Abstract

We use Omnès representations of the form factors $f_+$ and $f_0$ for exclusive semileptonic $B \to \pi$ decays, paying special attention to the treatment of the $B^*$ pole and its effect on $f_+$. We apply them to combine experimental partial branching fraction information with theoretical calculations of both form factors to extract $|V_{ub}|$. The precision we achieve is competitive with the inclusive determination and we do not find a significant discrepancy between our result, $|V_{ub}| = (3.90 \pm 0.32 \pm 0.18) \times 10^{-3}$, and the inclusive world average value, $(4.45 \pm 0.20 \pm 0.26) \times 10^{-3}$ [Heavy Flavor Averaging Group (HFAG), hep-ex/0603003].

1. Introduction

The magnitude of the Cabibbo–Kobayashi–Maskawa matrix element $V_{ub}$ can be determined from both inclusive and exclusive semileptonic $B$ meson decays. There has been a recent dramatic improvement in the quality of the experimental data for the exclusive decays [2–6], coupled with the appearance of the first dynamical lattice QCD and improved lightcone sum-rule calculations of the relevant form factors [7–12]. Dispersive approaches were combined with lattice results in [13] and with leading order heavy meson chiral perturbation theory and perturbative QCD inputs in [14]. The appearance of the first partial branching fraction measurements for $B \to \pi l \nu$ [2] made it possible [15] to combine dispersive constraints with experimental differential decay rate information and theoretical calculations of both form factors in limited regions of $q^2$ in order to improve the determination of $|V_{ub}|$. In [16] it was shown that the quality of the inputs now makes it possible for the exclusive determination to compete in precision with the inclusive one. Thus the compatibility of the two determinations becomes an interesting issue.

To perform the exclusive $|V_{ub}|$ extraction one needs a model-independent parametrisation of the form factors. In [16] a parametrisation inspired by dispersive bounds calculations was used. An alternative simple parametrisation using a multiply-subtracted Omnès representation for $f_+$, based on unitarity and analyticity properties, was employed in [18]. A shortcoming in the treatment of the $B^*$ was pointed out in [19]. In this Letter we have addressed this by improving the treatment of the $B^*$ within the Omnès framework. We have also incorporated the scalar form factor $f_0$ in a simultaneous analysis and examined the possible effects of correlations among lattice inputs. Finally, we have taken advantage of new experimental data from the BaBar 12-bin untagged analysis [6].

The outcome is that the precision achieved for $|V_{ub}|$ is indeed competitive with the inclusive determination and that we do not find a significant discrepancy between our result, $|V_{ub}| = (3.90 \pm 0.32 \pm 0.18) \times 10^{-3}$, and the inclusive world average value, $(4.45 \pm 0.20 \pm 0.26) \times 10^{-3}$ [1].

2. Omnès parametrisations

In our previous work [18,20] with the Omnès parametrisation [21,22] for the form factor $f_+(q^2)$, we treated the $B^*$ as a bound state and took the $B\pi$ elastic scattering phase shift to be $\pi$ at threshold, $s_{th} = (m_B + m_\pi)^2$. By using multiple sub-
tractions and approximating the phase shift by \( \pi \) from \( s_{th} \) to infinity, this led to a parametrisation:

\[
f_+(q^2) = \frac{1}{s_{th} - q^2} \prod_{i=0}^{n} \left[ f_+(s_i)(s_{th} - s_i) \right]^{\alpha_i(q^2)},
\]

(1)

with \( n + 1 \) subtractions at \( q^2 \in \{ s_0, s_1, \ldots, s_n \} \), below threshold (the \( q_i(q^2) \) are defined in Eq. (6) below). This parametrisation requires as input only the form factor values \( \{ f_+(q^2_i) \} \) at \( n + 1 \) positions \( q^2 \).

Using this parametrisation in a combined fit to experimental data and theoretical form-factor calculations (lattice QCD and lightcone sumrules) allows an extraction of data and theoretical form-factor calculations (lattice QCD and dashed lines show the Omnès fit of Eq. (1). The left-hand plot uses a ZE fit with three parameters and thus can be compared to Fig. 2 in [19], while the right-hand plot uses a four-parameter ZE fit. All Omnès fits use four parameters (four subtraction points). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this Letter.)

The ZE fit has been performed truncating the power series in Eq. (2) at \( n = 2 \), for comparison with Fig. 2 in [19], or \( n = 3 \), so that all fits have the same number of parameters. The green dashed lines show the Omnès fit using Eq. (1). The plot shows that once fluctuations are taken into account the differences are not significant.

Nevertheless, we show here that by treating the \( B^* \) explicitly as a pole of the form factor, we can understand and reduce the small deviation in the central fits at large \( q^2 \). This is illustrated by the solid blue lines in Fig. 1. We achieve this without altering the main results obtained for \( |V_{ub}| \) and \( f_+ \) in the \( q^2 \) region where theoretically calculated values lie. The new parametrisation, shown below in Eq. (7), is obtained from Eq. (1) by replacing \( s_{th} \) with \( m^2_{B^*} \). As before, the parametrisation relies only on very general properties of analyticity and unitarity and so, although simple, is well founded.

To obtain the new parametrisation we observe that, if \( f_+(q^2) \) has a pole at \( q^2 = m^2_{B^*} \), then \( \mathcal{F}(q^2) \equiv (m^2_{B^*} - q^2) f_+(q^2) \) has no poles and satisfies

\[
\frac{\mathcal{F}(s + i\epsilon)}{\mathcal{F}(s - i\epsilon)} = \exp(2i\delta_{1/2,1}(s)), \quad s \geq s_{th},
\]

(3)

where \( \delta_{IJ} \) is the phase-shift for elastic \( \pi B \to \pi B \) scattering in the isospin \( I \) and total angular momentum \( J \) channel. This is because \( f_+ \) satisfies a similar equation as required by Watson’s theorem [24] and we have multiplied it by a real function. An \((n + 1)\)-subtracted Omnès representation can now be written for \( \mathcal{F}(q^2) \), with \( q^2 < s_{th} \), which reads:

\[
\mathcal{F}(q^2) = \left( \prod_{j=0}^{n} (\mathcal{F}(s_j))^{\beta_j(q^2)} \right) \times \exp \left\{ I_{\delta}(q^2, s_0, \ldots, s_n) \prod_{j=0}^{n} (q^2 - s_j) \right\},
\]

(4)

\[
I_{\delta}(q^2, s_0, \ldots, s_n) = \frac{1}{\pi} \int_{s_{th}}^{+\infty} ds \frac{\delta_{1/2,1}(s)}{(s - s_0) \cdots (s - s_n)} \frac{1}{s - q^2},
\]

(5)
\[
\alpha_i(s) = \frac{1}{m_B^2 - q^2} \prod_{i=0}^{n} \left[ (f_+(s_i)(m_B^2 - s_i) \right]^{\alpha_i(q^2)}.
\]

This representation requires as input the phase shift \( \delta_{1/2,1}(s) \) plus the values \( \{F(s_i)\} \) at \( n + 1 \) positions \( s_i \) below the \( \pi B \) threshold. For sufficiently many subtractions, we can approximate \( \delta_{1/2,1}(s) \) by zero above threshold (see Appendix A). In this case we obtain,

\[
f_+(q^2) = \frac{1}{m_B^2 - q^2} \prod_{i=0}^{n} (f_+(s_i)(m_B^2 - s_i))^{\alpha_i(q^2)}.
\]

This amounts to finding an interpolating polynomial for \( \ln F(q^2) = \ln[(m_B^2 - q^2)f_+(q^2)] \) passing through the points \( F(s_i) = (m_B^2 - s_i)f_+(s_i) \). Similarly, our earlier parametrisation in Eq. (1) used an interpolating polynomial for \( \ln[q(q^2)f_0(q^2)] \) for a suitable function \( g(q^2) \), the derivation using the Omnès representation shows that taking \( g(q^2) = m_B^2 - q^2 \) (and equally \( s_i - q^2 \)) is physically motivated. From here onwards we will use Eq. (7) as our preferred parametrisation for \( f_+ \).

When using our parametrisation in the extraction of \( |V_{ub}| \), we make 4 subtractions. This is sufficient to justify using no information about the phase shift beyond its value at \( s_{th} \). To check this, we have put in a model for the \( B \pi \) phase shift \( \delta_B \) and confirmed that induced changes in our results are much smaller than the fluctuations produced by the errors in our inputs. This can be understood because with four evenly-spaced subtractions at \( 0, 1/3, 2/3, 1 \) \( q^2_{max} \), the factor \( \exp[I \times \prod_{i=0}^{n} (q^2 - s_i)] \) in Eq. (4) given by this model deviates from unity by no more than \( 6 \times 10^{-4} \) for \( 0 \leq q^2 \leq q^2_{max} \) (and, of course, is unity at each subtraction point).

Since \( f_+ \) and the scalar form factor \( f_0 \) satisfy the constraint \( f_+(0) = f_0(0) \) we will combine theoretical inputs for \( f_+ \) and \( f_0 \) with experimental \( B \to \pi l \nu \) partial branching fraction information to check the effect on the extracted value of \( |V_{ub}| \). We will investigate the effect of using the \( f_+ \) information alone or using both form factors.

For the scalar form factor \( f_0 \) there is no pole below threshold, so that we will use an Omnès formula like Eq. (4) for \( f_0(q^2) \), with \( F \to f_0 \) and \( \delta_{1/2,1} \to \delta_{1/2,0} \). For sufficiently many subtractions, we can approximate \( \delta_{1/2,0} \) by zero above threshold. Our recent analysis of the scalar form factor [25] for \( B \to \pi \) decays suggested the existence of a resonance with mass around 5.6 GeV. This could be incorporated in an Omnès parametrisation like that of Eq. (7), but (as we have confirmed) has negligible effect on \( |V_{ub}| \) and \( f_+ \) in our fit, producing only a small increase of around 7\% in the value of \( f_0 \) close to \( q^2_{max} \).

3. Application to \( |V_{ub}| \)

We have used experimental data for the partial branching fractions of \( B \to \pi l \nu \) decays in \( q^2 \) bins from both tagged and untagged analyses. The tagged analyses from CLEO [2], Belle [3] and BaBar [5] use three bins, while BaBar’s untagged analysis [6] uses twelve. CLEO and BaBar combine results for neutral and charged \( B \)-meson decays using isospin symmetry, while Belle quote separate values for \( B^0 \to \pi^- l^+ \nu_l \) and \( B^+ \to \pi^0 l^+ \nu_l \). For our analysis, for the three-bin data, we have combined the Belle charged and neutral \( B \)-meson results and subsequently combined these with the CLEO and BaBar results. The resulting input values can be found in Table 2 of [18].

Since the systematic errors of the three-bin data are small compared to the statistical ones, we have ignored correlations in the systematic errors and combined errors in quadrature. For the 12-bin BaBar data [6], complete correlation matrices are available in the EPAPS database [26] for both statistical and systematic errors and we have used these in our fits (we used the results corrected for final state radiation effects). We have assumed no correlation between the untagged and the tagged analyses.

When computing partial branching fractions, we have used \( \Gamma_{B^0} = 1/\Gamma_{Tot} = (1.527 \pm 0.008) \times 10^{-12} \text{ s}^{-1} \) for the \( B^0 \) lifetime.

Theoretical calculations provide information on \( f_+ \) and \( f_0 \). Since the effects of finite electron and muon masses are beyond current measurement precision, the experimental results provide information on the \( q^2 \) shape of \( f_+ \).

We use the lightcone sumrule (LCSR) result \( f_+(0) = f_0(0) = 0.258 \pm 0.031 \) [12] and lattice QCD results from dynamical simulations at larger \( q^2 \) from HPQCD [7] and FNAL-MILC [8–11]. The FNAL-MILC results [8–11] are still preliminary. Therefore we use the three \( f_+(q^2) \) values quoted in [16] and read off three values for \( f_0(q^2) \) at the same \( q^2 \) points from [9]. These are

\[
\begin{align*}
&f_0(15.87 \text{ GeV}^2) = 0.425 \pm 0.033, \\
&f_0(18.58 \text{ GeV}^2) = 0.506 \pm 0.037, \\
&f_0(24.09 \text{ GeV}^2) = 0.800 \pm 0.067.
\end{align*}
\]

The errors shown are statistical. A further 11\% systematic error should be added.

We implement the fitting procedure described in [18] using four evenly-spaced Omnès subtraction points at \( 0, 1/3, 2/3, 1 \) \( q^2_{max} \) (with \( \chi \)-squared function given in Eq. (10) of [18]), with the obvious changes to incorporate \( f_0 \). As before, we have assumed that the lattice input form factor data have independent statistical uncertainties and fully-correlated systematic errors. We have not assumed correlations between results for \( f_+ \) and \( f_0 \), though we will comment further on this below. Furthermore, we ignore possible correlations between the HPQCD and FNAL-MILC lattice inputs. These correlations are unknown and we showed in [18] that unless they are very strong they will have little effect on \( |V_{ub}| \).

The best-fit parameters are

\[
\begin{align*}
|V_{ub}| &= (3.90 \pm 0.32) \times 10^{-3}, \\
f_+(0) &= f_0(0) = 0.226 \pm 0.022, \\
f_+(q^2_{max}/3) &= 0.417 \pm 0.039, \\
f_+(2q^2_{max}/3) &= 0.941 \pm 0.064, \\
f_+(q^2_{max}) &= 7.29 \pm 1.28.
\end{align*}
\]
We have checked that the changes in the results for $f_+(0)$, $f_+(q_{max}^2/3)$ and $|V_{ub}|$ stem from using the updated BaBar untagged data.

- We have checked that the change in $f_+(q_{max}^2)$, which has little effect on the shape of the form factor in the $q^2$ range where experimental and theoretical information exists, arises from our use of the new Omnès parametrisation of Eq. (7) and reflects the existence of a pole in $f_+$ at $q^2 = m_{B^*}^2$.

- Since we do not know the correlations between the lattice input data we have also performed a fit neglecting all correlations in these inputs. We find that $|V_{ub}|$ increases by an amount $0.18 \times 10^{-3}$, which we will quote as a systematic error in our determination. We observe that knowledge of the correlations will be needed for more precise determinations of $|V_{ub}|$.

- The inclusion of $f_0$ in the analysis has no visible effect in our results for $f_+$ and $|V_{ub}|$. This is not surprising given that the number of input data affecting $f_+$ is much bigger than that affecting $f_0$ and that the parametrisation allows the data to determine each form factor independently apart from the constraint at $q^2 = 0$. The covariance matrix given in Appendix B shows this freedom, having negligible correlations between $f_+$ and $f_0$ at $q^2 \neq 0$. Correlations linking $f_+$ and $f_0$ in the lattice QCD inputs could modify the central values in (9) by an amount comparable to their errors as we have

\begin{align}
 f_0(q_{max}^2/3) &= 0.342 \pm 0.053, \\
 f_0(2q_{max}^2/3) &= 0.508 \pm 0.040, \\
 f_0(q_{max}^2) &= 1.09 \pm 0.21. \\
\end{align}

The fit has $\chi^2/dof = 0.62$ for 28 degrees of freedom, while the Gaussian correlation matrix can be found in Appendix B.

In Fig. 2 we show the fitted form factors, the differential decay rate calculated from our fit and the quantities $\log[(m_{B^*}^2 - q^2) f_+(q^2)/m_{B^*}^2]$ and $\Phi f_+$ where the details of the fit and inputs can better be seen. The dashed magenta curve in the $P\Phi f_+$ plot is a cubic polynomial fit in $z$ to the output from our analysis. We note that the sum of squares of the coefficients in this polynomial safely satisfies the dispersive constraint $\sum_n a_n^2 \leq 1$ [16].

The top left plot shows the two form factors with their error bands, the lattice and LCSR input points (dots: green LCSR, red HPQCD, blue FNAL-MILC) and ‘experimental’ points (black triangles, upward-pointing for tagged and downward pointing for untagged data) constructed by plotting at the centre of each bin the constant form factor that would reproduce the partial branching fraction in that bin. The top right plot shows the differential decay rate together with the experimental inputs. The bottom plots provide more details of the inputs and fits by showing on the left $\log[(m_{B^*}^2 - q^2) f_+(q^2)/m_{B^*}^2]$ as a function of $q^2$, and on the right $P\Phi f_+$ as a function of $-z$. The dashed magenta curve in the bottom right plot is a cubic polynomial fit in $z$ to the Omnès curve. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this Letter.)
confirmed by fully-correlating the systematic errors between them. As an example, for $|V_{ub}|$ we find a central value of $4.15 \times 10^{-3}$. Since we do not know the actual correlation information\(^2\) for the lattice data, we do not present these numbers.

- Because of the freedom allowed by the Omnès parametrisation of $f_+$ and $f_0$, one may wonder whether or not heavy quark symmetry (HQS) relations between the form factors at $q^2 = m_B^2$ are satisfied. Some earlier parametrisations were explicitly constructed to satisfy the HQS scaling relation $f_+(q_{\text{max}}^2)/f_0(q_{\text{max}}^2) \approx m_B$, for example dipole/pole forms [27–29], and these have been widely used. From our fit we calculate

$$\frac{1}{m_B} \left. \frac{f_+(q_{\text{max}}^2)}{f_0(q_{\text{max}}^2)} \right|_{B\pi} = 1.3 \pm 0.4 \text{ GeV}^{-1}$$

(10)

to be compared to the corresponding quantity in $D \rightarrow \pi$ exclusive semileptonic decays, $1.4 \pm 0.1 \text{ GeV}^{-1}$ extracted from the unquenched lattice QCD results in [30]. This agreement is reassuring but our determination of the ratio in $B \rightarrow \pi$ decays has a further uncertainty of around 10% arising from our incomplete knowledge of the correlations in the lattice inputs.

- Heavy quark effective theory in the soft-pion limit predicts [31],

$$f_0(m_B^2) = f_B/f_\pi + O(1/m_B^2) \approx 1.4(2)$$

(11)

where we have used $f_B = 189(27) \text{ MeV}$ [32]. Our fit for $f_0(q_{\text{max}}^2)$ in Eq. (9) is compatible within errors.

- Applying soft collinear effective theory (SCET) to $B \rightarrow \pi \pi$ decays allows a factorisation result to be derived which leads to a model-independent extraction of the form factor (multiplied by $|V_{ub}|$) at $q^2 = 0$ [33]. We quote the result from our fit,

$$|V_{ub}|f^+(0) = (8.8 \pm 0.8) \times 10^{-4},$$

(12)

which compares well with $|V_{ub}|f^+(0) = (7.6 \pm 1.9) \times 10^{-4}$ quoted in [17]. This also agrees with the value $|V_{ub}|f^+(0) = (9.1 \pm 0.7) \times 10^{-4}$ [19] obtained by fixing $|V_{ub}|$ from global CKM unitarity triangle fits and fitting to the BaBar 12-bin data [23].

- We noted above possible effects of correlations in the lattice data. Other sources of systematic variation in the result for $|V_{ub}|$ arising from uncertainties in the theoretical form factor inputs at or near $q^2 = 0$ were considered in [18] and shown to be safely covered by the statistical uncertainty.

4. Conclusion

We have updated our previous analysis of exclusive $B \rightarrow \pi$ semileptonic decays, based on Omnès dispersion relations. The principal change is to improve the treatment of the $B^*$ and its effect on the form factor $f_+$. We have also incorporated the scalar form factor $f_0$ in a simultaneous analysis and examined the possible effects of correlations among lattice inputs. Finally, we have taken advantage of new experimental data from the BaBar 12-bin untagged analysis [6]. We extract a value

$$|V_{ub}| = (3.90 \pm 0.32 \pm 0.18) \times 10^{-3}.$$  

(13)

The first error above is statistical arising from the chi-squared fit. The second is a systematic error to account for current partial knowledge of correlations in the lattice input data. The precision for $|V_{ub}|$ is comparable with that of the inclusive determination and we do not find a significant discrepancy between our result and the inclusive world average value, $(4.45 \pm 0.20 \pm 0.26) \times 10^{-3}$ [1].

Finally we would like to stress that the Omnès parametrisation is physically motivated and simple and provides a robust framework for a precise exclusive determination of $|V_{ub}|$.

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Appendix A. Choice of $\delta_{IJ}(s_{th})$ in the Omnès representation

In this appendix we provide more details on some aspects of the Omnès representation of the form factors. This builds on the discussion in the appendix of [34].

The scattering matrix $T$ depends on $\exp(2i\delta)$ and thus one has the freedom to add factors of $k\pi$ to the phase shift, for integer $k$, without modifying the $T$ matrix. However, the Omnès representation of the form factor certainly depends on the specific value of $k$. Indeed adding $k\pi$ to $\delta$ leads to

$$\exp\left\{I_{k\pi}\times \prod_{j=0}^{n} (q^2 - s_j)\right\} = \exp\left\{I_{\delta} \times \prod_{j=0}^{n} (q^2 - s_j)\right\} \cdot \left(\prod_{j=0}^{n} (s_{th} - s_j)^{a_j(q^2)} \right)^k \frac{k}{s_{th} - q^2}$$

(A.1)

which induces an unphysical $k$th order pole in the form factor at $s_{th}$.

Now consider $\mathcal{F}(q^2) = (m_{B^*}^2 - q^2)f_+(q^2)$, which has no poles in $0 \leq q^2 \leq s_{th}$. Its Omnès representation should not induce a pole at $s_{th}$ and therefore we should set $k = 0$ in Eq. (A.1) above. This is equivalent to setting $\delta_{1/2,1}(s_{th}) = 0$. With enough subtractions, we can then take $\delta_{1/2,1}(s) = 0$ inside the integral

\(^2\) It is reasonable to expect correlations not only in the systematic error but also in the statistical ones, since $f_+$ and $f_0$ are linear combinations of temporal and spatial components of vector current matrix elements.
because only the region close to \( s_{th} \) will be important, leading to the result presented in Eq. (7).

This choice for \( \delta_{1/2,1}(s_{th}) \) does not contradict Levinson’s theorem, which fixes only the difference

\[
\delta_{1/2,1}(\infty) - \delta_{1/2,1}(s_{th}) = \pi(n_z - n_p), \tag{A.2}
\]

where \( n_z \) (\( n_p \)) is the number of zeros (poles) of the scattering matrix \( T \) on the physical sheet. The usual convention [35] is to set \( \delta_{1/2,1}(s_{th}) = \pi n_p \) and \( \delta_{1/2,1}(\infty) = \pi n_z \). However, we use a different convention which follows from the discussion above on the effect of adding multiples of \( \pi \) to the phase shift. Our choice is \( \delta_{1/2,1}(s_{th}) = 0 \) which therefore also implies that \( \delta_{1/2,1}(\infty) = 0 \).

In our previous work [18,20,34], we assumed that \( f_+ \) had no poles. With the usual convention for Levinson’s theorem that \( \delta_{1/2,1}(s_{th}) = \pi \), we developed a pole for \( f_+ \) at \( s_{th} \) which was not discarded since it mimicked the \( B^* \) pole’s effects on the form factor because \( m_{B^*}^2 \approx s_{th} \). We already commented on this in the appendix of [34].

Appendix B. Correlation matrix

Here we give the correlation matrix of fitted parameters corresponding to the best-fit parameters in Eq. (9)

\[
\begin{pmatrix}
1 & -0.43 & -0.91 & -0.81 & -0.58 & -0.04 & 0.00 & 0.01 \\
1 & 0.20 & 0.50 & -0.04 & 0.10 & 0.00 & -0.02 & \\
1 & 0.76 & 0.61 & 0.02 & 0.00 & 0.00 & \\
1 & 0.36 & 0.05 & 0.00 & -0.01 & \\
1 & 0.00 & 0.00 & 0.00 & \\
1 & 0.32 & 0.83 & \\
1 & 0.22 & \\
1
\end{pmatrix} \tag{B.1}
\]

References