqslant n_j\}$, is a basis for H_{j_p} $(j = 1, \dots, k)$.

Definition 2.2. A *sub-Riemannian metric* $g_H = \langle\cdot,\cdot\rangle_H$ on N is a symmetric positive bilinear form on H. If (N, H) is a CC-space, then the *CC-distance* $d_H(p, q)$ between $p, q \in N$ is

$$d_H(p, q) := \inf \int \sqrt{\langle\dot\gamma, \dot\gamma\rangle_H}\, \mathrm{d}t,$$

where the infimum is taken over all piecewise-smooth horizontal paths γ joining p to q.

In fact, Chow's theorem (see [22,38]) implies that d_H is actually a metric on N, since any two points can be joined with (at least one) horizontal path; moreover the topology induced by the CC-metric turns out to be compatible with the given topology of N.

The general setting introduced above is the starting point of sub-Riemannian geometry. A nice and very large class of examples of these geometries is represented by *Carnot groups* which for many reasons play, in sub-Riemannian geometry, an analogous role to that of Euclidean spaces in Riemannian geometry. Below we will introduce their main features. For an introduction to the following topics, we suggest Helgason's book [26], and the survey paper by Milnor [33], regarding the geometry of Lie groups, and Gromov [22], Pansu [40,42], and Montgomery [38], specifically for sub-Riemannian geometry.

A *k-step Carnot group* $(\mathbb{G}, \bullet)$ is a n-dimensional, connected, simply connected, nilpotent and stratified Lie group (w.r.t. the multiplication $\bullet$) whose Lie algebra $\mathfrak{g} (\cong \mathbb{R}^n)$ satisfies:

$$\mathfrak{g} = H_1 \oplus \cdots \oplus H_k, \quad [H_1, H_{i-1}] = H_i \quad (i = 2, \dots, k), \quad H_{k+1} = \{0\}. \tag{2}$$

We shall denote by 0 the identity on $\mathbb{G}$ so that $\mathfrak{g} \cong T_0\mathbb{G}$. The smooth subbundle H_1 of the tangent bundle $T\mathbb{G}$ is said *horizontal* and henceforth denoted by H. We will set $V := H_2 \oplus \cdots \oplus H_k$ and call V the *vertical subbundle* of $T\mathbb{G}$. As before, we will assume that $\dim H_i = h_i$ $(i = 1, \ldots, k)$ and that H is generated by some basis of left-invariant horizontal vector fields $\underline{X_H} := \{X_1, \ldots, X_{h_1}\}$. This one can be completed to a global basis (frame) of left-invariant sections of $T\mathbb{G}$, $\underline{X} := \{X_i\colon i = 1, \ldots, n\}$, which is *graded* or *adapted to the stratification*. We set $n_l := h_1 + \cdots + h_l$ ($n_0 = h_0 := 0$, $n_k = n$), and

$$H_l = \operatorname{span}_{\mathbb{R}}\{X_i\colon n_{l-1} < i \leqslant n_l\}.$$

Note that the canonical basis $\{\mathrm{e}_i\colon i = 1, \ldots, n\}$ of $\mathbb{R}^n \cong \mathfrak{g}$ can be relabeled in such a way that it turns out to be adapted to the stratification. In this way, any vector field X_i of the frame $\underline{X}$ is given by $X_{ip} := L_{p_*}\mathrm{e}_i$ $(i = 1, \ldots, n)$.

Notation 2.3. In the sequel, we shall set $I_H := \{1, \ldots, h_1\}$, $I_{H_2} := \{h_1 + 1, \ldots, n_2 (= h_1 + h_2)\}, \ldots, I_{H_k} := \{n_{k-1} + 1, \ldots, n_k (= n)\}$, and $I_V := \{h_1 + 1, \ldots, n\}$. Moreover, we will use Latin letters $i, j, k, \ldots,$ for indices belonging to I_H and Greek letters $\alpha, \beta, \gamma, \ldots,$ for indices belonging to I_V. Unless otherwise specified, capital Latin letters $I, J, K, \ldots,$ may denote any generic index. Finally, we define the function $\operatorname{ord}\colon \{1, \ldots, n\} \to \{1, \ldots, k\}$ by $\operatorname{ord}(I) := i$ if, and only if, $n_{i-1} < I \leqslant n_i$ $(i = 1, \ldots, k)$.

If $p \in \mathbb{G}$ and $X \in \mathfrak{g}$ we set $\gamma_p^X(t) := \exp[tX](p)$ $(t \in \mathbb{R})$, i.e. γ_p^X is the integral curve of X starting from p and it is a 1-*parameter sub-group* of $\mathbb{G}$. The *Lie group exponential map* is then defined by:

$$\exp\colon \mathfrak{g} \mapsto \mathbb{G}, \quad \exp(X) := \exp[X](1).$$

It turns out that exp is an analytic diffeomorphism between $\mathfrak{g}$ and $\mathbb{G}$ whose inverse will be denoted by log. Moreover, we have:

$$\gamma_p^X(t) = p \bullet \exp(tX) \quad \forall t \in \mathbb{R}.$$

From now on we shall fix on $\mathbb{G}$ the so-called *exponential coordinates of 1st kind*, i.e. the coordinates associated to the map log.

As for any nilpotent Lie group, the *Baker–Campbell–Hausdorff formula* (see [10]) uniquely determines the group multiplication $\bullet$ of $\mathbb{G}$, from the "structure" of its own Lie algebra $\mathfrak{g}$. In fact, one has,

$$\exp(X) \bullet \exp(Y) = \exp(X \star Y) \quad (X, Y \in \mathfrak{g}),$$

where $\star\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the *Baker–Campbell–Hausdorff product* defined by:

$$X \star Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\big[X, [X, Y]\big] - \frac{1}{12}\big[Y, [X, Y]\big] + \text{brackets of length} \geqslant 3. \tag{3}$$

Using exponential coordinates, (3) implies that the group multiplication $\bullet$ of $\mathbb{G}$ is polynomial and explicitly computable (see [10]). Moreover, $0 = \exp(0, \ldots, 0)$ and the inverse of $p \in \mathbb{G}$ $(p = \exp(p_1, \ldots, p_n))$ is $p^{-1} = \exp(-p_1, \ldots, -p_n)$.

When we endow the horizontal subbundle with a metric $g_H = \langle \cdot, \cdot \rangle_H$, we say that $\mathbb{G}$ has a *sub-Riemannian structure*. Is important to note that it is always possible to define a left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ in such a way that the frame $\underline{X}$ turns out to be *orthonormal* and such that $g|_H = g_H$. For this, it is enough to choose a Euclidean metric on $\mathfrak{g} = T_0\mathbb{G}$ which can be left-translated to the whole tangent bundle. This way, the direct sum (2) becomes an orthogonal direct sum.

Since for Carnot groups the hypotheses of Chow's theorem trivially apply, the *Carnot–Carathéodory distance* d_H associated with g_H can be defined as before, and d_H makes $\mathbb{G}$ a complete metric space in which every couple of points can be joined by (at least) one d_H-geodesic.

We remark that Carnot groups are *homogeneous groups* (see [46]), i.e. they are equipped with a 1-parameter group of automorphisms $\delta_t\colon \mathbb{G} \to \mathbb{G}$ $(t > 0)$. In exponential coordinates, we have:

$$\delta_t p = \exp\bigg(\sum_{j, i_j} t^j p_{i_j} \mathrm{e}_{i_j}\bigg),$$

for all $p = \exp(\sum_{j,i_j} p_{i_j} e_{i_j}) \in \mathbb{G}$.[2] The *homogeneous dimension* of $\mathbb{G}$ is the integer $Q := \sum_{i=1}^{k} i h_i$, coinciding with the *Hausdorff dimension* of $(\mathbb{G}, d_H)$ as a metric space; see [35,38,22].

The introduction of a Riemannian metric will allow us to study Carnot groups in a Riemannian way. To this end, we define the left-invariant co-frame $\underline{\omega} := \{\omega_I\colon\ I = 1, \dots, n\}$ dual to $\underline{X}$. In particular, the *left-invariant* 1*-forms*[3] ω_i are uniquely determined by the condition

$$\omega_I(X_J) = \langle X_I, X_J\rangle = \delta_I^J \quad \text{(Kronecker)}\ (I, J = 1, \dots, n).$$

We remind that the *structural constants* of the Lie algebra $\mathfrak{g}$ associated with the (left invariant) frame $\underline{X}$ are defined by:

$$C^{\mathfrak{g}}{}^R_{IJ} := \big\langle [X_I, X_J], X_R\big\rangle \quad (I, J, R = 1, \dots, n).$$

They satisfy the customary properties:

(i) $C^{\mathfrak{g}}{}^R_{IJ} + C^{\mathfrak{g}}{}^R_{JI} = 0$ (skew-symmetry),
(ii) $\sum_{J=1}^{n} C^{\mathfrak{g}}{}^I_{JL} C^{\mathfrak{g}}{}^J_{RM} + C^{\mathfrak{g}}{}^I_{JM} C^{\mathfrak{g}}{}^J_{LR} + C^{\mathfrak{g}}{}^I_{JR} C^{\mathfrak{g}}{}^J_{ML} = 0$ (Jacobi's identity).

The stratification hypothesis on the Lie algebra implies the following further property:

$$X_i \in H_l,\ X_j \in H_m \quad \Longrightarrow \quad [X_i, X_j] \in H_{l+m}. \tag{4}$$

Therefore, if $i \in I_{H_s}$ and $j \in I_{H_r}$, one has:

$$C^{\mathfrak{g}}{}^m_{ij} \neq 0 \quad \Longrightarrow \quad m \in I_{H_{s+r}}. \tag{5}$$

Definition 2.4. Throughout this paper we shall make use of the following notation:

(i) $C_H^\alpha := [C^{\mathfrak{g}}{}^\alpha_{ij}]_{i,j \in I_H} \in \mathcal{M}_{h_1 \times h_1}(\mathbb{R})$ $(\alpha \in I_{H_2})$;
(ii) $C^\alpha := [C^{\mathfrak{g}}{}^\alpha_{IJ}]_{I,J=1,\dots,n} \in \mathcal{M}_{n \times n}(\mathbb{R})$ $(\alpha \in I_V)$.

The linear operators associated with these matrices will be denoted in the same manner.

Definition 2.5. The *ith curvature* of the distribution H $(i = 1, \dots, k)$ is the (antisymmetric, bilinear) map,

$$\Omega_{H_i} : H \otimes H_i \to H_{i+1}, \quad \Omega_{H_i}(X \otimes Y) := [X, Y] \bmod T^i \quad \forall X \in H,\ \forall Y \in H_i.$$

Obviously, we have that $\Omega_{H_k}(\cdot,\cdot) = 0$, by definition of k-step Carnot group.

Since the bracket map $[\cdot,\cdot] : H \otimes H_i \to H_{i+1}$ $(i = 1, \dots, k)$ is surjective, this definition turns out to be well posed. Notice that the 1st curvature $\Omega_H(\cdot,\cdot) := \Omega_{H_1}(\cdot,\cdot)$ of H is the customary curvature of a distribution; see [21,23,38].

Notation 2.6. If $Y \in T\mathbb{G}$ let us denote by $Y = (Y_1, \dots, Y_k)$ its canonical decomposition with respect to the grading of the tangent space, i.e. $Y = \sum_{i=1}^{k} p_{H_i}(Y)$, where p_{H_i} denotes the orthogonal projection onto H_i $(i = 1, \dots, k)$. Then we set $\Omega_V(X, Y) := \sum_{i=1}^{k-1} \Omega_{H_i}(X, Y_i)$ for $X \in H$ and $Y \in T\mathbb{G}$.

Lemma 2.7. *Let $X \in H$ and $Y, Z \in T\mathbb{G}$. Then we have*:

(i) $\langle \Omega_H(X, Y), Z\rangle = -\sum_{\alpha \in I_{H_2}} z_\alpha \langle C_H^\alpha X, Y\rangle$;
(ii) $\langle \Omega_V(X, Y), Z\rangle = -\sum_{\alpha \in I_V} z_\alpha \langle C^\alpha X, Y\rangle$.

[2] Here, $j \in \{1, \dots, k\}$ and $i_j \in I_{H_j} = \{n_{j-1} + 1, \dots, n_j\}$.
[3] That is, $L_p^* \omega_I = \omega_I$ for every $p \in \mathbb{G}$.

Proof. The proof is an immediate consequence of Definitions 2.5 and 2.4. □

In the sequel, we will give a quite general definition of connection which recovers the definitions of Riemannian, partial and non-holonomic connections. Classical notions of connection (linear, affine or Riemannian) and related topics can be found in [26,27] and [45]. Partial connections was introduced by Z. Ge in [21]; see also [23]. Non-holonomic connections were first used by É. Cartan in his studies on non-holonomic mechanics and then in a great number of works of the Russian school; see the survey by Vershik and Gershkovich [49], and also [29].

Definition 2.8. Let N be a $\mathbf{C}^\infty$ smooth manifold and let $\pi_E : E \to N$, $\pi_F : F \to N$ be smooth subbundles of TN. An *E-connection $\nabla^{(E,F)}$ on F* is a rule which assigns to each vector field $X \in \mathbf{C}^\infty(N, E)$ an $\mathbb{R}$-linear transformation $\nabla_X^{(E,F)} : \mathbf{C}^\infty(N, F) \to \mathbf{C}^\infty(N, F)$ such that

$$\text{(i) } \nabla^{(E,F)}_{fX+gY} Z = f\nabla^{(E,F)}_X Z + g\nabla^{(E,F)}_Y Z \quad \forall X, Y \in \mathbf{C}^\infty(N, E)\ \forall Z \in \mathbf{C}^\infty(N, F),\ \forall f, g \in \mathbf{C}^\infty(N);$$

$$\text{(ii) } \nabla^{(E,F)}_X fY = f\nabla^{(E,F)}_X Y + (Xf)Y \quad \forall X, Y \in \mathbf{C}^\infty(N, E)\ \forall f \in \mathbf{C}^\infty(N).$$

If $E = F$ we shall set $\nabla^E := \nabla^{(E,E)}$ and call ∇^E an *E-connection*. Any such connection will be called a *partial connection* of TN. If $E = TN$, then $\nabla^{(TN,F)}$ is called a *non-holonomic F-connection*.[4] If E has a positive definite inner product g_E, then an E-connection ∇^E is said *metric preserving* if

$$\text{(iii) } Zg_E(X, Y) = g_E(\nabla^E_Z X, Y) + g_E(X, \nabla^E_Z Y) \quad \forall X, Y, Z \in \mathbf{C}^\infty(N, E).$$

The *torsion* T_E associated to the E-connection ∇^E is defined by:

$$\mathrm{T}_E(X, Y) := \nabla^E_X Y - \nabla^E_Y X - p_E[X, Y] \quad \forall X, Y \in \mathbf{C}^\infty(N, E),$$

where $p_E : TN \to E$ denotes the orthogonal projection onto E. An E-connection is *torsion free* if $\mathrm{T}_E(X, Y) = 0$ for every $X, Y \in \mathbf{C}^\infty(N, E)$. We shall say that ∇^E is the *Levi-Civita E-connection* on E if it is metric preserving and torsion-free. Note that if $E = TN$, terminology and definitions adopted here are the customary ones and, in this case, we will denote by ∇ the (univocally determined) *Levi-Civita connection* on TN with respect to the canonical metric g on N.

We stress that the difference between the definitions of partial and non-holonomic connection is that the latter allows us to covariantly differentiate along any curve of N whereas using the first one only curves that are tangent to the subbundle E can be considered.

Definition 2.9. Henceforth, we shall denote by ∇ the (unique) *left-invariant Levi-Civita connection* on $\mathbb{G}$ associated with g. Moreover, if $X, Y \in \mathbf{C}^\infty(\mathbb{G}, H)(:= \mathfrak{X}(H))$, we shall set $\nabla^H_X Y := p_H(\nabla_X Y)$. We stress that ∇^H is an example of partial connection, called *horizontal H-connection*. For notational convenience, in the sequel we will denote by the same symbol the non-holonomic connection on $\mathbb{G}$, i.e. $\nabla^H = \nabla^{(T\mathbb{G},H)}$.

Remark 2.10. From Definition 2.9, using the properties of the structural constants of any Levi-Civita connection, we get that the horizontal connection ∇^H is *flat*, i.e.

$$\nabla^H_{X_i} X_j = 0 \quad (i, j \in I_H).$$

Note that the horizontal connection ∇^H is compatible with the sub-Riemannian metric g_H, i.e.

$$X\langle Y, Z\rangle_H = \langle \nabla^H_X Y, Z\rangle_H + \langle Y, \nabla^H_X Z\rangle_H \quad \forall X, Y, Z \in \mathfrak{X}(H).$$

This follows immediately from the very definition of ∇^H, by using the analogous property of the Levi-Civita connection ∇ on $\mathbb{G}$. Furthermore, ∇^H is torsion-free, i.e.

$$\nabla^H_X Y - \nabla^H_Y X - p_H[X, Y] = 0 \quad \forall X, Y \in \mathfrak{X}(H).$$

[4] This definition recovers the usual one of "vector bundle connection" (see [34]) where instead of a generic vector bundle $\pi : F \to N$ we make use of a subbundle of the tangent bundle.

Definition 2.11. If $\psi \in \mathbf{C}^{\infty}(\mathbb{G})$ we define the *horizontal gradient* of ψ, $\mathrm{grad}_H \psi$, as the (unique) horizontal vector field such that

$$\langle \mathrm{grad}_H \psi, X \rangle_H = \mathrm{d}\psi(X) = X\psi \quad \forall X \in \mathfrak{X}(H).$$

We will call *horizontal divergence* of $X \in \mathfrak{X}(H)$, $\mathrm{div}_H X$, the function given, at each point $p \in \mathbb{G}$ by:

$$\mathrm{div}_H X(p) := \mathrm{Trace}(Y \to \nabla^H_Y X)(p) \quad (Y \in H_p).$$

Later on, we will denote by $\mathcal{J}_H$ the Jacobian matrix of a vector-valued function, computed with respect to a given orthonormal frame $\underline{\tau}_H = \{\tau_1, \ldots, \tau_{h_1}\}$ for H.

For what concerns the theory of connections on Lie group and left-invariant differential forms, see [26]. Moreover, for many topics about the geometry of nilpotent Lie groups equipped with a left-invariant connection, see [33].

The *Cartan's structure equations* for the left-invariant co-frame $\underline{\omega}$ are given by:

$$\text{(I) } \mathrm{d}\omega_I = \sum_{J=1}^{n} \omega_{IJ} \wedge \omega_J, \quad \text{(II) } \mathrm{d}\omega_{JK} = \sum_{L=1}^{n} \omega_{JL} \wedge \omega_{LK} - \Omega_{JK} \quad (I, J, K = 1, \ldots, n),$$

where $\omega_{IJ}(X) = \langle \nabla_X X_I, X_J \rangle$ are the *connection* 1*-forms* for $\underline{\omega}$ while Ω_{JK} are the *curvature* 2*-forms*, defined by:

$$\Omega_{JK}(X, Y) = \omega_K\big(\mathrm{R}(X, Y)X_J\big) \quad \big(X, Y \in \mathfrak{X}(\mathbb{G})\big).$$

Here and in the sequel, R will denote the *Riemannian curvature tensor*, defined by:

$$\mathrm{R}(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \quad \big(X, Y, Z \in \mathfrak{X}(\mathbb{G})\big).$$

Both the connection 1-forms ω_{IJ} and the curvature 2-forms Ω_{IJ} are skew-symmetric in the lower indices. We explicitly remark that, with respect to the global frame $\underline{X} = \{X_1, \ldots, X_n\}$ of left-invariant vector fields on $\mathbb{G}$, it turns out that (see, for instance, [33]):

$$\nabla_{X_I} X_J = \frac{1}{2} \sum_{R=1}^{n} (C^{\mathfrak{g}}{}^R_{IJ} - C^{\mathfrak{g}}{}^I_{JR} + C^{\mathfrak{g}}{}^J_{RI}) X_R \quad (I, J = 1, \ldots, n). \tag{6}$$

In the sequel, by using this formula and condition (4)*, we will perform explicit computations in terms of the structural constants.* For instance, from (6) it follows that the 1*st structure equation* for the coframe $\underline{\omega}$, becomes:

$$\mathrm{d}\omega_R = -\frac{1}{2} \sum_{1 \leqslant I, J \leqslant n_{i-1}} C^{\mathfrak{g}}{}^R_{IJ} \omega_I \wedge \omega_J \quad \big(R \in I_{H_i} = \{j\colon n_{i-1} < j \leqslant n_i\},\ i = 1, \ldots, k\big). \tag{7}$$

We end this section with some examples.

Example 2.12 *(Heisenberg group $\mathbb{H}^n$).* Let $\mathfrak{h}_n := T_0\mathbb{H}^n = \mathbb{R}^{2n+1}$ denote the Lie algebra of the Heisenberg group $\mathbb{H}^n$ that is an important example of 2-step Carnot group. Its Lie algebra $\mathfrak{h}_n$ is defined by the rules,

$$[\mathrm{e}_i, \mathrm{e}_{i+n}] = \mathrm{e}_{2n+1} \quad (i = 1, \ldots, n),$$

and all other commutators are zero. We have $\mathfrak{h}_n = H \oplus \mathbb{R}\mathrm{e}_{2n+1}$ where $H = \mathrm{span}_{\mathbb{R}}\{\mathrm{e}_i\colon i = 1, \ldots, 2n\}$. In particular, the 2nd layer of the grading $\mathbb{R}\mathrm{e}_{2n+1}$ is the center of the Lie algebra $\mathfrak{h}_n$. These conditions determine the group law $\bullet$ via the Baker–Campbell–Hausdorff formula. More precisely, if $p, q \in \mathbb{H}^n$, then

$$p \bullet q = \exp\left(p_1 + q_1, \ldots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} + \frac{1}{2}\sum_{i=1}^{n}(p_i q_{i+n} - p_{i+n} q_i)\right).$$

Example 2.13 *(Engel group $\mathbb{E}^1$).* The Engel group is a simple but very important example of 3-step Carnot group; see, for instance, [38]. Its Lie algebra $\mathfrak{e}$ is 4-dimensional and is defined by the rules,

$$[\mathrm{e}_1, \mathrm{e}_2] = \mathrm{e}_3, \qquad [\mathrm{e}_1, \mathrm{e}_3] = [\mathrm{e}_2, \mathrm{e}_3] = \mathrm{e}_4,$$

and all other commutators vanish. We have $\mathfrak{e} = H \oplus \mathbb{R}\mathrm{e}_3 \oplus \mathbb{R}\mathrm{e}_4$, where $H = \mathrm{span}_{\mathbb{R}}\{\mathrm{e}_1, \mathrm{e}_2\}$ and the center of the Lie algebra $\mathfrak{e}$ is $\mathbb{R}\mathrm{e}_4$.

2.2. Hypersurfaces, H-regular submanifolds and measures

Throughout this paper we shall use many properties of differential forms for which we refer the reader, for instance, to [15,27,26,45].

In the sequel, $\mathcal{H}^m_{\mathbf{cc}}$ and $\mathcal{S}^m_{\mathbf{cc}}$ will denote, respectively, the usual and the spherical[5] m-dimensional Hausdorff measures on $\mathbb{G}$ associated with d_H, while $\mathcal{H}^m_{\mathbf{e}}$ will denote the (Euclidean) m-dimensional Hausdorff measure on $\mathbb{R}^n \cong \mathbb{G}$.[6] The (left-invariant) *Riemannian volume form* on $\mathbb{G}$ is defined as

$$\sigma^n_{\mathcal{R}} := \Lambda^n_{i=1}\omega_i \in \Lambda^n(T\mathbb{G}).$$

Remark 2.14. By integrating $\sigma^n_{\mathcal{R}}$ we obtain a measure $\mathrm{vol}^n_{\mathcal{R}}$, which is the so-called *Haar measure* of $\mathbb{G}$. Since the determinant of L_{p*} is equal to 1, this measure equals the measure induced on $\mathbb{G}$ by the push-forward of the n-dimensional Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n \cong \mathfrak{g}$. Moreover, up to a constant multiple, $\mathrm{vol}^n_{\mathcal{R}}$ equals the *Q-dimensional Hausdorff measure* $\mathcal{H}^Q_{\mathbf{cc}}$ on $\mathbb{G}$. This follows because they are both Haar measures for the group and therefore they are equal, up to a constant; see [38]. Here we assume this constant equal to 1.

In this paper we are mainly interested to the study of codimension 1 immersed[7] sub-manifolds (or hypersurfaces) of Carnot groups. Note that any hypersurface $S \subset \mathbb{R}^n (\cong \mathfrak{g})$ is identified, by means of the exponential map, with a hypersurface of $\mathbb{G}$, i.e. $S \cong \exp S$. A hypersurface S is $\mathbf{C}^r$-regular ($r = 1, \ldots, \infty$), if S is $\mathbf{C}^r$-regular as a Euclidean submanifold of $\mathbb{R}^n$.

In the study of hypersurfaces of Carnot groups we have to introduce the notion of *characteristic point*.

Definition 2.15. If $S \subset \mathbb{G}$ is a $\mathbf{C}^r$-regular ($r = 1, \ldots, \infty$) hypersurface, we say that S is *characteristic* at $p \in S$ if $\dim H_p = \dim(H_p \cap T_pS)$ or, equivalently, if $H_p \subset T_pS$. The *characteristic set* of S is denoted by C_S, i.e.

$$C_S := \{p \in S\colon \dim H_p = \dim(H_p \cap T_pS)\}.$$

A hypersurface $S \subset \mathbb{G}$, oriented by its unit normal vector ν, is *non-characteristic* if, and only if, the horizontal subbundle H is *transversal* to S ($H \pitchfork TS$). We have then,

$$H_p \pitchfork T_pS \iff p_H\nu_p \neq 0 \iff \exists X \in \mathfrak{X}(H)\colon \langle X_p, \nu_p\rangle \neq 0,$$

for all $p \in S$, where $p_H : T\mathbb{G} \to H$ denotes the orthogonal projection onto H.

Remark 2.16. *(Hausdorff measure of C_S; see [30].)* If $S \subset \mathbb{G}$ is a $\mathbf{C}^1$-regular hypersurface, then the $(Q-1)$-dimensional Hausdorff measure associated with d_H of the characteristic set C_S is zero, i.e.

$$\mathcal{H}^{Q-1}_{\mathbf{cc}}(C_S) = 0.$$

---

[5] We remind that

(i) $\mathcal{H}^m_{\mathbf{cc}}(S) = \lim_{\delta\to 0^+} \mathcal{H}^m_{\mathbf{cc},\delta}(S)$ where, up to a constant multiple,

$$\mathcal{H}^m_{\mathbf{cc},\delta}(S) = \inf\left\{\sum_i \big(\mathrm{diam}_H(C_i)\big)^m\colon S \subset \bigcup_i C_i;\ \mathrm{diam}_H(C_i) < \delta\right\},$$

and the infimum is taken with respect to any non-empty family of closed subsets $\{C_i\}_i \subset \mathbb{G}$;

(ii) $\mathcal{S}^m_{\mathbf{cc}}(S) = \lim_{\delta\to 0^+} \mathcal{S}^m_{\mathbf{cc},\delta}(S)$ where, up to a constant multiple,

$$\mathcal{S}^m_{\mathbf{cc},\delta}(S) = \inf\left\{\sum_i \big(\mathrm{diam}_H(B_i)\big)^m\colon S \subset \bigcup_i B_i;\ \mathrm{diam}_H(B_i) < \delta\right\},$$

and the infimum is taken with respect to closed d_H-balls B_i.

[6] Here and in the sequel, $\mathbb{G}$ is identified with $\mathbb{R}^n$ by means of the exponential map.

[7] If N^n is a manifold, then an *immersed m-submanifold* of N is a subset $M^m \subset N$ endowed with a m-manifold topology (not necessarily the subspace topology) together with a smooth structure such that the inclusion $\imath : M \to N$ is a smooth immersion (i.e. the push-forward $\imath_*$ is injective at each point, or equivalently, $\mathrm{rank}\,\imath_* = m$).

Remark 2.17 *(Riemannian measure on hypersurfaces).* Let $S \subset \mathbb{G}$ be a $\mathbf{C}^r$-regular hypersurface and let ν denote the unit normal vector along S. By definition, the $(n-1)$-dimensional Riemannian measure along S is given by:

$$\sigma_{\mathcal{R}}^{n-1} \llcorner S := (\nu \lrcorner \sigma_{\mathcal{R}}^{n})|_{S}, \tag{8}$$

where $\lrcorner$ denotes the "contraction" (or interior product) of a differential form.[8]

Since we shall study regular hypersurfaces, instead of the usual definition of H-perimeter measure[9] we now introduce a $(n-1)$-differential form which, by integration, coincides with the H-perimeter measure.

Definition 2.18 *(σ_H^{n-1}-measure on hypersurfaces).* Let $S \subset \mathbb{G}$ be a $\mathbf{C}^r$-regular non-characteristic hypersurface and let us denote by ν its unit normal vector. We will call *H-normal* along S, the normalized projection onto H of ν, i.e.

$$\nu_H := \frac{p_H \nu}{|p_H \nu|}.$$

We then define the $(n-1)$-dimensional measure σ_H^{n-1} along S to be the measure associated with the $(n-1)$-differential form $\sigma_H^{n-1} \in \Lambda^{n-1}(TS)$ given by the contraction of the volume form $\sigma_{\mathcal{R}}^n$ of $\mathbb{G}$ with the horizontal unit normal ν_H, i.e.

$$\sigma_H^{n-1} \llcorner S := (\nu_H \lrcorner \sigma_{\mathcal{R}}^{n})|_{S}. \tag{9}$$

If we allow S to have characteristic points we may trivially extend the definition of σ_H^{n-1} by setting $\sigma_H^{n-1} \llcorner C_S = 0$. Notice also that $\sigma_H^{n-1} \llcorner S = |p_H \nu| \cdot \sigma_{\mathcal{R}}^{n-1} \llcorner S$.

From this definition, we obtain:

$$\sigma_H^{n-1} \llcorner S = \sum_{i \in I_H} \nu_H^i (X_i \lrcorner \sigma_{\mathcal{R}}^n)|_S = \sum_{i \in I_H} (-1)^{i+1} \nu_H^i (\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_n)|_S,$$

where $\nu_H^i := \langle \nu_H, X_i \rangle$ $(i \in I_H)$. In the sequel, we will frequently use the next elementary lemma.

Lemma 2.19. *If $S \subset \mathbb{G}$ be a smooth non-characteristic hypersurface, then for every $X \in HS$ we have*

$$(X \lrcorner \sigma_{\mathcal{R}}^n)|_S = 0.$$

Proof. Since $X \in HS (\subset TS)$, we have $\langle X, \nu \rangle = 0$ and (8) implies $(X \lrcorner \sigma_{\mathcal{R}}^n)|_S = \langle X, \nu \rangle \sigma_{\mathcal{R}}^{n-1}|_S = 0$. □

The comparison among different notions of measures on submanifolds, is an interesting problem of the Geometric Measure Theory of Carnot–Carathéodory spaces. In the case of smooth hypersurfaces in Carnot groups, the problem is to compare the H-perimeter measure with the $(Q-1)$-dimensional Hausdorff measure associated with either the cc-distance d_H or with some suitable homogeneous distance. In general, thanks to a remarkable density estimate for σ_H^{n-1} proved in [1], we have the following:

---

[8] The linear map $\lrcorner : \Lambda^k(T\mathbb{G}) \to \Lambda^{k-1}(T\mathbb{G})$ is defined, for $X \in T\mathbb{G}$ and $\omega^k \in \Lambda^k(T\mathbb{G})$, by $(X \lrcorner \omega^k)(Y_1, \ldots, Y_{k-1}) := \omega^k(X, Y_1, \ldots, Y_{k-1})$; see [26,15].

[9] Let $U \subseteq \mathbb{G}$ be open and $f \in L^1(U)$. Then f has *H-bounded variation* in U if

$$|\nabla^H f|_H(U) := \sup\left\{ \int_U f \operatorname{div}_H Y \, d\mathcal{L}^n \colon Y \in \mathbf{C}_0^1(U, H),\ |Y|_H \leqslant 1 \right\} < \infty.$$

Let $HBV(U)$ denote the vector space of bounded H-variation in U. From Riesz's theorem it follows that $|\nabla^H f|_H$ is a *Radon measure* on U and that there exists a horizontal $|\nabla^H f|_H$-measurable section ν_f such that $|\nu_f|_H = 1$ for $|\nabla^H f|_H$-a.e. $p \in U$ and that

$$\int_U f \operatorname{div}_H Y \, d\mathcal{L}^n = \int_U \langle Y, \nu_f \rangle_H \, d|\nabla^H f|_H \quad \forall Y \in \mathbf{C}_0^1(U, H).$$

We say that a measurable set $E \subset \mathbb{G}$ has finite H-perimeter in U if $\chi_E \in HBV(U)$. The *H-perimeter* of E in U is the Radon measure $|\partial E|_H(U) := |\nabla^H \chi_E|_H(U)$. We call *generalized unit H-normal along ∂E* the Radon $\mathbb{R}^{h_1}$-measure $\nu_E := -\nu_{\chi_E}$; see [1,5,16–19].

Theorem 2.20. *If $S \subset \mathbb{G}$ is a $\mathbf{C}^1$-regular hypersurface which is locally the boundary of an open set E having* (*locally*) *finite H-perimeter* (*see footnote* 9)*, then*

$$|\partial E|_H(\mathcal{B}) = k_{Q-1}(\nu_H)\mathcal{S}^{Q-1}_{\mathbf{cc}} \llcorner (S \cap \mathcal{B}) \quad \forall \mathcal{B} \in \mathcal{B}or(\mathbb{G}), \tag{10}$$

where k_{Q-1} is a function depending on ν_H, called metric factor; *see* [30]. *It is important to stress that $|\partial E|_H(\mathcal{B}) = \sigma^{n-1}_H \llcorner (S \cap \mathcal{B})$ because of the regularity of ∂E.*

A proof of this theorem can be found in [30].

Remark 2.21. We would explicitly notice that

$$\sigma^{n-1}_H(S \cap U) = \int\limits_{S \cap U} \sqrt{\langle X_1, n_{\mathbf{e}}\rangle^2_{\mathbb{R}^n} + \cdots + \langle X_{m_1}, n_{\mathbf{e}}\rangle^2_{\mathbb{R}^n}}\, \mathrm{d}\mathcal{H}^{n-1}_{\mathbf{e}}, \tag{11}$$

where $n_{\mathbf{e}}$ denotes unit Euclidean normal along S,[10] and that its unit H-normal is given by:

$$\nu_H = \frac{(\langle X_1, n_{\mathbf{e}}\rangle_{\mathbb{R}^n}, \ldots, \langle X_{h_1}, n_{\mathbf{e}}\rangle_{\mathbb{R}^n})}{\sqrt{\langle X_1, n_{\mathbf{e}}\rangle^2_{\mathbb{R}^n} + \cdots + \langle X_{h_1}, n_{\mathbf{e}}\rangle^2_{\mathbb{R}^n}}}.$$

Here, the Euclidean normal $n_{\mathbf{e}}$ along S and the vector fields X_i ($i \in I_H$) of the horizontal left-invariant frame $\underline{X_H}$, are thought of as vectors in $\mathbb{R}^n \cong \mathbb{G}$, endowed with its canonical inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. We note that the (Riemannian) *unit normal* ν along S may be represented with respect to the global left-invariant frame $\underline{X}$ for $\mathbb{G}$, in terms of the Euclidean normal $n_{\mathbf{e}}$. More precisely, we have:

$$\nu(p) = \frac{(L_p \circ \exp)_* n_{\mathbf{e}}(\log p)}{|(L_p \circ \exp)_* n_{\mathbf{e}}(\log p)|} \quad (p \in S \subset \mathbb{G}),$$

where $L_{p_*}(q) = [X_1(q), \ldots, X_n(q)] \in \mathcal{M}_{n \times n}(\mathbb{R})$ $(p, q \in \mathbb{G})$.

Definition 2.22. If ν_H is the horizontal unit normal along S, at each regular point $p \in S \setminus C_S$ one has that $H_p = (\nu_H)_p \oplus H_p S$, where we have set:

$$H_p S := H_p \cap T_p S.$$

We call $H_p S$ the *horizontal tangent space* at p along S. Moreover, we define in the obvious way the associated subbundles $HS(\subset TS)$ and $\nu_H S$, called, respectively, *horizontal tangent bundle* and *horizontal normal bundle* of S.

If we consider an immersed submanifold $S^{n-i} \subset \mathbb{G}$ of codimension $i \geqslant 1$, the above construction can be generalized in the following way.

Definition 2.23. We say that a codimension i submanifold S^{n-i} of $\mathbb{G}$ is *geometrically H-regular* at $p \in S$ if there exist linearly independent vectors $\nu^1_H, \ldots, \nu^i_H \in H_p$ transversal along S at p. Without loss of generality, we may also suppose that these vectors be orthonormal at p. The horizontal tangent space at p is defined by:

$$H_p S := H_p \cap T_p S.$$

If this condition is independent of the point $p \in S$, we say that S is geometrically H-regular. In such case we may define the associated vector bundles $HS(\subset TS)$ and $\nu_H S$, called, respectively, *horizontal tangent bundle* and *horizontal normal bundle*. Therefore, one has

$$H_p := H_p S \oplus \mathbb{R}\nu^1_H \oplus \cdots \oplus \mathbb{R}\nu^i_H.$$

---

[10] If $S \subset \mathbb{R}^n$ has a $\mathbf{C}^r$-parametrization, $\Phi : B \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$, then we have:

$$n_{\mathbf{e}}\big(\Phi(\xi)\big) := \pm \frac{\Phi_{\xi_1} \wedge \cdots \wedge \Phi_{\xi_{n-1}}}{|\Phi_{\xi_1} \wedge \cdots \wedge \Phi_{\xi_{n-1}}|_{\mathbb{R}^n}}.$$

Definition 2.24 *(Characteristic set of S^{n-i}).* The *characteristic set* C_S of a $\mathbf{C}^1$-smooth i-codimensional submanifold $S^{n-i} \subset \mathbb{G}$ is defined by

$$C_S := \{p \in S\colon \dim H_p - \dim(H_p \cap T_pS) \leqslant i-1\}.$$

Remark 2.25 *(Hausdorff measure of $C_{S^{n-i}}$).* The above definition of C_S has been used in [31], where it was shown that every $\mathbf{C}^1$-smooth submanifold $S^{n-i} \subset \mathbb{G}$ has zero $(Q-i)$-dimensional Hausdorff measure, with respect to d_H, i.e.

$$\mathcal{H}_{\mathbf{cc}}^{Q-i}(C_S) = 0.$$

Definition 2.26 *(σ_H^{n-i}-measure on geometrically H-regular submanifolds).* Let $S^{n-i} \subset \mathbb{G}$ be a geometrically H-regular submanifold of codimension i. Let $\nu_H^1, \dots, \nu_H^i \in \nu_H S$ and assume that they are everywhere orthonormal. We set:

$$\nu_H := \nu_H^1 \wedge \cdots \wedge \nu_H^i \in \Lambda_i(T\mathbb{G}),$$

and define the $(n-i)$-dimensional measure σ_H^{n-i} along S to be the measure associated with the $(n-i)$-differential form $\sigma_H^{n-i} \in \Lambda^{n-i}(TS)$ given by the interior product of the volume form of $\mathbb{G}$ with the i-vector ν_H, i.e.[11]

$$\sigma_H^{n-i} \llcorner S := (\nu_H \lrcorner \sigma_{\mathcal{R}}^n)|_S. \tag{12}$$

Remark 2.27. The measure σ_H^{n-i} is *homogeneous of degree $Q-i$* with respect to Carnot dilations $\{\delta_t\}_{t>0}$, i.e. $\delta_t^* \sigma_H^{n-i} = t^{Q-i}\sigma_H^{n-i}$. This fact easily follows from the definitions. Moreover, it can be proved that the measure σ_H^{n-i} restricted to any geometrically H-regular submanifold S^{n-i} equals, up to a normalization constant, the $(Q-i)$-dimensional Hausdorff measure computed with respect to a some homogeneous distance on $\mathbb{G}$. Here, instead of proving the last statement, we shall refer the reader to the recent paper [32], where similar results are proved.

3. Geometry of HS and calculus on hypersurfaces

In this section we will study *non-characteristic hypersurfaces*, or equivalently, non-characteristic domains of a given hypersurface S. Some of the notions that we shall develop has been recently studied in [4,9,20,43,28,11,12].

We remark that, if ∇^{TS} denotes the induced connection on S from the Levi-Civita connection ∇ on $\mathbb{G}$,[12] then ∇^{TS} induces a partial connection ∇^{HS}, associated with the subbundle HS of TS, defined as follows[13]:

$$\nabla_X^{HS} Y := p_{HS}(\nabla_X^{TS} Y) \quad (X, Y \in HS).$$

Starting from the orthogonal decomposition $H = HS \oplus \nu_H S$ (see Definition 2.22), we could also define ∇^{HS} by mimicking the usual definition of "induced connection" on submanifolds (see, for instance, [6]). Indeed, it turns out that

$$\nabla_X^{HS} Y = \nabla_X^H Y - \langle \nabla_X^H Y, \nu_H \rangle_H \nu_H \quad (X, Y \in HS).$$

Definition 3.1. We will call *HS-gradient* of $\psi \in \mathbf{C}^\infty(S)$ the unique horizontal tangent section of HS, $\mathrm{grad}_{HS}\,\psi$, satisfying

$$\langle \mathrm{grad}_{HS}\,\psi, X \rangle_{HS} = \mathrm{d}\psi(X) = X\psi \quad \forall X \in HS.$$

We will denote by div_{HS} the divergence operator on HS, i.e. if $X \in HS$ and $p \in S$, then

$$\mathrm{div}_{HS}\, X(p) := \mathrm{Trace}(Y \to \nabla_Y^{HS} X)(p) \quad (Y \in H_p S).$$

[11] For the general definition of the operation $\lrcorner$ see [15], Chapter 1.

[12] Therefore, ∇^{TS} is the Levi-Civita connection on S (see [6]).

[13] The map $p_{HS}: TS \to HS$ denotes the orthogonal projection of TS onto HS.

We will also denote by Δ_{HS} the *HS-Laplacian*, i.e. the 2nd order differential operator given by:

$$\Delta_{HS}\psi := \mathrm{div}_{HS}(\mathrm{grad}_{HS}\,\psi) \quad \big(\psi \in \mathbf{C}^{\infty}(S)\big). \tag{13}$$

Finally, we will denote by $\mathcal{J}_{HS}$ the Jacobian matrix of any vector-valued function, computed with respect to any given orthonormal frame $\underline{\tau_{HS}} := \{\tau_2, \ldots, \tau_{h_1}\}$ for the subbundle HS.

Definition 3.2. We will call *sub-Riemannian horizontal IInd fundamental form* of S the map $\overline{B}_H : HS \times HS \to \nu_H S$ given by:

$$\overline{B}_H(X,Y) := \langle \nabla^H_X Y, \nu_H\rangle_H \nu_H \quad (X, Y \in HS).$$

Moreover we will denote by $\mathcal{H}_H \in \nu_H S$ the *horizontal mean curvature vector of* S, defined as the trace of $\overline{B}_H$, i.e. $\mathcal{H}_H = \mathrm{Tr}\,\overline{B}_H$. The *horizontal scalar mean curvature* of S, denoted by $\mathcal{H}^{\mathrm{sc}}_H$, is defined by $\mathcal{H}^{\mathrm{sc}}_H := \langle \mathcal{H}_H, \nu_H\rangle_H$. Finally, we shall set:

$$B_H(X,Y) := \langle \nabla^H_X Y, \nu_H\rangle_H \quad (X, Y \in HS).$$

Note that, in the previous definition, the trace Tr is computed with respect to the 1st sub-Riemannian fundamental form $g_{HS} = \langle\cdot,\cdot\rangle_{HS}$, which is the restriction to S of the metric g_H, i.e. $g_{HS} := g_H|_{HS} = g|_{HS}$.

By arguing as in the Riemannian case, we may prove that $B_H(X,Y)$ is a $\mathbf{C}^{\infty}(S)$*-bilinear form in* X *and* Y. More importantly, in general, B_H *is not symmetric*. The reason is the following: symmetry of B_H is easily seen to be equivalent to the following condition:

$$X, Y \in HS \quad \Longrightarrow \quad p_H[X,Y] \in HS.$$

But this condition fails to be true, in general. As a matter of fact, this is trivially true in the case of the Heisenberg group $\mathbb{H}^1$, being HS a 1-dimensional subbundle of TS, for any given non-characteristic surface $S \subset \mathbb{H}^1$. But, for example, the condition fails to hold, in general, in the case of $\mathbb{H}^n$ $(n > 1)$, as it can be easily proved, by using a dimensional argument.

According with Definition 2.8, we may give the following:

Definition 3.3. We define the *torsion* T_{HS} of the partial HS-connection ∇^{HS} by

$$\mathrm{T}_{HS}(X,Y) := \nabla^{HS}_X Y - \nabla^{HS}_Y X - p_H[X,Y] \quad (X, Y \in HS).$$

From this definition, it follows immediately that for every X, $Y \in \mathfrak{X}(HS)$ one has:

$$\mathrm{T}_{HS}(X,Y) = \overline{B}_H(Y,X) - \overline{B}_H(X,Y) = \big\langle p_H[Y,X], \nu_H\big\rangle_H \nu_H.$$

Note also that the mapping $HS \ni X \mapsto \nabla^H_X \nu_H$ is, in fact, the sub-Riemannian analogous of the usual Weingarten map; see [27], Chapter 2. In the case of hypersurfaces, using the compatibility of ∇^H with the metric g_H, we get that $(\nabla^H_X \nu_H)_p \in H_p S$. Indeed, by differentiating the identity $|\nu_H|^2 = 1$, we obtain:

$$X\langle \nu_H, \nu_H\rangle_H = 2\langle \nabla^H_X \nu_H, \nu_H\rangle_H = 0.$$

In the sequel, if $U \subset \mathbb{G}$ is open, we will set $\mathcal{U} := U \cap S$. Moreover we will assume that $\mathcal{U}$ is non-characteristic. We now introduce the notion of *adapted frame*, that will be used extensively throughout this paper. Roughly speaking, we shall “adapt” in the usual Riemannian way (see [45]) an orthonormal frame to the horizontal tangent space of a hypersurface.

Definition 3.4. We will call *adapted frame to* $\mathcal{U}$ *on* U any orthonormal frame on U $\underline{\tau} := (\tau_1, \ldots, \tau_n)$ such that

$$\text{(i) } \tau_1|_{\mathcal{U}} := \nu_H; \quad \text{(ii) } H_p\mathcal{U} = \mathrm{span}\big\{(\tau_2)_p, \ldots, (\tau_{h_1})_p\big\}\ (p \in \mathcal{U}); \quad \text{(iii) } \tau_\alpha := X_\alpha.$$

Remark 3.5. Let $\mathbf{GL}(\mathbb{R}^i)$ be the general linear group acting on $\mathbb{R}^i$ $(i = 1, \ldots, k)$ which we identify with the i-th layer H_i of $\mathfrak{g} = \mathtt{gr}(T\mathbb{G}) \cong \mathbb{R}^n$. We stress that any graded frame for $\mathbb{G}$ is naturally identified with an element of the subgroup[14] $\mathbf{GL}_h := \bigtimes_{i=1}^{k} \mathbf{GL}(\mathbb{R}^{h_i})$ of $\mathbf{GL}(\mathbb{R}^n)$. Using matrix notation, any element $A \in \mathbf{GL}_h(\mathbb{R}^n)$ is then a block diagonal matrix, i.e. $A_h = \mathrm{diag}[A_{h_1}, \dots, A_{h_k}]$. Furthermore, any graded orthonormal basis of $\mathfrak{g}$ may be identified with an element of the subgroup $\mathbf{O}_h(\mathbb{R}^n) := \bigtimes_{i=1}^{k} \mathbf{O}(\mathbb{R}^{h_i})$ of $\mathbf{O}(\mathbb{R}^n)$; see Definition 2.1.

Every adapted orthonormal frame to a hypersurface is a graded frame. In particular, given an adapted frame $\underline{\tau}$ for $\mathcal{U}$ on U, then at every $p \in \mathcal{U} \subset S$, there exists an orthogonal matrix,

$$A_h(p) = \left[A_I^J(p)\right]_{I,J} \in \mathbf{O}_n(\mathbb{R}) \quad (I, J = 1, \dots, n),$$

expressing the linear change of coordinates from the fixed left-invariant orthonormal frame $\underline{X}$ to the adapted one $\underline{\tau}$ such that

$$\tau_I(p) = \sum_{J=1}^{n} A_I^J(p) X_J \quad (I = 1, \dots, n).$$

Given an adapted frame $\underline{\tau}$, we will denote by $\underline{\phi} := (\phi_1, \dots, \phi_n)$, its dual co-frame. This means that

$$\phi_I(\tau_J) = \delta_I^J \quad \text{(Kronecker)}\ (I, J = 1, \dots, n).$$

Clearly, $\underline{\phi}$ satisfies its own Cartan's structural equations:

$$\text{(I)}\ \mathrm{d}\phi_I = \sum_{J=1}^{n} \phi_{IJ} \wedge \phi_J, \quad \text{(II)}\ \mathrm{d}\phi_{JK} = \sum_{L=1}^{n} \phi_{JL} \wedge \phi_{LK} - \Phi_{JK} \quad (I, J, K = 1, \dots, n),$$

where $\phi_{IJ}(X) := \langle \nabla_X \tau_J, \tau_I \rangle$ are the *connection* 1*-forms* for the co-frame $\underline{\phi}$ and Φ_{JK} denote its *curvature* 2*-forms*, defined by:

$$\Phi_{JK}(X, Y) := \phi_K\big(\mathrm{R}(X, Y)\tau_J\big) \quad \big(X, Y \in \mathfrak{X}(\mathbb{G})\big).$$

We have a basic identity between connection 1-forms and structural constants of $\underline{\tau}$, i.e.

$$C_{IJ}^K = \phi_{JK}(\tau_I) - \phi_{IK}(\tau_J) \quad (I, J, K = 1, \dots, n). \tag{14}$$

This can easily be proved using the fact that ∇ is torsion-free.

Notation 3.6. In the sequel, we shall frequently use the following notations:

(i) $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|}$ ($\alpha \in I_V$);
(ii) $\varpi := \sum_{\alpha \in I_V} \varpi_\alpha \tau_\alpha$;
(iii) $C_H := \sum_{\alpha \in I_{H_2}} \varpi_\alpha C_H^\alpha$;
(iv) $C := \sum_{\alpha \in I_V} \varpi_\alpha C^\alpha$.

Moreover, for any $\alpha \in I_{H_2}$, we shall set $C_{HS}^\alpha := C_H^\alpha|_{HS}$ to stress the fact that the linear operator C_{HS}^α only acts on horizontal tangent vectors, i.e. $(C_{HS}^\alpha)_{ij} := \langle C_H^\alpha \tau_j, \tau_i \rangle$ for $i, j \in I_{HS}$. Consequently, we set $C_{HS} := \sum_{\alpha \in I_{H_2}} \varpi_\alpha C_{HS}^\alpha$.

Remark 3.7. The horizontal mean curvature vector $\mathcal{H}_H$ can equivalently be written as follows:

$$\mathcal{H}_H = -\sum_{j \in I_{HS}} \langle \nabla_{\tau_j}^H \nu_H, \tau_j \rangle_{HS} \nu_H = -\sum_{j \in I_{HS}} \phi_{1j}(\tau_j) \nu_H = \mathcal{H}_H^{\mathrm{sc}} \nu_H.$$

We note that the symmetry of the sub-Riemannian horizontal II^a fundamental form would be equivalent to the symmetry of the connection 1-forms, i.e. $\phi_{1j}(\tau_i) = \phi_{1i}(\tau_j)$ $(i, j \in I_{HS})$. As already said, this is *false*, in general.

[14] The symbol "×" means *direct product* of groups.

Indeed, using the symmetry of the Riemannian II^a fundamental form and writing the unit normal vector along S w.r.t. $\underline{\tau}$, i.e. $\nu = \nu_1\tau_1 + \sum_{\alpha\in I_V}\nu_\alpha\tau_\alpha$, we see that

$$\phi_{1i}(\tau_j) = \phi_{1j}(\tau_i) + \sum_{\alpha\in I_{H_2}} \varpi_\alpha \langle C^\alpha_{HS}\tau_i, \tau_j\rangle_{HS} = \phi_{1j}(\tau_i) + \langle C_{HS}\tau_i, \tau_j\rangle_{HS} \quad (i, j \in I_{HS}).$$

Therefore B_H can be seen as a sum of two matrices, one symmetric and the other skew-symmetric, i.e. $B_H = S_H + A_H$, where the skew-symmetric matrix A_H is explicitly given by $A_H = \frac{1}{2}C_{HS}$.

3.1. Some preliminaries

The following lemma will be a useful tool in proving the second variation formula of σ^{n-1}_H.

Lemma 3.8. *Let $S \subset \mathbb{G}$ be an immersed hypersurface and let $U \subset \mathbb{G}$ be an open set having non-empty intersection with S and such that $\mathcal{U} := U \cap S$ is non-characteristic. Moreover, let us choose an adapted orthonormal moving frame $\underline{\tau} = \{\tau_1, \dots, \tau_n\}$ on U for $\mathcal{U}$ and fix $p_0 \in \mathcal{U}$. Then we claim that it is always possible to choose $\underline{\tau}$ so that the connection 1-forms of its dual co-frame $\underline{\phi} = \{\phi_1, \dots, \phi_n\}$ satisfy $\phi_{ij}(p_0) = 0$ whenever $i, j \in I_{HS} = \{2, \dots, h_1\}$.*

Proof. Consider a Riemannian orthonormal moving frame on U *adapted to* $\mathcal{U} = U \cap S$. This means that we have an orthonormal frame $\underline{\xi} = \{\xi_1, \dots, \xi_n\}$ on U, satisfying $\xi_1(p) = \nu(p)$ and such that

$$\underline{\xi^S} = \mathrm{span}_{\mathbb{R}}\{\xi_2(p), \dots, \xi_n(p)\} = T_pS$$

for every $p \in \mathcal{U} \subset S$. Moreover let us denote by $\underline{\varepsilon} = \{\varepsilon_1, \dots, \varepsilon_n\}$ its dual co-frame.

Claim 3.9. *It is always possible to choose another Riemannian orthonormal moving frame $\underline{\tilde{\xi}}$ for U adapted to $\mathcal{U}$ satisfying*:

(i) $\underline{\tilde{\xi}}(p_0) = \underline{\xi}(p_0)$;
(ii) *The connection 1-forms $\tilde{\varepsilon}_{IJ} = \langle\nabla\tilde{\xi}_I, \tilde{\xi}_J\rangle$ $(I, J = 1, \dots, n)$ for $\underline{\tilde{\xi}}$ satisfies $\tilde{\varepsilon}_{ij}(p_0) = 0$ for every $i, j = 2, \dots, n$.*

Here again, $\underline{\tilde{\xi}^S} = \{\tilde{\xi}_2, \dots, \tilde{\xi}_n\}$ is a tangent orthonormal frame for $\mathcal{U}$. We stress that the proof of this claim is standard and it can be found, for instance, in [45], pp. 517–519, Eq. (17). Therefore, from this fact the thesis easily follows by assuming that at p_0 the frame $\underline{\xi}$ satisfy $\xi_i(p_0) = \tau_i(p_0)$ for every $i \in I_{HS}$, i.e. the set of vectors $\{\xi_2(p_0), \dots, \xi_{h_1}(p_0)\}$ is an orthonormal basis of the horizontal tangent space $H_{p_0}S$ at p_0, coinciding with that given at the beginning. In this case we get, in particular, that

$$\tilde{\varepsilon}_{ij}(p_0) = \langle\nabla_{X_{p_0}}\tilde{\xi}_i, \tilde{\xi}_j\rangle(p_0) = 0 \quad \text{for every } i, j \in I_{HS}.$$

By extending the orthonormal frame $\{\tilde{\xi}_2, \dots, \tilde{\xi}_{h_1}\}$ for the horizontal tangent space to a full adapted frame $\underline{\tau}$ in the sense of Definition 3.4 we get our initial claim. □

Definition 3.10. From now on we shall set:

$$\tau^S_\alpha := \tau_\alpha - \frac{\nu_\alpha}{|p_H\nu|}\nu_H \quad (\alpha \in I_V).$$

Note that $HS^\perp = \mathrm{span}_{\mathbb{R}}\{\tau^S_\alpha\colon\ \alpha \in I_V\}$, where $HS^\perp$ denotes the orthogonal complement of HS in TS, i.e. $TS = HS \oplus HS^\perp$.

Remark 3.11. If $X \in \mathfrak{X}(\mathbb{G})$ we shall set $X_V := p_V(X)$. It is readily seen that[15]

$$\sum_{\alpha\in I_V} \tau^S_\alpha(x_\alpha) = \mathrm{div}(X_V) - \langle\mathcal{J}_H(X_V)\nu_H, \varpi\rangle = \mathrm{div}(X_V) - \frac{\langle\mathcal{J}_H(X_V)\nu_H, \nu_V\rangle}{|p_H\nu|}. \tag{15}$$

[15] Here and in the sequel $\nu_V := p_V\nu$, where $p_V : T\mathbb{G} \to V$ denotes the orthogonal projection onto V.

If $X = X_{HS} + X_{HS}^{\perp} \in \mathfrak{X}(S)$,[16] by differentiating the identity $\langle X, \nu\rangle = x_1\nu_1 + \sum_{\alpha\in I_V} x_\alpha \nu_\alpha = 0$, we get:

$$\sum_{\alpha\in I_V} \tau_\alpha^S(x_\alpha) = x_1\mathcal{H}_H^{\mathrm{sc}} + \mathrm{div}(X_{HS}^{\perp}) + \sum_{\alpha\in I_V} x_\alpha \frac{\partial \varpi_\alpha}{\partial \nu_H};$$

note also that $\mathrm{div}(X_{HS}^{\perp}) = -x_1\mathcal{H}_H^{\mathrm{sc}} + \frac{\partial x_1}{\partial \nu_H} + \sum_{\alpha\in I_V} \tau_\alpha(x_\alpha)$.

In the following two lemmas we collect some useful identities for the sequel.

Lemma 3.12. *The following identities hold*:

(i) $\phi_{1i}(\tau_j) = \phi_{1j}(\tau_i) + \langle C_{HS}\tau_i, \tau_j\rangle_H$ $(i, j \in I_{HS})$;
(ii) $\phi_{1i}(\tau_\alpha^S) = \tau_i(\varpi_\alpha) + \frac{1}{2}\langle C_H^\alpha \tau_1, \tau_i\rangle_H - \langle C\tau_\alpha^S, \tau_i\rangle_H$ $(i \in I_H,\ \alpha \in I_V)$;
(iii) $\phi_{i\alpha}(\tau_j) = \phi_{j\alpha}(\tau_i) + \langle C_{HS}^\alpha \tau_i, \tau_j\rangle_H$ $(i, j \in I_H,\ \alpha \in I_V)$;
(iv) $\tau_\alpha^S(\varpi_\beta) - \tau_\beta^S(\varpi_\alpha) = \langle C\tau_\beta^S, \tau_\alpha^S\rangle$ $(\alpha, \beta \in I_V)$.

Proof. The proof is an elementary exercise based on the definitions and on the fact that the bracket of tangent vectors at regular points of S is again a tangent vector to S. For instance, to prove (i) it is enough to use the identity $\langle [\tau_i, \tau_j], \nu\rangle = 0$ $(i, j \in I_{HS})$. Moreover, (ii), (iv) follow from the identity $\langle [\tau_i, \tau_\alpha^S], \nu\rangle = 0$ $(i \in I_{HS},\ \alpha \in I_V)$ and $\langle [\tau_\alpha^S, \tau_\beta^S], \nu\rangle = 0$ $(\alpha, \beta \in I_V)$, respectively. Finally, (iii) is just a reformulation of the fact that ∇ is torsion free. Note also that (i) says that the partial connection ∇^{HS} has, in general, non-zero torsion. □

Lemma 3.13. *For every $i, j \in I_H$ and every $\alpha \in I_V$, the following identities hold*:

(i) $\phi_{i\alpha}(\tau_\alpha) = 0$;
(ii) $\phi_{\alpha i}(\tau_i) = 0$;
(iii) $\phi_{i\alpha}(\tau_j) = \frac{1}{2}\langle C_H^\alpha \tau_i, \tau_j\rangle$.

Proof. Set $\tau_I = \sum_J A_I^J X_J$ where at each $p \in U$ we have set $A(p) = [A_I^J(p)] \in \mathbf{O}_n(\mathbb{R})$. We first prove (i). We have:

$$\phi_{i\alpha}(\tau_\alpha) = \langle \nabla_{\tau_\alpha}\tau_i, \tau_\alpha\rangle = \sum_{l\in I_H} A_i^l \langle \nabla_{X_\alpha} X_l, X_\alpha\rangle = \frac{1}{2}\sum_{l\in I_H} A_i^l \left(C^{\mathfrak{g}}{}_{\alpha l}^{\alpha} - C^{\mathfrak{g}}{}_{l\alpha}^{\alpha} + C^{\mathfrak{g}}{}_{\alpha l}^{\alpha}\right) = 0 \quad (\alpha \in I_V),$$

by (6) and (5) of Section 2.1. To prove (ii), we use again (6) and (5) of Section 2.1. We have:

$$\begin{aligned}\phi_{\alpha i}(\tau_i) = \langle \nabla_{\tau_i}\tau_\alpha, \tau_i\rangle &= \sum_{l,m\in I_H} A_i^l A_i^m \langle \nabla_{X_l} X_\alpha, X_m\rangle = \frac{1}{2}\sum_{l,m\in I_H} A_i^l A_i^m \left(C^{\mathfrak{g}}{}_{l\alpha}^{m} - C^{\mathfrak{g}}{}_{\alpha m}^{l} + C^{\mathfrak{g}}{}_{ml}^{\alpha}\right)\\ &= \frac{1}{2}\sum_{l,m\in I_H} A_i^l A_i^m C^{\mathfrak{g}}{}_{ml}^{\alpha} = \frac{1}{2}\langle C^\alpha \tau_i, \tau_i\rangle = 0 \quad \big(\text{by skew-symmetry of any } C^\alpha\ (\alpha \in I_V)\big).\end{aligned}$$

Clearly, the identity (iii) can be proved in the same way. More precisely, we have:

$$\begin{aligned}\phi_{i\alpha}(\tau_j) = \langle \nabla_{\tau_j}\tau_i, \tau_\alpha\rangle &= \sum_{l,m\in I_H} A_j^l A_i^m \langle \nabla_{X_l} X_m, X_\alpha\rangle\\ &= \frac{1}{2}\sum_{l,m\in I_H} A_j^l A_i^m \left(C^{\mathfrak{g}}{}_{lm}^{\alpha} - C^{\mathfrak{g}}{}_{m\alpha}^{l} + C^{\mathfrak{g}}{}_{\alpha l}^{m}\right) = \frac{1}{2}\langle C_H^\alpha \tau_i, \tau_j\rangle. \qquad \square\end{aligned}$$

[16] Note that $TS \ni X = \sum_{i=1}^n x_i\tau_i = x_1\tau_1 + \sum_{i\in I_{HS}} x_i\tau_i + \sum_{\alpha\in I_V} x_\alpha\tau_\alpha = X_{HS} + \sum_{\alpha\in I_V} x_\alpha \tau_\alpha^S$.

Here below we make some computations involving the (Riemannian) curvature 2-forms Φ_{IJ} associated with the orthonormal co-frame $\underline{\phi}$ (dual of $\underline{\tau}$). More precisely, we are interested in computing the quantity:

$$\sum_{j\in I_{HS}} \Phi_{1j}(X,\tau_j) = \sum_{j\in I_{HS}} \big\langle \mathrm{R}(X,\tau_j)\tau_1,\tau_j\big\rangle_H.$$

Note that for $X\in \nu_H S$ this is nothing but the Ricci curvature for the partial HS-connection ∇^{HS}.

Lemma 3.14. *We have*:

(i) $\langle \mathrm{R}(\tau_i,\tau_j)\tau_h,\tau_k\rangle_H = -\frac{3}{4}\sum_{\alpha\in I_{H_2}}\langle C_H^\alpha \tau_i,\tau_j\rangle_H\langle C_H^\alpha\tau_h,\tau_k\rangle_H$ $(i,j,h,k\in I_H)$;
(ii) $\langle \mathrm{R}(\tau_\beta,\tau_i)\tau_j,\tau_k\rangle_H = -\frac{1}{4}\sum_{\alpha\in I_{H_2}}\langle C_H^\alpha \tau_j,\tau_k\rangle_H\langle C^\beta\tau_\alpha,\tau_i\rangle_H$ $(i,j,k\in I_H,\ \beta\in I_{H_3})$.

Proof. By linearity of the curvature tensor, we may compute these quantities with respect to the fixed frame $\underline{X}$ of left-invariant vector fields. More precisely, to prove (i), we first compute:

$$\mathrm{R}_{abcd} := \big\langle \mathrm{R}(X_a,X_b)X_c,X_d\big\rangle_H \quad (a,b,c,d\in I_H),$$

and then we deduce the result by observing that, if $\tau_i=\sum_{a\in I_H} A_i^a X_a$ $(i\in I_H)$, one has:

$$\big\langle \mathrm{R}(\tau_i,\tau_j)\tau_h,\tau_k\big\rangle_H = \sum_{a,b,c,d\in I_H} A_i^a A_j^b A_h^c A_k^d \big\langle \mathrm{R}(X_a,X_b)X_c,X_d\big\rangle_H.$$

Now we claim that

$$\big\langle \mathrm{R}(X_a,X_b)X_c,X_d\big\rangle_H = \sum_{\beta\in I_{H_2}}\bigg(\frac{1}{4}C^{\mathfrak{g}}{}^\beta_{ac}C^{\mathfrak{g}}{}^\beta_{db} - \frac{1}{4}C^{\mathfrak{g}}{}^\beta_{bc}C^{\mathfrak{g}}{}^\beta_{da} - \frac{1}{2}C^{\mathfrak{g}}{}^\beta_{ba}C^{\mathfrak{g}}{}^\beta_{dc}\bigg).$$

This formula can be proved directly from the definition of R, by using (5) and (6) of Section 2.1. The computation of (ii) can be done analogously, by linearity, but we need to compute preliminarily the quantity $\mathrm{R}_{\beta abc} := \langle \mathrm{R}(X_\beta,X_a)X_b,X_c\rangle_H$ $(\beta\in I_3,\ a,b,c\in I_H)$. It can be easily shown that

$$\mathrm{R}_{\beta abc} = -\frac{1}{4}\sum_{\alpha\in I_{H_2}}\big(C^{\mathfrak{g}}{}^\beta_{b\alpha}C^{\mathfrak{g}}{}^\alpha_{ca} + C^{\mathfrak{g}}{}^\alpha_{ab}C^{\mathfrak{g}}{}^\beta_{c\alpha}\big).$$

By (5) of Section 2.1, this quantity is different from zero only if $\beta\in I_3$. □

Proposition 3.15. *For every $X(=x\nu_H)\in\mathfrak{X}(\nu_H S)$, we have*:

$$\mathrm{Ric}_{HS}(X) := \sum_{j\in I_{HS}}\big\langle \mathrm{R}(X,\tau_j)\nu_H,\tau_j\big\rangle_{HS} = -\frac{3}{4}x\sum_{\alpha\in I_{H_2}}|C_H^\alpha\nu_H|_{HS}^2.$$

Moreover, for every $X(=X_H+X_V)\in\mathfrak{X}(\mathbb{G})$, $X\pitchfork S$, one has:

$$\sum_{j\in I_{HS}}\Phi_{1j}(X,\tau_j) = -\frac{3}{4}\sum_{\alpha\in I_{H_2}}\langle C_H^\alpha\nu_H, C_H^\alpha X_H\rangle_{HS} - \frac{1}{4}\sum_{\alpha\in I_{H_2}}\sum_{\beta\in I_{H_3}} x_\beta\langle C_H^\alpha\nu_H, C^\beta\tau_\alpha\rangle_{HS}.$$

Proof. Use Lemma 3.14. □

3.2. Integration by parts on hypersurfaces

The aim of this section is to write down explicit integration by parts formulas for non-characteristic hypersurfaces of any Carnot group, endowed with the measure σ_H^{n-1}.

If $X\in\mathfrak{X}(S)$, by the very definition of σ_H^{n-1} using the *Riemannian Divergence Formula* (see [45]), we get:

$$\begin{aligned}\mathrm{d}(X \lrcorner \sigma_H^{n-1})|_{\mathcal{U}} &= \mathrm{d}\big(|p_H\nu| X \lrcorner \sigma^{n-1}\big) = \mathrm{div}_{TS}\big(|p_H\nu|\, X\big)\sigma^{n-1}\\ &= \left(\mathrm{div}_{TS}\, X + \left\langle X, \frac{\mathrm{grad}_{TS}\,|p_H\nu|}{|p_H\nu|}\right\rangle\right)\sigma_H^{n-1} \llcorner \mathcal{U},\end{aligned}$$

where grad_{TS} e div_{TS} are, respectively, the (Riemannian) tangential gradient and the tangential divergence operator on $\mathcal{U} \subset S$. However this formula is not so "explicit", from a sub-Riemannian point of view. The notion of adapted frame has been introduced so far to bypass this inconvenience.

So let $\underline{\tau}$ be an adapted frame to $\mathcal{U} \subset S$ on the open set U and let us denote by $\underline{\phi} := \{\phi_1, \ldots, \phi_n\}$ its *dual co-frame*, obtained by means of the metric g. It is immediate to see that the H-perimeter σ_H^{n-1} on $\mathcal{U}$ is given by:

$$\begin{aligned}\sigma_H^{n-1} \llcorner \mathcal{U} &= (\nu_H \lrcorner \sigma^n)|_{\mathcal{U}} = (\tau_1 \lrcorner \phi_1 \wedge \cdots \wedge \phi_n)|_{\mathcal{U}} = (\phi_2 \wedge \cdots \wedge \phi_n)|_{\mathcal{U}}\\ &= (-1)^{\alpha+1}\big((\varpi_\alpha)^{-1}\phi_1 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n\big)\big|_{\mathcal{U}} \quad (\alpha \in I_V),\end{aligned}$$

where the last identity makes sense only if $\nu_\alpha \neq 0$.[17] By direct computations based on the 1st structure equation of $\underline{\phi}$, we will obtain divergence-type formulas and some easy but useful corollaries.

Remark 3.16 *(Measure on the boundary $\partial\mathcal{U}$).* Before stating these results we have to make a preliminary comment on the topological boundary $\partial\mathcal{U}$ of $\mathcal{U}$. We first assume, as in the Riemannian case that $\partial\mathcal{U}$ is a $(n-2)$-dimensional Riemannian manifold, oriented by the unit normal vector η. Let us denote by $\sigma_{\mathcal{R}}^{n-2}$ the usual Riemannian measure on $\partial\mathcal{U}$, which can be written as

$$\sigma_{\mathcal{R}}^{n-2} \llcorner \partial\mathcal{U} = (\eta \lrcorner \sigma_{\mathcal{R}}^{n-1})|_{\partial\mathcal{U}}.$$

This means that if $X \in \mathfrak{X}(T\mathcal{U})$, then

$$(X \lrcorner \sigma_H^{n-1})|_{\partial\mathcal{U}} = \langle X, \eta\rangle |p_H\nu| \sigma_{\mathcal{R}}^{n-2} \llcorner \partial\mathcal{U}.$$

Now suppose that $\partial\mathcal{U}$ is *geometrically H-regular*. As it can be easily seen, this is equivalent to require that the projection onto HS of the unit (Riemannian) normal η along $\partial\mathcal{U}$ is non-singular, i.e. $|p_{HS}(\eta_p)| \neq 0$, for every $p \in \partial\mathcal{U}$. In the sequel, we shall denote by $C_{\partial\mathcal{U}}$ the characteristic set of $\partial\mathcal{U}$, which turns out to be given by $C_{\partial\mathcal{U}} = \{p \in \partial\mathcal{U}\colon |p_{HS}(\eta_p)| = 0\}$. From Definition 2.26 it follows that

$$\sigma_H^{n-2} \llcorner \partial\mathcal{U} = \left(\frac{p_{HS}\eta}{|p_{HS}\eta|} \lrcorner \sigma_H^{n-1}\right)\bigg|_{\partial\mathcal{U}},$$

or, equivalently, that $\sigma_H^{n-2} \llcorner \partial\mathcal{U} = |p_H\nu| \cdot |p_{HS}\eta| \sigma_{\mathcal{R}}^{n-2} \llcorner \partial\mathcal{U}$. Setting $\eta_{HS} := \frac{p_{HS}\eta}{|p_{HS}\eta|}$, we will call η_{HS} the *unit horizontal normal* along $\partial\mathcal{U}$. We then get:

$$(X \lrcorner \sigma_H^{n-1})|_{\partial\mathcal{U}} = \langle X, \eta_{HS}\rangle \sigma_H^{n-2} \llcorner \partial\mathcal{U} \quad \forall X \in \mathbf{C}^\infty(S, HS).$$

We now state the main results of this section.

Theorem 3.17 *(Horizontal Divergence Theorem). Let $\mathbb{G}$ be a k-step Carnot group. Let $S \subset \mathbb{G}$ be an immersed hypersurface and $\mathcal{U} \subset S \setminus C_S$ be a non-characteristic relatively compact open set. Assume that $\partial\mathcal{U}$ is $\mathbf{C}^\infty$-regular, $(n-2)$-dimensional manifold oriented by its unit normal vector η. Then, for every $X \in \mathbf{C}^\infty(S, HS)$ one has*

$$\int_{\mathcal{U}} \big(\mathrm{div}_{HS}\, X + \langle C_H\nu_H, X\rangle_{HS}\big)\sigma_H^{n-1} = \int_{\partial\mathcal{U}\setminus C_{\partial\mathcal{U}}} \langle X, \eta_{HS}\rangle_{HS} \sigma_H^{n-2}.$$

If $\partial\mathcal{U}$ is geometrically H-regular we have that $C_{\partial\mathcal{U}} = \{\emptyset\}$.

From this formula we obtain the following Green's type-formulas:

[17] We remind that, w.r.t. the adapted frame $\underline{\tau}$, the Riemannian unit normal ν_H is given by $\nu = \nu_1\tau_1 + \sum_{\alpha\in I_V} \nu_\alpha\tau_\alpha$ and that $\tau_1 := \nu_H$ and $\nu_1 := |p_H\nu|$.

Theorem 3.18 *(Horizontal Green's formulas). Under the hypotheses of Theorem* 3.17*, let us assume that* $\phi_1, \phi_2 \in \mathbf{C}^\infty(S)$ *and that at least one of them be compactly supported on* $\mathcal{U}$*. Then*

$$\int_{\mathcal{U}} \{\phi_1 \Delta_{HS}\phi_2 + \langle \operatorname{grad}_{HS}\phi_1, \operatorname{grad}_{HS}\phi_2\rangle_{HS} + \phi_1\langle C_H\nu_H, \operatorname{grad}_{HS}\phi_2\rangle_{HS}\}\sigma_H^{n-1} = \int_{\partial\mathcal{U}\setminus C_{\partial\mathcal{U}}} \phi_1\langle \operatorname{grad}_{HS}\phi_2, \eta_{HS}\rangle_{HS}\sigma_H^{n-2}.$$

Moreover, we have:

$$\int_{\mathcal{U}} \{(\phi_1\Delta_{HS}\phi_2 - \phi_2\Delta_{HS}\phi_1) + \langle C_H\nu_H, (\phi_1 \operatorname{grad}_{HS}\phi_2 - \phi_2 \operatorname{grad}_{HS}\phi_1)\rangle_{HS}\}\sigma_H^{n-1} = 0.$$

Proof. Use Theorem 3.17 with $X = \phi_1 \operatorname{grad}_{HS}\phi_2$ for the first claim. Analogously, the second claim follows since $\phi_1\Delta_{HS}\phi_2 - \phi_2\Delta_{HS}\phi_1 = \operatorname{div}_{HS}(\phi_1 \operatorname{grad}_{HS}\phi_2) - \operatorname{div}_{HS}(\phi_2 \operatorname{grad}_{HS}\phi_1)$. □

Corollary 3.19 *(Horizontal integration by parts). Under the hypotheses Theorem* 3.17*, for any* $X \in \mathfrak{X}(H)$ *we have*:

$$\int_{\mathcal{U}} (\operatorname{div}_{HS} X + \langle C_H\nu_H, X\rangle_{HS})\sigma_H^{n-1} = -\int_{\mathcal{U}} \langle X, \mathcal{H}_H\rangle_H\sigma_H^{n-1} + \int_{\partial\mathcal{U}\setminus C_{\partial\mathcal{U}}} \langle X, \eta_{HS}\rangle_{HS}\sigma_H^{n-2}.$$

Proof. It follows by Theorem 3.17 and Definition 3.2. □

Theorem 3.20 *(Divergence Theorem). Let* $\mathbb{G}$ *be a k-step Carnot group. Let* $S \subset \mathbb{G}$ *be an immersed hypersurface and* $\mathcal{U} \subset S \setminus C_S$ *be a non-characteristic relatively compact open set. Assume that* $\partial\mathcal{U}$ *is* $\mathbf{C}^\infty$*-regular,* $(n-2)$*-dimensional manifold oriented by its unit normal vector* η*. Set* $\varpi = \frac{p_V\nu}{|p_H\nu|}$ *and choose* $X \in \mathfrak{X}(S)$, $X = X_{HS} + X_{HS^\perp}$*. Then we have*:

$$\int_{\mathcal{U}} \left\{\operatorname{div}_{HS}(X_{HS}) + \operatorname{div}(X_{HS^\perp}) - \frac{\partial\langle X, \nu_H\rangle}{\partial\nu_H} + \langle [X, \nu_H], \varpi\rangle\right\}\sigma_H^{n-1} = \int_{\partial\mathcal{U}} \langle X, \eta\rangle|p_H\nu|\sigma_{\mathcal{R}}^{n-2}.$$

We remark that the previous formula can also be written as follows:

$$\int_{\mathcal{U}} \left\{\operatorname{div}_{HS}(X_{HS}) - \langle \mathcal{H}_H, X\rangle + \langle C\nu_H, X\rangle + \sum_{\alpha\in I_V} \tau_\alpha^S(x_\alpha)\right\}\sigma_H^{n-1} = \int_{\partial\mathcal{U}} \langle X, \eta\rangle|p_H\nu|\sigma_{\mathcal{R}}^{n-2},$$

where $X = \langle X, \nu_H\rangle\nu_H + X_{HS} + \sum_{\alpha\in I_V} x_\alpha\tau_\alpha$ and $\tau_\alpha^S = \tau_\alpha - \varpi_\alpha\nu_H$ $(\alpha \in I_V)$; see Eq. (24) below.

We stress that denoting by $\operatorname{Div}_{\sigma_H^{n-1}}(X)$ the *Lie divergence of X with respect to* σ_H^{n-1} $(X \in \mathfrak{X}(\mathbb{G}))$, i.e.

$$\operatorname{Div}_{\sigma_H^{n-1}}(X)\sigma_H^{n-1} := \mathcal{L}_X\sigma_H^{n-1},$$

one could shortly rewrite the previous divergence-type theorems; see, for instance, [6], p. 139.

3.3. Divergence-type theorems: proofs

Proof. For $X \in \mathbf{C}^\infty(S, HS)$, we have to compute the exterior derivative of the contraction by X of σ_H^{n-1}, i.e. $\mathrm{d}(X \lrcorner\, \sigma_H^{n-1})|_S$. So if $X = \sum_{J=1}^n x_J\tau_J$, then

$$\begin{aligned}\mathrm{d}(X \lrcorner\, \sigma_H^{n-1})|_S &= \sum_{J=1}^n \mathrm{d}(x_J\,\tau_J \lrcorner\, \sigma_H^{n-1})|_S = \sum_{J=1}^n \mathrm{d}(x_J\,\tau_J \lrcorner\, \phi_2 \wedge \cdots \wedge \phi_n)|_S \\ &= \sum_{J=2}^n \left(\tau_J(x_J)\,\sigma_H^{n-1}|_S - \tau_1(x_J)(\tau_J \lrcorner\, \sigma_{\mathcal{R}}^n)|_S + x_J\,\mathrm{d}(\tau_J \lrcorner\, \sigma_H^{n-1})|_S\right)\end{aligned}$$

$$= \underbrace{\sum_{J=2}^{n}\big(\tau_J(x_J)\,\sigma_H^{n-1}|_S - \tau_1(x_J)(\tau_J \lrcorner \sigma_{\mathcal{R}}^n)|_S\big)}_{:=I} + \sum_{i\in I_{HS}} x_i\,\mathrm{d}(\tau_i \lrcorner \sigma_H^{n-1})|_S$$
$$+ \sum_{\alpha\in I_V} x_\alpha \mathrm{d}(\tau_\alpha \lrcorner \sigma_H^{n-1})|_S. \tag{16}$$

Notice that, using Lemma 2.19, we get:

$$I = \Bigg(\sum_{i\in I_{HS}} \tau_i(x_i) + \sum_{\alpha\in I_V}\big(\tau_\alpha(x_\alpha) - \varpi_\alpha \tau_1(x_\alpha)\big)\Bigg)\sigma_H^{n-1}|_S$$
$$= \Bigg(\sum_{i\in I_{HS}} \tau_i(x_i) + \sum_{\alpha\in I_V} \tau_\alpha^S(x_\alpha)\Bigg)\sigma_H^{n-1}|_S. \tag{17}$$

Claim 3.21. *We claim that*

$$\mathrm{d}(\tau_i \lrcorner \sigma_H^{n-1})|_S = \Bigg(\sum_{j\in I_{HS}} \phi_{ij}(\tau_j) + \sum_{\beta\in I_{H_2}} \varpi_\beta \langle C_H^\beta \tau_1, \tau_i\rangle_H\Bigg)\sigma_H^{n-1}|_S. \tag{18}$$

Proof. Since $\mathrm{d}(\tau_i \lrcorner \sigma_H^{n-1})|_S = (-1)^i\,\mathrm{d}(\phi_2 \wedge \cdots \wedge \widehat{\phi_i} \wedge \cdots \wedge \phi_n)|_S$, without loss of generality we assume that $i = 2$. We have:

$$I := \mathrm{d}(\phi_3 \wedge \cdots \wedge \phi_n) = \sum_{J=3}^{n}(-1)^{J+1}\phi_3 \wedge \cdots \wedge \mathrm{d}\phi_J \wedge \cdots \wedge \phi_n$$
$$= \sum_{J=3}^{n}(-1)^{J+1}\phi_3 \wedge \cdots \wedge \underbrace{\Bigg(\sum_{I=1}^{n}\phi_I \wedge \phi_{IJ}\Bigg)}_{J\text{th place}} \wedge \cdots \wedge \phi_n$$
$$= -\sum_{J=3}^{n}(-1)^{J+1}\Bigg(\sum_{I=1}^{2}\phi_{IJ} \wedge \phi_I\Bigg) \wedge \widehat{\phi_2} \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \cdots \wedge \phi_n$$
$$= \underbrace{-\sum_{J=3}^{n}(-1)^{J+1}(\phi_{1J} \wedge \phi_1) \wedge \widehat{\phi_2} \wedge \phi_3 \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \cdots \wedge \phi_n}_{:=II}$$
$$\underbrace{-\sum_{J=3}^{n}(-1)^{J+1}(\phi_{2J} \wedge \phi_2) \wedge \widehat{\phi_2} \wedge \phi_3 \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \ldots \wedge \phi_n}_{:=III}. \tag{19}$$

Here above, we have used the first structure equation of the adapted coframe $\underline{\phi}$. The generic term of *II* is given by:

$$(\phi_{1J} \wedge \phi_1) \wedge \widehat{\phi_2} \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \phi_n = \big(\phi_{1J}(\tau_2)\,\phi_2 + \phi_{1J}(\tau_J)\,\phi_J\big) \wedge \phi_1 \wedge \widehat{\phi_2} \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \phi_n$$
$$= -\phi_{1J}(\tau_2)\phi_1 \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \phi_n$$
$$+ (-1)^J \phi_{1J}(\tau_J)\,\phi_1 \wedge \widehat{\phi_2} \wedge \cdots \wedge \phi_n. \tag{20}$$

Now, if $J \in I_{HS}$, Lemma 2.19 says that $(\phi_1 \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \phi_n)|_S$ is zero and that it is different from zero only if $J \in I_V$. Furthermore, Lemma 2.19 says that the second addend is zero, when restricted to S. Analogously, for the generic term of *III*, we have:

$$(\phi_{2J} \wedge \phi_2) \wedge \widehat{\phi_2} \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \phi_n = \big(\phi_{2J}(\tau_1)\,\phi_1 + \phi_{2J}(\tau_J)\,\phi_J\big) \wedge \phi_2 \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \wedge \phi_n$$
$$= \phi_{2J}(\tau_1)\,\phi_1 \wedge \cdots \wedge \widehat{\phi_J} \wedge \cdots \phi_n + (-1)^J \phi_{2J}(\tau_J)\phi_2 \wedge \cdots \wedge \phi_n. \tag{21}$$

Arguing as above, by using again Lemma 2.19, we get that the first term of (21) is different from zero only if $J \in I_V$, while the second one is different from zero only if $J \in I_H$, because $\phi_{i\alpha}(\tau_\alpha) = 0$. From (19), (20) and (21) we get:

$$\begin{aligned} I &= \sum_{J=3}^{n} \phi_{2J}(\tau_J)(\phi_2 \wedge \cdots \wedge \phi_n)|_S + \sum_{\beta \in I_V} (-1)^{\beta+1}\big(\phi_{1\beta}(\tau_2) - \phi_{2\beta}(\tau_1)\big)(\phi_1 \wedge \cdots \wedge \cdots \widehat{\phi_\beta} \wedge \cdots \wedge \phi_n)|_S \\ &= \sum_{J=3}^{n} \phi_{2J}(\tau_J)(\sigma_H^{n-1})|_S + \sum_{\beta \in I_2} \big(\phi_{1\beta}(\tau_2) - \phi_{2\beta}(\tau_1)\big)(\tau_\beta \,\lrcorner\, \sigma_{\mathcal{R}}^n)|_S \\ &= \Bigg(\sum_{j \in I_{HS}} \phi_{2j}(\tau_j) + \sum_{\beta \in I_2} \varpi_\beta\big(\phi_{1\beta}(\tau_2) - \phi_{2\beta}(\tau_1)\big)\Bigg)\sigma_H^{n-1}|_S. \end{aligned}$$

Since $\phi_{1\beta}(\tau_2) - \phi_{2\beta}(\tau_1) = -C_{12}^{\beta}$, we get our initial claim, by using Definition 2.4. □

Claim 3.22. *We claim that*

$$\mathrm{d}(\tau_\alpha \,\lrcorner\, \sigma_H^{n-1})|_S = -\Bigg(\sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} \varpi_\gamma C_{1\alpha}^{\gamma} + \varpi_\alpha \sum_{j \in I_{HS}} \phi_{1j}(\tau_j)\Bigg)\sigma_H^{n-1}|_S \quad (\alpha \in I_V), \tag{22}$$

where $C_{IJ}^{K} = \langle [\tau_I, \tau_J], \tau_K \rangle$ $(I, J, K = 1, \ldots, n)$ are the structural constants of the adapted frame $\underline{\tau}$.

Proof. We have $\mathrm{d}(\tau_\alpha \,\lrcorner\, \sigma_H^{n-1}) = (-1)^\alpha\, \mathrm{d}(\phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n)$ and so

$$\begin{aligned} \mathrm{d}(\phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n) &= \underbrace{\sum_{j \in I_{HS}} (-1)^j \phi_2 \wedge \cdots \wedge \mathrm{d}\phi_j \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n}_{:=I} \\ &+ \underbrace{\sum_{\substack{\gamma \in I_V \\ \gamma < \alpha}} (-1)^\gamma \phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \mathrm{d}\phi_\gamma \wedge \phi_n}_{:=II} + \underbrace{\sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} (-1)^{\gamma+1} \phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \mathrm{d}\phi_\gamma \wedge \phi_n}_{:=III}. \end{aligned}$$

As above, we shall make use of the 1st structure equation for the co-frame $\underline{\phi}$ and of Lemma 2.19. For the first summation, since $\mathrm{d}\phi_j = \sum_{K \neq j} \phi_K \wedge \phi_{Kj}$ $(K = 1, \ldots, n)$, we get:

$$\begin{aligned} I &= \sum_{j \in I_{HS}} (-1)^j \phi_2 \wedge \cdots \wedge \mathrm{d}\phi_j \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n \\ &= \sum_{j \in I_{HS}} \sum_{K \neq j} (-1)^j \phi_2 \wedge \cdots \wedge \underbrace{(\phi_K \wedge \phi_{Kj})}_{j\text{th place}} \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n \\ &= \sum_{j \in I_{HS}} (-1)^j \phi_2 \wedge \cdots \wedge \underbrace{(\phi_1 \wedge \phi_{1j} + \phi_\alpha \wedge \phi_{\alpha j})}_{j\text{th place}} \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n. \end{aligned}$$

Using this expression, Lemma 2.19 and the fact that $\phi_{\alpha j}(\tau_j) = 0$ (see Section 3.1), we obtain:

$$\begin{aligned} I &= \sum_{j \in I_{HS}} (-1)^j \phi_2 \wedge \cdots \wedge (\phi_1 \wedge \phi_{1j}) \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n \\ &= \sum_{j \in I_{HS}} (-1)^j (-1)^{j-2} \phi_{1j}(\tau_j) \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_j \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \phi_n \\ &= (-1)^{\alpha+1} \sum_{j \in I_{HS}} \phi_{1j}(\tau_j)(\tau_\alpha \,\lrcorner\, \sigma_{\mathcal{R}}^n)|_S = (-1)^{\alpha+1} \varpi_\alpha \sum_{j \in I_{HS}} \phi_{1j}(\tau_j) \sigma_H^{n-1}|_S. \end{aligned}$$

Moreover the second and the third summations can be computed as follows. First, we note that, using the 1st structural equation for the coframe $\underline{\phi}$ and Lemma 2.19, the term $\phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \mathrm{d}\phi_\gamma \wedge \phi_n$ is given, up to sign, by:

$$\begin{aligned}\pm\phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \mathrm{d}\phi_\gamma \wedge \phi_n &= \pm \sum_{L \neq \gamma} \phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \underbrace{(\phi_L \wedge \phi_{L\gamma})}_{\gamma\text{th place}} \wedge \cdots \wedge \phi_n \\ &= \pm\phi_2 \wedge \cdots \wedge \widehat{\phi_\alpha} \wedge \cdots \wedge \underbrace{(\phi_1 \wedge \phi_{1\gamma} + \phi_\alpha \wedge \phi_{\alpha\gamma})}_{\gamma\text{th place}} \wedge \cdots \wedge \phi_n \\ &= \pm\big(\phi_{1\gamma}(\tau_\alpha) - \phi_{\alpha\gamma}(\tau_1)\big)\phi_1 \wedge \cdots \wedge \widehat{\phi_\gamma} \wedge \cdots \wedge \mathrm{d}\phi_\gamma \wedge \phi_n = \pm\big\langle[\tau_1, \tau_\alpha], \tau_\gamma\big\rangle(\tau_\gamma \lrcorner\, \sigma^n_{\mathcal{R}})|_S.\end{aligned}$$

Using this fact, by an easy computation of the signs and the fact that $C^\gamma_{1\alpha} = \langle[\tau_1, \tau_\alpha], \tau_\gamma\rangle$, we see that

$$\begin{aligned}II + III &= \sum_{\substack{\gamma \in I_V \\ \gamma < \alpha}} (-1)^{\gamma+\alpha} C^\gamma_{1\alpha}\phi_1 \wedge \cdots \wedge \widehat{\phi_\gamma} \wedge \cdots \wedge \phi_n + \sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} (-1)^{\gamma+\alpha-1} C^\gamma_{1\alpha}\phi_1 \wedge \cdots \wedge \widehat{\phi_\gamma} \wedge \cdots \wedge \phi_n \\ &= \Bigg[(-1)^{\alpha+1} \sum_{\substack{\gamma \in I_V \\ \gamma < \alpha}} C^\gamma_{1\alpha} - (-1)^\alpha \sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} C^\gamma_{1\alpha}\Bigg](\tau_\gamma \lrcorner\, \sigma^n_{\mathcal{R}})|_S = (-1)^{\alpha+1} \sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} \varpi_\gamma C^\gamma_{1\alpha}\sigma^{n-1}_H|_S,\end{aligned}$$

where we have used the identity[18] $C^\gamma_{1\alpha} = 0$ if $\mathrm{ord}(\gamma) \leqslant \mathrm{ord}(\alpha)$. Putting all together we obtain (22).

At this point we may achieve the proof, using (16), (17) and the previous claims. We have:

$$\begin{aligned}\mathrm{d}(X \lrcorner\, \sigma^{n-1}_H)|_S = \Bigg\{ &\sum_{i \in I_{HS}} \Bigg[\tau_i(x_i) + x_i\Bigg(\sum_{j \in I_{HS}} \phi_{ij}(\tau_j) + \sum_{\beta \in I_{H_2}} \varpi_\beta \langle C^\beta_H \tau_1, \tau_i\rangle\Bigg)\Bigg] \\ &+ \sum_{\alpha \in I_V} \Bigg[\tau^S_\alpha(x_\alpha) - x_\alpha\Bigg(\sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} \varpi_\gamma C^\gamma_{1\alpha} + \varpi_\alpha \sum_{j \in I_{HS}} \phi_{1j}(\tau_j)\Bigg)\Bigg]\Bigg\}\sigma^{n-1}_H|_S. \end{aligned} \tag{23}$$

Claim 3.23. *Let $X = X_{HS} + X^\perp_{HS} (= X_{HS} + \sum_{\alpha \in I_V} x_\alpha \tau^S_\alpha)$. Then, we have*:

(i) *the HS-divergence of $X_{HS} (= p_{HS}X)$ turns out to be given by*:

$$\mathrm{div}_{HS} X_{HS} = \sum_{i \in I_{HS}} \Bigg(\tau_i(x_i) + x_i \sum_{j \in I_{HS}} \phi_{ij}(\tau_j)\Bigg);$$

(ii) *if $X \in TS$, then $x_1\nu_1 + \sum_{\alpha \in I_V} x_\alpha \nu_\alpha = 0$, and $x_1 = -\sum_{\alpha \in I_V} \varpi_\alpha x_\alpha$. By differentiating this identity, we get that $-\sum_{\alpha \in I_V} \varpi_\alpha \tau_1(x_\alpha) = \tau_1(x_1) + \frac{x_1\tau_1(\nu_1)}{\nu_1} + \sum_{\alpha \in I_V} \frac{x_\alpha \tau_1(\nu_\alpha)}{\nu_1}$*;

(iii) $\langle [X, \tau_1], \varpi\rangle = \langle C\tau_1, X\rangle - \sum_{\alpha \in I_V} \varpi_\alpha \tau_1(x_\alpha)$;

(iv) $\sum_{\alpha \in I_V} x_\alpha \sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} \varpi_\gamma C^\gamma_{1\alpha} = -\sum_{\alpha \in I_V} x_\alpha \sum_{\substack{\gamma \in I_V \\ \gamma > \alpha}} \varpi_\gamma \langle C^\gamma \tau_1, \tau_\alpha\rangle = -\langle C\tau_1, X^\perp_{HS}\rangle$.

Note that, if $X \in \mathbf{C}^\infty(S, HS)$, then from the very definition of C and C_H (see Notation 3.6*), we obtain*:

$$\langle C\tau_1, X\rangle = \langle C_H\tau_1, X\rangle_{HS}.$$

[18] We have:

$$C^\gamma_{1\alpha} = \big\langle[\tau_1, \tau_\alpha], \tau_\gamma\big\rangle = \sum_{l \in I_H} \big\langle[A^l_1 X_l, X_\alpha], X_\gamma\big\rangle = \sum_{l \in I_H} A^l_1 C^{\mathfrak{g}\,\gamma}_{\ l\alpha},$$

and the last term is different from zero only if $\mathrm{ord}(\gamma) = \mathrm{ord}(\alpha) + 1$, by (5) of Section 2.1.

Therefore Theorem 3.17 follows from (23), by applying (i) of Claim 3.23 together with the very definition of C_H and setting $x_\alpha = 0$ ($\alpha \in I_V$). Moreover, to prove Theorem 3.20, it is enough to apply Claim 3.23 into (23). Indeed, from equation (23) by using (i), (ii), (iv) above, we get:

$$\mathrm{d}(X \lrcorner \sigma_H^{n-1})|_S = \Big(\mathrm{div}_{HS} X_{HS} - x_1 \mathcal{H}_H^{\mathrm{sc}} + \langle C\tau_1, X\rangle + \sum_{\alpha\in I_V} \tau_\alpha^S(x_\alpha)\Big)\sigma_H^{n-1}|_S. \tag{24}$$

Therefore, using (iii) of Claim 3.23, Remark 3.11 and (ii) of Claim 3.23, we get the thesis. □

4. Variational formulas: 1st and 2nd variation of σ_H^{n-1}

4.1. 1st variation of σ_H^{n-1}

In this section, we will compute the 1st variation of σ_H^{n-1}, by adapting to the sub-Riemannian setting of Carnot groups, some classical differential-geometric methods based on the use of moving frames and differential forms. As references for these topics in the Riemannian case we mention Spivak's book [45] and also the paper by Hermann [24].

As before, let $\mathbb{G}$ be a k-step Carnot group and let $S \subset \mathbb{G}$ be a non-characteristic hypersurface oriented by its unit normal vector ν. Moreover, let $\mathcal{U} \subset S \setminus C_S$ be a relatively compact open set which is assumed to be *non-characteristic* and let us assume that the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ is a $(n-2)$-dimensional $\mathbf{C}^\infty$-regular submanifold oriented by its outward unit normal vector η.

Definition 4.1. Let $\iota : \mathcal{U} \to \mathbb{G}$ denote the inclusion of $\mathcal{U}$ in $\mathbb{G}$ and let $\vartheta : (-\varepsilon, \varepsilon) \times \mathcal{U} \to \mathbb{G}$ be a $\mathbf{C}^\infty$ map. Then ϑ is a *smooth variation* of ι if

(i) every $\vartheta_t := \vartheta(t, \cdot) : \mathcal{U} \to \mathbb{G}$ is an immersion;
(ii) $\vartheta_0 = \iota$.

Moreover, we say that the variation ϑ *keeps the boundary* $\partial\mathcal{U}$ *fixed* if

(iii) $\vartheta_t|_{\partial\mathcal{U}} = \iota|_{\partial\mathcal{U}}$ for every $t \in (-\varepsilon, \varepsilon)$.

The *variation vector* of ϑ, is defined by $W := \frac{\partial\vartheta}{\partial t}|_{t=0} = \vartheta_* \frac{\partial}{\partial t}|_{t=0}$.

Later on we shall set $\widetilde{W} := \frac{\partial\vartheta}{\partial t} = \vartheta_* \frac{\partial}{\partial t}$ and we will assume that $\widetilde{W}$ is defined in a neighborhood of $\mathrm{Im}(\vartheta)$. For any $t \in (-\varepsilon, \varepsilon)$, we will denote by ν^t the unit normal vector along $\mathcal{U}_t := \vartheta_t(\mathcal{U})$ and by $(\sigma_{\mathcal{R}}^{n-1})_t$ the Riemannian measure on $\mathcal{U}_t$. Note that if $\mathcal{U}$ and ε are small enough, then $\mathcal{U}_t = \vartheta_t(\mathcal{U})$ turns out to be immersed and non-characteristic for every $t \in (-\varepsilon, \varepsilon)$. So let us define the differential $(n-1)$-form $(\sigma_H^{n-1})_t$ along $\mathcal{U}_t$ by:

$$(\sigma_H^{n-1})_t|_{\mathcal{U}_t} = (\nu_H^t \lrcorner \sigma_{\mathcal{R}}^n)|_{\mathcal{U}_t} \in \Lambda^{n-1}(T\mathcal{U}_t),$$

for $t \in (-\varepsilon, \varepsilon)$, where

$$\nu_H^t := \frac{p_H \nu^t}{|p_H \nu^t|}.$$

By setting

$$\Gamma(t) := \vartheta_t^*(\sigma_H^{n-1})_t \in \Lambda^{n-1}(T\mathcal{U}), \quad t \in (-\varepsilon, \varepsilon),$$

we get that $\Gamma(t)$ is a $\mathbf{C}^\infty$ 1-parameter family of $(n-1)$-forms along $\mathcal{U}$. Thus, in order to determine the 1st variation $I_{\mathcal{U}}(W, \sigma_H^{n-1})$ of σ_H^{n-1}, we have to compute:

$$I_{\mathcal{U}}(W, \sigma_H^{n-1}) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_{\mathcal{U}} \Gamma(t) = \int_{\mathcal{U}} \dot{\Gamma}(0). \tag{25}$$

So we will need to preliminarily compute $\dot{\Gamma}(0)$. Notice that the derivative under the integral sign can be done by the well-known *Leibnitz's rule* (see, for instance, [45], p. 417). Thus making use of the *Cartan's formula* for the Lie derivative of a differential form, we may prove the following:

Theorem 4.2 *(1st variation of σ_H^{n-1}). Under the previous hypotheses we have*:

$$I_{\mathcal{U}}(W,\sigma_H^{n-1}) = -\int_{\mathcal{U}} \mathcal{H}_H^{\mathrm{sc}} \frac{\langle W,\nu\rangle}{|p_H\nu|}\sigma_H^{n-1} + \int_{\partial\mathcal{U}} \langle W,\eta\rangle |p_H\nu| \sigma_{\mathcal{R}}^{n-2}. \tag{26}$$

Notice that from this result it follows immediately that a necessary condition for *minimality* of any smooth non-characteristic hypersurface is given by the vanishing of the *scalar horizontal mean curvature* $\mathcal{H}_H^{\mathrm{sc}}$. This justifies the fact that the equation,

$$-\mathcal{H}_H^{\mathrm{sc}} = \mathrm{div}_{HS}\,\nu_H = \mathrm{div}\,\nu_H = 0,$$

is the right sub-Riemannian generalization of the Riemannian one. In this respect, we would note that the Riemannian scalar mean curvature $\mathcal{H}_{\mathcal{R}}^{\mathbf{sc}}$ and that horizontal $\mathcal{H}_H^{\mathrm{sc}}$ are related by the identity:

$$\mathcal{H}_{\mathcal{R}}^{\mathbf{sc}} = |p_H\nu|\mathcal{H}_H^{\mathrm{sc}} - \frac{\partial|p_H\nu|}{\partial\nu_H} - \mathrm{div}(p_V\nu).$$

Analogously to the Riemannian case, the terms in the 1st variation formula are two, the first one—the integral along $\mathcal{U}$—only depending on the normal component of the variation vector W, and the second one—the integral along the boundary $\partial\mathcal{U}$—which only depends on the tangential component of W. This fact relies on a general principle of the Calculus of Variations on manifolds, for which we refer the reader to [25]. It is also clear that, if we allow the variation vector to be horizontal, then (26) becomes more "intrinsic". Indeed, if $W\in\mathbf{C}^\infty(S,H)$, $W=\langle W,\nu_H\rangle_H\nu_H + W_{HS}$, then we get the following:

Theorem 4.3 *(Horizontal 1st variation of σ_H^{n-1}). Under the previous hypotheses, let us assume that the variation vector W of ϑ be horizontal, i.e. $W\in\mathbf{C}^\infty(S,H)$. Then we have*:

$$I_{\mathcal{U}}(W,\sigma_H^{n-1}) = -\int_{\mathcal{U}} \langle\mathcal{H}_H, W\rangle_H\,\sigma_H^{n-1} + \int_{\partial\mathcal{U}\setminus C_{\partial\mathcal{U}}} \langle W,\eta_{HS}\rangle_{HS}\,\sigma_H^{n-2}. \tag{27}$$

Proof. Use Theorem 4.2 and Remark 3.16.

Therefore, in the case of horizontal variations, by remembering Corollary 3.19, we get:

$$I_{\mathcal{U}}(W,\sigma_H^{n-1}) = \int_{\mathcal{U}} \big(\mathrm{div}_{HS}\,W + \langle C_H\nu_H, W\rangle_{HS}\big)\,\sigma_H^{n-1}.$$

We stress that, also in the case of horizontal variations, the 1st variation formula (27) is given by two terms, the first of which only depends on the horizontal normal component of W, while the second one only depends on its horizontal tangential component.

Remark 4.4 *(Boundary integrals).* The integrals along the boundary $\partial\mathcal{U}$ of the domain $\mathcal{U}\subset S$ are zero in the following two cases:

(i) $W\in\mathbf{C}_0^\infty(\mathcal{U},T\mathbb{G})$, i.e. we assume that the vector variation be compactly supported on $\mathcal{U}$;
(ii) The smooth variation ϑ of $\mathcal{U}$ keeps the boundary $\partial\mathcal{U}$ fixed; see Definition 4.1.

Note also that, from (26) (resp. (27)) it follows that the boundary integral is zero whenever we choose $W\in\mathfrak{X}(\nu S)$ (resp. $W\in\mathfrak{X}(\nu_H S)$).

As a corollary of the 1st variation formula we obtain a necessary condition for a smooth domain to be *sub-Riemannian isoperimetric*. To this end, let us consider the *sub-Riemannian isoperimetric functional*:

$$J_H(D) = \frac{\sigma_H^{n-1}(\partial D)}{\mathrm{vol}^n_{\mathcal{R}}(D)^{1-Q}}, \tag{28}$$

where D varies over bounded domains in $\mathbb{G}$ having smooth (at least $\mathbf{C}^2$) boundary. We stress that, we do not need any assumption about the characteristic set of ∂D, since $C_{\partial D}$ is a set of zero σ_H^{n-1}-measure.

Corollary 4.5. *Let $D \subset \mathbb{G}$ be a bounded domain with smooth boundary that is a critical point of the functional* (28). *Then, at every point of $\partial D \setminus C_{\partial D}$ we have that $\mathcal{H}_H^{\mathrm{sc}}$ is constant.*

Proof. The proof is analogous to the Riemannian case (see, for instance, [7]). Indeed, let us choose a volume-preserving vector field $W \in \mathfrak{X}(\mathbb{G})$. Then the flow $\vartheta_t : (-\varepsilon, \varepsilon) \times \mathbb{G} \to \mathbb{G}$ generated by W does not change the volume, i.e. $\mathrm{vol}^n_{\mathcal{R}}(\vartheta_t(D)) = \mathrm{vol}^n_{\mathcal{R}}(D)$ for every $t \in (-\varepsilon, \varepsilon)$. So, by the Riemannian Divergence Theorem, we get:

$$\int_D \mathrm{div}\, W \,\mathrm{d}\,\mathrm{vol}^n_{\mathcal{R}} = \int_{\partial D} \langle W, \nu \rangle \sigma^{n-1}_{\mathcal{R}} = 0,$$

for any such W. By differentiating (28) along the flow ϑ_t, using Theorem 27 we get:

$$\frac{\mathrm{d}}{\mathrm{d}t} J_H\big(\vartheta_t(D)\big)\Big|_{t=0} = -\frac{1}{\mathrm{vol}^n_{\mathcal{R}}(D)^{1-Q}} \int_{\partial D} \mathcal{H}_H^{\mathrm{sc}} \langle W, \nu \rangle \sigma^{n-1}_{\mathcal{R}} - \frac{Q-1}{Q} \int_{\partial D} \langle W, \nu \rangle \sigma^{n-1}_{\mathcal{R}} = 0,$$

since D is an extremal of (28). Therefore, $\int_{\partial D} \mathcal{H}_H^{\mathrm{sc}} \langle W, \nu \rangle \sigma_H^{n-1} = 0$ for every volume-preserving vector field $W \in \mathfrak{X}(\mathbb{G})$. A standard argument now implies that $\mathcal{H}_H^{\mathrm{sc}}$ must be constant. □

4.2. 1st variation of σ_H^{n-1}: proof of Theorem 4.2

Proof. Let us choose an orthonormal moving frame $\underline{\tau}$ on the open set $U \subset \mathbb{G}$ satisfying:

$$\text{(i) } \tau_1|_{\mathcal{U}_t} := \nu_H^t; \quad \text{(ii) } HT_p\mathcal{U}_t = \mathrm{span}\big\{(\tau_2)_p, \dots, (\tau_{h_1})_p\big\}\ (p \in \mathcal{U}_t); \quad \text{(iii) } \tau_\alpha := X_\alpha.$$

Let $\underline{\phi} := \{\phi_1, \dots, \phi_n\}$ be its dual co-frame (i.e. $\phi_I(\tau_J) = \delta_I^J$ $(I, J = 1, \dots, n)$). We have:

$$(\sigma_H^{n-1})_t \llcorner\, \mathcal{U}_t = (\tau_1 \rfloor \phi_1 \wedge \dots \wedge \phi_n)|_{\mathcal{U}_t} = (\phi_2 \wedge \dots \wedge \phi_n)|_{\mathcal{U}_t}$$

and $\Gamma(t) = \vartheta_t^*(\phi_2 \wedge \dots \wedge \phi_n)$. We stress that the variation vector field W on $\mathcal{U}$ can be seen as the restriction to $\mathcal{U}$ of the vector field $\widetilde{W} = \frac{\partial \vartheta}{\partial t}$. Clearly the integral curve of $\widetilde{W}$ that starts at a point $p \in \mathcal{U}$ is just $t \mapsto \vartheta_t(p)$.

Claim 1. *We claim that $\dot{\Gamma}(0) = \imath^*(\mathcal{L}_{\widetilde{W}}((\sigma_H^{n-1})_t)) = \imath^*(\mathcal{L}_{\widetilde{W}}(\phi_2 \wedge \dots \wedge \phi_n))$.*

Proof of Claim 1. The proof of this fact is standard; see, for instance, [45]. For the sake of completeness we shall report it below. Denote by $\theta_t(p)$ the integral path of $\widetilde{W}$ starting at $p \in U$. If $p \in \mathcal{U}$ and $Y \in T_p\mathcal{U}$ we have $\theta_{t*}(\imath_* Y) = \vartheta_{t*} Y$. So let $Y_1, \dots, Y_{n-1}$ be tangent vectors of $\mathcal{U}$. Then

$$\begin{aligned}
\dot{\Gamma}(0)(Y_1, \dots, Y_{n-1}) &= \lim_{t \to 0} \frac{1}{t} \big\{\Gamma(t)(Y_1, \dots, Y_{n-1}) - \Gamma(0)(Y_1, \dots, Y_{n-1})\big\} \\
&= \lim_{t \to 0} \frac{1}{t} \big\{{\vartheta_t}^*(\sigma_H^{n-1})_t(Y_1, \dots, Y_{n-1}) - \imath^*(\sigma_H^{n-1})_t(Y_1, \dots, Y_{n-1})\big\} \\
&= \lim_{t \to 0} \frac{1}{t} \big\{(\sigma_H^{n-1})_t(\vartheta_{t*}Y_1, \dots, \vartheta_{t*}Y_{n-1}) - (\sigma_H^{n-1})_t(\imath_*Y_1, \dots, \imath_*Y_{n-1})\big\} \\
&= \lim_{t \to 0} \frac{1}{t} \big\{(\sigma_H^{n-1})_t\big(\theta_{t*}(\imath_*Y_1), \dots, \theta_{t*}(\imath_*Y_{n-1})\big) - (\sigma_H^{n-1})_t(\imath_*Y_1, \dots, \imath_*Y_{n-1})\big\} \\
&= \mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t(\imath_*Y_1, \dots, \imath_*Y_{n-1}) \quad \text{(by definition of Lie derivative).} \qquad \square
\end{aligned}$$

By Cartan's formula for the Lie derivative we get $\mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t = \widetilde{W} \lrcorner \mathrm{d}(\sigma_H^{n-1})_t + \mathrm{d}(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t)$ and therefore, by Claim 1, we get:

$$\dot{\Gamma}(0) = \imath^*\big(\widetilde{W} \lrcorner \mathrm{d}(\sigma_H^{n-1})_t + \mathrm{d}\big(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\big)\big). \tag{29}$$

Now we have:

$$\begin{aligned}
\mathrm{d}(\sigma_H^{n-1})_t = \mathrm{d}(\phi_2 \wedge \cdots \wedge \phi_n) &= \sum_{I=2}^{n} (-1)^I \phi_2 \wedge \cdots \wedge \mathrm{d}\phi_I \wedge \cdots \wedge \phi_n \\
&= \sum_{I=2}^{n} (-1)^I \phi_2 \wedge \cdots \wedge \left(-\sum_{J=1}^{n} \phi_{JI} \wedge \phi_J\right) \wedge \cdots \wedge \phi_n \qquad (30)\\
&= -\sum_{I=2}^{n} (-1)^I \phi_2 \wedge \cdots \wedge (\phi_{1I} \wedge \phi_1) \wedge \cdots \wedge \phi_n. \qquad (31)
\end{aligned}$$

Note that (30) is the 1st structure equation of the coframe $\underline{\phi} = \{\phi_1, \ldots, \phi_n\}$, while (31) comes from the fact that J can only be equal to 1. Since $\phi_{1I} = \sum_{K=1}^{n} \phi_{1I}(\tau_K)\,\phi_K$, we get:

$$\begin{aligned}
\mathrm{d}(\sigma_H^{n-1})_t &= -\sum_{I=2}^{n} (-1)^I (-1)^{I-1} \phi_1 \wedge \cdots \wedge \phi_{1I} \wedge \cdots \wedge \phi_n \\
&= \sum_{I=2}^{n} \phi_{1I}(\tau_I)\,\phi_1 \wedge \cdots \wedge \phi_I \wedge \cdots \wedge \phi_n \quad \text{(since } K \text{ must be equal to } I\text{)} \\
&= \sum_{i \in I_{HS}} \phi_{1i}(\tau_i)\,\phi_1 \wedge \cdots \wedge \phi_n, \qquad (32)
\end{aligned}$$

where (32) follows because $\phi_{i\alpha}(\tau_\alpha) = 0$; see Lemma 3.13. Thus we get:

$$\begin{aligned}
\imath^*\big(\widetilde{W} \lrcorner \mathrm{d}(\sigma_H^{n-1})_t\big) &= \imath^*\left(\sum_{i \in I_{HS}} \phi_{1i}(\tau_i)(\widetilde{W} \lrcorner \phi_1 \wedge \cdots \wedge \phi_n)\right) = \left(\sum_{i \in I_{HS}} \phi_{1i}(\tau_i)\,\langle \widetilde{W}, \nu^t\rangle\,(\sigma_{\mathcal{R}}^{n-1})_t\right)\Bigg|_{\mathcal{U}} \\
&= -\mathcal{H}_H^{\mathrm{sc}}\langle W, \nu\rangle\,\sigma^{n-1} \llcorner \mathcal{U}. \qquad (33)
\end{aligned}$$

The second term in (29) is given by $\imath^*(\mathrm{d}(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t)) = \mathrm{d}(\imath^*(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t))$. Moreover,

$$\imath^*\big(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\big) = \imath^*\big(\widetilde{W} \lrcorner |p_H \nu^t|\,(\sigma_{\mathcal{R}}^{n-1})_t\big) = \big(W \lrcorner |p_H \nu|\,\sigma_{\mathcal{R}}^{n-1}\big)\big|_{\partial\mathcal{U}} = |p_H \nu|\,(W \lrcorner \sigma_{\mathcal{R}}^{n-1})|_{\partial\mathcal{U}}.$$

Using the last computation and equalities (29) and (33) we get:

$$\dot{\Gamma}(0) = -\mathcal{H}_H^{\mathrm{sc}}\langle W, \nu\rangle\,\sigma_{\mathcal{R}}^{n-1} + \mathrm{d}\big(|p_H \nu|(W \lrcorner \sigma_{\mathcal{R}}^{n-1})\big). \tag{34}$$

The thesis now easily follows using (25), Leibnitz's rule, and then integrating along $\mathcal{U}$ both sides of (34). Clearly, for the second term, we use Stokes' theorem and the fact that

$$(W \lrcorner \sigma_{\mathcal{R}}^{n-1})|_{\partial\mathcal{U}} = \langle W, \eta\rangle\,(\sigma_{\mathcal{R}}^{n-2})|_{\partial\mathcal{U}}. \qquad \square$$

Remark 4.6. By analyzing (29) we see that, if $W \in \mathbf{C}^\infty(S, H)$, the Lie derivative of σ_H^{n-1} along the flow of W can be thought of as the sum of two terms, one only depending on the horizontal normal component of W, the other only depending on its horizontal tangential component. Analogously, in the case of an arbitrary vector variation $W \in \mathbf{C}^\infty(S, T\mathbb{G})$, (29) says that the Lie derivative of σ_H^{n-1} along the flow of W, is the sum of two terms, the first one only depending on the normal component of W, and the second one only depending on its tangential component.

4.3. 2nd variation of σ_H^{n-1}

In this section we illustrate the main result of this paper, that is, a complete formula for the 2nd variation of the measure σ_H^{n-1} on non-characteristic hypersurfaces, with or without boundary, having constant horizontal mean curvature $\mathcal{H}_H^{\mathrm{sc}}$. From what we have seen in Section 4.1 we have that

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \int_{\mathcal{U}} \Gamma(t) = \int_{\mathcal{U}} \ddot{\Gamma}(0), \tag{35}$$

and so we have to compute $\ddot{\Gamma}(0)$. We preliminarily note that

$$\ddot{\Gamma}(t) = \vartheta_t^*\big(\mathcal{L}_{\widetilde{W}}\big(\widetilde{W} \lrcorner \, \mathrm{d}(\sigma_H^{n-1})_t\big) + \mathcal{L}_{\widetilde{W}}\, \mathrm{d}\big(\widetilde{W} \lrcorner \, (\sigma_H^{n-1})_t\big)\big),$$

and, as in the Riemannian case, the hard part of the computation is in the first addend of the above formula. We note that, just in the case of 3-dimensional contact manifolds, for which the Heisenberg group $\mathbb{H}^1$ constitutes a noteworthy example, a similar formula for the 2nd variation of the H-perimeter measure on minimal surfaces (i.e. $\mathcal{H}_H^{\mathrm{sc}} = 0$), has been proved in [9]. This formula, in the case of minimal surfaces of $\mathbb{H}^1$, also appears in [12]; compare with Example 4.10 below.

The next result gives the second variation of σ_H^{n-1} in a particularly important special case.

Corollary 4.7 *(Horizontal normal 2nd variation). Under the hypotheses of Section* 4.1*, let ϑ be a smooth variation of $\mathcal{U} \subset S$ having variation vector $W = \vartheta_* \frac{\partial}{\partial t}|_{t=0}$ such that $W \in \nu_H S$, i.e. $W = w\, \nu_H$, where $w \in \mathbf{C}^\infty(S)$. Then we have*:

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \left\{ -\mathcal{H}_H^{\mathrm{sc}} w \frac{\partial w}{\partial \nu_H} + |\mathrm{grad}_{HS}\, w|^2 + w^2 \left[(2\, \mathrm{Tr}_2\, B_H) - \sum_{\alpha \in I_V} \big\langle \big(2\, \mathrm{grad}_{HS}(\varpi_\alpha) - C\tau_\alpha^S\big), C^\alpha \nu_H \big\rangle \right] \right\} \sigma_H^{n-1}$$
$$- \int_{\partial\mathcal{U} \setminus C_{\partial\mathcal{U}}} w \langle \mathrm{grad}_{HS}\, w, \eta_{HS} \rangle_{HS}\, \sigma_H^{n-2},$$

where we remind that $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|}$ and that $\tau_\alpha^S := \tau_\alpha - \varpi_\alpha \nu_H$ $(\alpha \in I_V)$. Moreover, if we assume that $W \in \mathbf{C}_0^\infty(\mathcal{U}, \nu_H S)$, or equivalently, that W keeps the boundary fixed, the boundary integral in the previous formula is identically zero.

Note that in the previous corollary we do not assume that $\mathcal{H}_H^{\mathrm{sc}}$ is constant. A more general statement for the second variation formula of σ_H^{n-1} in the horizontal case can be given; see Corollary 4.25. Actually, the proof of Corollary 4.7 is an immediate consequence of Corollary 4.25; see Section 4.5.

The next theorem is perhaps the main result of this paper and its proof will be given in Section 4.6.

Theorem 4.8 *(General 2nd variation of σ_H^{n-1} for hypersurfaces with $\mathcal{H}_H^{\mathrm{sc}}$ constant). Under the hypotheses of Section* 4.1*, let ϑ be a smooth variation of $\mathcal{U} \subset S$ having variation vector $W = \vartheta_* \frac{\partial}{\partial t}|_{t=0}$ and let us denote by $\widetilde{W} := \vartheta_* \frac{\partial}{\partial t}$ any extension of W to a neighborhood of* $\mathrm{Im}(\vartheta)$*. Finally, let us set $w := \frac{\langle W, \nu \rangle}{|\mathcal{P}_H \nu|}$. If $\mathcal{H}_H^{\mathrm{sc}} = \mathrm{const.}$ along $\mathcal{U}$, then we have*:

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \left\{ -W(w)\mathcal{H}_H^{\mathrm{sc}} + |\mathrm{grad}_{HS}\, w|^2 + w^2 \left[(2\, \mathrm{Tr}_2\, B_H) - \sum_{\alpha \in I_V} \big\langle \big(2\, \mathrm{grad}_{HS}(\varpi_\alpha) - C\tau_\alpha^S\big), C^\alpha \nu_H \big\rangle \right] \right\} \sigma_H^{n-1}$$
$$+ \int_{\partial\mathcal{U}} \big\{ \big\langle \big(-w\, \mathrm{grad}_{HS}\, w + [\widetilde{W}^{\nu^t}, \widetilde{W}^T]^T|_{t=0}\big), \eta \big\rangle |\mathcal{P}_H \nu|$$
$$+ \big(\mathrm{div}_{TS}\big(|\mathcal{P}_H \nu| W^T\big) - \mathcal{H}_H^{\mathrm{sc}} \langle W, \nu \rangle\big) \langle W^T, \eta \rangle \big\} \sigma_{\mathcal{R}}^{n-2},$$

where we remind that $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|}$ and that $\tau_\alpha^S := \tau_\alpha - \varpi_\alpha \nu_H$ $(\alpha \in I_V)$. Obviously, if we assume that $W \in \mathbf{C}_0^\infty(\mathcal{U}, T\mathbb{G})$ the boundary integral in the previous formula is identically zero.

It should be noted that we will prove this theorem as a consequence of a more general statement in which we do not require that $\mathcal{H}_H^{\mathrm{sc}}$ is constant along $\mathcal{U}$; see Proposition 4.13 in the next section.

Remark 4.9. We have used the notation Tr_2 for the sum of the principal minors of order 2 of the matrix representing a linear operator. In our case we have $\mathrm{Tr}_2 B_H = \frac{1}{2}\sum_{i,j\in I_{HS}}(\phi_{1i}(\tau_i)\phi_{1j}(\tau_j) - \phi_{1i}(\tau_j)\phi_{1j}(\tau_i))$. Moreover we remind that, in general, the following identity holds (see [15], Chapter 1, p. 36):

$$\mathrm{Tr}_2 B_H = \frac{1}{2}\big((\mathrm{Tr}\, B_H)^2 - \mathrm{Tr}(B_H \circ B_H)\big).$$

By a simple calculation using Remark 3.7, we then get:

$$\mathrm{Tr}_2\, B_H = \frac{1}{2}\left(\mathcal{H}_H^{\mathrm{sc}2} - \|S_H\|_{\mathrm{Gr}}^2 - \frac{1}{4}\|C_{HS}\|_{\mathrm{Gr}}^2\right),$$

where we have denoted by $\|\cdot\|_{\mathrm{Gr}}$ the Gram norm of a linear operator.

Notice that $\mathrm{Tr}_2\, B_H = 0$ if $\dim HS = 1$. This is the case, for instance, of the 3-dimensional Heisenberg group $\mathbb{H}^1$ and of the Engel group $\mathbb{E}^1$ on $\mathbb{R}^4$.

Example 4.10 *(Heisenberg group $\mathbb{H}^1$).* Let $\{X, Y, T\}$ be the standard set of generators for the Lie algebra $\mathfrak{h}_1$ of $\mathbb{H}^1$. They satisfy $[X, Y] = T$ with all other commutators zero. In particular, T is the center of $\mathfrak{h}_1$. Under the hypotheses of Theorem 4.8, we have:

$$II_{\mathcal{U}}(W, \sigma_H^2) = \int_{\mathcal{U}} \left\{-W(w)\mathcal{H}_H^{\mathrm{sc}} + \left(\frac{\partial w}{\partial \nu_H^{\perp}}\right)^2 + w^2\left[2\frac{\partial \varpi}{\partial \nu_H^{\perp}} - \varpi^2\right]\right\}\sigma_H^2,$$

for every vector variation W compactly supported on $\mathcal{U}$, where as before $w = \frac{\langle W,\nu\rangle}{|p_H\nu|}$ and, if $\nu = (\nu_X, \nu_Y, \nu_T)$ denotes the Riemannian unit normal, then $\varpi := \frac{\nu_T}{\sqrt{\nu_X^2+\nu_Y^2}}$. In the previous formula $\nu_H^{\perp}$ denotes the unique horizontal tangent vector of HS satisfying $|\nu_H^{\perp}| = 1$ and such that $\det[\nu_H, \nu_H^{\perp}, T] = 1$.

Example 4.11 *(Heisenberg group $\mathbb{H}^n$).* Let $\{X_1, \ldots, X_{2n}, X_{2n+1}\}$ be the standard set of generators for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$. We have $[X_i, X_{i+n}] = X_{2n+1}$ $(i = 1, \ldots, n)$ with all other commutators zero. The center of $\mathfrak{h}_1$ is X_{2n+1}; see Example 2.12. Under the hypotheses of Theorem 4.8, one has:

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \big\{-W(w)\mathcal{H}_H^{\mathrm{sc}} + |\mathrm{grad}_{HS}\, w|^2 + w^2\big[(2\,\mathrm{Tr}_2\, B_H) - \langle 2\,\mathrm{grad}_{HS}(\varpi), C_H^{2n+1}\nu_H\rangle - \varpi^2\big]\big\}\sigma_H^{n-1},$$

for every vector variation W compactly supported on $\mathcal{U}$, where $w = \frac{\langle W,\nu\rangle}{|p_H\nu|}$ and, if $\nu = (\nu_1, \ldots, \nu_{2n}, \nu_{2n+1})$ is the Riemannian unit normal, then $\varpi := \frac{\nu_{2n+1}}{|p_H\nu|}$. With respect to the canonical basis of $\mathbb{H}^n$, we have:

$$C_H^{2n+1}\nu_H = (\nu_H^2, -\nu_H^1, \nu_H^4, -\nu_H^3, \ldots, \nu_H^{2n}, -\nu_H^{2n-1}, 0) =: -\nu_H^{\perp},$$

where $\nu_H = (\nu_H^1, \ldots, \nu_H^{2n}, 0)$. Note that $\|C_{HS}^{2n+1}\|_{gr}^2 = 2(n-1)$ and therefore that

$$2\,\mathrm{Tr}_2\, B_H = \mathcal{H}_H^{\mathrm{sc}\,2} - \|S_H\|_{\mathrm{Gr}}^2 - \frac{1}{4}\|C_{HS}\|_{\mathrm{Gr}}^2 = \mathcal{H}_H^{\mathrm{sc}\,2} - \|S_H\|_{\mathrm{Gr}}^2 - \frac{2(n-1)}{4}\varpi^2.$$

So we finally obtain

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \left\{-W(w)\mathcal{H}_H^{\mathrm{sc}} + |\mathrm{grad}_{HS}\, w|^2 + w^2\left[\mathcal{H}_H^{\mathrm{sc}\,2} - \|S_H\|_{\mathrm{Gr}}^2 + 2\frac{\partial \varpi}{\partial \nu_H^{\perp}} - \frac{n+1}{2}\varpi^2\right]\right\}\sigma_H^{n-1}.$$

Example 4.12 *(Engel's group $\mathbb{E}^1$).* Let $\{X_1, X_2, X_3, X_4\}$ be the set of generators for the Lie algebra $\mathfrak{e}_1$ of $\mathbb{E}^1$ satisfying $[X_1, X_2] = X_3$, $[X_1, X_3] = [X_2, X_3] = X_4$ and such that all other commutators vanish. In particular, X_4 is the center of $\mathfrak{h}_1$. Under the hypotheses of Theorem 4.8, we have:

$$II_{\mathcal{U}}(W,\sigma_H^{n-1}) = \int_{\mathcal{U}} \bigg\{ -W(w)\mathcal{H}_H^{\text{sc}} + \left(\frac{\partial w}{\partial \nu_H^{\perp}}\right)^2 + w^2\bigg[\left(2\frac{\partial \varpi_3}{\partial \nu_H^{\perp}} - \varpi_3^2\right) - \varpi_4^2\big[(\nu_H^2)^2 - (\nu_H^1)^2 - 2\nu_H^1\nu_H^2\big]^2 - \varpi_4\big[(\nu_H^2)^2 - (\nu_H^1)^2 + 2\nu_H^1\nu_H^2\big]\bigg]\bigg\}\sigma_H^3,$$

for every vector variation W compactly supported on $\mathcal{U}$, where as above $w = \frac{\langle W,\nu\rangle}{|p_H\nu|}$. Here $\varpi_3 := \frac{\nu_3}{|p_H\nu|}$ and $\varpi_4 := \frac{\nu_4}{|p_H\nu|}$ where $\nu = (\nu_1,\nu_2,\nu_3,\nu_4)$ denotes the Riemannian unit normal. Moreover $\nu_H^{\perp}$ denotes the unique horizontal tangent vector of HS satisfying $|\nu_H^{\perp}| = 1$ and such that $\det[\nu_H, \nu_H^{\perp}, X_3, X_4] = 1$. Thus in canonical coordinates we have $\nu_H^{\perp} = (-\nu_H^2, \nu_H^1, 0, 0) \in HS$, where $\nu_H = (\nu_H^1, \nu_H^2, 0, 0)$. Note that we have used $\langle C\tau_3, C_H^3\nu_H\rangle = -\varpi_4\langle C^4\tau_3, \nu_H^{\perp}\rangle = \varpi_4[(\nu_H^2)^2 - (\nu_H^1)^2 + 2\nu_H^1\nu_H^2]$ and $|C^4\nu_H| = ((\nu_H^2)^2 - (\nu_H^1)^2 - 2\nu_H^1\nu_H^2)$. Using polar coordinates on H in such a way that $\nu_H = e^{i\psi}$, $\psi := \arg(\nu_H) \in [0, 2\pi]$, we get

$$II_{\mathcal{U}}(W,\sigma_H^{n-1}) = \int_{\mathcal{U}} \bigg\{ -W(w)\mathcal{H}_H^{\text{sc}} + \left(\frac{\partial w}{\partial \nu_H^{\perp}}\right)^2 + w^2\bigg[\left(2\frac{\partial \varpi_3}{\partial \nu_H^{\perp}} - \varpi_3^2\right) - \varpi_4^2(1+\sin 4\psi) + \sqrt{2}\varpi_4\cos\left(2\psi + \frac{\pi}{4}\right)\bigg]\bigg\}\sigma_H^3.$$

4.4. 2nd-variation of σ_H^{n-1}: proof

In this section we will prove all the results stated in the previous section. Our proof will closely follow that of the 1st variation of σ_H^{n-1} and so we will use the notations previously adopted in Section 4.2. We stress that in the following computations, we shall sometimes omit the subscripts H and HS from the notations of inner products and norms.

Our first step in proving the results introduced before is the following, more general:

Proposition 4.13 *(General 2nd variation of σ_H^{n-1}). Under the hypotheses of Section* 4.1*, let ϑ be a smooth variation of $\mathcal{U} \subset S$ having variation vector $W = \vartheta_* \frac{\partial}{\partial t}|_{t=0}$ and let us set $w := \frac{\langle W,\nu\rangle}{|p_H\nu|}$. Then*

$$\begin{aligned} II_{\mathcal{U}}(W,\sigma_H^{n-1}) = \int_{\mathcal{U}} \bigg\{ &-\mathcal{H}_H^{\text{sc}}\big[W(w) + w\big(\operatorname{div}_{HS} W_{HS} + \operatorname{div}(W_V) - \big\langle \mathcal{J}_H(W_V)\nu_H, \varpi\big\rangle + 2\langle C\nu_H, W\rangle\big)\big] \\ &+ w\bigg[-\Delta_{HS}w_1 - \sum_{\alpha\in I_V}\big(\varpi_\alpha\Delta_{HS}w_\alpha + \big\langle(\operatorname{grad}_{HS} w_\alpha + C^\alpha W), (2\operatorname{grad}_{HS}\varpi_\alpha - C\tau_\alpha^S)\big\rangle\big) \\ &+ w_1(2\operatorname{Tr}_2 B_H) + \operatorname{Tr}\big(B_H \circ [\mathcal{J}_{HS}W_{HS}]^{\text{tr}}\big) + \operatorname{Tr}\big(C \circ [\mathcal{J}_{HS}W\,\mathbf{0}]^{\text{tr}}\big) \\ &- \frac{1}{2}\sum_{\alpha\in I_{H_2}}\big(w_\alpha \operatorname{Tr}(B_H \circ C_{HS}^\alpha) + \big\langle C_H^\alpha\nu_H, \operatorname{grad}_{HS} w_\alpha\big\rangle\big)\bigg]\bigg\}\sigma_H^{n-1} \\ &+ \int_{\partial\mathcal{U}} \big\{\big\langle[\widetilde{W}^{\nu^t}, \widetilde{W}^T]^T|_{t=0}, \eta\big\rangle|p_H\nu| + \big(\operatorname{div}_{TS}\big(|p_H\nu|W^T\big) - \mathcal{H}_H^{\text{sc}}\langle W,\nu\rangle\big)\langle W^T,\eta\rangle\big\}\sigma_{\mathcal{R}}^{n-2}. \end{aligned}$$

Finally, if we assume that $W \in \mathbf{C}_0^\infty(\mathcal{U}, T\mathbb{G})$, then the boundary integral in the previous formula is identically zero.

Here above, $\mathbf{0} := \mathbf{0}_{n\times n-h_1+1}$ denotes the zero matrix in $\mathcal{M}_{n\times n-h_1+1}(\mathbb{R})$ and so $[\mathcal{J}_{HS}W\,\mathbf{0}] \in \mathcal{M}_{n\times n}(\mathbb{R})$. Remind that, if $\underline{\tau} = \{\tau_1,\ldots,\tau_n\}$ is an adapted moving frame for $\mathcal{U} \subset S$ on U, we have $\tau_1 := \nu_H$ and $\tau_\alpha^S := \tau_\alpha - \frac{\nu_\alpha}{|p_H\nu|}\nu_H$ ($\alpha \in I_V$) along $\mathcal{U}$. Moreover, by definition, $\varpi = \sum_{\alpha\in I_V}\varpi_\alpha\tau_\alpha = \frac{p_V\nu}{|p_H\nu|}$. We also remember that $S \subset \mathbb{G}$ is a smooth immersed hypersurface and $U \subset \mathbb{G}$ is an open set such that $\mathcal{U} = U \cap S$ is a relatively compact subset of S with smooth $(n-2)$-dimensional boundary $\partial\mathcal{U}$ oriented by its unit normal η.

Proof of Proposition 4.13. The proof below can be seen as a continuation of the proof of Theorem 4.2. Throughout this section we will choose, as in Section 4.2, an orthonormal moving frame $\underline{\tau}$ on $U \subset \mathbb{G}$ satisfying for every $t \in (-\varepsilon, \varepsilon)$:

$$\text{(i) } \tau_1|_{\mathcal{U}_t} := \nu_H^t; \quad \text{(ii) } HT_p\mathcal{U}_t = \operatorname{span}\{(\tau_2)_p, \dots, (\tau_{h_1})_p\}\ (p \in \mathcal{U}_t); \quad \text{(iii) } \tau_\alpha := X_\alpha.$$

From now on we also assume that the variation vector field $W \in \mathbf{C}^\infty(S, T\mathbb{G})$ of ϑ is transversal along $\mathcal{U}$. We already know that, in order to compute the 2nd variation of σ_H^{n-1}, we have to compute, in a fixed point $p_0 \in \mathcal{U}$, the quantity $\ddot{\Gamma}(0)$. (We stress that, in the next computations, we shall drop the dependence on the "initial" point $p_0 \in \mathcal{U}$.) Therefore we will first compute

$$\ddot{\Gamma}(t) = \vartheta_t^*\big\{\mathcal{L}_{\widetilde{W}}\big(\widetilde{W} \lrcorner \operatorname{d}(\sigma_H^{n-1})_t\big) + \mathcal{L}_{\widetilde{W}} \operatorname{d}\big(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\big)\big\} \quad (:= A + B). \tag{36}$$

Remark 4.14. From (36), making use of Stoke's theorem, we see that

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = II_{\mathcal{U}}^{\text{Int.}}(W, \sigma_H^{n-1}) + II_{\mathcal{U}}^{\text{Bound.}}(W, \sigma_H^{n-1}),$$

where

$$II_{\mathcal{U}}^{\text{Int.}}(W, \sigma_H^{n-1}) := \int_{\mathcal{U}} \imath^*\big(\mathcal{L}_{\widetilde{W}}(\widetilde{W} \lrcorner \operatorname{d}\big(\sigma_H^{n-1}\big)_t)\big)$$

and

$$II_{\mathcal{U}}^{\text{Bound.}}(W, \sigma_H^{n-1}) := \int_{\partial\mathcal{U}} \imath^*\big(\mathcal{L}_{\widetilde{W}}\big(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\big)\big|_{\partial\mathcal{U}_t}\big).$$

By setting:

$$w := \frac{\langle W, \nu\rangle}{|p_H\nu|}, \qquad w_t := \frac{\langle \widetilde{W}, \nu^t\rangle}{|p_H\nu^t|}, \tag{37}$$

we obtain, using what we have proved in Section 4.2, that the first term A in (36) is given by

$$A = \sum_{j\in I_{HS}} \mathcal{L}_{\widetilde{W}}(w_t\,\phi_2 \wedge \cdots \wedge \underbrace{\phi_{1j}}_{j\text{th place}} \wedge \cdots \wedge \phi_n)|_{\mathcal{U}_t}. \tag{38}$$

Remark 4.15 *(Boundary terms).* Since the Lie derivative commutes with exterior differentiation, using Stoke's theorem we get that the second term in (36), is given by $B = \vartheta_t^*\mathcal{L}_{\widetilde{W}}(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t)|_{\partial\mathcal{U}_t}$. Using well-known properties of the Lie derivative, B can be computed in the following way:

(i) If $\widetilde{W} \in \mathfrak{X}(\mathbb{G})$, we may decompose the variation vector as $\widetilde{W} = \widetilde{W}^T + \widetilde{W}^{\nu^t}$ (tangent and normal components of $\widetilde{W}$ with respect to $\mathcal{U}_t$) and we get:

$$\begin{aligned} B &= \mathcal{L}_{\widetilde{W}}\big(\widetilde{W}^T \lrcorner (\sigma_H^{n-1})_t\big)\big|_{\partial\mathcal{U}_t} = \big([\widetilde{W}, \widetilde{W}^T]^T \lrcorner (\sigma_H^{n-1})_t + \widetilde{W}^T \lrcorner \big(\mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t\big)\big)\big|_{\partial\mathcal{U}_t} \\ &= \big([\widetilde{W}^{\nu^t}, \widetilde{W}^T]^T \lrcorner (\sigma_H^{n-1})_t + \widetilde{W}^T \lrcorner \big(\widetilde{W}^{\nu^t} \lrcorner \operatorname{d}(\sigma_H^{n-1})_t\big) + \widetilde{W}^T \lrcorner \operatorname{d}\big(\widetilde{W}^T \lrcorner (\sigma_H^{n-1})_t\big)\big)\big|_{\partial\mathcal{U}_t}, \end{aligned}$$

where we have used the fact that the bracket of tangent vector fields is still a tangent vector and Cartan's formula for the Lie derivative. By integrating B along $\partial\mathcal{U}_t$ and setting $t = 0$, we obtain:

$$\begin{aligned} II_{\mathcal{U}}^{\text{Bound.}}(W, \sigma_H^{n-1}) &= \int \imath_{\partial\mathcal{U}}^*\big(\mathcal{L}_{\widetilde{W}}\big(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\big)\big) \\ &= \int_{\partial\mathcal{U}} \big\{\big\langle [\widetilde{W}^{\nu^t}, \widetilde{W}^T]^T|_{t=0}, \eta\big\rangle |p_H\nu| + \big(\operatorname{div}_{TS}\big(|p_H\nu| W^T\big) - \mathcal{H}_H^{\text{sc}}\langle W, \nu\rangle\big)\big\langle W^T, \eta\big\rangle\big\}\sigma_{\mathcal{R}}^{n-2}. \end{aligned} \tag{39}$$

(ii) If $\widetilde{W} \in \mathfrak{X}(H)$, we may write the variation vector as $\widetilde{W} = \widetilde{W}_{\nu_H} + \widetilde{W}_{HS}$, where $\widetilde{W}_{\nu_H}$ and $\widetilde{W}_{HS}$ are respectively, the horizontal normal component and the horizontal tangential component of $\widetilde{W}$ along $\mathcal{U}_t$. In this case we have:

$$\begin{aligned} B &= \mathcal{L}_{\widetilde{W}}\big(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\big)\big|_{\partial\mathcal{U}_t} = \mathcal{L}_{\widetilde{W}}\big(\widetilde{W}_{HS} \lrcorner (\sigma_H^{n-1})_t\big)\big|_{\partial\mathcal{U}_t} \\ &= \big([\widetilde{W}, \widetilde{W}_{HS}]^T \lrcorner (\sigma_H^{n-1})_t + \widetilde{W}_{HS} \lrcorner \big(\mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t\big)\big)\big|_{\partial\mathcal{U}_t} \\ &= \big([\widetilde{W}, \widetilde{W}_{HS}]^T \lrcorner (\sigma_H^{n-1})_t + \widetilde{W}_{HS} \lrcorner (\widetilde{W}_{\nu_H} \lrcorner \mathrm{d}(\sigma_H^{n-1})_t) + \widetilde{W}_{HS} \lrcorner \mathrm{d}\big(\widetilde{W}_{HS} \lrcorner (\sigma_H^{n-1})_t\big)\big)\big|_{\partial\mathcal{U}_t}. \end{aligned}$$

By integrating B along $\partial\mathcal{U}_t$, using Theorem 3.17, and setting $t = 0$, we get:

$$\begin{aligned} II_{\mathcal{U}}^{\text{Bound.}}(W, \sigma_H^{n-1}) &= \int \iota_{\partial\mathcal{U}}^*\big(\mathcal{L}_{\widetilde{W}}\big(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\big)\big) \\ &= \int_{\partial\mathcal{U}} \big\{\big\langle[\widetilde{W}, \widetilde{W}_{HS}]^T|_{t=0}, \eta\big\rangle + \big[\operatorname{div}_{HS} W_{HS} + \langle C_H \nu_H, W_{HS}\rangle_{HS} \\ &\quad - \mathcal{H}_H^{\text{sc}}\langle W, \nu_H\rangle_H\big]\langle W_{HS}, \eta\rangle_{HS}\big\}|p_H\nu|\sigma_{\mathcal{R}}^{n-2}. \end{aligned} \tag{40}$$

We start with the computation of (38) by first computing the following quantities:

(i) $\mathcal{L}_{\widetilde{W}}(\phi_h)$ for $h \in I_{HS} = \{2, \ldots, h_1\}$;
(ii) $\mathcal{L}_{\widetilde{W}}(\phi_{1j})$ for $j \in I_{HS}$;
(iii) $\mathcal{L}_{\widetilde{W}}(\phi_\alpha)$ for $\alpha \in I_V = \{h_1 + 1, \ldots, n\}$.

This can be done using Cartan's formula and the structure equations for our coframe $\underline{\phi} = \{\phi_1, \ldots, \phi_n\}$. For the term appearing in (i) we get:

$$\mathcal{L}_{\widetilde{W}}(\phi_h) = \widetilde{W} \lrcorner \mathrm{d}\phi_h + \mathrm{d}\phi_h(\widetilde{W}) = \sum_L (\widetilde{W} \lrcorner \phi_{hL} \wedge \phi_L) + \mathrm{d}\widetilde{w}_h,$$

and so

$$\mathcal{L}_{\widetilde{W}}(\phi_h) = \sum_{L \neq h} \big(\phi_{hL}(\widetilde{W})\phi_L - \widetilde{w}_L\phi_{hL}\big) + \mathrm{d}\widetilde{w}_h. \tag{41}$$

Analogously, for the term in (ii), using the 2nd structure equation for $\underline{\phi}$, we get:

$$\mathcal{L}_{\widetilde{W}}(\phi_{1j}) = \widetilde{W} \lrcorner \mathrm{d}\phi_{1j} + \mathrm{d}\phi_{1j}(\widetilde{W}) = \sum_L \big(\widetilde{W} \lrcorner (-\widetilde{\Phi}_{1j} + \phi_{1L} \wedge \phi_{Lj})\big) + \mathrm{d}\phi_{1j}(\widetilde{W}),$$

and therefore

$$\mathcal{L}_{\widetilde{W}}(\phi_{1j}) = -\Phi_{1j}(\widetilde{W}, \cdot) + \sum_{L \neq 1, j} \big(\phi_{1L}(\widetilde{W})\phi_{Lj} - \phi_{Lj}(\widetilde{W})\phi_{1L}\big) + \mathrm{d}\phi_{1j}(\widetilde{W}). \tag{42}$$

Finally, for the term in (iii), we get:

$$\mathcal{L}_{\widetilde{W}}(\phi_\alpha) = \widetilde{W} \lrcorner \mathrm{d}\phi_\alpha + \mathrm{d}\phi_\alpha(\widetilde{W}) = \sum_{L \neq \alpha} (\widetilde{W} \lrcorner \phi_{\alpha L} \wedge \phi_L) + \mathrm{d}\widetilde{w}_\alpha,$$

and so

$$\mathcal{L}_{\widetilde{W}}(\phi_\alpha) = \sum_{L \neq \alpha} \big(\phi_{\alpha L}(\widetilde{W})\phi_L - \widetilde{w}_L\phi_{\alpha L}\big) + \mathrm{d}\widetilde{w}_\alpha. \tag{43}$$

Now we may compute A. We have:

$$A = \mathcal{L}_{\widetilde{W}}\big(\widetilde{W} \lrcorner \mathrm{d}(\sigma_H^{n-1})_t\big) = -\widetilde{W}(w_t)(\mathcal{H}_H^{\text{sc}})_t(\sigma_H^{n-1})_t + w_t \sum_{j \in I_{HS}} \mathcal{L}_{\widetilde{W}}(\phi_2 \wedge \cdots \wedge \phi_{1j} \wedge \cdots \wedge \phi_n)$$

$$= -\widetilde{W}(w_t)(\mathcal{H}_H^{\mathrm{sc}})_t(\sigma_H^{n-1})_t + \sum_{j,h\in I_{HS}} \underbrace{w_t(\phi_2\wedge\cdots\wedge\phi_{1j}\wedge\cdots\wedge\mathcal{L}_{\widetilde{W}}\phi_h\wedge\cdots\wedge\phi_n)}_{=:A_1}$$

$$+ \sum_{j\in I_{HS}} \underbrace{w_t(\phi_2\wedge\cdots\wedge\mathcal{L}_{\widetilde{W}}\phi_{1j}\wedge\cdots\wedge\phi_n)}_{=:A_2} + \sum_{j\in I_{HS}}\sum_{\alpha\in I_2} \underbrace{w_t(\phi_2\wedge\cdots\wedge\phi_{1j}\wedge\cdots\wedge\mathcal{L}_{\widetilde{W}}\phi_\alpha\wedge\cdots\wedge\phi_n)}_{=:A_3}.$$

By using (41) and Lemma 2.19, the term A_1 can be easily computed as follows:

$$A_1 = w_t\left\{\phi_2\wedge\cdots\wedge\big(\phi_{1j}(\tau_j)\phi_j + \phi_{1j}(\tau_h)\phi_h\big)\wedge\cdots\wedge\left[\sum_{L\neq h}\big(\phi_{hL}(\widetilde{W})\phi_L - \widetilde{w}_L\phi_{hL}\big) + \mathrm{d}\widetilde{w}_h\right]\wedge\cdots\wedge\phi_n\right\}$$

$$= w_t\left\{\phi_{1j}(\tau_j)\left[\tau_h(\widetilde{w}_h) - \sum_{L\neq h}\widetilde{w}_L\phi_{hL}(\tau_h)\right] - \phi_{1j}(\tau_h)\left[\phi_{hj}(\widetilde{W}) + \tau_j(\widetilde{w}_h) - \sum_{L\neq h}\widetilde{w}_L\phi_{hL}(\tau_j)\right]\right\}(\sigma_H^{n-1})_t$$

$$= w_t\left\{\phi_{1j}(\tau_j)\left[\tau_h(\widetilde{w}_h) - \sum_{l\neq h}\widetilde{w}_l\phi_{hl}(\tau_h)\right] - \phi_{1j}(\tau_h)\left[\tau_j(\widetilde{w}_h) + \sum_{L\neq h}\widetilde{w}_L C^h_{jL}\right]\right\}(\sigma_H^{n-1})_t,$$

where we have used the identity $\phi_{h\alpha}(\tau_h) = 0$ (see (ii) in Lemma 3.13) and also (14) to compute the last term; see Section 3. For the term A_2, by means of (42) and Lemma 2.19, we get:

$$A_2 = w_t\left\{-\Phi_{1j}(\widetilde{W},\tau_j) + \tau_j\big(\phi_{1j}(\widetilde{W})\big) + \sum_{L\neq 1,j}\big[\phi_{1l}(\widetilde{W})\phi_{lj}(\tau_j) - \phi_{lj}(\widetilde{W})\phi_{1l}(\tau_j)\big]\right\}(\sigma_H^{n-1})_t.$$

Analogously, the term A_3 is computed by means of (43) and Lemma 2.19 as follows:

$$A_3 = w_t\left\{\phi_2\wedge\cdots\wedge\phi_{1j}\wedge\cdots\wedge\left[\sum_{L\neq\alpha}\big(\phi_{\alpha L}(\widetilde{W})\phi_L - \widetilde{w}_L\phi_{\alpha L}\big) + \mathrm{d}\widetilde{w}_\alpha\right]\wedge\cdots\wedge\phi_n\right\}$$

$$= w_t\left\{\phi_2\wedge\cdots\wedge\left(\sum_K\phi_{1j}(\tau_K)\phi_K\right)\wedge\cdots\wedge\left[\sum_{L\neq\alpha}\sum_M\big(\phi_{\alpha L}(\widetilde{W})\phi_L - \widetilde{w}_L\phi_{\alpha L}(\tau_M)\phi_M\big) + \tau_M(\widetilde{w}_\alpha)\phi_M\right]\wedge\cdots\wedge\phi_n\right\}$$

$$= w_t\left\{\phi_{1j}(\tau_j)\left[\tau_\alpha(\widetilde{w}_\alpha) - \varpi^t_\alpha\left(\tau_1(\widetilde{w}_\alpha) + \phi_{\alpha 1}(\widetilde{W}) - \sum_{L\neq 1,\alpha}\widetilde{w}_L\phi_{\alpha L}(\tau_1)\right)\right]\right.$$

$$+ \varpi^t_\alpha\phi_{1j}(\tau_1)\left[\tau_j(\widetilde{w}_\alpha) + \phi_{\alpha j}(\widetilde{W}) - \sum_{L\neq j,\alpha}\widetilde{w}_L\phi_{\alpha L}(\tau_j)\right]$$

$$\left. - \phi_{1j}(\tau_\alpha)\left[\tau_j(\widetilde{w}_\alpha) + \phi_{\alpha j}(\widetilde{W}) - \sum_{L\neq j,\alpha}\widetilde{w}_L\phi_{\alpha L}(\tau_j)\right]\right\}(\sigma_H^{n-1})_t$$

$$= w_t\left\{\phi_{1j}(\tau_j)\left[\tau^S_\alpha(\widetilde{w}_\alpha) + \varpi^t_\alpha\sum_{L\neq 1,\alpha}\widetilde{w}_L C^\alpha_{L1}\right] - \phi_{1j}(\tau^S_\alpha)\left[\tau_j(\widetilde{w}_\alpha) + \sum_{L\neq j,\alpha}\widetilde{w}_L C^\alpha_{jL}\right]\right\}(\sigma_H^{n-1})_t.$$

Here we have used the notation $\varpi^t_\alpha := \frac{\nu^t_\alpha}{\nu^t_1} = \frac{\nu^t_\alpha}{|\mathcal{P}_H\nu^t|}$. We also stress that, in the above computations, we have used the fact that $\phi_{\alpha L}(\tau_\alpha) = 0$ for every L and that $\phi_{\alpha j}(\tau_j) = 0$ for $j\in I_H$; see Lemma 3.13. Now, by using these expressions, identity (14), and rearranging a little bit we obtain:

$$A = \left\{-\widetilde{W}(w_t)(\mathcal{H}_H^{\mathrm{sc}})_t + w_t\left[\sum_{j\in I_{HS}}\phi_{1j}(\tau_j)\left[\sum_{l\in I_H}\sum_{\substack{h\in I_{HS}\\ h\neq l}}\big(\tau_h(\widetilde{w}_h) + \widetilde{w}_l\phi_{lh}(\tau_h)\big)\right.\right.\right.$$

$$\left.\left. + \sum_{\alpha\in I_V}\left(\tau^S_\alpha(\widetilde{w}_\alpha) + \varpi^t_\alpha\sum_{L\neq 1,\alpha}\widetilde{w}_L C^\alpha_{L1}\right)\right] - \sum_{j,h\in I_{HS}}\phi_{1j}(\tau_h)\left[\tau_j(\widetilde{w}_h) + \sum_{L\neq h}\widetilde{w}_L C^h_{jL}\right]\right.$$

$$- \sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\phi_{1j}(\tau^S_\alpha)\left[\tau_j(\widetilde{w}_\alpha) + \sum_{L\neq j,\alpha}\widetilde{w}_L C^\alpha_{jL}\right]$$

$$+\sum_{j\in I_{HS}}\Big[\big(-\Phi_{1j}(\widetilde{W},\tau_j)+\tau_j\big(\phi_{1j}(\widetilde{W})\big)\big)+\sum_{\alpha\in I_V}\phi_{1\alpha}(\tau_j)\phi_{j\alpha}(\widetilde{W})\Big]$$
$$+\sum_{L\neq 1,j}\big[\phi_{1l}(\widetilde{W})\phi_{lj}(\tau_j)-\phi_{lj}(\widetilde{W})\phi_{1l}(\tau_j)\big]\Big]\Big\}(\sigma_H^{n-1})_t.$$

Remark 4.16. From now on we will extensively make use of Lemma 3.8. Roughly speaking, Lemma 3.8 says that, if we fix a point $p_0\in\mathcal{U}=U\cap S$, we can always choose our moving frame $\underline{\tau}$ for U adapted to $\mathcal{U}$, in such a way that its dual coframe $\underline{\phi}$ satisfies $\phi_{ij}(p_0)=0$, whenever $i,j\in I_{HS}$. Since our computation is actually done in a fixed point $p_0\in\mathcal{U}$, making use of this fact will greatly simplify our next computations.

Thus, in the sequel, we shall restrict to $\mathcal{U}\subset S$ the above expression. We have then,

$$(\iota^*A)_{p_0}=\bigg\{-W(w)\mathcal{H}_H^{\rm sc}+w\bigg\{\sum_{j\in I_{HS}}\phi_{1j}(\tau_j)\bigg[-\mathcal{H}_H^{\rm sc}\big(\tau_h(w_h)+w_1\phi_{1h}(\tau_h)\big)+\sum_{\alpha\in I_V}\bigg(\tau_\alpha^S(w_\alpha)+\varpi_\alpha\sum_{L\neq 1,\alpha}w_LC_{L1}^\alpha\bigg)\bigg]$$
$$-\sum_{j,h\in I_{HS}}\phi_{1j}(\tau_h)\bigg(\tau_j(w_h)+\sum_{L\neq h}w_LC_{jL}^h\bigg)-\sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\phi_{1j}(\tau_\alpha^S)\bigg(\tau_j(w_\alpha)+\sum_{L\neq j,\alpha}w_LC_{jL}^\alpha\bigg)$$
$$+\sum_{j\in I_{HS}}\bigg[\big(-\Phi_{1j}(W,\tau_j)+\tau_j\big(\phi_{1j}(W)\big)\big)+\sum_{\alpha\in I_V}\phi_{1\alpha}(\tau_j)\phi_{j\alpha}(W)\bigg]\bigg\}\bigg\}\bigg|_{p_0}\sigma_H^{n-1}(p_0).$$

By using again Lemma 3.8, together with (5) of Section 2.1 and (14) of Section 3, we get that

$$\sum_{L\neq h}w_LC_{jL}^h=w_1\phi_{1h}(\tau_j)+\sum_{\alpha\in I_V}w_\alpha\phi_{\alpha h}(\tau_j)\quad\text{at }p_0$$

and therefore that

$$(\iota^*A)_{p_0}=\bigg\{-\mathcal{H}_H^{\rm sc}\bigg[W(w)+w\sum_{\alpha\in I_V}\bigg(\tau_\alpha^S(w_\alpha)+\varpi_\alpha\sum_{L\neq 1,\alpha}w_LC_{L1}^\alpha\bigg)\bigg]$$
$$+w\bigg[\sum_{j,h\in I_{HS}}\Big[w_1\big(\phi_{1j}(\tau_j)\phi_{1h}(\tau_h)-\phi_{1j}(\tau_h)\phi_{1h}(\tau_j)\big)+\big(\phi_{1j}(\tau_j)\tau_h(w_h)-\phi_{1j}(\tau_h)\tau_j(w_h)\big)\Big]$$
$$-\sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\phi_{1j}(\tau_\alpha^S)\bigg(\tau_j(w_\alpha)-\sum_{L\neq j,\alpha}w_LC_{Lj}^\alpha\bigg)+\sum_{j\in I_{HS}}\bigg[\big(-\Phi_{1j}(W,\tau_j)+\tau_j\big(\phi_{1j}(W)\big)\big)$$
$$+\sum_{\alpha\in I_V}\bigg(\phi_{1\alpha}(\tau_j)\phi_{j\alpha}(W)+\sum_{\substack{h\in I_{HS}\\ h\neq j}}w_\alpha\phi_{1j}(\tau_h)\phi_{h\alpha}(\tau_j)\bigg)\bigg]\bigg]\bigg\}\bigg|_{p_0}\sigma_H^{n-1}(p_0).$$

In Proposition 3.15 we have computed some of the curvature 2-forms. In particular, it was shown that

$$\sum_{j\in I_{HS}}\Phi_{1j}(W,\tau_j)=-\frac{3}{4}\sum_{\alpha\in I_{H_2}}\langle C_H^\alpha\nu_H,C_H^\alpha W_H\rangle-\frac{1}{4}\sum_{\alpha\in I_{H_2}}\sum_{\beta\in I_{H_3}}w_\beta\langle C_H^\alpha\nu_H,C^\beta\tau_\alpha\rangle.$$

Substituting this identity into the previous formula gives us:

$$(\iota^*A)_{p_0}=\bigg\{-\mathcal{H}_H^{\rm sc}\bigg[W(w)+w\sum_{\alpha\in I_V}\bigg(\tau_\alpha^S(w_\alpha)+\varpi_\alpha\sum_{L\neq 1,\alpha}w_LC_{L1}^\alpha\bigg)\bigg]$$
$$+w\bigg[w_1\sum_{j,h\in I_{HS}}\big(\phi_{1j}(\tau_j)\phi_{1h}(\tau_h)-\phi_{1j}(\tau_h)\phi_{1h}(\tau_j)\big)+\sum_{j,h\in I_{HS}}\big(\phi_{1j}(\tau_j)\tau_h(w_h)-\phi_{1j}(\tau_h)\tau_j(w_h)\big)$$
$$+\sum_{j\in I_{HS}}\tau_j\big(\phi_{1j}(W)\big)-\sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\phi_{1j}(\tau_\alpha^S)\bigg(\tau_j(w_\alpha)-\sum_{L\neq j,\alpha}w_LC_{Lj}^\alpha\bigg)+\sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\phi_{1\alpha}(\tau_j)\phi_{j\alpha}(W)$$

$$+\sum_{\substack{j,h\in I_{HS}\\ j\neq h}}\sum_{\alpha\in I_V} w_\alpha\phi_{1j}(\tau_h)\phi_{h\alpha}(\tau_j)+\frac{3}{4}\sum_{\alpha\in I_{H_2}}\langle C_H^\alpha\tau_1, C_H^\alpha W_H\rangle$$

$$+\frac{1}{4}\sum_{\alpha\in I_{H_2}}\sum_{\beta\in I_{H_3}} w_\beta\langle C_H^\alpha\tau_1, C^\beta\tau_\alpha\rangle\Bigg]\Bigg\}\Bigg|_{(t,p)=(0,p_0)}\sigma_H^{n-1}(p_0). \tag{44}$$

Claim 4.17. *The following hold*:

(i) *Let* $[\mathcal{J}_{HS}W_{HS}]^{\text{tr}}$ *denote the transposed matrix of the horizontal tangent Jacobian of* W_{HS}.[19] *Then, using Lemma* 3.8*, we see that at* p_0*, one has*

$$\sum_{j,h\in I_{HS}}\big(\phi_{1j}(\tau_j)\tau_h(w_h)-\phi_{1j}(\tau_h)\tau_j(w_h)\big)=-\mathcal{H}_H^{\text{sc}}\operatorname{div}_{HS}W_{HS}+\operatorname{Tr}\big(B_H\circ[\mathcal{J}_{HS}W_{HS}]^{\text{tr}}\big).$$

(ii) *Since* $C_{L1}^\alpha=\langle[\tau_L,\tau_1],\tau_\alpha\rangle=\langle C^\alpha\tau_1,\tau_L\rangle$*, it follows that*

$$-\mathcal{H}_H^{\text{sc}}\sum_{\alpha\in I_V}\left(\varpi_\alpha\sum_L w_L C_{L1}^\alpha\right)=-\mathcal{H}_H^{\text{sc}}\langle C\tau_1, W\rangle.$$

(iii) *Since* $\phi_{1\alpha}(\tau_j)=\frac{1}{2}\langle C_H^\alpha\tau_1,\tau_j\rangle$*, and since* $\phi_{j\alpha}(W)=-\frac{1}{2}(\langle C^\alpha W,\tau_j\rangle+\sum_{\beta\in I_V}w_\beta\langle C^\beta\tau_\alpha,\tau_j\rangle)$*, as is easily seen, we find that*

$$\begin{aligned}\sum_{j\in I_{HS}}\phi_{1\alpha}(\tau_j)\phi_{j\alpha}(W)&=-\frac{1}{4}\sum_{j\in I_{HS}}\langle C_H^\alpha\tau_1,\tau_j\rangle\left(\langle C^\alpha W,\tau_j\rangle+\sum_{\beta\in I_V}w_\beta\langle C^\beta\tau_\alpha,\tau_j\rangle\right)\\&=-\frac{1}{4}\left(\langle C_H^\alpha\tau_1, C^\alpha W\rangle+\sum_{\beta\in I_V}w_\beta\langle C_H^\alpha\tau_1, C^\beta\tau_\alpha\rangle\right).\end{aligned}$$

(iv) *We have*:

$$\begin{aligned}\sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\sum_{L\neq j}\phi_{1j}(\tau_\alpha^S)w_L C_{Lj}^\alpha&=-\sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\sum_{L\neq j}\phi_{1j}(\tau_\alpha^S)w_L\langle C^\alpha\tau_L,\tau_j\rangle=-\sum_{\alpha\in I_V}\langle\nabla_{\tau_\alpha^S}\tau_1, C^\alpha W\rangle\\&=-\sum_{\alpha\in I_V}\big(\langle\nabla_{\tau_\alpha}\tau_1, C^\alpha W\rangle-\varpi_\alpha\langle\nabla_{\tau_1}\tau_1, C^\alpha W\rangle\big)\\&=\langle\nabla_{\tau_1}^H\tau_1, CW\rangle-\sum_{\alpha\in I_V}\left(\big\langle p_{HS}[\tau_\alpha,\tau_1], C^\alpha W\big\rangle+\frac{1}{2}\langle C_H^\alpha\tau_1, C^\alpha W\rangle\right),\end{aligned}$$

where we have used the identity $\phi_{1j}(\tau_\alpha)=\langle\nabla_{\tau_\alpha}\tau_1,\tau_j\rangle=\langle[\tau_\alpha,\tau_1],\tau_j\rangle+\frac{1}{2}\langle C_H^\alpha\tau_1,\tau_j\rangle$.

(v) $\sum_{j\in I_{HS}}\phi_{1j}(\tau_\alpha^S)\tau_j(w_\alpha)=\langle\nabla_{\tau_\alpha^S}^H\tau_1,\operatorname{grad}_{HS}w_\alpha\rangle$.

(vi) *By using* (iii) *of Lemma* 3.13 *and the very definition of* B_H*, we get*:

$$\sum_{j,h\in I_{HS}}\phi_{1j}(\tau_h)\phi_{h\alpha}(\tau_j)=\frac{1}{2}\sum_{h\in I_{HS}}\langle\nabla_{\tau_h}^H\tau_1, C_H^\alpha\tau_h\rangle=-\frac{1}{2}\sum_{j,h\in I_{HS}}B_H(\tau_h,\tau_j)\langle C_H^\alpha\tau_h,\tau_j\rangle=-\frac{1}{2}\operatorname{Tr}(B_H\circ C_{HS}^\alpha);$$

we have set $C_{HS}^\alpha:=C_H^\alpha|_{HS}$ *to stress the fact that* C_H^α *acts here only on horizontal tangent vectors*; *see Notation* 3.6*.*

(vii) *By using Definition* 2.4 *and* (5) *of Section* 2.1*, we see that* $C_H^\alpha\neq 0$ *if and only if* $\alpha\in I_{H_2}$ *and, in this case,* $C^\alpha=C_H^\alpha$*. Analogously, we also infer that* $\langle C_H^\alpha\tau_1, C^\alpha\tau_\beta\rangle=0$ $(\alpha,\beta\in I_V)$*.*

Now we have to compute the term $\sum_{j\in I_{HS}}\tau_j(\phi_{1j}(W))$ and, to this aim we need an extra little work. We start with the following:

[19] $[\mathcal{J}_{HS}W_{HS}]=[\tau_j(w_h)]_{(h,j)\in I_{HS}\times I_{HS}}$.

Claim 4.18. *We claim that* $\langle[\widetilde{W}, \vartheta_{t*}X], \nu^t\rangle = 0$ *for every* $X \in \mathbf{C}^\infty(S, HS)$.

Proof. A proof of this claim can also be found in Spivak [45], Chapter 9, pp. 521–522.

First, we remind that $\widetilde{W}(t, p) := \frac{\partial\vartheta}{\partial t}(t, p)$ for any $(t, p) \in (-\varepsilon, \varepsilon) \times \mathcal{U}$. Now let $u_1, \dots, u_{n-1}$ be a system of local coordinates around $p_0 \in \mathcal{U}$. Thus $X(\bar{u}) = \sum_{i=1}^{n-1} a_i(\bar{u}) \frac{\partial\vartheta}{\partial u_i}$, where each a_i is a function of $\bar{u} = (u_1, \dots, u_{n-1})$. We therefore have:

$$\left[\widetilde{W}, \frac{\partial\vartheta}{\partial u_i}\right] = \left[\frac{\partial\vartheta}{\partial t}, \frac{\partial\vartheta}{\partial u_i}\right] = \vartheta_*\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u_i}\right] = 0,$$

because $[\frac{\partial}{\partial t}, \frac{\partial}{\partial u_i}] = 0$. Therefore:

$$[\widetilde{W}, \vartheta_{t*}X] = \left[\widetilde{W}, \sum_{i=1}^{n-1} a_i(\bar{u}) \frac{\partial\vartheta}{\partial u_i}\right] = \left(\sum_{i=1}^{n-1} \widetilde{W}(a_i) \frac{\partial\vartheta}{\partial u_i}\right)$$

and this shows that $[\widetilde{W}, \vartheta_{t*}X]$ is tangent to $\mathcal{U}_t$ which is the claim. □

Claim 4.19. *Let us set* $C^t := \sum_{\alpha\in I_V} \varpi^t_\alpha C^\alpha_H$. *Then we have*:

$$\nabla^H_{\widetilde{W}} \tau_1 = -\operatorname{grad}_{HS} \widetilde{w}_1 - \sum_{\alpha\in I_V} \varpi^t_\alpha \operatorname{grad}_{HS} \widetilde{w}_\alpha - p_{HS}(C^t \widetilde{W}). \tag{45}$$

Proof. Using the previous Claim 4.18 we get $\langle[\widetilde{W}, \tau_j], \nu^t\rangle = 0$ for any $j \in I_{HS}$, and so:

$$\langle\nabla_{\widetilde{W}} \tau_j, \nu^t\rangle = \langle\nabla_{\tau_j} \widetilde{W}, \nu^t\rangle.$$

This implies that:

$$\begin{aligned}
-\langle\nabla^H_{\widetilde{W}} \nu^t_H, \tau_j\rangle &= \langle\nabla_{\tau_j} \widetilde{W}, \nu^t_H\rangle + \sum_{\alpha\in I_V} \varpi^t_\alpha\big(\langle\nabla_{\tau_j} \widetilde{W}, \tau_\alpha\rangle - \langle\nabla_{\widetilde{W}} \tau_j, \tau_\alpha\rangle\big)\\
&= \tau_j(\widetilde{w}_1) + \sum_{\alpha\in I_V} \varpi^t_\alpha \tau_j(\widetilde{w}_\alpha) + \sum_{\alpha\in I_V} \sum_I \widetilde{w}_I \varpi^t_\alpha\big(\langle\nabla_{\tau_j} \tau_I, \tau_\alpha\rangle - \langle\nabla_{\tau_I} \tau_j, \tau_\alpha\rangle\big)\\
&= \tau_j(\widetilde{w}_1) + \sum_{\alpha\in I_V} \varpi^t_\alpha \tau_j(\widetilde{w}_\alpha) + \sum_{\alpha\in I_V} \sum_I \widetilde{w}_I \varpi^t_\alpha C^\alpha_{jI}\\
&= \tau_j(\widetilde{w}_1) + \sum_{\alpha\in I_V} \varpi^t_\alpha \tau_j(\widetilde{w}_\alpha) + \langle C^t \widetilde{W}, \tau_j\rangle \quad \big(j \in I_{HS} = \{2, \dots, h_1\}\big)
\end{aligned}$$

which is equivalent to the claim. □

Claim 4.20. *At* p_0 *we have*:

$$\sum_{j\in I_{HS}} \tau_j\big(\phi_{1j}(W)\big)(p_0) = -\Delta_{HS} w_1 - \sum_{\alpha\in I_V} \big(\varpi_\alpha \Delta_{HS} w_\alpha + \langle\operatorname{grad}_{HS} w_\alpha, \operatorname{grad}_{HS} \varpi_\alpha\rangle\big) - \operatorname{div}_{HS}(CW). \tag{46}$$

Proof. We have:

$$\sum_{j\in I_{HS}} \tau_j\big(\phi_{1j}(\widetilde{W})\big) = \operatorname{div}_{HS_t}(\nabla^H_{\widetilde{W}} \tau_1) - \sum_{j,l\in I_{HS}} \phi_{lj}(\tau_j)\langle\nabla^H_{\widetilde{W}} \tau_1, \tau_l\rangle \tag{47}$$

and the thesis follows by applying Lemma 3.8 (which says that the sum in the previous identity (47) vanishes at $(0, p_0)$) and Claim 4.19. More precisely, we have:

$$\begin{aligned}
\sum_{j\in I_{HS}} \tau_j\big(\phi_{1j}(W)\big)(p_0) &= \left(\operatorname{div}_{HS_t}\left(-\operatorname{grad}_{HS} \widetilde{w}_1 - \sum_{\alpha\in I_V} \varpi^t_\alpha \operatorname{grad}_{HS} \widetilde{w}_\alpha - C^t \widetilde{W}\right)\right)\Bigg|_{(t,p)=(0,p_0)}\\
&= -\Delta_{HS} w_1 - \sum_{\alpha\in I_V} \big(\varpi_\alpha \Delta_{HS} w_\alpha + \langle\operatorname{grad}_{HS} w_\alpha, \operatorname{grad}_{HS} \varpi_\alpha\rangle\big) - \operatorname{div}_{HS}(CW) \quad \text{at } p_0. \qquad \square
\end{aligned}$$

Remark 4.21. To compute the last term in the previous sum we may proceed as follows:

$$\begin{aligned}\mathrm{div}_{HS}(CW)(p_0) &= \sum_{j\in I_{HS}} \big\langle \nabla^H_{\tau_j}(CW), \tau_j\big\rangle\big|_{p_0} = \sum_{j\in I_{HS}}\sum_{\alpha\in I_V}\big\langle \nabla^H_{\tau_j}\varpi_\alpha(C^\alpha W), \tau_j\big\rangle\big|_{p_0}\\ &= \sum_{\alpha\in I_V}\Big[\langle \mathrm{grad}_{HS}\,\varpi_\alpha, C^\alpha W\rangle + \sum_I \varpi_\alpha\big(\langle C^\alpha\tau_I, \mathrm{grad}_{HS}\, w_I\rangle + w_I\,\mathrm{div}_{HS}(C^\alpha\tau_I)\big)\Big]\Big|_{p_0}\\ &= \sum_{\alpha\in I_V}\bigg[\langle \mathrm{grad}_{HS}\,\varpi_\alpha, C^\alpha W\rangle + \sum_{I,L}\varpi_\alpha\big(\langle C^\alpha\tau_I, \mathrm{grad}_{HS}\, w_I\rangle + w_I\langle C^\alpha\tau_I, \tau_L\rangle\mathrm{div}_{HS}(\tau_L)\big)\bigg]\bigg|_{p_0}\\ &= \sum_{\alpha\in I_V}\langle \mathrm{grad}_{HS}\,\varpi_\alpha, C^\alpha W\rangle + \sum_I\langle C\tau_I, \mathrm{grad}_{HS}\, w_I\rangle + \mathcal{H}^{\mathrm{sc}}_H\langle C\tau_1, W\rangle \quad \text{at } p_0.\end{aligned}$$

We stress that in this computation we have used the fact that $C^\alpha \in \mathbf{GL}(\mathbb{R}^n)$ is a linear operator and that, by Lemma 3.8 and (ii) of Lemma 3.13, it turns out that $\mathrm{div}_{HS}(\tau_L)(p_0) \neq 0$ only if $L = 1$ and, in this case, we have $\mathrm{div}_{HS}(\tau_1)(p_0) = -\mathcal{H}^{\mathrm{sc}}_H(p_0)$.

Claim 4.22. *We have*:

$$\langle \nabla^H_{\tau_1}\tau_1, CW\rangle - \sum_{\alpha\in I_V}\big\langle p_{HS}[\tau_\alpha, \tau_1], C^\alpha W\big\rangle = -\sum_{\alpha\in I_V}\big\langle(\mathrm{grad}_{HS}\,\varpi_\alpha - C\tau^S_\alpha), C^\alpha W\big\rangle.$$

Proof. We need identity (ii) of Lemma 3.12 which can be written as follows:

$$\langle \nabla^H_{\tau^S_\alpha}\tau_1, \tau_j\rangle = \tau_j(\varpi_\alpha) + \frac{1}{2}\langle C^\alpha_H\tau_1, \tau_j\rangle - \langle C\tau^S_\alpha, \tau_j\rangle \quad (j\in I_{HS}, \alpha\in I_V). \tag{48}$$

Moreover, note that $\langle p_{HS}[\tau_\alpha, \tau_1], \tau_j\rangle = \langle \nabla^H_{\tau_\alpha}\tau_1, \tau_j\rangle - \frac{1}{2}\langle C^\alpha_H\tau_1, \tau_j\rangle$ $(j\in I_{HS})$. So we get:

$$\begin{aligned}\langle \nabla^H_{\tau_1}\tau_1, CW\rangle - \sum_{\alpha\in I_V}\big\langle p_{HS}[\tau_\alpha, \tau_1], C^\alpha W\big\rangle &= \sum_{\alpha\in I_V}\Big(-\langle \nabla^H_{\tau_\alpha}\tau_1, C^\alpha W\rangle + \frac{1}{2}\langle C^\alpha_H\tau_1, C^\alpha W\rangle + \varpi_\alpha\langle \nabla^H_{\tau_1}\tau_1, C^\alpha W\rangle\Big)\\ &= \sum_{\alpha\in I_V}\Big(-\langle \nabla^H_{\tau^S_\alpha}\tau_1, C^\alpha W\rangle + \frac{1}{2}\langle C^\alpha_H\tau_1, C^\alpha W\rangle\Big)\\ &= -\sum_{\alpha\in I_V}\big\langle(\mathrm{grad}_{HS}\,\varpi_\alpha - C\tau^S_\alpha), C^\alpha W\big\rangle. \qquad \square\end{aligned}$$

Claim 4.23. *We have*:

$$\sum_{\alpha\in I_V}\langle \nabla^H_{\tau^S_\alpha}\tau_1, \mathrm{grad}_{HS}\, w_\alpha\rangle = \langle \mathrm{grad}_{HS}\,\varpi_\alpha, \mathrm{grad}_{HS}\, w_\alpha\rangle + \frac{1}{2}\langle C^\alpha_H\tau_1, \mathrm{grad}_{HS}\, w_\alpha\rangle - \langle C\tau^S_\alpha, \mathrm{grad}_{HS}\, w_\alpha\rangle.$$

Proof. This follows once again from (ii) of Lemma 3.12; see (48). $\square$

We may now accomplish the computation of our second variation formula of σ^{n-1}_H. Indeed, by applying Remark 4.9 together with Claims 4.17 and 4.20 into (45) and rearranging we get:

$$\begin{aligned}\imath^*(A)_{p_0} = \bigg\{&-\mathcal{H}^{\mathrm{sc}}_H\bigg[W(w) + w\bigg(\mathrm{div}_{HS}\,W_{HS} + \sum_{\alpha\in I_V}\tau^S_\alpha(w_\alpha) + \langle C\nu^t_H, W\rangle\bigg)\bigg]\\ &+ w\bigg[\mathrm{div}_{HS_t}(\nabla^H_{\widetilde{W}}\nu^t_H) + w_1(2\,\mathrm{Tr}_2\,B_H) + \mathrm{Tr}\big(B_H\circ[\mathcal{J}_{HS}W_{HS}]^{\mathrm{tr}}\big) - \frac{1}{2}\sum_{\alpha\in I_V}w_\alpha\mathrm{Tr}(B_H\circ C^\alpha_{HS})\\ &+ \langle \nabla^H_{\nu^t_H}\nu^t_H, CW\rangle - \sum_{\alpha\in I_V}\big(\big\langle p_{HS}[\tau_\alpha, \nu^t_H], C^\alpha W\big\rangle + \langle \nabla^H_{\tau^S_\alpha}\nu^t_H, \mathrm{grad}_{HS}\, w_\alpha\rangle\big)\bigg]\bigg\}\bigg|_{(t,p)=(0,p_0)}\sigma^{n-1}_H(p_0).\end{aligned} \tag{49}$$

Starting from (49), making use of identity (15) (see Remark 3.11), Claim 4.20, Remark 4.21, Claim 4.22 and Claim 4.23 we get:

$$\begin{aligned}
\iota^*(A)_{p_0} = \Bigg\{ & -\mathcal{H}_H^{\mathrm{sc}}\big[W(w) + w\big(\mathrm{div}_{HS} W_{HS} + \mathrm{div}(W_V) - \big\langle(\mathcal{J}_H W_V)\nu_H, \varpi\big\rangle + \langle C\nu_H, W\rangle\big)\big] \\
& + w\Bigg[-\Delta_{HS} w_1 - \sum_{\alpha\in I_V}\big(\varpi_\alpha \Delta_{HS} w_\alpha + \langle \mathrm{grad}_{HS} w_\alpha, \mathrm{grad}_{HS}\varpi_\alpha\rangle + \langle \mathrm{grad}_{HS}\varpi_\alpha, C^\alpha W\rangle\big) \\
& - \sum_I \langle C\tau_I, \mathrm{grad}_{HS} w_I\rangle - \mathcal{H}_H^{\mathrm{sc}}\langle C\nu_H, W\rangle + w_1(2\,\mathrm{Tr}_2 B_H) + \mathrm{Tr}\big(B_H \circ [\mathcal{J}_{HS} W_{HS}]^{\mathrm{tr}}\big) \\
& - \frac{1}{2}\sum_{\alpha\in I_V} w_\alpha \,\mathrm{Tr}(B_H \circ C^\alpha_{HS}) \\
& - \sum_{\alpha\in I_V}\Bigg[\big\langle(\mathrm{grad}_{HS}\varpi_\alpha - C\tau^S_\alpha), C^\alpha W\big\rangle + \langle \mathrm{grad}_{HS}\varpi_\alpha, \mathrm{grad}_{HS} w_\alpha\rangle + \frac{1}{2}\langle C^\alpha_H \nu_H, \mathrm{grad}_{HS} w_\alpha\rangle \\
& - \langle C\tau^S_\alpha, \mathrm{grad}_{HS} w_\alpha\rangle\Bigg]\Bigg]\Bigg\}\sigma_H^{n-1}(p_0) \\
= \Bigg\{ & -\mathcal{H}_H^{\mathrm{sc}}\big[W(w) + w\big(\mathrm{div}_{HS} W_{HS} + \mathrm{div}(W_V) - \big\langle(\mathcal{J}_H W_V)\nu_H, \varpi\big\rangle + 2\langle C\nu_H, W\rangle\big)\big] \\
& + w\Bigg[-\Delta_{HS} w_1 - \sum_{\alpha\in I_V}\Bigg(\varpi_\alpha \Delta_{HS} w_\alpha + 2\Bigg\langle(\mathrm{grad}_{HS} w_\alpha + C^\alpha W), \Bigg(\mathrm{grad}_{HS}\varpi_\alpha - \frac{1}{2}C\tau^S_\alpha\Bigg)\Bigg\rangle\Bigg) \\
& - \mathrm{Tr}(\mathcal{J}_{HS} W \circ C) + w_1(2\,\mathrm{Tr}_2 B_H) + \mathrm{Tr}\big(B_H \circ [\mathcal{J}_{HS} W_{HS}]^{\mathrm{tr}}\big) \\
& - \frac{1}{2}\sum_{\alpha\in I_{H_2}}\big(w_\alpha \,\mathrm{Tr}(B_H \circ C^\alpha_{HS}) + \langle C^\alpha_H \nu_H, \mathrm{grad}_{HS} w_\alpha\rangle\big)\Bigg]\Bigg\}\sigma_H^{n-1}(p_0).
\end{aligned}$$

Remark 4.24. Note that here above we have set $\mathrm{Tr}(\mathcal{J}_{HS} W \circ C) := \sum_I \langle C\tau_I, \mathrm{grad}_{HS} w_I\rangle$. However there is a slight abuse of notation here and, in fact, we have $\mathrm{Tr}(\mathcal{J}_{HS} W \circ C) := \mathrm{Tr}([\mathcal{J}_{HS} W \mathbf{0}] \circ C)$ where $\mathbf{0} := \mathbf{0}_{n\times n-h_1+1}$ denotes the zero matrix in $\mathcal{M}_{n\times n-h_1+1}(\mathbb{R})$.

Now from the last expression, using Remarks 4.14 and 4.15, we finally get:

$$\begin{aligned}
II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \Bigg\{ & -\mathcal{H}_H^{\mathrm{sc}}\big[W(w) + w\big(\mathrm{div}_{HS} W_{HS} + \mathrm{div}(W_V) - \big\langle(\mathcal{J}_H W_V)\nu_H, \varpi\big\rangle + 2\langle C\nu_H, W\rangle\big)\big] \\
& + w\Bigg[-\Delta_{HS} w_1 - \sum_{\alpha\in I_V}\big(\varpi_\alpha \Delta_{HS} w_\alpha + \big\langle(\mathrm{grad}_{HS} w_\alpha + C^\alpha W), (2\,\mathrm{grad}_{HS}\varpi_\alpha - C\tau^S_\alpha)\big\rangle\big) \\
& + w_1(2\,\mathrm{Tr}_2 B_H) + \mathrm{Tr}\big(B_H \circ [\mathcal{J}_{HS} W_{HS}]^{\mathrm{tr}}\big) + \mathrm{Tr}\big(C \circ [\mathcal{J}_{HS} W\,\mathbf{0}]^{\mathrm{tr}}\big) \\
& - \frac{1}{2}\sum_{\alpha\in I_{H_2}}\big(w_\alpha \,\mathrm{Tr}(B_H \circ C^\alpha_{HS}) + \big\langle C^\alpha_H \nu_H, \mathrm{grad}_{HS} w_\alpha\big\rangle\big)\Bigg]\Bigg\}\sigma_H^{n-1} \\
& + \int_{\partial\mathcal{U}} \big\{\big\langle[\widetilde{W}^{\nu^t}, \widetilde{W}^T]^T|_{t=0}, \eta\big\rangle |p_H\nu| + \big(\mathrm{div}_{TS}\big(|p_H\nu| W^T\big) - \mathcal{H}_H^{\mathrm{sc}}\langle W, \nu\rangle\big)\langle W^T, \eta\rangle\big\}\sigma_{\mathcal{R}}^{n-2}
\end{aligned}$$

and the proof of Proposition 4.13 is complete. □

4.5. Case $W \in \mathbf{C}^{\infty}(S, H)$

Now we find the expression for the 2nd variation of σ_H^{n-1} relatively to arbitrary horizontal variations. If $W = W_H \in \mathbf{C}^{\infty}(S, H)$ $(W_H = w_1\nu_H + W_{HS})$, then w is equal to w_1 by (37). Thus, using Proposition 4.13 and (40) of Remark 4.15 we immediately obtain the following expression for $II_{\mathcal{U}}(W, \sigma_H^{n-1})$:

$$\begin{aligned} II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \Big\{ &-\mathcal{H}_H^{\mathrm{sc}}\big[W_H(w) + w\big(\operatorname{div}_{HS} W_{HS} + 2\langle C_H\nu_H, W_{HS}\rangle\big)\big] \\ &+ w\bigg[-\Delta_{HS} w - \sum_{\alpha\in I_V}\big\langle (2\operatorname{grad}_{HS}\varpi_\alpha - C\tau_\alpha^S), C^\alpha W_H\big\rangle + w(2\operatorname{Tr}_2 B_H) \\ &+ \operatorname{Tr}\big(B_H \circ [\mathcal{J}_{HS} W_{HS}]^{\mathrm{tr}}\big) + \operatorname{Tr}\big(C \circ [\mathcal{J}_{HS} W_H \mathbf{0}]^{\mathrm{tr}}\big)\bigg]\Big\}\sigma_H^{n-1} \\ &+ \int_{\partial\mathcal{U}} \big\{\big\langle [\widetilde{W}_H, \widetilde{W}_{HS}]^T|_{t=0}, \eta\big\rangle + \big(\operatorname{div}_{HS} W_{HS} + \langle C_H\nu_H, W_{HS}\rangle \\ &- w\mathcal{H}_H^{\mathrm{sc}}\big)\langle W_{HS}, \eta\rangle\big\}|p_H\nu|\sigma_{\mathcal{R}}^{n-2}. \end{aligned} \tag{50}$$

Starting from (50) we may easily obtain the following general version of the second variation formula relatively to arbitrary horizontal variations:

Corollary 4.25 *(Horizontal 2nd variation). Under the hypotheses of Proposition* 4.13 *let us assume that* $W \in \mathbf{C}^{\infty}(S, H)$, $W = w\nu_H + W_{HS}$. *Then we have*:

$$\begin{aligned} II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \Big\{ &-\mathcal{H}_H^{\mathrm{sc}}\big[W_H(w) + w\big(\operatorname{div}_{HS} W_{HS} + 2\langle C_H\nu_H, W_{HS}\rangle\big)\big] \\ &+ |\operatorname{grad}_{HS} w|^2 + w\bigg[w(2\operatorname{Tr}_2 B_H) + \operatorname{Tr}(B_H \circ \mathcal{J}_{HS} W_{HS}) \\ &- \sum_{\alpha\in I_V}\big\langle (2\operatorname{grad}_{HS}\varpi_\alpha - C\tau_\alpha^S), C^\alpha W_H\big\rangle\bigg]\Big\}\sigma_H^{n-1} \\ &+ \int_{\partial\mathcal{U}} \big\{\big\langle\big(-w\operatorname{grad}_{HS} w + [\widetilde{W}, \widetilde{W}_{HS}]|_{t=0}\big), \eta\big\rangle \\ &+ \big(\operatorname{div}_{HS} W_{HS} + \langle C_H\nu_H, W_{HS}\rangle - w\mathcal{H}_H^{\mathrm{sc}}\big)\langle W_{HS}, \eta\rangle\big\}|p_H\nu|\sigma_{\mathcal{R}}^{n-2}. \end{aligned} \tag{51}$$

Proof. First, note that:

$$\operatorname{Tr}\big([\mathcal{J}_{HS} W_H\, \mathbf{0}] \circ C\big) = \operatorname{Tr}\big([\mathcal{J}_{HS} W_H\, \mathbf{0}] \circ C_H\big) = \langle \operatorname{grad}_{HS} w, C_H\nu_H\rangle + \operatorname{Tr}(\mathcal{J}_{HS} W_{HS} \circ C_{HS}). \tag{52}$$

We therefore have:

$$\begin{aligned} &\operatorname{Tr}\big(B_H \circ [\mathcal{J}_{HS} W_{HS}]^{\mathrm{tr}}\big) + \operatorname{Tr}\big(C_{HS} \circ [\mathcal{J}_{HS} W_{HS}]^{\mathrm{tr}}\big) \\ &\quad = \sum_{i,j\in I_{HS}} \big(\langle \nabla^H_{\tau_i}\tau_j, \nu_H\rangle\langle \operatorname{grad}_{HS} w_i, \tau_j\rangle + \langle C_{HS}\operatorname{grad}_{HS} w_i, \tau_i\rangle\big) \\ &\quad = \sum_{i,j\in I_{HS}} \langle \operatorname{grad}_{HS} w_i, \tau_j\rangle\big(\phi_{j1}(\tau_i) - \langle \tau_i, C_{HS}\tau_j\rangle\big) \\ &\quad = \sum_{i,j\in I_{HS}} \tau_j(w_i)\phi_{i1}(\tau_j) \quad \big(\text{by (i) of Lemma 3.12}\big) \\ &\quad = \operatorname{Tr}(B_H \circ \mathcal{J}_{HS} W_{HS}). \end{aligned} \tag{53}$$

Thus, we will get the thesis by using the following application of Theorem 3.17:

$$\int_{\mathcal{U}} \big(w\,\Delta_{HS} w + |\mathrm{grad}_{HS}\, w|^2 + w\,\langle C_H \nu_H, \mathrm{grad}_{HS}\, w\rangle\big)\,\sigma_H^{n-1} = \int_{\partial\mathcal{U}\setminus C_{\partial\mathcal{U}}} w\,\langle \mathrm{grad}_{HS}\, w, \eta_{HS}\rangle_{HS}\,\sigma_H^{n-2}. \tag{54}$$

Indeed, from (50), (52) and (53) we have:

$$\begin{aligned} II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \Big\{ &-\mathcal{H}_H^{\mathrm{sc}}\big[W_H(w) + w\big(\mathrm{div}_{HS}\, W_{HS} + 2\langle C_H \nu_H, W_{HS}\rangle\big)\big] \\ &+ w\Big[-\Delta_{HS} w - \langle \mathrm{grad}_{HS}\, w, C_H \nu_H\rangle - \sum_{\alpha\in I_V} \big\langle (2\,\mathrm{grad}_{HS}\,\varpi_\alpha - C\tau_\alpha^S), C^\alpha W_H\big\rangle \\ &+ w(2\,\mathrm{Tr}_2\, B_H) + \mathrm{Tr}(B_H \circ \mathcal{J}_{HS} W_{HS})\Big]\Big\}\sigma_H^{n-1} \\ &+ \int_{\partial\mathcal{U}} \big\{\big\langle [\widetilde{W}_H, \widetilde{W}_{HS}]^T|_{t=0}, \eta\big\rangle + \big(\mathrm{div}_{HS}\, W_{HS} + \langle C_H \nu_H, W_{HS}\rangle - w\mathcal{H}_H^{\mathrm{sc}}\big)\langle W_{HS}, \eta\rangle\big\}|p_H \nu|\sigma_{\mathcal{R}}^{n-2} \end{aligned}$$

and the thesis follows by applying (54). □

Proof of Corollary 4.7. Starting from Corollary 4.25 the proof is quite immediate. Indeed, it is enough to substitute $W_{HS} = 0$ into (51). □

4.6. Proof of the main result: the case $\mathcal{H}_H^{\mathrm{sc}} = \mathrm{const}$.

In this section we shall prove Theorem 4.8. To this aim we remark that the hypothesis that $\mathcal{H}_H^{\mathrm{sc}}$ be constant along S is crucial to obtain a more simple expression for the second variation formula of σ_H^{n-1}. In Appendix A, an analogous remark will be made in the particular case that $\mathcal{H}_H^{\mathrm{sc}} = 0$.

Let us preliminarily set $(\mathcal{H}_H^{\mathrm{sc}})_t := -\sum_{j\in I_{HS}} \phi_{1j}(\tau_j) = \sum_{j\in I_{HS}} \langle \nabla^H_{\tau_j}\tau_j, \nu_H^t\rangle$ to denote the horizontal scalar mean curvature of $\mathcal{U}_t = \vartheta_t(\mathcal{U})$, $t \in (-\varepsilon, \varepsilon)$.

Remark 4.26. If we assume that $\mathcal{H}_H^{\mathrm{sc}} = \mathrm{const.}$ along S, we immediately get that $\mathcal{L}_X \mathcal{H}_H^{\mathrm{sc}} = 0$ along S whenever $X \in \mathbf{C}^\infty(S, HS)$. If W denotes the variation vector of ϑ, we see that:

$$\imath^*\big(\mathcal{L}_{\widetilde{W}_{HS}}(\mathcal{H}_H^{\mathrm{sc}})_t\big) = \mathcal{L}_{W_{HS}}\mathcal{H}_H^{\mathrm{sc}} = 0.$$

Analogously, we see that $\imath^*(\mathcal{L}_{\tau_\alpha^S}(\mathcal{H}_H^{\mathrm{sc}})_t) = \mathcal{L}_{\tau_\alpha^S}\mathcal{H}_H^{\mathrm{sc}} = 0$ $(\alpha \in I_V)$ and this implies that:

$$\imath^*\big(\mathcal{L}_{\tau_\alpha}(\mathcal{H}_H^{\mathrm{sc}})_t\big) = \imath^*\big(\mathcal{L}_{\varpi_\alpha^t \nu_H^t}(\mathcal{H}_H^{\mathrm{sc}})_t\big) \quad (\alpha \in I_V). \tag{55}$$

We have already noted (see Remark 4.14 in Section 4.4) that:

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = II_{\mathcal{U}}^{\mathrm{Int.}}(W, \sigma_H^{n-1}) + II_{\mathcal{U}}^{\mathrm{Bound.}}(W, \sigma_H^{n-1}). \tag{56}$$

We stress that the hypothesis $\mathcal{H}_H^{\mathrm{sc}} = \mathrm{const.}$ can be used to compute the first addend in (56) in a slightly different way with respect to what we have done in Section 4.4 throughout the proof of Proposition 4.13. More precisely, we have the following

Claim 4.27. *Let $\mathcal{U}$ be such that $\mathcal{H}_H^{\mathrm{sc}}$ is constant. Then we have*:

$$II_{\mathcal{U}}^{\mathrm{Int.}}(W, \sigma_H^{n-1}) = II_{\mathcal{U}}^{\mathrm{Int.}}(w\,\nu_H, \sigma_H^{n-1}) + \int_{\mathcal{U}} \Big\{-W(w) + w\frac{\partial w}{\partial \nu_H}\Big\}\mathcal{H}_H^{\mathrm{sc}}\sigma_H^{n-1}. \tag{57}$$

Proof of Claim 4.27. We remind the notations $w := \frac{\langle W,\nu\rangle}{|p_H\nu|}$ and $w_t := \frac{\langle \widetilde{W},\nu^t\rangle}{|p_H\nu^t|}$. Note that the first addend in the first variation formula (27) can be written as follows

$$I_{\mathcal{U}}^{\text{Int.}}(W,\sigma_H^{n-1}) = -\int_{\mathcal{U}} w\,\mathcal{H}_H^{\text{sc}}\sigma_H^{n-1}.$$

So we easily get that

$$II_{\mathcal{U}}^{\text{Int.}}(W,\sigma_H^{n-1}) = \int_{\mathcal{U}} \iota^*\bigl\{\mathcal{L}_{\widetilde{W}}\bigl(-w_t(\mathcal{H}_H^{\text{sc}})_t(\sigma_H^{n-1})_t\bigr)\bigr\} = \int_{\mathcal{U}} \bigl\{w(\mathcal{H}_H^{\text{sc}})^2 - W(w)\mathcal{H}_H^{\text{sc}} - w\iota^*\bigl(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H^{\text{sc}})_t\bigr)\bigr\}\sigma_H^{n-1}. \tag{58}$$

Now we make use of Remark 4.26 to compute $\iota^*(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H^{\text{sc}})_t)$. Setting $W_{\perp(HS)} := w_1\nu_H + W_V$, we have:

$$\begin{aligned}
\iota^*\bigl(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H^{\text{sc}})_t\bigr) &= \iota^*\bigl(\mathcal{L}_{\widetilde{W}_{HS}}(\mathcal{H}_H^{\text{sc}})_t\bigr) + \iota^*\bigl(\mathcal{L}_{\widetilde{W}_{\perp(HS)}}(\mathcal{H}_H^{\text{sc}})_t\bigr)\\
&= \mathcal{L}_{W_{HS}}\mathcal{H}_H^{\text{sc}} + \iota^*\bigl(\mathcal{L}_{\widetilde{W}_{\perp(HS)}}(\mathcal{H}_H^{\text{sc}})_t\bigr)\\
&= \iota^*\bigl(\mathcal{L}_{\widetilde{W}_{\perp(HS)}}(\mathcal{H}_H^{\text{sc}})_t\bigr) \quad \text{(by Remark 4.26)}\\
&= \iota^*\bigl(\mathcal{L}_{\widetilde{w}_1\nu_H^t}(\mathcal{H}_H^{\text{sc}})_t\bigr) + \sum_{\alpha\in I_V}\iota^*\bigl(\mathcal{L}_{\widetilde{w}_\alpha\tau_\alpha}(\mathcal{H}_H^{\text{sc}})_t\bigr)\\
&= \iota^*\bigl(\mathcal{L}_{\widetilde{w}_1\nu_H^t}(\mathcal{H}_H^{\text{sc}})_t\bigr) + \sum_{\alpha\in I_V}\iota^*\bigl(\mathcal{L}_{\widetilde{w}_\alpha\varpi_\alpha^t\nu_H^t}(\mathcal{H}_H^{\text{sc}})_t\bigr) \quad \bigl(\text{by (55)}\bigr)\\
&= \iota^*\bigl(\mathcal{L}_{w_t\,\nu_H^t}(\mathcal{H}_H^{\text{sc}})_t\bigr).
\end{aligned} \tag{59}$$

Therefore, from (58) we get:

$$II_{\mathcal{U}}^{\text{Int.}}(W,\sigma_H^{n-1}) = \int_{\mathcal{U}} \bigl\{(w\,\mathcal{H}_H^{\text{sc}})^2 - W(w)\mathcal{H}_H^{\text{sc}} - w\iota^*\bigl(\mathcal{L}_{w_t\,\nu_H^t}(\mathcal{H}_H^{\text{sc}})_t\bigr)\bigr\}\sigma_H^{n-1},$$

and the thesis easily follows by observing that:

$$II_{\mathcal{U}}^{\text{Int.}}(w\,\nu_H,\sigma_H^{n-1}) = \int_{\mathcal{U}} \left\{(w\,\mathcal{H}_H^{\text{sc}})^2 - w\,\frac{\partial w}{\partial\nu_H}\mathcal{H}_H^{\text{sc}} - w\iota^*\mathcal{L}_{w\,\nu_H}(\mathcal{H}_H^{\text{sc}})_t\right\}\sigma_H^{n-1}. \qquad \square$$

Proof of Theorem 4.8. At this point the proof of Theorem 4.8 is very simple. Indeed, using (56) and Claim 4.27 we get:

$$\begin{aligned}
II_{\mathcal{U}}(W,\sigma_H^{n-1}) &= II_{\mathcal{U}}^{\text{Int.}}(W,\sigma_H^{n-1}) + II_{\mathcal{U}}^{\text{Bound.}}(W,\sigma_H^{n-1})\\
&= II_{\mathcal{U}}^{\text{Int.}}(w\,\nu_H,\sigma_H^{n-1}) + \int_{\mathcal{U}}\left\{-W(w) + w\frac{\partial w}{\partial\nu_H}\right\}\mathcal{H}_H^{\text{sc}}\sigma_H^{n-1} + II_{\mathcal{U}}^{\text{Bound.}}(W,\sigma_H^{n-1}).
\end{aligned}$$

The first addend can be computed using Corollary 4.7 with $w = \frac{\langle W,\nu\rangle}{|p_H\nu|}$, while the third addend has been already computed in the general case; see (39) in Remark 4.15. Putting all together we therefore get:

$$\begin{aligned}
II_{\mathcal{U}}(W,\sigma_H^{n-1}) = \int_{\mathcal{U}} \Bigg\{&-\mathcal{H}_H^{\text{sc}}w\frac{\partial w}{\partial\nu_H} + \left[-W(w) + w\frac{\partial w}{\partial\nu_H}\right]\mathcal{H}_H^{\text{sc}}\\
&+ |\text{grad}_{HS}\,w|^2 + w^2\left[(2\,\text{Tr}_2\,B_H) - \sum_{\alpha\in I_V}\bigl\langle\bigl(2\,\text{grad}_{HS}(\varpi_\alpha) - C\tau_\alpha^S\bigr), C^\alpha\nu_H\bigr\rangle\right]\Bigg\}\sigma_H^{n-1}\\
&+ \int_{\partial\mathcal{U}} \bigl\{\bigl\langle\bigl(-w\,\text{grad}_{HS}\,w + [\widetilde{W}^{\nu^t}, \widetilde{W}^T]^T|_{t=0}\bigr), \eta\bigr\rangle|p_H\nu|\\
&+ \bigl(\text{div}_{TS}\bigl(|p_H\nu|W^T\bigr) - \mathcal{H}_H^{\text{sc}}\langle W,\nu\rangle\bigr)\langle W^T,\eta\rangle\bigr\}\sigma_{\mathcal{R}}^{n-2}
\end{aligned}$$

$$= \int_{\mathcal{U}} \left\{ -W(w)\mathcal{H}_H^{sc} + |\mathrm{grad}_{HS}\, w|^2 + w^2\left[(2\,\mathrm{Tr}_2\, B_H) - \sum_{\alpha\in I_V} \big\langle \big(2\,\mathrm{grad}_{HS}(\varpi_\alpha) - C\tau_\alpha^S\big), C^\alpha \nu_H \big\rangle \right] \right\} \sigma_H^{n-1}$$
$$+ \int_{\partial\mathcal{U}} \big\{ \big\langle \big(-w\,\mathrm{grad}_{HS}\, w + [\widetilde{W}^{\nu^t}, \widetilde{W}^T]^T|_{t=0}\big), \eta \big\rangle |p_H \nu|$$
$$+ \big(\mathrm{div}_{TS}\big(|p_H\nu| W^T\big) - \mathcal{H}_H^{sc}\langle W, \nu\rangle\big)\langle W^T, \eta\rangle \big\} \sigma_{\mathcal{R}}^{n-2}.$$

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Appendix A. Remark about the case $\mathcal{H}_H^{sc} = 0$

As before, let $S \subset \mathbb{G}$ be an immersed hypersurface and $U \subset \mathbb{G}$ be an open set having non-empty intersection with S. Let $\mathcal{U} := U \cap S$ be non-characteristic with smooth boundary $\partial\mathcal{U}$ and denote by $\imath$ the inclusion of $\mathcal{U}$ in $\mathbb{G}$. Assume that $\mathcal{U}$ is an extremal of the H-perimeter functional (25), so that its scalar horizontal mean curvature $\mathcal{H}_H^{sc}$ is identically zero. We set:

$$\mathcal{SV}_{\mathcal{U}}(X, Y) := X \lrcorner\, \mathrm{d}(Y \lrcorner\, \mathrm{d}\sigma_H^{n-1}) \quad \text{for } X, Y \in \mathfrak{X}(\mathbb{G}).$$

Lemma A.1. [24] *With the previous hypotheses we have* $\int_{\mathcal{U}} \imath^*(\mathcal{SV}_{\mathcal{U}}(X, Y)) = 0$ *if either* X *or* Y *is tangent to* $\mathcal{U}$.

This lemma appears in [24] in a more general setting.[20] Now we explicitly remark that if the variation vector W of ϑ is compactly supported on $\mathcal{U}$, we have:

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \imath^*\big(\mathcal{SV}_{\mathcal{U}}(\widetilde{W}, \widetilde{W})\big),$$

[20] *Proof of Lemma A.1.* First note that, using standard properties of the Lie derivative and the hypothesis $\mathcal{H}_H^{sc} = 0$, it turns out that

$$\int_{\mathcal{U}} \imath^*\big(\mathcal{SV}_{\mathcal{U}}(X, Y)\big) = \int_{\mathcal{U}} \imath^*\big(\mathcal{SV}_{\mathcal{U}}(Y, X)\big) + \int_{\partial\mathcal{U}} \imath_{\partial\mathcal{U}}^*\big(Y \lrcorner\, X \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big). \tag{A.1}$$

Indeed:

$$\imath^*\big(\mathcal{SV}_{\mathcal{U}}(X, Y)\big) = \big(X \lrcorner\, \mathrm{d}\big(Y \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big)\big)\big|_{\mathcal{U}} = \big(-[Y, X] \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big)\big|_{\mathcal{U}} + \big(\mathcal{L}_Y\big(X \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big)\big)\big|_{\mathcal{U}}$$
$$= \big(-[Y, X] \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big)\big|_{\mathcal{U}} + \big(\mathrm{d}\big(Y \lrcorner\, X \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big)\big)\big|_{\mathcal{U}} + \imath^*\big(\mathcal{SV}_{\mathcal{U}}(Y, X)\big),$$

and the first addend is zero since $\mathcal{U}$ is an extremal of (25) (i.e. $\mathcal{H}_H^{sc} = 0$). So (A.1) follows using Stoke's theorem. Now suppose that X is tangent to $\mathcal{U}$. We have $\mathcal{SV}_{\mathcal{U}}(X, Y) = (\mathcal{L}_X(Y \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t))|_{\mathcal{U}} - \mathrm{d}(X \lrcorner\, Y \lrcorner\, (\sigma_H^{n-1})_t)|_{\mathcal{U}}$. Note that $(Y \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t)|_{\mathcal{U}} = 0$ again because $\mathcal{U}$ is an extremal of (25); since X is tangent to $\mathcal{U}$, we also get:

$$\imath^*\big(\mathcal{L}_X\big(Y \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big)\big) = \big(\mathcal{L}_X \imath^*\big(Y \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big)\big) = 0.$$

Then

$$\int_{\mathcal{U}} \imath^*\big(\mathcal{SV}_{\mathcal{U}}(X, Y)\big) = \int_{\partial\mathcal{U}} \imath_{\partial\mathcal{U}}^*\big(Y \lrcorner\, X \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t\big) = 0,$$

where the second equality follows because $(Y \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t)|_{\mathcal{U}} = 0$ and X is tangent to $\mathcal{U}$. Finally, if Y is tangent to $\mathcal{U}$, the right-hand side of (A.1) vanishes because $(X \lrcorner\, \mathrm{d}(\sigma_H^{n-1})_t)|_{\mathcal{U}} = 0$ and we may use the previous case. □

where, as usual, $\widetilde{W}$ denotes any extension of W to a neighborhood of $\mathrm{Im}(\vartheta)$. Denote by $\widetilde{W}^{\nu}$ the normal component of $\widetilde{W}$ along $\mathcal{U}_t = \vartheta_t(\mathcal{U})$ and set $w = \frac{\langle W, \nu\rangle}{|\mathcal{P}_H \nu|}$. Therefore, using Lemma A.1 and arguing as in Claim 4.27 (see (59)) we get that:

$$II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} \imath^*\big(\mathcal{SV}_{\mathcal{U}}(\widetilde{W}^{\nu}, \widetilde{W}^{\nu})\big) = \int_{\mathcal{U}} -w\, \imath^*\big(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H^{\mathrm{sc}})_t\big)\sigma_H^{n-1}$$
$$= \int_{\mathcal{U}} -w\imath^*\big(\mathcal{L}_{w_t \nu_H^t}(\mathcal{H}_H^{\mathrm{sc}})_t\big)\sigma_H^{n-1}.$$

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