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ORIGINAL ARTICLE

Approximation Space on Novel Granulations

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Abstract Granulation of a universe involves grouping of similar elements into granules. With granulated views, we deal with approximations of concepts, represented by subsets of the universe, in terms of granules. This paper examines the problem of approximations with respect to various granulations of the universe. The granulation structures used by rough set theory, neighborhood systems and topological space and the corresponding approximation structures, are studied.

1. Introduction

Granular computing may be regarded to as a label of the family of theories, methodologies, and techniques that make use of granules, i.e., groups, classes, or clusters of a universe, in the process of problem solving [1]. The basic ideas of granular computing have appeared in many fields, such as interval analysis, quantization, rough set theory, Dempster–Shafer theory of belief functions, divide and conquer, cluster analysis, machine learning, databases, information retrieval, and many others [2,3]. There are many reasons for the study of granular computing [2]. The practical necessity and simplicity in problem solving are perhaps some of the main reasons. When a problem involves incomplete, uncertain, or vague information, it may be difficult to differentiate distinct elements and one is forced to consider granules. Although detailed information may be available, it may be sufficient to use granules in order to have an efficient and practical solution. Very precise solutions may not be required for many practical problems. The use of granules generally leads to simplification of practical problems. The acquisition of precise information may be too costly, and coarse-grained information reduces cost. There is clearly a need for the systematic studies of granular computing. It is expected that granular computing will play an important role in the design and implementation of efficient and practical intelligent information systems. Lin [4] and Yao [5] studied granular computing using neighborhood systems for the interpretation of granules. Pawlak [6], Polkowski and Skowron [7], and Skowron and Stepaniuk [8] examined granular computing in connection with the theory of rough sets. The theories of rough sets and neighborhood systems provide convenient and effective tools for granulation, and deal with some fundamental granulation structures. In the rough set theory, one starts with an equivalence relation. A universe is divided into a family of disjoint subsets. The granulation structure adopted is a partition of the universe. By weakening the requirement of equivalence relations, we can have more general granulation structures such as coverings of the universe. Neighborhood systems provide an even more general granulation structure. For each element of a universe, one associates it with a nonempty family of neigh-

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borhood granules, which is called a neighborhood system. It offers a multi-layered granulation of the universe, which is a natural generalization of the singlelayered granulation structure used by rough set theory. With the granulation of universe, one considers elements within a granule as a whole rather than individually [3]. The loss of information through granulation implies that some subsets of the universe can only be approximately described. Topology is also a mathematical tool to study information systems and rough sets [9,10]. In theory of rough sets, a pair of lower and upper approximation is typically used. The approximations are expressed in terms of granules according to their overlaps with the set to be approximated. Based on this idea, the main objective of the paper is to study the three related issues of granulation and approximation. The granulation structures used by theories of rough sets, neighborhood systems and topological space are analyzed and compared, and the corresponding approximation structures are investigated.

2. Granulations and approximations

From view of points of rough sets, this section examines connections between granulations and approximations.

2.1. Rough sets: granulation by partitions

Let U be a finite and nonempty set called the universe, and let $E \subseteq U \times U$ denote an equivalence relation on U. The pair $apr = (U, E)$ is called an approximation space. The equivalence relation E partitions the set U into disjoint subsets. This partition of the universe is denoted by U/E. The equivalence relation is the available information or knowledge about the objects under consideration. If two elements x, y in U belong to the same equivalence class, we say that x and y are indistinguishable. Each equivalence class may be viewed as a granule consisting of indistinguishable elements, and it is also referred to as an equivalence granule. The granulation structure induced by an equivalence relation is a partition of the universe. An arbitrary set $X \subseteq U$ may not necessarily be a union of some equivalence classes. This implies that one may not be able to describe X precisely using the equivalence classes of E. In this case, one may characterize X by a pair of lower and upper approximations:

$$\underline{apr}(X) = \bigcup_{[x]_E \subseteq X} [x]_E,$$

$$\overline{apr}(X) = \bigcup_{[x]_E \cap X \neq \phi} [x]_E,$$

where $[x]_E = \{y | xEy\}$, is the equivalence class containing x. The lower approximation $\underline{apr}(X)$ is the union of all the equivalence granules which are subsets of X. The upper approximation $\overline{apr}(X)$ is the union of all the equivalence granules which have a nonempty intersection with X.

Equivalence classes of the partition U/E are called the elementary granules. They represent the available information. All knowledge we have about the universe are about these elementary granules, instead of about individual elements. With this interpretation, we also have knowledge about the union of some elementary granules. The empty set ϕ and the union of one or more elementary sets are usually called definable, observable, measurable, or composed sets. In this study, we call them granules. The set of all granules is denoted $GK(U)$, which is a subset of the power set 2^U. By extending equivalence class of x as given above to a subset $X \subseteq U$, we have:

$$[X]_E = \bigcup_{x \in X} [x]_E.$$

Thus, each element of $GK(U)$ may be viewed as the equivalence granule containing a subset of U, and the set $GK(U)$ is defined by:

$$GK(U) = \{[X]_E | X \subseteq U\}.$$

The set of granules $GK(U)$ is closed under both set intersection and union.

For an element $G \in GK(U)$, we have:

$$\underline{apr}(G) = G = \overline{apr}(G).$$

For an arbitrary subset $X \subseteq U$, we have the following equivalent definition of rough set approximations:

$$\underline{apr}(X) = \bigcup \{G | G \subseteq X, G \in GK(U)\},$$

$$\overline{apr}(X) = \bigcap \{G | X \subseteq G, G \in GK(U)\}.$$

This definition offers another interesting interpretation. The lower approximation is the largest granule contained in X, where the upper approximation is the smallest granule containing X. They therefore represent the best approximation of X from below and above using granules.

2.2. Generalized rough sets: granulation by coverings

Granulation of the universe by family of disjoint subsets is a simple and easy to analyze case. One may consider general cases by extending partitions to coverings of the universe, or by extending equivalence relations to arbitrary binary relations. In this section, we use the covering induced by a reflexive binary relation. Let $R \subseteq U \times U$ be a binary relation on U. For two elements x, y in U, if xRy, we say that y is R related to x. A binary relation may be more conveniently represented using right neighborhoods:

$$xR = \{y \in U | xRy\}.$$

But we will use a minimal neighborhood of a point x [11] in the form:

$$\langle x \rangle R = \bigcap_{x \in yR} yR.$$

When R is an equivalence relation, $\langle x \rangle R$ is the equivalence class containing x. When R is a reflexive relation, the family of the neighborhoods $U/R = \{\langle x \rangle R | x \in U\}$ is a covering of U, namely, $\cup_{x \in U} \langle x \rangle R = U$. The binary relation R represents the similarity between elements of a universe. It is reasonable to assume that similarity is at least reflexive, but not necessarily symmetric and transitive [12]. For the granulation induced by the covering U/R, rough set approximations can be defined by generalizing Pawlak definitions. The equivalence class $[x]_E$ may be replaced by the minimal neighborhood of $x \langle x \rangle R$ as the following:

$$\underline{apr}(X) = \bigcup_{\langle x \rangle R \subseteq X} \langle x \rangle R,$$

$$\begin{aligned}\overline{apr}(X) &= [\underline{apr}(X^c)]^c, = \{x \in U | \exists y [x \in \langle y \rangle R, \langle y \rangle R \subseteq X^c]\}^c \\ &= \{x \in U | \forall y [x \in \langle y \rangle R \Rightarrow \langle y \rangle R \nsubseteq X^c]\} \\ &= \{x \in U | \forall y [x \in \langle y \rangle R \Rightarrow \langle y \rangle R \cap X \neq \phi]\}.\end{aligned}$$

In this definition, we generalize the lower approximation and define the upper approximation through duality, where X^c denotes the complementation of X in U. In general, $\overline{apr}(X)$ is different from the straightforward generalization $\overline{apr}(X) = \bigcup_{\langle x \rangle R \cap X \neq \phi} \langle x \rangle R$. While the lower approximation is the union of some new successor neighborhoods, the upper approximation cannot be expressed in this way. Similar to the case of partition, we call the elements of a covering elementary granules. The empty set ϕ or the union of some elementary granules is referred to as a granule. For a subset $X \subseteq U$, we define:

$$\langle X \rangle R = \bigcup_{x \in X} \langle x \rangle R,$$

which is the neighborhood of X. The set of all such neighborhoods is given by:

$$GK(U) = \{\langle X \rangle R | X \subseteq U\}.$$

Proposition 1. *The set $GK(U)$ is closed under both set intersection and union.*

Proof. Let G_1, G_2 in $GK(U)$ we want to show that $G_1 \cap G_2$, $G_1 \cup G_2$ in $GK(U)$. Firstly, we have $G_1 = \langle X_1 \rangle R$ and $G_2 = \langle X_2 \rangle R$, if $x \in G_1 \cap G_2$ then, $x \in \langle X_1 \rangle R$ and $x \in \langle X_2 \rangle R$, hence $\langle x \rangle R \subseteq \langle X_1 \rangle R$ and $\langle x \rangle R \subseteq \langle X_2 \rangle R$ for all $x \in G_1 \cap G_2$, i.e., $\langle x \rangle R \subseteq (\langle X_1 \rangle R \cap \langle X_2 \rangle R)$ for all $x \in G_1 \cap G_2$, thus $G_1 \cap G_2 \in GK(U)$. Secondly, if $x \in G_1 \cup G_2$ then, $x \in \langle X_1 \rangle R$ or $x \in \langle X_2 \rangle R$, hence $\langle x \rangle R \subseteq \langle X_1 \rangle R$ or $\langle x \rangle R \subseteq \langle X_2 \rangle R$ for all $x \in G_1 \cup G_2$, i.e., $\langle x \rangle R \subseteq (\langle X_1 \rangle R \cup \langle X_2 \rangle R)$ for all $x \in G_1 \cup G_2$, thus $G_1 \cup G_2 \in GK(U)$. $\square$

The complemented system:

$$GK^c(U) = \{G^c | G \in GK(U)\}$$

is also closed under both set intersection and union. In fact, $GK^c(U)$ is a closure system. For an element $G \in GK(U)$, i.e., $G^c \in GK^c(U)$, we have:

$$\underline{apr}(G) = G,$$
$$\overline{apr}(G^c) = G^c.$$

In general, $G = \underline{apr}(G) \neq \overline{apr}(G)$ and $\underline{apr}(G^c) \neq \overline{apr}(G^c) = G^c$ for an arbitrary $G \in GK(U)$. By these properties, we refer to the elements of $GK(U)$ as inner definable granules, and the elements of $GK^c(U)$ as outer definable granules. Using these granules, we have another equivalent definition:

$$\underline{apr}(X) = \bigcup\{G | G \subseteq X, G \in GK(U)\},$$
$$\overline{apr}(X) = \bigcap\{G | X \subseteq G, G \in GK^c(U)\}.$$

The lower approximation is the largest inner definable granule contained in X, and the upper approximation is the smallest outer definable granules containing X. They are related to the definition for the case of partitions, in which $GK(U)$ and $GK^c(U)$ are the same set. For a covering, the set $GK(U) \cap GK^c(U)$ consists of both inner and outer definable granules. Obviously, ϕ, $U \in GK(U) \cap GK^c(U)$.

Proposition 2. *Let R be a reflexive binary relation, then the lower and the upper approximations,*

$$\underline{apr}(X) = \bigcup\{G | G \subseteq X, G \in GK(U)\},$$
$$\overline{apr}(X) = \bigcap\{G | X \subseteq G, G \in GK^c(U)\}.$$

Satisfy the following condition:

$L1$. $\underline{apr}(X) = [\overline{apr}(X^c)]^c$.
$L2$. $\underline{apr}(U) = U$.
$L3$. $\underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y)$.
$L4$. $\underline{apr}(X \cup Y)$ $\underline{apr}(X) \cup \underline{apr}(Y)$.
$L5$. $X \subset Y \Rightarrow \underline{apr}(X) \subset \underline{apr}(Y)$.
$L6$. $\underline{apr}(\phi) = \phi$.
$L7$. $\underline{apr}(X) \subset X$.
$L9$. $\underline{apr}(X) \subset \underline{apr}$ $(\underline{apr}(X))$.
$U1$. $\overline{apr}(X) = [\underline{apr}(X^c)]^c$.
$U2$. $\overline{apr}(\phi) = \phi$.
$U3$. $\overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y)$.
$U4$. $\overline{apr}(X \cap Y) \subset \overline{apr}(X) \cap \overline{apr}(Y)$.
$U5$. $X \subset Y \Rightarrow \overline{apr}(X) \subset \overline{apr}(Y)$.
$U6$. $\overline{apr}(U) = U$.
$U7$. $X \subset \overline{apr}(X)$.
$U9$. $\overline{apr}(\overline{apr}(X)) \subset \overline{apr}(X)$.

Proof. We give only the proves of ($L1$–$L7$) and ($L9$).

(L1)

$$\begin{aligned}[\overline{apr}(X^c)]^c &= \left[\bigcap\{G | X^c \subseteq G, G \in GK^c(U)\}\right]^c \\ &= \bigcup\{G | X^c \subseteq G, G \in GK^c(U)\}^c \\ &= \bigcap\{G | X^c \subseteq G^c, G \in GK(U)\} \\ &= \bigcap\{G | G \subseteq X, G \in GK(U)\} = \underline{apr}(X).\end{aligned}$$

(L2) Since $\underline{apr}(U) \subseteq U$, we want to show that $U \subseteq \underline{apr}(X)$. Let $x \in U$, since $U \in GK(U)$ and $U \in U$, then $x \in \underline{apr}(X)$, i.e., $U \subseteq \underline{apr}(X)$.

(L3)

$$\begin{aligned}\underline{apr}(X \cap Y) &= \bigcup\{G | G \subseteq X \cap Y, G \in GK(U)\} \\ &= \bigcup\{G | G \subseteq X \text{ and } G \subseteq Y, G \in GK(U)\} \\ &= \left(\bigcup\{G | G \subseteq X, G \in GK(U)\}\right) \\ &\quad \cap \left(\bigcup\{G | G \subseteq Y, G \in GK(U)\}\right) \\ &= \underline{apr}(X) \cap \underline{apr}(Y).\end{aligned}$$

(L4) Suppose $x \notin \underline{apr}(X \cup Y)$, there is no $G \in GK(U)$ and $x \in G$ such that $G \subseteq X \cup Y$. So there is no $G \in GK(U)$ and $x \in G$ such that $G \subseteq X$ and $G \subseteq Y$, hence $x \notin \underline{apr}(X) \cup \underline{apr}(Y)$. Thus we have $\underline{apr}(X) \cup \underline{apr}(Y) \subseteq \underline{apr}(X \cup Y)$.

(L5) Assume that $X \subseteq Y$. If $x \in \underline{apr}(X)$, then there is $G \in GK(U)$ and $x \in G$ such that $G \subseteq X$. But $X \subseteq Y$, thus $G \subseteq Y$ and so $x \in \underline{apr}(Y)$, i.e., $\underline{apr}(X) \subseteq \underline{apr}(Y)$.

(L6) Since $\phi \subseteq \underline{apr}(\phi)$ we want to show that $\underline{apr}(\phi) \subseteq \phi$. Let $x \in \underline{apr}(\phi)$, then there is $G \in GK(U)$ and $x \in G$ such that $G \subseteq \phi$, which is a contradiction, i.e., $\underline{apr}(\phi) = \phi$.

(L7) Let $x \in \underline{apr}(X)$, then there is $G \in GK(U)$ and $x \in G$ such that $G \subseteq X$, hence $x \in X$, thus, $\underline{apr}(X) \subseteq X$.

(L9) Since $\underline{apr}(X) = \cup\{G | G \subseteq X, G \in GK(U)\}$, then from Proposition 1 we have $\underline{apr}(X) \in GK(U)$ and for every $G \subseteq X$ we get $G \subseteq \underline{apr}(X)$, thus, $\underline{apr}(X) = \cup\{G | G \subseteq \underline{apr}(X), G \in GK(U)\} = \underline{apr}(\underline{apr}(X))$. $\square$

Proposition 3. *Let R be a reflexive binary relation, then the following properties do not hold generally:*

(L8) $X \subset \underline{apr}(\overline{apr}(X))$.
(L10) $\overline{apr}(X) \subset \underline{apr}(\overline{apr}(X))$.
(U8) $\overline{apr}(\underline{apr}(X)) \subset X$.
(U10) $\overline{apr}(\underline{apr}(X)) \subset \underline{apr}(X)$.

Example 1 (*Examples of the above properties do not hold*). Let $R = \{(a, a), (a, b), (b, b), (c, b), (c, c), (d, e), (d, d), (e, e), (e, c)\}$ be any reflexive binary relation on a nonempty set $U = \{a, b, c, d, e\}$. Then, $\langle a\rangle R = \{a, b\}$, $\langle b\rangle R = \{b\}$, $\langle c\rangle R = \{c\}$, $\langle d\rangle R = \{d, e\}$ and $\langle e\rangle R = \{e\}$. Thus, $GK(U) = \{\phi, U, \{b\}, \{c\}, \{e\}, \{a, b\}, \{d, e\}, \{b, c\}, \{b, e\}, \{c, e\}, \{a, b, c\}, \{a, b, e\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{b, c, d, e\}\}$ and $GK^c(U) = \{\phi, U, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{d, e\}, \{a, b, d\}, \{a, c, d\}, \{a, d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, d, e\}, \{a, c, d, e\}\}$.

(L8) Let $X = \{b, c, d\}$, we have $\overline{apr}(X) = \{a, b, c, d\}$ and $\underline{apr}(\overline{apr}(X)) = \{a, b, c\}$, so, $X \not\subseteq \underline{apr}(\overline{apr}(X))$.
(L10) For $X = \{b, c, d\}$, $\overline{apr}(X) = \{a, b, c, d\}$. But $\underline{apr}(\overline{apr}(X)) = \{a, b, c\}$, so, $\overline{apr}(X) \not\subseteq \underline{apr}(\overline{apr}(X))$.
(U8) Let $X = \{b, c, d\}$, we get $\underline{apr}(X) = \{b, c\}$ and $\overline{apr}(\underline{apr}(X)) = \{a, b, c\}$, so, $\overline{apr}(\underline{apr}(X)) \not\subseteq X$.
(U10) For $X = \{b, c, d\}$, $\underline{apr}(X) = \{b, c\}$. But $\overline{apr}(\underline{apr}(X)) = \{a, b, c\}$, so, $\overline{apr}(\underline{apr}(X)) \not\subseteq \underline{apr}(X)$.

Definition 1 13. Let (U, τ) be a topological space, a closure (resp. interior) operator cl: $U \to 2^U$ (resp. int: $U \to \tau$) satisfy the Kuratowski axioms iff for every $X, Y \in U$ the following hold:

(1) $cl(\phi) = \phi$ (resp. $int(U) = U$),
(2) $cl(X \cup Y) = cl(X) \cup cl(Y)$ (resp. $int(X \cap Y) = int(X) \cap int(Y)$),
(3) $X \subseteq cl(X)$ (resp. $int(X) \subseteq X$),
(4) $cl(cl(X)) = cl(X)$ (resp. $int(X) = int(int(X))$).

Theorem 1. *The pair of new lower and upper approximations are a pair of interior and closure operators satisfying Kuratowski axioms.*

Proof. The proof follows from Definition 1 and Propositions 2. □

3. Granulations and neighborhood systems

In the theory of rough sets, single-layered granulation structures of the universe are used. The granulated view of the universe is based on a binary relation representing the simplest type of relationships between elements of a universe. Two elements are either related or unrelated. The notion of neighborhood systems is used to derive more general granulation structures on the universe. Two granulation structures are defined from a neighborhood system. One is a single covering of the universe, and the other is a layered family of coverings of the universe.

The concept of neighborhood systems was originally introduced by Sierpenski and Krieger [13] for the study of Féchet (V) spaces. Lin [14,4] adopted it for describing relationships between objects in database systems. Yao [5] used the notion for granular computing by focusing on the granulation structures induced by neighborhood systems.

For an element x of a finite universe U, one associates with it a subset $n(x) \subseteq U$ called the neighborhood of x. Intuitively speaking, elements in a neighborhood of an element are somewhat indiscernible or at least not noticeably distinguishable from x. A neighborhood of x may or may not contain x. A neighborhood of x containing x is called a reflexive neighborhood. We are only interested in reflexive neighborhoods of x to accommodate the intuitive interpretation of neighborhoods. A neighborhood system $NS(x)$ of x is a nonempty family of neighborhoods of x. Distinct neighborhoods of x consist of elements having different types of, or various degrees of, similarity to x. A neighborhood system is reflexive, if every neighborhood in it is reflexive. Let $NS(U)$ denote the collection of neighborhood systems for all elements in U. It determines a Féchet (V) space, written $(U, NS(U))$. There is no additional requirements on neighborhood systems.

Neighborhood systems can be used to describe more general types of relationships between elements of a universe [5,16]. A binary relation can be interpreted in terms of 1-neighborhood systems, in which each neighborhood system contains only one neighborhood. More precisely, the neighborhood system of x is given by

$$NS(x) = \langle x\rangle R.$$

If R is a reflexive relation, one obtains a reflexive neighborhood system which is the covering U/R. If R is an equivalence relation, the neighborhood $\langle x\rangle R$ is the equivalence class containing x, and the neighborhood system is the partition U/R. With the introduction of multi-neighborhood, we consider various granulations and the corresponding approximations.

A simple method for defining approximations is to construct a covering of the universe by using all neighborhoods in every reflexive neighborhood system:

$$C_0 = \bigcup_{x \in U} NS(x) = \{n(x) | n(x) \in NS(x), x \in U\}.$$

Each granule in C_0 is a neighborhood of an element of U. The approximations are defined by:

$$\underline{apr}_{C_0}(X) = \bigcup_{n(x) \subseteq X} n(x),$$
$$\overline{apr}_{C_0}(X) = (\underline{apr}_{c_0}(X^c))^c.$$

A disadvantage of this formulation is that it uses a single-layered granulation structure, and does not make full use of the information provided by neighborhood systems. In a neighborhood system, different neighborhoods represent different types or degrees of similarity. Such information should be taken into consideration in the approximation. From a neighborhood system of the universe, we may construct a family of coverings of the universe. Instead of using all neighborhoods, each covering is obtained by selecting one particular neighborhood of each element, i.e.,

$$C = \{n(x), \ldots, n(y), n(z)\},$$

where $n(x) \in NS(x), \ldots, n(y) \in NS(y), n(z) \in NS(z)$ for $x, \ldots, y, z \in U$. In this way, we transform a neighborhood system into a family of 1-neighborhood systems $FC(U)$. An order relation $\preceq$ on $FC(U)$ can be defined as follows, for $C_1, C_2 \in FC(U)$,

$C_1 \preceq C_2 \Longleftrightarrow n_{C_1}(x) \subseteq n_{C_2}(x), \quad \text{for all x } \in U.$

The covering C_1 is finer than C_2, or C_2 is coarser than C_1. For each granule in C_2, one can find a granule in C_1 which is at least as small as the former. It can be verified that $\preceq$ is reflexive, transitive, and antisymmetric. In other words, $\preceq$ is a partial order, and the set $FC(U)$ is a poset. Thus, we have obtained a family of multi-layered coverings, which in turn produces multi-layered granulations of the universe.

For each covering $C \in FC(U)$, we can define a pair of lower and upper approximations:

$$\underline{apr}_C(X) = \bigcup_{G \in C, G \subseteq X} G,$$
$$\overline{apr}_C(X) = (\underline{apr}_C(X^c))^c.$$

With the poset $FC(U)$, we obtain multi-layered approximations. Approximations in various layers satisfy the property: $C_1 \preceq C_2 \Rightarrow$

$$\underline{apr}_{C_2}(X) \subseteq \underline{apr}_{C_1}(X),$$
$$\overline{apr}_{C_1}(X) \subseteq \overline{apr}_{C_2}(X).$$

A finer covering C_1 produces a better approximation than a coarser covering C_2. In the above formulation, we have transformed general reflexive neighborhood systems into a family of reflexive 1-neighborhood systems. This enables us to apply the results about approximations from the theory of rough sets. Our formulation is indeed based on two basic granulation structures, i.e., partitions and coverings of the universe. They are interpreted by using equivalence and reflexive relations. Consequently, two types of approximations are examined.

The use of nested sequences of binary relation has also been discussed by many authors. Marek and Rasiowa [15] considered gradual approximations of sets based on a descending sequence of equivalence relations. Pomykala [16] used a sequence of tolerance relations (i.e., reflexive and symmetric relations). Some recent results on this topic were given by Yao and Lin [17]. The results reported in this paper are more general.

4. Granulations and topological space

In this section, we introduce the connections between granulations and approximations from the topological point of view.

Zhu in [18] defined a new type of covering-based rough sets from a topological concept called neighborhood. The authors in [11] introduced a new definition for binary relation-based rough sets.

But if we consider the finite intersections of right neighborhoods as granule, the set of granules form a classical topology (in other words, right neighborhood is a sub-base). So, we present a new type of covering-based granulation from view of points of topological space.

Let us consider the pair (U, B), where $B = \{R1, R2, \ldots, Rn\}$ is a family of general binary relations on the universe U. When B is a family of equivalence relations, Pawlak call it knowledge base and Lin call the general case binary knowledge base in [4]. As the term "knowledge base" often means something else, Lin begin to use the generic name granular structure [5,6]. We will use knowledge structure and granular structure interchangeably.

Next, we will consider the topological space for each binary relation; we will call it the topological space of the binary relation. We denote the base $\beta_R = \{\langle x \rangle R \colon x \in U\}$ that is generated by the binary relation R. In this case, one may characterize X by a pair of lower and upper approximations:

$$\underline{apr}(X) = \bigcup_{\langle x \rangle_R \subseteq X} \langle x \rangle_R,$$
$$\overline{apr}(X) = [\underline{apr}(X^c)]^c = \left(\bigcup\{\langle x \rangle R | \langle x \rangle R \subseteq X^c\}\right)^c$$
$$= \bigcap\{\langle x \rangle R | \langle x \rangle R \subseteq X^c\}^c$$
$$= \bigcap\{U - \langle x \rangle R | X \subseteq U - \langle x \rangle R\}.$$

where $\langle x \rangle R$ is an element of the base β_R of the topology τ_R, which generated by the binary relation R. Obviously, if R is an equivalence relation, $\langle x \rangle R = [x]_R$ and these definitions are equivalent to the original Pawlak's definitions.

Lemma 1. *For any binary relation R on U if $x \in \langle y \rangle R$, then $\langle x \rangle R \subseteq \langle y \rangle R$.*

Proof. Let $z \in \langle x \rangle R = \cap_{x \in wR}(wR)$. Then z is contained in any wR which contains x, and since also x is contained in any uR which contains y, then z is contained in any uR which contains y, i.e., $z \in \langle y \rangle R$. Then $\langle x \rangle R \subseteq \langle y \rangle R$. □

Proposition 4. *Let R be a reflexive binary relation, then the lower and the upper approximations,*

$$\underline{apr}(X) = \bigcup_{\langle x \rangle_R \subseteq X} \langle x \rangle_R,$$
$$\overline{apr}(X) = \bigcap\{U - \langle x \rangle R | X \subseteq U - \langle x \rangle R\}.$$

Satisfy the following condition:

(L1) $\underline{apr}(X) = [\overline{apr}(X^c)]^c$.
(L2) $\underline{apr}(U) = U$.
(L3) $\underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y)$.
(L4) $\underline{apr}(X \cup Y)\ \underline{apr}(X) \cup \underline{apr}(Y)$.
(L5) $X \subset Y \Rightarrow \underline{apr}(X) \subset \underline{apr}(Y)$.
(L6) $\underline{apr}(\phi) = \phi$.
(L7) $\underline{apr}(X) \subset X$.
(L9) $\underline{apr}(X) \subset \underline{apr}\ (\underline{apr}(X))$.
(U1) $\overline{apr}(X) = [\underline{apr}(X^c)]^c$.
(U2) $\overline{apr}(\phi) = \phi$.
(U3) $\overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y)$.
(U4) $\overline{apr}(X \cap Y) \subset \overline{apr}(X) \cap \overline{apr}(Y)$.
(U5) $X \subset Y \Rightarrow \overline{apr}(X) \subset \overline{apr}(Y)$.
(U6) $\overline{apr}(U) = U$.
(U7) $X \subset \overline{apr}(X)$.
(U9) $\overline{apr}(\overline{apr}(X)) \subset \overline{apr}(X)$.

Proof. The proof is similar to that of Proposition 2. □

Proposition 5. *Let R be a reflexive binary relation, then the following properties do not hold generally:*

(L8) $X \subset \underline{apr}(\overline{apr}(X))$.
(L10) $\overline{apr}(X) \subset \underline{apr}(\overline{apr}(X))$.
(U8) $\overline{apr}(\underline{apr}(X)) \subset X$.
(U10) $\overline{apr}(\underline{apr}(X)) \subset \underline{apr}(X)$.

Example 1 (*Examples of the above properties do not hold*). Let $U = \{a, b, c, d, e\}$ and R be a reflexive binary relations on U, where $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, c), (c, d), (d, d), (d, e), (e, e)\}$, then $\langle a\rangle R = \langle b\rangle R = \{a, b, c\}$, $\langle c\rangle R = \{c\}$, $\langle d\rangle R = \{d\}$ and $\langle e\rangle R = \{e\}$. So, $\beta_R = \{\{a, b, c\}, \{c\}, \{d\}, \{e\}\}$, hence the topology $\tau_R = \{U, \phi, \{c\}, \{d\}, \{e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}\}$.

(L8) Let $X = \{a, e\}$, we have $\overline{apr}(X) = \{a, b, e\}$ and $\underline{apr}(\overline{apr}(X)) = \{e\}$, so, $X \not\subseteq \underline{apr}(\overline{apr}(X))$.
(L10) For $X = \{a, e\}$, $\overline{apr}(X) = \{a, b, e\}$. But $\underline{apr}(\overline{apr}(X)) = \{e\}$, so, $\overline{apr}(X) \not\subseteq \underline{apr}(\overline{apr}(X))$.
(U8) Let $X = \{b, c, d\}$, we get $\underline{apr}(X) = \{c, d\}$ and $\overline{apr}(\underline{apr}(X)) = \{a, b, c, d\}$, so, $\overline{apr}(\underline{apr}(X)) \not\subseteq X$.
(U10) For $X = \{b, c, d\}$, $\underline{apr}(X) = \{c, d\}$. But $\overline{apr}(\underline{apr}(X)) = \{a, b, c, d\}$, so, $\overline{apr}(\underline{apr}(X)) \not\subseteq \underline{apr}(X)$.

Theorem 2. *Suppose R is a reflexive binary relation on a finite set U. Then, the pair of lower and upper approximations is a pair of interior and closure operators satisfying Kuratowski axioms.*

Proof. The proof follows from Definition 1 and Propositions 4. □

The complemented system: $\tau_R^c = \{G^c | G \in \tau_R\}$ is a closure system. For an element $G \in \tau_R$, i.e., $G^c \in \tau_R^c$, we have: $\underline{apr}(G) = G$, $\overline{apr}(G^c) = G^c$. In general, $G = \underline{apr}(G) \neq \overline{apr}(G)$ and $\underline{apr}(G^c) \neq \overline{apr}(G^c) = G^c$ for an arbitrary $G \in GK(U)$. By these properties, we refer to the elements of τ_R as inner definable granules, and the elements of τ_R^c as outer definable granules. Also, the lower approximation is the largest inner definable granule contained in X, and the upper approximation is the smallest outer definable granules containing X. Every subset of the universe is approximated from below by inner definable granules, and from above by outer definable granules.

They are related to the definition for the case of partitions, in which τ_R and τ_R^c are the same set, or in which R is an equivalent relation and in this case τ_R is called a quasi-discrete topology. For a covering, the set $\tau_R \cap \tau_R^c$ consists of both inner and outer definable granules. Obviously, $\phi, U \in \tau_R \cap \tau_R^c$.

5. Conclusion

The use of granulation in problem solving can be described as on old and new method in the same time. It is used on great wide range from simple problem of clothes classification to choose a uniform for specific purpose to international problem such as taking a political decision according to a classification for countries in view of their decision. The work presented in this paper utilized the granulation generated by general binary relation to get a specific type of right neighborhood and applying it to get more accurate approximations, this approach is more general than that of Pawlak and Yao. Granulation structures and the corresponding approximation structures introduced in this paper provide a starting point for further study of granulation and approximation. Investigations in this direction may produce interesting and useful results.

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