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Nonlinear supersymmetry, brane–bulk interactions and super-Higgs without gravity

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Abstract

We derive the coupling of a hypermultiplet of $N=2$ global supersymmetry to the Dirac–Born–Infeld Maxwell theory with linear $N=1$ and a second nonlinear supersymmetry. At the level of global supersymmetry, this construction corresponds to the interaction with Maxwell brane fields of bulk hypermultiplets, such as the universal dilaton of type IIB strings compactified on a Calabi–Yau manifold. It displays in particular the active role of a four-form field. Constrained $N=1$ and $N=2$ superfields and the formulation of the hypermultiplet in its single-tensor version are used to derive the nonlinear realization, allowing a fully off-shell description. Exact results with explicit symmetries and supersymmetries are then obtained. The electric–magnetic dual version of the theory is also derived and the gauge structure of the interaction is exemplified with $N=2$ nonlinear QED of a charged hypermultiplet. Its Higgs phase describes a novel super-Higgs mechanism without gravity, where the goldstino is combined with half of the hypermultiplet into an $N=1$ massive vector multiplet.

Keywords: Dirac–Born–Infeld; Supersymmetry; Duality

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1. Introduction

It is notorious that (linear) $N = 2$ supersymmetry, global or local, forbids a dependence on hypermultiplet scalars of gauge kinetic terms. For instance, in $N = 2$ supergravity, the scalar manifold is the product of a quaternion-Kähler (Einstein) manifold, for hypermultiplet scalars [1], and a Kähler manifold of a special type for vector multiplet scalars [2]. In global $N = 2$ supersymmetry, the quaternion-Kähler manifold of hypermultiplet scalars is replaced by a Ricci-flat hyperkähler space [3].

If however (at least) one of the supersymmetries is nonlinearly realized, these restrictions on the action are expected to change. For instance, string theory indicates that the Dirac–Born–Infeld (DBI) Lagrangian describing kinetic terms of brane gauge fields may interact with the dilaton and with its hypermultiplet partners. Moreover, if the dilaton supermultiplet is formulated with one or two antisymmetric tensors, more involved interactions dictated by the gauge symmetries of the theory are certainly allowed. An interesting problem is then to construct an interaction Lagrangian in which, when the second supersymmetry turns nonlinear, both the DBI Lagrangian and its necessary dilaton dependence are simultaneously generated. In other words, if we consider a theory with a broken, nonlinear supersymmetry realized in a goldstino mode, another unbroken linear supersymmetry and a DBI super-Maxwell system coupled to hypermultiplet fields, we certainly expect that the allowed Lagrangians are severely restricted. Analyzing these restrictions is the main motivation of this paper.

In this work, we construct an action invariant under $N = 2$ global supersymmetry, one of them being nonlinearly realized, involving the Maxwell goldstino multiplet of the nonlinear supersymmetry coupled to a single-tensor $N = 2$ multiplet [4–6], or equivalently to a hypermultiplet with one Abelian (shift) isometry. In the absence of this multiplet, the action reduces to the standard super-Maxwell DBI theory, derived in the past from the same symmetry principle [7–9]. The coupling of the two multiplets is shown to arise from an $N = 2$ Chern–Simons (CS) term which, under electric–magnetic duality, amounts to shifting the gauge field strength by the antisymmetric tensor. Moreover, under Poincaré duality of the antisymmetric tensor to a pseudoscalar, the CS coupling becomes a Stückelberg gauging of the pseudoscalar axionic symmetry.

An important property of the single-tensor multiplet is that it admits an off-shell (superspace) formulation, unlike the generic hypermultiplet that can be formulated off-shell only at the cost of introducing infinite number of auxiliary fields in the context of harmonic superspace [10]. Thus, our formalism using the single-tensor $N = 2$ multiplet allows to construct off-shell supersymmetric Lagrangians. By an appropriate change of variables from the $N = 2$ single-tensor multiplet, one finds an action that couples the goldstino vector multiplet (of the linear supersymmetry) to an $N = 2$ charged hypermultiplet, describing the low-energy limit of a theory with partial spontaneous supersymmetry breaking from $N = 2$ to $N = 1$ [11,12].

The vacuum of this theory exhibits an interesting novel feature: the goldstino is ‘absorbed’ into a massive vector multiplet of $N = 1$ linear supersymmetry, leaving a massless $N = 1$ chiral multiplet associated to flat directions of the scalar potential. The goldstino assembles with one of the two Weyl fermions in the single-tensor multiplet to form a massive Dirac spinor. At one particular point along the flat directions, the vector multiplet becomes massless and the $U(1)$ is restored. This phenomenon is known from D-brane dynamics, where the $U(1)$ world-volume field becomes generically massive due to the CS coupling. A crucial role for the invariance of the action under nonlinear supersymmetry is played by a non-dynamical four-form gauge potential, known again from D-brane dynamics. Hence, a globally supersymmetric combination of Higgs and super-Higgs mechanisms, in the presence of a four-form field, eliminates any massless goldstino fermion related to partial supersymmetry breaking. This interesting new mechanism can be studied in the context of nonlinear $N = 2$ quantum electrodynamics with one charged hypermultiplet, which after a holomorphic field redefinition and a duality transformation, is equivalent to our setup.

In type IIB superstrings compactified to four dimensions with eight residual supercharges, the dilaton scalar (associated to the string coupling) belongs to a universal hypermultiplet, together with the (Neveu–Schwarz) NS–NS antisymmetric tensor and the (Ramond) R–R scalar and two-form. Its natural basis is therefore a double-tensor supermultiplet,[1] having three perturbative isometries associated to the two axionic shifts of the antisymmetric tensors and an extra shift of the R–R scalar. These isometries form a Heisenberg algebra, which at the string tree-level is enhanced to the quaternion-Kähler and Kähler space $SU(2,1)/SU(2) \times U(1)$. At the level of global $N = 2$, imposing the Heisenberg algebra of isometries determines a unique hyperkähler manifold of dimension four, depending on a single parameter, in close analogy with the local case of a quaternionic space where the corresponding parameter is associated to the one-loop correction [15]. This manifold is *not* trivially flat and should describe the rigid limit of the universal hypermultiplet.

The plan of the paper is as follows. In Section 2, we review the construction of the $N = 2$ simple-tensor and Maxwell supermultiplets in terms of $N = 1$ superfields and we describe their interaction in a Chern–Simons term, as was earlier partly done in Ref. [9]. In addition we explain how the intricate web of gauge variations in the Stückelberg coupling of the Maxwell and single-tensor supermultiplets leads to the interpretation of one (non-propagating) component of the single-tensor as a four-form field. In Section 3, we reformulate the supermultiplets in chiral $N = 2$ superspace and then demonstrate how this construction can be used to describe electric–magnetic duality in a manifestly $N = 2$ covariant way. In Section 4, we first review the construction of the Dirac–Born–Infeld theory from constrained $N = 2$ superfields describing the goldstino of one nonlinear supersymmetry and then extend it to construct its coupling to a single-tensor supermultiplet, engineered by a CS term. We also perform an electromagnetic duality to determine the 'magnetic' version of the theory. With the dilaton hypermultiplet of type IIB superstrings in mind, we impose the Heisenberg algebra of perturbative isometries to our theory. In Section 5, we derive the coupling of the Maxwell goldstino multiplet to a charged hypermultiplet and make a detailed analysis of the vacuum structure of $N = 2$ super-QED with partial supersymmetry breaking. We conclude in Section 6 and two appendices present our conventions and the resolution of a quadratic constraint applied on an $N = 2$ chiral superfield.

2. The linear $N = 2$ Maxwell-dilaton system

Our first objective is to describe, in the context of linear $N = 2$ supersymmetry, the coupling of the single-tensor multiplet to $N = 2$ super-Maxwell theory. Since these two supermultiplets admit off-shell realizations, they can be described in superspace without reference to a particular Lagrangian. Gauge transformations of the Maxwell multiplet use a single-tensor multiplet, we then begin with the latter.

[1] This representation of $N = 2$ global supersymmetry has been only recently explicitly constructed [13]. See also Ref. [14].

2.1. The single-tensor multiplet

In global $N=1$ supersymmetry, a real antisymmetric tensor field $b_{\mu\nu}$ is described by a chiral, spinorial superfield χ_α with 8_B+8_F fields [16][2]:

$$\chi_\alpha = -\frac{1}{4}\theta_\alpha\left(C+iC'\right)+\frac{1}{4}\left(\theta\sigma^\mu\bar\sigma^\nu\right)_\alpha b_{\mu\nu}+\cdots \quad (\bar D_{\dot\alpha}\chi_\alpha=0), \tag{2.1}$$

C and C' being the real scalar partners of $b_{\mu\nu}$. The curl $h_{\mu\nu\rho}=3\partial_{[\mu}b_{\nu\rho]}$ is described by the real superfield

$$L = D^\alpha\chi_\alpha - \bar D_{\dot\alpha}\bar\chi^{\dot\alpha}. \tag{2.2}$$

Chirality of χ_α implies linearity of L: $DDL=\overline{DD}L=0$. The linear superfield L is invariant under the supersymmetric gauge transformation[3]

$$\chi_\alpha \;\rightarrow\; \chi_\alpha + \frac{i}{4}\overline{DD}D_\alpha\Delta, \qquad \bar\chi_{\dot\alpha} \;\rightarrow\; \bar\chi_{\dot\alpha}+\frac{i}{4}DD\bar D_{\dot\alpha}\Delta, \tag{2.3}$$

of χ_α: this is the supersymmetric extension of the invariance of $h_{\mu\nu\rho}$ under $\delta b_{\mu\nu}=2\partial_{[\mu}\Lambda_{\nu]}$. Considering bosons only, the gauge transformation (2.3) eliminates three of the six components of $b_{\mu\nu}$ and the scalar field C'. Accordingly, L only depends on the invariant curl $h_{\mu\nu\rho}$ and on the invariant real scalar C. The linear L describes then 4_B+4_F fields. Using either χ_α or L, we will find two descriptions of the single-tensor multiplet of global $N=2$ supersymmetry [4–6].

In the gauge-invariant description using L, the $N=2$ multiplet is completed with a chiral superfield Φ (8_B+8_F fields in total). The second supersymmetry transformations (with parameter η_α) are

$$\delta^*L = -\frac{i}{\sqrt2}(\eta D\Phi+\overline{\eta D}\bar\Phi), \qquad \delta^*\Phi = i\sqrt2\overline{\eta D}L, \qquad \delta^*\bar\Phi = i\sqrt2\eta DL, \tag{2.4}$$

where D_α and $\bar D_{\dot\alpha}$ are the usual $N=1$ supersymmetry derivatives verifying $\{D_\alpha,\bar D_{\dot\alpha}\} = -2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu$. It is easily verified that the $N=2$ supersymmetry algebra closes on L and Φ.

We may try to replace L by χ_α with second supersymmetry transformation $\delta^*\chi_\alpha=-\frac{i}{\sqrt2}\Phi\eta_\alpha$, as suggested when comparing Eqs. (2.2) and (2.4). However, with superfields χ_α and Φ only, the $N=2$ algebra only closes up to a gauge transformation (2.3). This fact, and the unusual number 12_B+12_F of fields, indicate that (χ_α,Φ) is a gauge-fixed version of the off-shell $N=2$ multiplet. We actually need another chiral $N=1$ superfield Y to close the supersymmetry algebra. The second supersymmetry variations are

$$\begin{aligned}
&\delta^*Y = \sqrt2\eta\chi,\\
&\delta^*\chi_\alpha = -\frac{i}{\sqrt2}\Phi\eta_\alpha - \frac{\sqrt2}{4}\eta_\alpha\overline{DD}\bar Y - \sqrt2 i\left(\sigma^\mu\bar\eta\right)_\alpha\partial_\mu Y,\\
&\delta^*\Phi = 2\sqrt2 i\left[\frac14\overline{DD\eta\chi}+i\partial_\mu\chi\sigma^\mu\bar\eta\right].
\end{aligned} \tag{2.5}$$

One easily verifies that the Y-dependent terms in $\delta^*\chi_\alpha$ induce a gauge transformation (2.3). Hence, the linear L and its variation δ^*L do not feel Y. The superfields χ_α, Φ and Y have

[2] The notation m_B+n_F stands for ‘m bosonic and n fermionic fields’.

[3] Δ is an arbitrary real superfield.

$16_B + 16_F$ field components. Gauge transformation (2.3) eliminates $4_B + 4_F$ fields. To further eliminate $4_B + 4_F$ fields, a new gauge variation

$$Y \rightarrow Y - \frac{1}{2}\overline{DD}\Delta', \tag{2.6}$$

with Δ' real, is then postulated. We will see below that this variation is actually dictated by $N=2$ supersymmetry. There exists then a gauge in which $Y=0$ but in this gauge the supersymmetry algebra closes on χ_α only up to a transformation (2.3). This is analogous to the Wess–Zumino gauge of $N=1$ supersymmetry, but in our case, this particular gauge respects $N=1$ supersymmetry and gauge symmetry (2.3).

Two remarks should be made at this point. Firstly, the superfield Y will play an important role in the construction of the Dirac–Born–Infeld interaction with nonlinear $N=2$ supersymmetry. As we will see later on,[4] it includes a four-index antisymmetric tensor field in its highest component. Secondly, a constant (θ-independent) background value $\langle\Phi\rangle$ breaks the second supersymmetry only, $\delta^*\chi_\alpha = -\frac{i}{\sqrt{2}}\langle\Phi\rangle\eta_\alpha + \cdots$. It is a natural source of partial supersymmetry breaking in the single-tensor multiplet. Notice that the condition $\delta^*\langle\Phi\rangle = 0$ is equivalent to $\bar{D}_{\dot\alpha}(D\chi - \overline{D\chi}) = 0$.

An invariant kinetic action for the gauge-invariant single-tensor multiplet involves an arbitrary function solution of the three-dimensional Laplace equation (for the variables L, Φ and $\bar\Phi$) [5]:

$$\mathcal{L}_{ST} = \int d^2\theta\, d^2\bar\theta\, \mathcal{H}(L, \Phi, \bar\Phi), \qquad \frac{\partial^2\mathcal{H}}{\partial L^2} + 2\frac{\partial^2\mathcal{H}}{\partial\Phi\partial\bar\Phi} = 0. \tag{2.7}$$

In the dual hypermultiplet formulation the Laplace equation is replaced by a Monge–Ampère equation. We will often insist on theories with axionic shift symmetry $\delta\Phi = ic$ (c real), dual to a double-tensor theory. In this case, $\mathcal{H}$ is a function of L and $\Phi + \bar\Phi$ so that the general solution of Laplace equation is

$$\mathcal{L}_{ST} = \int d^2\theta\, d^2\bar\theta\, H(\mathcal{V}) + \text{h.c.}, \quad \mathcal{V} = L + \frac{i}{\sqrt{2}}(\Phi + \bar\Phi), \tag{2.8}$$

with an arbitrary analytic function $H(\mathcal{V})$.

2.2. The Maxwell multiplet, Fayet–Iliopoulos terms

Take two real vector superfields V_1 and V_2. Variations

$$\delta^* V_1 = -\frac{i}{\sqrt{2}}[\eta D + \overline{\eta D}]V_2, \qquad \delta^* V_2 = \sqrt{2}i[\eta D + \overline{\eta D}]V_1 \tag{2.9}$$

provide a representation of $N=2$ supersymmetry with $16_B + 16_F$ fields. We may reduce the supermultiplet by imposing on V_1 and V_2 constraints consistent with the second supersymmetry variations: for instance, the single-tensor multiplet is obtained by requiring $V_1 = L$ and $V_2 = \Phi + \bar\Phi$. Another option is to impose a gauge invariance: we may impose that the theory is invariant under[5]

[4] See Section 2.4.

[5] For clarity, we use the following convention for field variations: δ^* refers to the second ($N=2$) supersymmetry variations of the superfields and component fields; $\delta_{U(1)}$ indicates the Maxwell gauge variations; δ appears for gauge variations of superfields or field components related (by supersymmetry) to $\delta b_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]}$.

$$\delta_{U(1)} V_1 = \Lambda_\ell, \qquad \delta_{U(1)} V_2 = \Lambda_c + \bar{\Lambda}_c, \tag{2.10}$$

where Λ_ℓ and Λ_c form a single-tensor multiplet,

$$\Lambda_\ell = \bar{\Lambda}_\ell, \qquad DD\Lambda_\ell = 0, \qquad \bar{D}_{\dot{\alpha}} \Lambda_c = 0, \tag{2.11}$$

with transformations (2.4). Defining the gauge invariant superfields[6]

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\overline{DD} D_\alpha V_2, \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} DD\bar{D}_{\dot{\alpha}} V_2, \\ X &= \frac{1}{2}\overline{DD} V_1, \qquad \bar{X} = \frac{1}{2} DDV_1, \end{aligned} \tag{2.12}$$

the variations (2.9) imply[7]

$$\begin{aligned} \delta^* X &= \sqrt{2} i \eta^\alpha W_\alpha, \qquad \delta^* \bar{X} = \sqrt{2} i \bar{\eta}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}, \\ \delta^* W_\alpha &= \sqrt{2} i \left[\frac{1}{4} \eta_\alpha \overline{DD} \bar{X} + i \left(\sigma^\mu \bar{\eta} \right)_\alpha \partial_\mu X \right], \\ \delta^* \bar{W}_{\dot{\alpha}} &= \sqrt{2} i \left[\frac{1}{4} \bar{\eta}_{\dot{\alpha}} DDX - i \left(\eta \sigma^\mu \right)_{\dot{\alpha}} \partial_\mu \bar{X} \right]. \end{aligned} \tag{2.13}$$

While (V_1, V_2) describes the $N = 2$ supersymmetric extension of the gauge potential A_μ, (W_α, X) is the multiplet of the gauge curvature $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ [17].

The $N = 2$ gauge-invariant Lagrangian depends on the derivatives of a holomorphic prepotential $\mathcal{F}(X)$:

$$\begin{aligned} \mathcal{L}_{Max.} &= \frac{1}{4} \int d^2\theta \left[\mathcal{F}''(X) WW - \frac{1}{2} \mathcal{F}'(X) \overline{DD} \bar{X} \right] + \text{c.c.} \\ &= \frac{1}{4} \int d^2\theta\, \mathcal{F}''(X) WW + \text{c.c.} + \frac{1}{2} \int d^2\theta\, d^2\bar{\theta} \left[\mathcal{F}'(X) \bar{X} + \bar{\mathcal{F}}'(\bar{X}) X \right] + \partial_\mu(\ldots). \end{aligned} \tag{2.14}$$

In the construction of the Maxwell-multiplet in terms of X and W_α, one expects a triplet of Fayet–Iliopoulos terms,

$$\mathcal{L}_{FI} = -\frac{1}{4}(\xi_1 + ia) \int d^2\theta\, X - \frac{1}{4}(\xi_1 - ia) \int d^2\bar{\theta}\, \bar{X} + \xi_2 \int d^2\theta\, d^2\bar{\theta}\, V_2, \tag{2.15}$$

with real parameters ξ_1, ξ_2 and a. They may generate background values of the auxiliary components f_X and d_2 of X and V_2 which in general break both supersymmetries:

$$\delta^* X = \sqrt{2} i \eta\theta \langle d_2 \rangle + \cdots, \qquad \delta^* W_\alpha = \sqrt{2} i \eta_\alpha \langle \bar{f}_X \rangle + \cdots. \tag{2.16}$$

In terms of V_1 and V_2 however, the relation $X = \frac{1}{2}\overline{DD} V_1$ implies that $\operatorname{Im} f_X$ is the curl of a three-index antisymmetric tensor (see Section 2.4) and that its expectation value is turned into an integration constant of the tensor field equation [18,19]. As a consequence,

$$-\frac{1}{4}(\xi_1 + ia) \int d^2\theta\, X - \frac{1}{4}(\xi_1 - ia) \int d^2\bar{\theta}\, \bar{X} = \xi_1 \int d^2\theta\, d^2\bar{\theta}\, V_1 + \text{derivative}$$

6 Remember that with this (standard) convention, $\bar{W}_{\dot{\alpha}}$ is *minus* the complex conjugate of W_α.

7 There is a phase choice in the definition of X: a phase rotation of X can be absorbed in a phase choice of η.

and the Fayet–Iliopoulos Lagrangian becomes

$$\mathcal{L}_{FI} = \int d^2\theta\, d^2\bar\theta\, [\xi_1 V_1 + \xi_2 V_2], \tag{2.17}$$

with two real parameters only.

The Maxwell multiplet with superfields (X, W_α) and the single-tensor multiplet (Y, χ_α, Φ) have a simple interpretation in terms of chiral superfields on $N = 2$ superspace. We will use this formalism to construct their interacting Lagrangians in Section 3.

2.3. The Chern–Simons interaction

With a Maxwell field $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ (in W_α) and an antisymmetric tensor $b_{\mu\nu}$ (in χ_α or L), one may expect the presence of a $b \wedge F$ interaction

$$\epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} F_{\rho\sigma} = 2\epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu b_{\rho\sigma} + \text{derivative}.$$

This equality suggests that its $N = 2$ supersymmetric extension also exists in two forms: either as an integral over chiral superspace of an expression depending on χ_α, W_α, X, Φ and Y, or as a real expression using L, $\Phi + \bar\Phi$, V_1 and V_2.

In the 'real' formulation, the $N = 2$ Chern–Simons term is[8]

$$\mathcal{L}_{CS} = -g \int d^2\theta\, d^2\bar\theta\, \big[L V_2 + (\Phi + \bar\Phi) V_1\big], \tag{2.18}$$

with a real coupling constant g. It is invariant (up to a derivative) under the gauge transformations (2.10) of V_1 and V_2 with L and Φ left inert. Notice that the introduction of Fayet–Iliopoulos terms for V_1 and V_2 corresponds respectively to the shifts $\Phi + \bar\Phi \to \Phi + \bar\Phi - \xi_1/g$ and $L \to L - \xi_2/g$ in the Chern–Simons term.

The 'chiral' version uses the spinorial prepotential χ_α instead of L. Turning expression (2.18) into a chiral integral and using $X = \frac{1}{2}\overline{DD} V_1$ leads to

$$\mathcal{L}_{CS,\chi} = g \int d^2\theta\, \Big[\chi^\alpha W_\alpha + \frac{1}{2}\Phi X\Big] + g \int d^2\bar\theta\, \Big[-\bar\chi_{\dot\alpha} \bar W^{\dot\alpha} + \frac{1}{2}\bar\Phi \bar X\Big], \tag{2.19}$$

which differs from $\mathcal{L}_{CS}$ by a derivative. The chiral version of the Chern–Simons term $\mathcal{L}_{CS,\chi}$ transforms as a derivative under the gauge variation (2.3) of χ_α. Its invariance under constant shift symmetry of $\operatorname{Im}\Phi$ follows from $X = \frac{1}{2}\overline{DD} V_1$. It does not depend on Y.

The consistent Lagrangian for the Maxwell–single-tensor system with Chern–Simons interaction is then

$$\mathcal{L}_{ST} + \mathcal{L}_{Max.} + \mathcal{L}_{CS} \quad \text{or} \quad \mathcal{L}_{ST} + \mathcal{L}_{Max.} + \mathcal{L}_{CS,\chi}. \tag{2.20}$$

The first two contributions include the kinetic terms and self-interactions of the multiplets while the third describes how they interact. Each of the three terms is separately $N = 2$ supersymmetric.

Using an $N = 1$ duality, a linear multiplet can be transformed into a chiral superfield with constant shift symmetry and the opposite transformation of course exists. Hence, performing both transformations, a single-tensor multiplet Lagrangian (L, Φ) with constant shift symmetry

[8] The dimensions in mass unit of our superfields are as follows: V_1, V_2: 0, X, Y: 1, W_α, χ_α: 3/2, Φ, L: 2. The coupling constant g is then dimensionless.

of the chiral Φ has a 'double-dual' second version. Suppose that we start with a Lagrangian where Maxwell gauge symmetry acts as a Stückelberg gauging of the single-tensor multiplet[9]:

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\, \mathcal{H}(L - gV_1, \Phi + \bar\Phi - gV_2). \tag{2.21}$$

The shift symmetry of $\mathrm{Im}\,\Phi$ has been gauged and $\mathcal{L}$ is invariant under gauge transformations (2.10) combined with

$$\delta_{U(1)} L = g\Lambda_\ell, \qquad \delta_{U(1)}\Phi = g\Lambda_c, \tag{2.22}$$

and under $N=2$ supersymmetry if $\mathcal{H}$ verifies Laplace equation (2.7). If we perform a double dualization $(L, \Phi + \bar\Phi) \to (\tilde\Phi + \bar{\tilde\Phi}, \tilde L)$, we obtain the dual theory

$$\tilde{\mathcal{L}} = \int d^2\theta\, d^2\bar\theta\, \tilde{\mathcal{H}}(\tilde L, \tilde\Phi + \bar{\tilde\Phi}) + g\int d^2\theta \left[\tilde\chi^\alpha W_\alpha + \frac{1}{2}\tilde\Phi X\right] + \text{c.c.}, \tag{2.23}$$

where $\tilde{\mathcal{H}}$ is the result of the double Legendre transformation

$$\tilde{\mathcal{H}}(\tilde y, \tilde x) = \mathcal{H}(x, y) - \tilde x x - \tilde y y. \tag{2.24}$$

The dual theory is then the sum of the ungauged Lagrangian (2.7) and of the Chern–Simons coupling (2.18). This *single-tensor–single-tensor* duality is actually $N=2$ covariant: if $\mathcal{H}$ solves Laplace equation, so does $\tilde{\mathcal{H}}$, and every intermediate step of the duality transformation can be formulated with explicit $N=2$ off-shell supersymmetry.

We have then found two classes of couplings of Maxwell theory to the single-tensor multiplet. Firstly, using the supersymmetric extension of the $b\wedge F$ coupling, as in Eqs. (2.20). Secondly, using a Stückelberg gauging (2.21) of the single-tensor kinetic terms. The first version only is directly appropriate to perform an electric–magnetic duality transformation. However, since the second version can always be turned into the first one by a single-tensor–single-tensor duality, electric–magnetic duality of the second version requires this preliminary step: both theories have the same 'magnetic' dual.

2.4. The significance of V_1, X and Y

In the description of the $N=2$ Maxwell multiplet in terms of two $N=1$ real superfields, V_2 describes as usual the gauge potential A_μ, a gaugino λ_α and a real auxiliary field d_2 (in Wess–Zumino gauge). We wish to clarify the significance and the field content of the superfields V_1 and $X = \frac{1}{2}\overline{DD}V_1$, as well as the related content of the chiral superfield Y used in the description in terms of the spinorial potential χ_α of the single-tensor multiplet (Y, χ_α, Φ).

The vector superfield V_1 has the $N=2$ Maxwell gauge variation $\delta_{U(1)}V_1 = \Lambda_\ell$, with a real linear parameter superfield Λ_ℓ. In analogy with the Wess–Zumino gauge commonly applied to V_2, there exists then a gauge where

$$V_1(x,\theta,\bar\theta) = \theta\sigma^\mu\bar\theta v_{1\mu} - \frac{1}{2}\theta\theta\bar x - \frac{1}{2}\overline{\theta\theta}x - \frac{1}{\sqrt2}\theta\theta\overline{\theta\psi}_X - \frac{1}{\sqrt2}\overline{\theta\theta}\theta\psi_X + \frac{1}{2}\theta\theta\overline{\theta\theta}d_1. \tag{2.25}$$

[9] Strictly speaking, the coupling constant g in this theory has dimension $(\text{energy})^2$. There is an irrelevant energy scale involved in the duality transformation of a dimension two L into a dimension two chiral superfield. Hence, g in Eq. (2.23) is again dimensionless.

This gauge leaves a residual invariance acting on the vector field $v_{1\mu}$ only:

$$\delta_{U(1)} v_1^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu \Lambda_{\rho\sigma}. \tag{2.26}$$

This indicates that the vector v_1^μ is actually a three-index antisymmetric tensor,

$$v_1^\mu = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma} A_{\nu\rho\sigma}, \tag{2.27}$$

with Maxwell gauge invariance

$$\delta_{U(1)} A_{\mu\nu\rho} = 3\partial_{[\mu}\Lambda_{\nu\rho]}. \tag{2.28}$$

By construction, $X = \frac{1}{2}\overline{DD}V_1$ is gauge invariant. In chiral variables,

$$X(y,\theta) = x + \sqrt{2}\theta\psi_X - \theta\theta\big(d_1 + i\partial_\mu v_1^\mu\big). \tag{2.29}$$

Hence, while $\mathrm{Re}\, f_X = d_1$,

$$\mathrm{Im}\, f_X = \partial_\mu v_1^\mu = \frac{1}{24}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}, \qquad F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A_{\nu\rho\sigma]} \tag{2.30}$$

is the gauge-invariant curl of $A_{\mu\nu\rho}$. It follows that the field content (in Wess–Zumino gauge) of V_1 is the second gaugino ψ_X, the complex scalar of the Maxwell multiplet x, a real auxiliary field d_1 and the three-form field $A_{\mu\nu\rho}$, which corresponds to a single, non-propagating component field. The gauge-invariant chiral X includes the four-form curvature $F_{\mu\nu\rho\sigma}$.

At the Lagrangian level, the implication of relations (2.30) is as follows. Suppose that we compare two theories with the same Lagrangian $\mathcal{L}(u)$ but either with $u = \phi$, a real scalar, or with $u = \partial_\mu V^\mu$, as in Eq. (2.30). Since $\mathcal{L}(\phi)$ does not depend on $\partial_\mu\phi$, the scalar ϕ is auxiliary. The field equations for both theories are

$$\frac{\partial}{\partial\phi}\mathcal{L}(\phi) = 0, \qquad \partial_\nu \frac{\partial}{\partial u}\mathcal{L}(u)\bigg|_{u=\partial_\mu V^\mu} = 0.$$

The second case allows a supplementary integration constant k related to the possible addition of a 'topological' term proportional to $\partial_\mu V^\mu$ to the Lagrangian [18,19]:

$$\frac{\partial}{\partial u}\mathcal{L}(u)\bigg|_{u=\partial_\mu V^\mu} = k.$$

In the first case, the same integration constant appears if one considers the following modified theory and field equation:

$$\mathcal{L}(\phi) - k\phi \;\rightarrow\; \frac{\partial}{\partial\phi}\mathcal{L}(\phi) = k.$$

Returning to our super-Maxwell case, the relation is $\phi = \mathrm{Im}\, f_X$ and the modification of the Lagrangian is then

$$-k\,\mathrm{Im}\, f_X = -\frac{ik}{2}\int d^2\theta\, X + \mathrm{c.c.} \tag{2.31}$$

This is the third Fayet–Iliopoulos term, which becomes a 'hidden parameter' [18] when using V_1 instead of X.

Table 1

$N=1$ superfield	Field	Gauge invariance	Number of fields
χ_α	$b_{\mu\nu}$	$\delta b_{\mu\nu}=2\partial_{[\mu}\Lambda_{\nu]}$	$6_B-3_B=3_B$
	C		1_B
	χ_α		4_F
Φ	Φ		2_B
	f_Φ		2_B (auxiliary)
	ψ_Φ		4_F
Y	$C_{\mu\nu\rho\sigma}$	$\delta C_{\mu\nu\rho\sigma}=4\partial_{[\mu}\Lambda_{\nu\rho\sigma]}$	$1_B-1_B=0_B$

Consider finally the single-tensor multiplet (Y, χ_α, Φ) and the supersymmetric extension of the antisymmetric-tensor gauge symmetry, as given in Eqs. (2.3) and (2.6):

$$\delta Y=-\frac{1}{2}\overline{DD}\Delta', \qquad \delta\chi_\alpha=\frac{i}{4}\overline{DD}D_\alpha\Delta, \qquad \delta\Phi=0.$$

Using expansion (2.29), there is a gauge in which Y reduces simply to

$$Y=-i\theta\theta\,\mathrm{Im}\,f_Y \tag{2.32}$$

and one should identify $\mathrm{Im}\,f_Y$ as a four-index antisymmetric tensor field,

$$\mathrm{Im}\,f_Y=\frac{1}{24}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}, \tag{2.33}$$

with residual gauge invariance

$$\delta C_{\mu\nu\rho\sigma}=4\partial_{[\mu}\Lambda_{\nu\rho\sigma]}. \tag{2.34}$$

The antisymmetric tensor $C_{\mu\nu\rho\sigma}$ describes a single field component which can be gauged away using $\Lambda_{\nu\rho\sigma}$. Applying this extended Wess–Zumino gauge to the $N=2$ multiplet (Y, χ_α, Φ), the fields described by these $N=1$ superfields are as given in Table 1.

The propagating bosonic fields $b_{\mu\nu}$, C and Φ (four bosonic degrees of freedom) have kinetic terms defined by Lagrangian $\mathcal{L}_{ST}$, Eq. (2.7).

3. Chiral $N=2$ superspace

Many results of the previous section can be reformulated in terms of chiral superfields on $N=2$ superspace. We now turn to a discussion of this framework, including an explicitly $N=2$ covariant formulation of electric–magnetic duality.

3.1. Chiral $N=2$ superfields

A chiral superfield on $N=2$ superspace can be written as a function of y^μ, θ, $\tilde\theta$:

$$\bar D_{\dot\alpha}\mathcal{Z}=\bar{\tilde D}_{\dot\alpha}\mathcal{Z}=0 \quad\longrightarrow\quad \mathcal{Z}=\mathcal{Z}(y,\theta,\tilde\theta) \tag{3.1}$$

with $y^\mu=x^\mu-i\theta\sigma^\mu\bar\theta-i\tilde\theta\sigma^\mu\bar{\tilde\theta}$ and $\bar D_{\dot\alpha}y^\mu=\bar{\tilde D}_{\dot\alpha}y^\mu=0$. Its second supersymmetry variations are

$$\delta^*\mathcal{Z}=i(\eta\tilde Q+\bar\eta\bar{\tilde Q})\mathcal{Z}, \tag{3.2}$$

with supercharge differential operators $\tilde{Q}_\alpha$ and $\bar{\tilde{Q}}_{\dot\alpha}$ which we do not need to explicitly write. It includes four $N=1$ chiral superfields and 16_B+16_F component fields and we may use the expansions

$$\begin{aligned}\mathcal{Z}(y,\theta,\tilde\theta) &= Z(y,\theta)+\sqrt{2}\tilde\theta^\alpha\omega_\alpha(y,\theta)-\tilde\theta\tilde\theta F(y,\theta)\\ &= Z(y,\theta)+\sqrt{2}\tilde\theta^\alpha\omega_\alpha(y,\theta)-\tilde\theta\tilde\theta\Big[\frac{i}{2}\Phi_{\mathcal{Z}}(y,\theta)+\frac{1}{4}\overline{DD}\bar{Z}(y,\theta)\Big],\end{aligned}\tag{3.3}$$

where $\tilde\theta$ and $\tilde{D}_\alpha$ are the Grassmann coordinates and the super-derivatives associated with the second supersymmetry. The second supersymmetry variations (3.2) are easily obtained by analogy with the $N=1$ chiral supermultiplet:

$$\begin{aligned}&\delta^* Z=\sqrt{2}\eta\omega,\\ &\delta^*\omega_\alpha=-\sqrt{2}\big[F\eta_\alpha+i\big(\sigma^\mu\bar\eta\big)_\alpha\partial_\mu Z\big]=-\frac{i}{\sqrt{2}}\Phi_{\mathcal{Z}}\eta_\alpha-\frac{\sqrt{2}}{4}\eta_\alpha\overline{DD}\bar{Z}-\sqrt{2}i\big(\sigma^\mu\bar\eta\big)_\alpha\partial_\mu Z,\\ &\delta^* F=-\sqrt{2}i\partial_\mu\omega\sigma^\mu\bar\eta,\\ &\delta^*\Phi_{\mathcal{Z}}=2\sqrt{2}i\left[\frac{1}{4}\overline{DD}\overline{\eta\omega}+i\partial_\mu\omega\sigma^\mu\bar\eta\right].\end{aligned}\tag{3.4}$$

We immediately observe that the second expansion (3.3) leads to the second supersymmetry variations (2.5) of a single-tensor multiplet ($Y=Z$, $\chi=\omega$, $\Phi=\Phi_{\mathcal{Z}}$). Similarly, the expansion

$$\mathcal{W}(y,\theta,\tilde\theta)=X(y,\theta)+\sqrt{2}i\tilde\theta W(y,\theta)-\tilde\theta\tilde\theta\frac{1}{4}\overline{DD}\bar{X}(y,\theta),\tag{3.5}$$

which is obtained by imposing $\Phi_{\mathcal{Z}}=0$ in expansion (3.3), leads to the Maxwell supermultiplet (2.13) [20]. The Bianchi identity $D^\alpha W_\alpha=\bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$ is required by $\delta^*\Phi_{\mathcal{Z}}=0$. The $N=2$ Maxwell Lagrangian (2.14) rewrites then as an integral over chiral $N=2$ superspace,

$$\mathcal{L}_{Max.}=\frac{1}{2}\int d^2\theta\int d^2\tilde\theta\,\mathcal{F}(\mathcal{W})+\text{c.c.},\tag{3.6}$$

and the Fayet–Iliopoulos terms (2.17) can be written [21]

$$\mathcal{L}_{FI}=\int d^2\theta\, d^2\bar\theta\,[\xi_1V_1+\xi_2V_2]=-\frac{1}{4}\int d^2\theta\int d^2\tilde\theta\,[\tilde\theta\tilde\theta\xi_1-\sqrt{2}i\theta\tilde\theta\xi_2]\mathcal{W}+\text{c.c.}\tag{3.7}$$

Considering the unconstrained chiral superfield (3.3) with 16_B+16_F fields, the reduction to the 8_B+8_F components of the single-tensor multiplet is done by imposing gauge invariance (2.3) and (2.6). In terms of $N=2$ chiral superfields, this gauge symmetry is simply

$$\delta\mathcal{Y}=-\hat{\mathcal{W}},\tag{3.8}$$

where $\hat{\mathcal{W}}$ is a Maxwell $N=2$ superfield parameter (3.5). In terms of $N=1$ superfields, this is

$$\delta Y=-\hat{X},\qquad \delta\chi_\alpha=-i\hat{W}_\alpha,\qquad \delta\Phi=0,\tag{3.9}$$

as in Eqs. (2.3) and (2.6). Hence, a single-tensor superfield $\mathcal{Y}$ is a chiral superfield $\mathcal{Z}$ with the second expansion (3.3) and with gauge symmetry (3.8).

The chiral version of the Chern–Simons interaction (2.19) can be easily written on $N=2$ superspace. Using $\mathcal{Y}$ with gauge invariance (3.8) and $\mathcal{W}$ to respectively describe the single-tensor and the Maxwell multiplets. Then

$$\mathcal{L}_{CS,\chi} = ig \int d^2\theta \int d^2\tilde{\theta}\, \mathcal{Y}\mathcal{W} + \text{c.c.} \tag{3.10}$$

It is gauge-invariant since for any pair of Maxwell superfields

$$i \int d^2\theta \int d^2\tilde{\theta}\, \mathcal{W}\hat{\mathcal{W}} + \text{c.c.} = \text{derivative.} \tag{3.11}$$

Notice that the lowest component superfield Y of $\mathcal{Y}$ does not contribute to the field equations derived from $\mathcal{L}_{CS,\chi}$: it only contributes to this Lagrangian with a derivative.

Finally, a second method to obtain an interactive Lagrangian for the Maxwell–single-tensor system is then obvious. Firstly, a generic $N = 2$ chiral superfield $\mathcal{Z}$ can always be written as

$$\mathcal{Z} = \mathcal{W} + 2g\mathcal{Y}. \tag{3.12}$$

It is invariant under the single-tensor gauge variation (3.8) if one also postulates that

$$\delta\mathcal{W} = 2g\hat{\mathcal{W}}, \tag{3.13}$$

which amounts to an $N = 2$ Stückelberg gauging of the symmetry of the antisymmetric tensor. With this decomposition, $F_{\mu\nu}$ and $b_{\mu\nu}$ only appear in the $\theta_\alpha\tilde{\theta}_\beta$ component of $\mathcal{Z}$ through the gauge-invariant combination $F_{\mu\nu} - gb_{\mu\nu}$. The chiral integral

$$\mathcal{L} = \frac{1}{2} \int d^2\theta \int d^2\tilde{\theta}\, \mathcal{F}(\mathcal{W} + 2g\mathcal{Y}) + \text{c.c.} + \mathcal{L}_{ST} \tag{3.14}$$

provides an $N = 2$ invariant Lagrangian describing $16_B + 16_F$ (off-shell) interacting fields. There exists a gauge in which $\mathcal{W} = 0$, in which case theory (3.14) describes a massive chiral $N = 2$ superfield.

Theory (3.14) is actually related to the Chern–Simons Lagrangian (2.20) by electric–magnetic duality, as will be shown below.

3.2. Electric-magnetic duality

The description in chiral $N = 2$ superspace of the Maxwell multiplet allows to derive an $N = 2$ covariant version of electric–magnetic duality. The Maxwell Lagrangian (2.14) supplemented by the Chern–Simons coupling (2.19) can be written

$$\mathcal{L}_{electric} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{2}\mathcal{F}(\mathcal{W}) + ig\mathcal{Y}\mathcal{W}\right] + \text{c.c.}, \tag{3.15}$$

adding Eqs. (3.6) and (3.10). Replace then $\mathcal{W}$ by an unconstrained chiral superfield $\hat{\mathcal{Z}}$ (with $N = 1$ superfields $\hat{Z}$, $\hat{\omega}_\alpha$ and $\hat{\Phi}$) and introduce a new Maxwell multiplet $\tilde{\mathcal{W}}$ (with $N = 1$ superfields $\tilde{X}$ and $\tilde{W}_\alpha$). Using

$$\tilde{X} = \frac{1}{2}\overline{DD}\tilde{V}_1, \qquad \tilde{W}_\alpha = -\frac{1}{4}\overline{DD}D_\alpha\tilde{V}_2,$$

we have

$$\begin{aligned} i \int d^2\theta \int d^2\tilde{\theta}\, \tilde{\mathcal{W}}\hat{\mathcal{Z}} + \text{c.c.} &= \int d^2\theta \left[\frac{1}{2}\hat{\Phi}\tilde{X} + \hat{\omega}\tilde{W}\right] + \text{c.c.} \\ &= -\int d^2\theta\, d^2\bar{\theta} \left[\tilde{V}_1(\hat{\Phi} + \bar{\hat{\Phi}}) + \tilde{V}_2\big(D^\alpha\hat{\omega}_\alpha - \bar{D}_{\dot{\alpha}}\bar{\hat{\omega}}^{\dot{\alpha}}\big)\right]. \end{aligned} \tag{3.16}$$

Consider now the Lagrangian

$$\mathcal{L} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{2}\mathcal{F}(\hat{\mathcal{Z}}) + \frac{i}{2}\hat{\mathcal{Z}}(\tilde{\mathcal{W}} + 2g\mathcal{Y}) \right] + \text{c.c.} \tag{3.17}$$

Invariance under the gauge transformation of the single-tensor superfield, Eq. (3.8), requires a compensating gauge variation of $\tilde{\mathcal{W}}$, as in Eq. (3.13). Eliminating $\tilde{\mathcal{W}}$ leads back to theory (3.15) with $\hat{\mathcal{Z}} = \mathcal{W}$. This can be seen in two ways. Firstly, the condition

$$i \int d^2\theta \int d^2\tilde{\theta}\, \tilde{\mathcal{W}}\hat{\mathcal{Z}} + \text{c.c.} = \text{derivative}$$

leads to $\hat{\mathcal{Z}} = \mathcal{W}$, an $N = 2$ Maxwell superfield, up to a background value. Secondly, using Eqs. (3.16), we see that $\tilde{V}_2$ imposes the Bianchi identity on $\hat{\omega}$ while $\tilde{V}_1$ cancels $\hat{\Phi}$ up to an imaginary constant.[10] We will come back to the (important) role of a nonzero background value in the next section. For the moment we disregard it.

On the other hand, we may prefer to eliminate $\hat{\mathcal{Z}}$, using its field equation

$$\mathcal{F}'(\hat{\mathcal{Z}}) = -i\mathcal{V}, \quad \mathcal{V} \equiv \tilde{\mathcal{W}} + 2g\mathcal{Y}, \tag{3.18}$$

which corresponds to a Legendre transformation exchanging variables $\hat{\mathcal{Z}}$ and $\mathcal{V}$. Defining

$$\tilde{\mathcal{F}}(\mathcal{V}) = \mathcal{F}(\hat{\mathcal{Z}}) + i\mathcal{V}\hat{\mathcal{Z}}, \tag{3.19}$$

we have

$$\tilde{\mathcal{F}}'(\mathcal{V}) = i\hat{\mathcal{Z}}, \qquad \mathcal{F}'(\hat{\mathcal{Z}}) = -i\mathcal{V}, \qquad \tilde{\mathcal{F}}''(\mathcal{V})\mathcal{F}''(\hat{\mathcal{Z}}) = 1. \tag{3.20}$$

The dual (Legendre-transformed) theory is then

$$\tilde{\mathcal{L}}_{magnetic} = \frac{1}{2}\int d^2\theta \int d^2\tilde{\theta}\, \tilde{\mathcal{F}}(\tilde{\mathcal{W}} + 2g\mathcal{Y}) + \text{c.c.} \tag{3.21}$$

or, expressed in $N = 1$ superspace,[11]

$$\begin{aligned}\tilde{\mathcal{L}}_{magnetic} = \frac{1}{4}\int d^2\theta \big[& \tilde{\mathcal{F}}''(\tilde{X} + 2gY)(\tilde{W} - 2ig\chi)^\alpha(\tilde{W} - 2ig\chi)_\alpha \\ & - \frac{1}{2}\tilde{\mathcal{F}}'(\tilde{X} + 2gY)\overline{DD}(\bar{\tilde{X}} + 2g\bar{Y}) - 2ig\tilde{\mathcal{F}}'(\tilde{X} + 2gY)\Phi \big] + \text{c.c.}\end{aligned} \tag{3.22}$$

We then conclude that the presence of the Chern–Simons term in the electric theory induces a Stückelberg gauging in the dual magnetic theory.

As explained in Ref. [21], the situation changes when Fayet–Iliopoulos terms (3.7) are present in the electric theory. In the magnetic theory coupled to the single-tensor multiplet, with Lagrangian (3.22), the gauging $\delta\tilde{\mathcal{W}} = 2g\hat{\mathcal{W}}$ forbids Fayet–Iliopoulos terms for the magnetic Maxwell superfields $\tilde{V}_1$ and $\tilde{V}_2$. Spontaneous supersymmetry breaking by Fayet–Iliopoulos terms in the electric theory finds then a different origin in the magnetic dual.

For our needs, we only consider the Fayet–Iliopoulos term induced by V_1, *i.e.* we add

$$\mathcal{L}_{FI} = \xi_1 \int d^4\theta\, V_1 = -\frac{1}{4}\xi_1 \int d^2\theta \int d^2\tilde{\theta}\, \tilde{\theta}\tilde{\theta}\mathcal{W} + \text{c.c.} \tag{3.23}$$

10 An unconstrained $\tilde{X}$ would forbid this constant.

11 The free, canonically-normalized theory corresponds to $\mathcal{F}(\mathcal{W}) = \frac{1}{2}\mathcal{W}^2$ and $\tilde{\mathcal{F}}(\mathcal{V}) = \frac{1}{2}\mathcal{V}^2$.

to $\mathcal{L}_{electric}$, Eq. (3.15). In turn, this amounts to add

$$-\frac{1}{4}\xi_1 \int d^2\theta \int d^2\tilde{\theta}\, \tilde{\theta}\tilde{\theta}\hat{\mathcal{Z}} + \text{c.c.}$$

to theory (3.17). But, in contrast to expression (3.23), this modification is not invariant under the second supersymmetry: according to the first Eq. (3.4), its δ^* variation

$$-\frac{\sqrt{2}}{4}\xi_1 \int d^2\theta\, \eta\omega + \text{c.c.}$$

is not a derivative.[12] To restore $N = 2$ supersymmetry, we must deform the δ^* variation of $\tilde{W}_\alpha - 2ig\chi_\alpha$ into

$$\delta^*_{deformed}(\tilde{W}_\alpha - 2ig\chi_\alpha) = \frac{1}{\sqrt{2}}\xi_1\eta_\alpha + \delta^*(\tilde{W}_\alpha - 2ig\chi_\alpha), \tag{3.24}$$

the second term being the usual, undeformed, variations (2.13) and (2.5). Hence, the magnetic theory has a goldstino fermion and linear $N = 2$ supersymmetry partially breaks to $N = 1$, as a consequence of the electric Fayet–Iliopoulos term. Concretely, the magnetic theory is now

$$\begin{aligned}
\tilde{\mathcal{L}}_{magnetic} &= \frac{1}{2}\int d^2\theta \int d^2\tilde{\theta}\, \tilde{\mathcal{F}}\left(\tilde{\mathcal{W}} + 2g\mathcal{Y} + \frac{i}{2}\xi_1\tilde{\theta}\tilde{\theta}\right) + \text{c.c.} \\
&= \frac{1}{2}\int d^2\theta \int d^2\tilde{\theta}\left[\tilde{\mathcal{F}}(\tilde{\mathcal{W}} + 2g\mathcal{Y}) + \frac{i}{2}\xi_1\tilde{\theta}\tilde{\theta}\tilde{\mathcal{F}}'(\tilde{\mathcal{W}} + 2g\mathcal{Y})\right] + \text{c.c.} \\
&= \left[\frac{1}{2}\int d^2\theta \int d^2\tilde{\theta}\, \tilde{\mathcal{F}}(\tilde{\mathcal{W}} + 2g\mathcal{Y}) + \frac{i}{4}\xi_1 \int d^2\theta\, \tilde{\mathcal{F}}'(\tilde{X} + 2gY)\right] + \text{c.c.}
\end{aligned} \tag{3.25}$$

One easily checks that $N = 2$ supersymmetry holds, using the deformed variations (3.24).

4. Nonlinear $N = 2$ supersymmetry and the DBI action

In the previous sections, we have developed various aspects of the coupling of a Maxwell multiplet to a single-tensor multiplet in linear $N = 2$ supersymmetry. With these tools, we can now address our main subject: show how a Dirac–Born–Infeld Lagrangian (DBI) coupled to the single-tensor multiplet arises from nonlinearization of the second supersymmetry.

It has been observed that the DBI Lagrangian with nonlinear second supersymmetry can be derived by solving a constraint invariant under $N = 2$ supersymmetry imposed on the super-Maxwell theory [7,8]. We start with a summary of this result, following mostly Roček and Tseytlin [8], and we then generalize the method to incorporate the fields of the single-tensor multiplet.

4.1. The $N = 2$ super-Maxwell DBI theory

The constraint imposed on the $N = 2$ Maxwell chiral superfield $\mathcal{W}$ is [8][13]

$$\mathcal{W}^2 - \frac{1}{\kappa}\tilde{\theta}\tilde{\theta}\mathcal{W} = \left(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right)^2 = 0. \tag{4.1}$$

[12] It would be a derivative if ω_α would be replaced by the Maxwell superfield W_α, as in Eq. (3.23).

[13] See also Ref. [22] and very recently Ref. [23] in the context of $N = 1$ supersymmetry.

It imposes a relation between the super-Maxwell Lagrangian superfield $\mathcal{W}^2$ and the Fayet–Iliopoulos 'superfield' $\tilde{\theta}\tilde{\theta}\mathcal{W}$, Eq. (3.23). The real scale parameter κ has dimension (energy)$^{-2}$. In terms of $N = 1$ superfields, the constraint is equivalent to

$$X^2 = 0, \qquad X W_\alpha = 0, \qquad WW - \frac{1}{2} X \overline{DD} \bar{X} = \frac{1}{\kappa} X. \tag{4.2}$$

The third equality leads to

$$X = \frac{2WW}{\frac{2}{\kappa} + \overline{DD}\bar{X}} \tag{4.3}$$

which, since $W_\alpha W_\beta W_\gamma = 0$, implies the first two conditions. Solving the third constraint amounts to express X as a function of WW [7].[14] The DBI theory is then obtained using as Lagrangian the Fayet–Iliopoulos term (3.23) properly normalized:

$$\mathcal{L}_{DBI} = \frac{1}{4\kappa} \int d^2\theta\, X + \text{c.c.} = \frac{1}{8\kappa^2}\Big[1 - \sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}\Big] + \cdots. \tag{4.4}$$

The constraints (4.1) and (4.2) are not invariant under the second linear supersymmetry, with variations δ^*. However, one easily verifies that the three constraints (4.2) are invariant under the deformed, nonlinear variation

$$\delta^*_{deformed} W_\alpha = \sqrt{2} i \left[\frac{1}{2\kappa}\eta_\alpha + \frac{1}{4}\eta_\alpha \overline{DD}\bar{X} + i\left(\sigma^\mu \bar{\eta}\right)_\alpha \partial_\mu X\right], \tag{4.5}$$

with $\delta^* X$ unchanged. The deformation preserves the $N = 2$ supersymmetry algebra. It indicates that the gaugino spinor in $W_\alpha = -i\lambda_\alpha + \cdots$ transforms inhomogeneously, $\delta^*\lambda_\alpha = -\frac{1}{\sqrt{2}\kappa}\eta_\alpha + \cdots$, like a goldstino for the breaking of the second supersymmetry. In other words, at the level of the $N = 2$ chiral superfield $\mathcal{W}$,

$$\delta^*_{deformed}\mathcal{W} = -\frac{1}{\kappa}\tilde{\theta}\eta + i(\eta\tilde{Q} + \bar{\eta}\bar{\tilde{Q}})\mathcal{W} = i(\eta\tilde{Q} + \bar{\eta}\bar{\tilde{Q}})\left(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right).$$

The deformed second supersymmetry variations $\delta^*_{deformed}$ act on $\mathcal{W}$ as the usual variations δ^* act on the shifted superfield $\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}$. In fact, this superfield transforms like a chiral $N = 2$ superfield (3.3) with $Z = X$, $\omega_\alpha = i W_\alpha$ verifying the Bianchi identity and with $\Phi_{\mathcal{Z}} = -i/\kappa$. The latter background value of $\Phi_{\mathcal{Z}}$ may be viewed as the source of the partial breaking of linear supersymmetry.

Hence, the scale parameter κ introduced in the nonlinear constraint (4.1) appears as the scale parameter of the DBI Lagrangian and also as the order parameter of partial supersymmetry breaking. The Fayet–Iliopoulos term (4.4) has in principle an arbitrary coefficient $-\xi_1/4$, as in Eq. (2.17). We have chosen $\xi_1 = -\kappa^{-1}$ to canonically normalize gauge kinetic terms.

The DBI Lagrangian is invariant under electric–magnetic duality.[15] In our $N = 2$ case, the invariance is easily established in the language of $N = 2$ superspace. We first include the constraint as a field equation of the Lagrangian:

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\kappa}\tilde{\theta}\tilde{\theta}\mathcal{W} + \frac{1}{4}\Lambda\left(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right)^2\right] + \text{c.c.} \tag{4.6}$$

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[14] See Appendix B.

[15] For instance, in the context of D3-branes of IIB superstrings, see Ref. [24]. Our procedure is inspired by Ref. [8].

The field equation of the $N = 2$ superfield Λ enforces (4.1). We then introduce two unconstrained $N = 2$ chiral superfields U and Υ and the modified Lagrangian

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\kappa}\tilde{\theta}\tilde{\theta}\mathcal{W} + \frac{1}{4}\Lambda U^2 - \frac{1}{2}\Upsilon\left(U - \mathcal{W} + \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right)\right] + \text{c.c.}$$

Since the Lagrange multiplier Υ imposes $U = \mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}$, the equivalence with (4.6) is manifest. But we may also eliminate $\mathcal{W}$ which only appears linearly in the last version of the theory. The result is

$$\Upsilon = -i\tilde{\mathcal{W}} - \frac{1}{2}\left(\frac{1}{\kappa} - i\zeta\right)\tilde{\theta}\tilde{\theta}$$

where $\tilde{\mathcal{W}}$ is a Maxwell $N = 2$ superfield dual to $\mathcal{W}$ and ζ an arbitrary real constant. As in Section 3.2, $N = 2$ supersymmetry of the theory with a Fayet–Iliopoulos term requires a nonlinear deformation of the δ^* variation of $\tilde{\mathcal{W}}$: $\tilde{\mathcal{W}} - \frac{i}{2}(\frac{1}{\kappa} - i\zeta)\tilde{\theta}\tilde{\theta}$ should be a ‘good’ $N = 2$ chiral superfield. Replacing Υ in the Lagrangian and taking $\zeta = 0$ leads to

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4}\Lambda U^2 + \frac{i}{2}U[\tilde{\mathcal{W}} - \frac{i}{2\kappa}\tilde{\theta}\tilde{\theta}] + \frac{i}{4\kappa}\tilde{\mathcal{W}}\tilde{\theta}\tilde{\theta}\right] + \text{c.c.}$$

Finally, eliminating U gives the magnetic dual

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\Lambda}(\tilde{\mathcal{W}} - \frac{i}{2\kappa}\tilde{\theta}\tilde{\theta})^2 + \frac{i}{4\kappa}\tilde{\mathcal{W}}\tilde{\theta}\tilde{\theta}\right] + \text{c.c.} \tag{4.7}$$

One easily verifies that the resulting theory has the same expression as the initial ‘electric’ theory (4.4). The Lagrange multiplier Λ^{-1} imposes constraint (4.1) to $-i\tilde{\mathcal{W}}$, which reduces to Eq. (4.3) applied to $-i\tilde{X}$. The Lagrangian is then given by the Fayet–Iliopoulos term for this superfield.

4.2. Coupling the DBI theory to a single-tensor multiplet: A super-Higgs mechanism without gravity

The $N = 2$ super-Maxwell DBI theory is given by a Fayet–Iliopoulos term for a Maxwell superfield submitted to the quadratic constraint (4.1), which also provides the source of partial supersymmetry breaking. The second supersymmetry is deformed by the constraint: it is $\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}$ which transforms as a regular $N = 2$ chiral superfield. Instead of expression (3.10), we are thus led to consider the following Chern–Simons interaction with the single-tensor multiplet:

$$\begin{aligned}\mathcal{L}_{CS,def.} &= ig\int d^2\theta \int d^2\tilde{\theta}\, \mathcal{Y}\left(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right) + \text{c.c.} \\ &= g\int d^2\theta \left[\frac{1}{2}\Phi X + \chi^\alpha W_\alpha - \frac{i}{2\kappa}Y\right] + \text{c.c.} + \text{derivative.}\end{aligned} \tag{4.8}$$

The new term induced by the deformation of $\delta^* W_\alpha$ is proportional to the four-form field described by the chiral superfield Y, as explained in Section 2.4 [see Eq. (2.33)]. This modified Chern–Simons interaction, invariant under the deformed second supersymmetry variations, may be simply added to the Maxwell DBI theory (4.6). We then consider the Lagrangian

$$\begin{aligned}\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} &\left[ig\mathcal{Y}\left(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right) - \frac{1}{4}\xi_1\tilde{\theta}\tilde{\theta}\mathcal{W} + \frac{1}{2}\Lambda\left(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right)^2\right] \\ &+ \text{c.c.},\end{aligned} \tag{4.9}$$

for the constrained Maxwell and single-tensor multiplets, keeping the Fayet–Iliopoulos coefficient ξ_1 arbitrary. For a coherent theory with a propagating single-tensor multiplet, a kinetic Lagrangian $\mathcal{L}_{ST}$ [Eq. (2.7)] should also be added. Since

$$\int d^2\theta \int d^2\tilde{\theta} \left[ig\mathcal{Y}\mathcal{W} - \frac{1}{4}\xi_1\tilde{\theta}\tilde{\theta}\mathcal{W}\right] + \text{c.c.}$$
$$= \int d^2\theta \left[g\chi W + \frac{g}{2}\Phi X - \frac{1}{4}\xi_1 X\right] + \text{c.c.} + \text{derivative},$$

we see that the Fayet–Iliopoulos term is equivalent to a constant real shift of Φ which, according to variations (2.5), partially breaks supersymmetry. We will choose to expand Φ around $\langle\Phi\rangle = 0$ and keep $\xi_1 \neq 0$.

Again, the constraint (4.1) imposed by the Lagrange multiplier Λ can be solved to express X as a function of WW: $X = X(WW)$. The result is [7]

$$X(WW) = \kappa WW - \kappa^3\overline{DD}\left[\frac{WW\overline{WW}}{1 + \kappa^2 A + \sqrt{1 + 2\kappa^2 A + \kappa^4 B^2}}\right], \tag{4.10}$$

where A and B are defined in Appendix B. The DBI Lagrangian coupled to the single-tensor multiplet reads then

$$\mathcal{L}_{DBI} = \int d^2\theta \left[\frac{1}{4}(2g\Phi - \xi_1)X(WW) + g\chi^\alpha W_\alpha - \frac{ig}{2\kappa}Y\right] + \text{c.c.} + \mathcal{L}_{ST}. \tag{4.11}$$

The bosonic Lagrangian depends on a single auxiliary field,[16] d_2 in W_α or V_2:

$$\mathcal{L}_{DBI,bos.} = \frac{1}{8\kappa}(2g\,\mathrm{Re}\,\Phi - \xi_1)\left(1 - \sqrt{-8\kappa^2 d_2^2 - \det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}\right) - \frac{g}{2}Cd_2$$
$$+ g\epsilon^{\mu\nu\rho\sigma}\left(\frac{\kappa}{4}\,\mathrm{Im}\,\Phi\, F_{\mu\nu}F_{\rho\sigma} - \frac{1}{4}b_{\mu\nu}F_{\rho\sigma} + \frac{1}{24\kappa}C_{\mu\nu\rho\sigma}\right) + \mathcal{L}_{ST,bos.}. \tag{4.12}$$

The real scalar field C is the lowest component of the linear superfield L. Contrary to $\langle\Phi\rangle$, its background value is allowed by $N = 2$ supersymmetry. However, a nonzero $\langle C\rangle$ would induce a nonzero $\langle d_2\rangle$ which would spontaneously break the residual $N = 1$ linear supersymmetry. This is visible in the bosonic action which, after elimination of

$$d_{2,bos.} = \frac{gC}{2\kappa}\sqrt{\frac{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}{(2g\,\mathrm{Re}\,\Phi - \xi_1)^2 + 2g^2C^2}}, \tag{4.13}$$

becomes

$$\mathcal{L}_{DBI,bos.} = \frac{1}{8\kappa}(2g\,\mathrm{Re}\,\Phi - \xi_1)\left[1 - \sqrt{1 + \frac{2g^2C^2}{(2g\,\mathrm{Re}\,\Phi - \xi_1)^2}}\sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}\right]$$
$$+ g\epsilon^{\mu\nu\rho\sigma}\left(\frac{\kappa}{4}\,\mathrm{Im}\,\Phi\, F_{\mu\nu}F_{\rho\sigma} - \frac{1}{4}b_{\mu\nu}F_{\rho\sigma} + \frac{1}{24\kappa}C_{\mu\nu\rho\sigma}\right) + \mathcal{L}_{ST,bos.}. \tag{4.14}$$

First of all, as expected, the theory includes a DBI Lagrangian for the Maxwell field strength $F_{\mu\nu}$, with scale $\sim\kappa$. With the Chern–Simons coupling to the single-tensor multiplet, the DBI term acquires a field-dependent coefficient,

[16] Since $X(WW)|_{\theta=0}$ is a function of fermion bilinears, the auxiliary f_Φ does not contribute to the bosonic Lagrangian and χ_α does not include any auxiliary field.

$$-\frac{1}{8\kappa}\sqrt{(2g\,\mathrm{Re}\,\Phi-\xi_1)^2+2g^2C^2}\sqrt{-\det(\eta_{\mu\nu}+2\sqrt{2}\kappa F_{\mu\nu})}. \tag{4.15}$$

It also includes an $F \wedge F$ term which respects the axionic shift symmetry of $\mathrm{Im}\,\Phi$, a $b \wedge F$ coupling induced by (linear) $N = 2$ supersymmetry and a 'topological' C_4 term induced by the nonlinear deformation. These terms are strongly reminiscent of those found when coupling a D-brane Lagrangian to IIB supergravity. The contribution of the four-form can be eliminated by a gauge choice of the single-tensor symmetry (2.34). We have however insisted on keeping off-shell (deformed) $N = 2$ supersymmetry, hence the presence of this term.

The theory also includes a semi-positive scalar potential[17]

$$V(C,\mathrm{Re}\,\Phi)=\frac{2g\,\mathrm{Re}\,\Phi-\xi_1}{8\kappa}\left[\sqrt{1+\frac{2g^2C^2}{(2g\,\mathrm{Re}\,\Phi-\xi_1)^2}}-1\right] \tag{4.16}$$

which vanishes only if C is zero.[18] The scalar potential determines then $\langle C\rangle = 0$ but leaves $\mathrm{Re}\,\Phi$ arbitrary. Since

$$\langle d_2\rangle=\frac{g\langle C\rangle}{2\kappa}\big\langle(2g\,\mathrm{Re}\,\Phi-\xi_1)^2+2g^2C^2\big\rangle^{-1/2},$$

the vacuum line $\langle C\rangle = 0$ is compatible with linear $N = 1$ and deformed second supersymmetry. While Φ is clearly massless, C has a mass term

$$-\frac{1}{2}M_C^2C^2=-\frac{g^2}{4\kappa(2\,\mathrm{Re}\,\Phi-\xi_1)}C^2.$$

The same mass is acquired by the $U(1)$ gauge field coupled to the antisymmetric tensor $b_{\mu\nu}$, and by the goldstino (the $U(1)$ gaugino in W_α) that forms a Dirac spinor with the fermion of the linear multiplet χ_α. In other words, the Chern–Simons coupling χW pairs the Maxwell goldstino with the linear multiplet to form a massive vector, while the chiral multiplet Φ remains massless with no superpotential.

At $\langle C\rangle = \langle\mathrm{Re}\,\Phi\rangle = 0$, gauge kinetic terms are canonically normalized if $\xi_1 = -\kappa^{-1}$. The Maxwell DBI theory (4.4) is of course recovered when the Chern–Simons interaction decouples with $g = 0$. Notice finally that the kinetic terms $\mathcal{L}_{ST}$ of the single-tensor multiplet are given by Eq. (2.7), as with linear $N = 2$ supersymmetry. Since the nonlinear deformation of the second supersymmetry does not affect δ^*L or $\delta^*\Phi$ even if $\langle\mathrm{Re}\,\Phi\rangle \neq 0$, the function $\mathcal{H}$ remains completely arbitrary.

The phenomenon described above provides a first instance of a super-Higgs mechanism without gravity: the nonlinear goldstino multiplet is 'absorbed' by the linear multiplet to form a massive vector $N = 1$ superfield. One may wonder how this can happen without gravity; normally one expects that the goldstino can be absorbed only by the gravitino in local supersymmetry. The reason of this novel mechanism is that the goldstino sits in the same multiplet of the linear supersymmetry as a gauge field which has a Chern–Simons interaction with the tensor multiplet. This will become clearer in Section 5, where we will show by a change of variables that this coupling is equivalent to an ordinary gauge interaction with a charged hypermultiplet, providing non-derivative gauge couplings to the goldstino. Actually, this particular super-Higgs mechanism is an explicit realization of a phenomenon known in string theory where the $U(1)$ field of

[17] We only consider $2g\,\mathrm{Re}\,\Phi - \xi_1 > 0$, in order to have well-defined positive gauge kinetic terms.

[18] With respect to $\mathrm{Re}\,\Phi$, the potential is stationary, $\frac{\partial V}{\partial\,\mathrm{Re}\,\Phi} = 0$, only if $C = 0$. All local minima are then characterized by $C = 0$ and $\mathrm{Re}\,\Phi$ arbitrary and are then (supersymmetric) global minima.

the D-brane world-volume becomes in general massive due to a Chern–Simons interaction with the R–R antisymmetric tensor of a bulk hypermultiplet.[19]

We have chosen a description in terms of the single-tensor multiplet because it admits an off-shell formulation well adapted to our problem. Our DBI Lagrangian (4.9), supplemented with kinetic terms $\mathcal{L}_{ST}$, admits however several duality transformations. Firstly, since it only depends on $\mathcal{W}$, we may perform an electric–magnetic duality transformation, as described in Section 4.4. Then, for any choice of $\mathcal{L}_{ST}$, we can transform the linear $N=1$ superfield L into a chiral Φ'. The resulting theory is a hypermultiplet formulation with superfields (Φ, Φ') and $N=2$ supersymmetry realized only on-shell. As already explained in Section 2.3, the $b \wedge F$ interaction is replaced by a Stückelberg gauging of the axionic shift symmetry of the new chiral Φ': the Kähler potential of the hypermultiplet formulation is a function of $\Phi' + \bar{\Phi}' - gV_2$. Explicit formulae are given in the next subsection and in Section 5 we will use this mechanism in the case of non-linear $N=2$ QED. Finally, if kinetic terms $\mathcal{L}_{ST}$ also respect the shift symmetry of $\operatorname{Im}\Phi$, the chiral Φ can be turned into a second linear superfield L', leading to two formulations which are also briefly described below.

4.3. Hypermultiplet, double-tensor and single-tensor dual formulations

As already mentioned, using the single-tensor multiplet is justified by the existence of an off-shell $N=2$ formulation. The hypermultiplet formulation, with two $N=1$ chiral superfields, is however more familiar and the first purpose of this subsection is to translate our results into this formalism. In the DBI theory (4.11), the linear superfield L only appears in

$$\mathcal{L}_{ST} + g\int d^2\theta\, \chi^\alpha W_\alpha + \text{c.c.} = \int d^2\theta\, d^2\bar{\theta}\, \big[\mathcal{H}(L, \Phi, \bar{\Phi}) + gLV_2\big] + \text{derivative}.$$

These contributions are not invariant under δ^* variations: the nonlinear deformation acts on W_α and on V_2. Nevertheless, the linear superfield can be transformed into a new chiral superfield Φ'. The resulting 'hypermultiplet formulation' has Lagrangian

$$\mathcal{L}_{DBI,hyper.} = \int d^2\theta\, d^2\bar{\theta}\, \mathcal{K}\big(\Phi' + \bar{\Phi}' - gV_2, \Phi, \bar{\Phi}\big) + \int d^2\theta \left[\frac{1}{4}(2g\Phi - \xi_1)X(WW) - \frac{ig}{2\kappa}Y\right] + \text{c.c.} \tag{4.17}$$

The Kähler potential is given by the Legendre transformation

$$\mathcal{K}\big(\Phi' + \bar{\Phi}', \Phi, \bar{\Phi}\big) = \mathcal{H}(U, \Phi, \bar{\Phi}) - U\big(\Phi' + \bar{\Phi}'\big), \tag{4.18}$$

where U is the solution of

$$\frac{\partial}{\partial U}\mathcal{H}(U, \Phi, \bar{\Phi}) = \Phi' + \bar{\Phi}'. \tag{4.19}$$

In the single-tensor formulation, $N=2$ supersymmetry implies that $\mathcal{H}$ solves Laplace equation. As a result of the Legendre transformation, the determinant of $\mathcal{K}$ is constant and the metric is hyperkähler [5]. It should be noted that the Legendre transformation defines the new auxiliary scalar $f_{\Phi'}$ of Φ' according to

[19] This can be avoided in the orientifold case: the $N=2$ bulk supermultiplets are truncated by the orientifold projection.

$$f_{\Phi'} = \left(\frac{\partial^2 \mathcal{H}}{\partial U\, \partial \Phi}\right)_{\theta=0} f_\Phi. \tag{4.20}$$

Hence, the hypermultiplet formulation has the same number of independent auxiliary fields as the single-tensor version: d_2 and f_Φ.

The second supersymmetry variation δ^* of Φ' is also defined by transformation (4.19): in the hypermultiplet formulation, $N = 2$ is realized on-shell only, using the Lagrangian function. The nonlinear deformation of variations δ^* acts on V_2. Since $W_\alpha = -\frac{1}{4}\overline{DD}D_\alpha V_2$, Eq. (4.5) indicates that

$$\delta^* V_2 = \frac{i}{\sqrt{2}\kappa}(\overline{\theta\theta}\theta\eta - \theta\theta\overline{\theta\eta}) + \sqrt{2}i(\eta D + \overline{\eta D})V_1.$$

The κ-dependent term in the δ^* variation of the Kähler potential term in $\mathcal{L}_{DBI,hyper.}$ is then the same as the κ-dependent part in $g\delta^* \int d^2\theta\, \chi^\alpha W_\alpha + \text{c.c.}$, which is compensated by the variation of the four-form field. This can again be shown using Eqs. (4.18) and (4.19). This hypermultiplet formulation will be used in Section 5, on the example of nonlinear DBI QED with a charged hypermultiplet.

For completeness, let us briefly mention two further formulations of the same DBI theory, using either a double-tensor, or a dual single-tensor $N = 2$ multiplet. These possibilities appear if Lagrangian (4.11) has a second shift symmetry of $\text{Im}\,\Phi$. This is the case if the single-tensor kinetic Lagrangian has this isometry:

$$\mathcal{L}_{ST} = \int d^2\theta\, d^2\bar{\theta}\, \mathcal{H}(L, \Phi + \bar{\Phi}).$$

We may then transform Φ into a linear superfield L' using an $N = 1$ duality transformation. Keeping L and turning Φ into L' leads to a double-tensor formulation with superfields (L, L'). The Lagrangian has the form

$$\mathcal{L}_{DT} = \int d^2\theta\, d^2\bar{\theta}\, \mathcal{G}\big(L, L' - gV_1(WW)\big) - \int d^2\theta \left[\frac{1}{4}\xi_1 X(WW) - g\chi^\alpha W_\alpha + \frac{ig}{2\kappa}Y\right] + \text{c.c.} \tag{4.21}$$

The function $\mathcal{G}$ is the Legendre transform of $\mathcal{H}$ with respect to its second variable $\Phi + \bar{\Phi}$ and the real superfield $V_1(WW)$ is defined by the equation

$$X(WW) = \frac{1}{2}\overline{DD}V_1(WW). \tag{4.22}$$

It includes the DBI gauge kinetic term in its d_1 component and the Lagrangian depends on the new tensor $b'_{\mu\nu}$ through the combination $3\partial_{[\mu}b'_{\nu\rho]} - g\omega_{\mu\nu\rho}$, where $\omega_{\mu\nu\rho} = 3A_{[\mu}F_{\nu\rho]}$ is the Maxwell Chern–Simons form.

Finally, turning Φ and L into L' and Φ', leads to another single-tensor theory with a Stückelberg gauging of both Φ' and L', as in theory (2.21). In this case, the Lagrangian is

$$\mathcal{L}_{ST'} = \int d^2\theta\, d^2\bar{\theta}\, \tilde{\mathcal{H}}\big(\Phi' + \bar{\Phi}' - gV_2, L' - gV_1(WW)\big) - \int d^2\theta \left[\frac{1}{4}\xi_1 X(WW) + \frac{ig}{2\kappa}Y\right] + \text{c.c.} \tag{4.23}$$

While in the double-tensor theory (4.21) the second nonlinear supersymmetry only holds on-shell, it is valid off-shell in theory (4.23). The function $\tilde{\mathcal{H}}$ verifies Laplace equation, as required by $N=2$ linear supersymmetry.[20] Using the supersymmetric Legendre transformation, one can show that the nonlinear deformation of $\delta^* V_2$, which affects $\delta^*\tilde{\mathcal{H}}$, is again balanced by the variation of the four-form superfield Y.

4.4. The magnetic dual

To perform electric–magnetic duality on theory (4.9), we first replace it with

$$\begin{aligned}\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \Big[ig\mathcal{Y}\Big(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\Big) - \frac{1}{4}\xi_1\tilde{\theta}\tilde{\theta}\mathcal{W} \\ + \frac{1}{4}\Lambda U^2 - \frac{1}{2}\Upsilon\Big(U - \mathcal{W} + \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\Big)\Big] + \text{c.c.} + \mathcal{L}_{ST}.\end{aligned} \tag{4.24}$$

Both U and Υ are unconstrained chiral $N=2$ superfields. The Lagrange multiplier Υ imposes $U = \mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}$, which leads again to theory (4.9). The first two terms, which have gauge and $N=2$ invariance properties related to the Maxwell character of $\mathcal{W}$ are left unchanged. The term quadratic in $\mathcal{W}$ has been turned into a linear one using the Lagrange multiplier. Hence, the Maxwell superfield $\mathcal{W}$, which contributes to Lagrangian (4.24) by

$$\int d^2\theta \int d^2\tilde{\theta}\, \mathcal{W}\Big(ig\mathcal{Y} + \frac{1}{2}\Upsilon - \frac{1}{4}\xi_1\tilde{\theta}\tilde{\theta}\Big) + \text{c.c.}, \tag{4.25}$$

can as well be eliminated: Υ should be such that this contribution is a derivative. In terms of $N=1$ chiral superfields, $\mathcal{W}$ has components X and W_α and since there exists two real superfields V_1 and V_2 such that $X = \frac{1}{2}\overline{DD}V_1$ and $W_\alpha = -\frac{1}{4}\overline{DD}D_\alpha V_2$, we actually need to eliminate V_1 and V_2 with result

$$\Upsilon = -i\tilde{\mathcal{W}} - 2ig\mathcal{Y} + \frac{1}{2}(\xi_1 + i\zeta)\tilde{\theta}\tilde{\theta}. \tag{4.26}$$

In this expression, $\tilde{\mathcal{W}}$ is a Maxwell $N=2$ superfield, the 'magnetic dual' of the eliminated $\mathcal{W}$. There is a new arbitrary real deformation parameter ζ, allowed by the field equation of V_2. Notice however that $\xi_1 + i\zeta$ can be eliminated by a constant complex shift of Φ. Invariance of Υ under the single-tensor gauge variation (3.8) implies that $\delta\tilde{\mathcal{W}} = 2g\hat{\mathcal{W}} = -2g\delta\mathcal{Y}$ and

$$\mathcal{Z} \equiv \tilde{\mathcal{W}} + 2g\mathcal{Y} \tag{4.27}$$

is then a gauge-invariant chiral superfield. As already mentioned, any unconstrained chiral $N=2$ superfield can be decomposed in this way and our theory may as well be considered as a description of the chiral superfields $\mathcal{Z}$ and $\mathcal{Y}$ with Lagrangian

$$\begin{aligned}\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \Big[\frac{1}{4}\Lambda U^2 + iU\Big(\frac{1}{2}\mathcal{Z} + \frac{i}{4}(\xi_1 + i\zeta)\tilde{\theta}\tilde{\theta}\Big) + \frac{i}{4\kappa}\tilde{\theta}\tilde{\theta}(\mathcal{Z} - 2g\mathcal{Y})\Big] \\ + \text{c.c.} + \mathcal{L}_{ST}.\end{aligned} \tag{4.28}$$

Invariance under the second supersymmetry implies that $\mathcal{Z} + \frac{i}{2}(\xi_1 + i\zeta)\tilde{\theta}\tilde{\theta}$ transforms as a standard $N=2$ chiral superfield and then

$$\delta^*_{deformed}\mathcal{Z} = i(\xi_1 + i\zeta)\tilde{\theta}\eta + i(\eta\tilde{Q} + \bar{\eta}\bar{\tilde{Q}})\mathcal{Z}. \tag{4.29}$$

[20] See Eq. (2.7).

Eliminating U leads finally to

$$\tilde{\mathcal{L}}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\Lambda} \left(\mathcal{Z} + \frac{i}{2}(\xi_1 + i\zeta)\tilde{\theta}\tilde{\theta} \right)^2 + \frac{i}{4\kappa}\tilde{\theta}\tilde{\theta}(\mathcal{Z} - 2g\mathcal{Y}) \right] + \text{c.c.} + \mathcal{L}_{ST}, \tag{4.30}$$

which is the electric–magnetic dual of theory (4.9).[21] The Lagrange multiplier superfield Λ^{-1} implies now the constraint

$$0 = \left(\mathcal{Z} + \frac{i}{2}(\xi_1 + i\zeta)\tilde{\theta}\tilde{\theta} \right)^2 = \mathcal{Z}^2 + i(\xi_1 + i\zeta)\tilde{\theta}\tilde{\theta}\mathcal{Z}. \tag{4.31}$$

Using the expansion (3.3),

$$\mathcal{Z}(y, \theta, \tilde{\theta}) = Z(y, \theta) + \sqrt{2}\tilde{\theta}\omega(y, \theta) - \tilde{\theta}\tilde{\theta}\left[\frac{i}{2}\Phi_{\mathcal{Z}}(y, \theta) + \frac{1}{4}\overline{DD}\bar{Z}(y, \theta) \right],$$

with $Z = \tilde{X} + 2gY$, $\omega_\alpha = i\tilde{W}_\alpha + 2g\chi_\alpha$ and $\Phi_{\mathcal{Z}} = 2g\Phi$, this constraint corresponds to

$$Z^2 = 0, \qquad Z\omega_\alpha = 0, \qquad \frac{1}{2}Z\overline{DD}\bar{Z} + \omega\omega = -iZ\left[\Phi_{\mathcal{Z}} - (\xi_1 + i\zeta)\right].$$

In this case, and in contrast to the electric case, the constraint leading to the DBI theory is due to the scale $\langle\Phi_{\mathcal{Z}}\rangle = 2g\langle\Phi\rangle$: we will actually choose $\zeta = 0$, absorb ξ_1 into $\Phi_{\mathcal{Z}}$ and consider the constraint $\mathcal{Z}^2 = 0$ with a nonzero background value $\langle\Phi_{\mathcal{Z}}\rangle$ breaking the second supersymmetry. Our magnetic theory is then

$$\tilde{\mathcal{L}}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\Lambda}\mathcal{Z}^2 + \frac{i}{4\kappa}\tilde{\theta}\tilde{\theta}(\mathcal{Z} - 2g\mathcal{Y}) \right] + \text{c.c.} + \mathcal{L}_{ST}, \tag{4.32}$$

with constraints

$$Z^2 = 0, \qquad Z\omega_\alpha = 0, \qquad \frac{1}{2}Z\overline{DD}\bar{Z} + \omega\omega = -iZ\Phi_{\mathcal{Z}}, \tag{4.33}$$

the DBI scale arising from $\Phi_{\mathcal{Z}} = \phi_{\mathcal{Z}} + \langle\Phi_{\mathcal{Z}}\rangle$. As in the Maxwell case, the third equation, which also reads

$$Z = \frac{i\omega\omega}{\Phi_{\mathcal{Z}} - \frac{i}{2}\overline{DD}\bar{Z}}, \tag{4.34}$$

implies $Z\omega_\alpha = Z^2 = 0$ and allows to express Z as a function of $\omega\omega$ and Φ, $Z = Z(\omega\omega, \Phi)$, using $\Phi_{\mathcal{Z}} = 2g\Phi - \xi_1$. The magnetic theory (4.32) is then simply

$$\tilde{\mathcal{L}}_{DBI} = -\frac{1}{2\kappa}\,\text{Im}\int d^2\theta \left[Z(\omega\omega, \Phi) - 2gY\right] + \mathcal{L}_{ST}. \tag{4.35}$$

It is the electric–magnetic dual of expression (4.11). At this point, it is important to recall that ω and Φ are actually $N = 1$ superfields components of $\mathcal{Z} = \tilde{\mathcal{W}} + 2g\mathcal{Y}$, *i.e.*

$$\omega_\alpha = i\tilde{W}_\alpha + 2g\chi_\alpha. \tag{4.36}$$

The kinetic terms for the single-tensor multiplet (L, Φ), $L = D\chi - \bar{D}\bar{\chi}$, are included in $\mathcal{L}_{ST}$ while $Z(\omega\omega, \Phi)$ includes the DBI kinetic terms for the Maxwell $N = 1$ superfield $\tilde{W}_\alpha$. As in the

[21] It reduces to Eq. (4.7) if $g = 0$.

electric case, the magnetic theory has a contribution proportional to the four-form field included in Y.

The third constraint (4.33) is certainly invariant under the variations (3.4), using $Z\omega_\alpha = 0$. But with a nonzero background value $\Phi = \phi + \langle\Phi\rangle$, the spinor ω_α transforms nonlinearly, like a goldstino[22]:

$$\delta^*\omega_\alpha = -\frac{i}{\sqrt{2}}\langle\Phi\rangle\eta_\alpha - \frac{i}{\sqrt{2}}\phi\eta_\alpha - \frac{\sqrt{2}}{4}\eta_\alpha\overline{DD}\bar{Z} - \sqrt{2}i\big(\sigma^\mu\bar{\eta}\big)_\alpha\partial_\mu Z. \tag{4.37}$$

The solution of the constraint (4.34) is given in Appendix B. The bosonic Lagrangian included in the magnetic theory (4.35) is

$$\begin{aligned}\tilde{\mathcal{L}}_{DBI,bos.} = {} & \frac{\operatorname{Re}\Phi_{\mathcal{Z}}}{8\kappa} - \frac{\operatorname{Re}\Phi_{\mathcal{Z}}}{8\kappa|\Phi_{\mathcal{Z}}|^2}\big\{-|\Phi_{\mathcal{Z}}|^4\det\big[\eta_{\mu\nu} - 2\sqrt{2}|\Phi_{\mathcal{Z}}|^{-1}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})\big] \\ & - 8\tilde{d}_2^2\big(|\Phi_{\mathcal{Z}}|^2 + 2g^2C^2\big) + 2g^2C^2|\Phi_{\mathcal{Z}}|^2 \\ & + 8gC\tilde{d}_2\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma})\big\}^{1/2} \\ & - \frac{\operatorname{Im}\Phi_{\mathcal{Z}}}{8\kappa|\Phi_{\mathcal{Z}}|^2}\big[\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) - 4gC\tilde{d}_2\big] \\ & + \frac{g}{24\kappa}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \mathcal{L}_{ST,bos.}.\end{aligned} \tag{4.38}$$

It depends on a single auxiliary field, the Maxwell real scalar $\tilde{d}_2$, with field equation

$$\begin{aligned}\tilde{d}_{2,bos.} = {} & -\frac{gC}{2(|\Phi_{\mathcal{Z}}|^2 + 2g^2C^2)}\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) \\ & - \frac{gC\operatorname{Im}\Phi_{\mathcal{Z}}}{2|\Phi_{\mathcal{Z}}|^2}\frac{\sqrt{-\det(\eta_{\mu\nu} + \frac{2\sqrt{2}}{\sqrt{2g^2C^2+|\Phi_{\mathcal{Z}}|^2}}(\tilde{F}_{\mu\nu} - gb_{\mu\nu}))}}{\sqrt{(\operatorname{Re}\Phi_{\mathcal{Z}})^2 + 2g^2C^2}}.\end{aligned} \tag{4.39}$$

Eliminating $\tilde{d}_2$ and using $\Phi_{\mathcal{Z}} = 2g\Phi - \xi_1$ to reintroduce the superfield Φ of the single-tensor multiplet and the 'original' Fayet–Iliopoulos term ξ_1, we finally obtain the magnetic, bosonic Lagrangian

$$\begin{aligned}\tilde{\mathcal{L}}_{DBI,bos.} = {} & \frac{2g\operatorname{Re}\Phi - \xi_1}{8\kappa} - \frac{1}{8\kappa}\sqrt{(2g\operatorname{Re}\Phi - \xi_1)^2 + 2g^2C^2} \\ & \times\sqrt{-\det\left(\eta_{\mu\nu} - \frac{2\sqrt{2}}{\sqrt{2g^2C^2 + |2g\Phi - \xi_1|^2}}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})\right)} \\ & - \frac{g\operatorname{Im}\Phi}{4\kappa(2g^2C^2 + |2g\Phi - \xi_1|^2)}\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) \\ & + \frac{g}{24\kappa}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \mathcal{L}_{ST,bos.}.\end{aligned} \tag{4.40}$$

As in the electric case, the DBI term has a field-dependent coefficient,

[22] See Eq. (4.29).

$$-\frac{1}{8\kappa}\sqrt{(2g\,\mathrm{Re}\,\Phi-\xi_1)^2+2g^2C^2}\sqrt{-\det\left(\eta_{\mu\nu}-\frac{1}{\sqrt{2g^2C^2+|2g\Phi-\xi_1|^2}}(\tilde{F}_{\mu\nu}-gb_{\mu\nu})\right)}, \tag{4.41}$$

and, as expected, the scalar potentials of the magnetic and electric [Eq. (4.16)] theories are identical.

Define the complex dimensionless field

$$S=\kappa\sqrt{(2g\,\mathrm{Re}\,\Phi-\xi_1)^2+2g^2C^2}+2i\kappa g\,\mathrm{Im}\,\Phi, \tag{4.42}$$

for which $\kappa^{-2}|S|^2=|2g\Phi-\xi_1|^2+2g^2C^2$. In terms of S, the magnetic theory (4.40) rewrites as

$$\begin{aligned}\tilde{\mathcal{L}}_{DBI,bos.}&=\frac{2g\,\mathrm{Re}\,\Phi-\xi_1}{8\kappa}-\frac{1}{8\kappa^2}\mathrm{Re}\,\frac{1}{S}\sqrt{-\det\bigl(|S|\eta_{\mu\nu}-2\sqrt{2}\kappa(\tilde{F}_{\mu\nu}-gb_{\mu\nu})\bigr)}\\&\quad+\frac{1}{8}\mathrm{Im}\,\frac{1}{S}\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu}-gb_{\mu\nu})(\tilde{F}_{\rho\sigma}-gb_{\rho\sigma})+\frac{g}{24\kappa}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}+\mathcal{L}_{ST,bos.}\\&=\frac{2g\,\mathrm{Re}\,\Phi-\xi_1}{8\kappa}-\frac{1}{8\kappa^2}\mathrm{Re}\,S\sqrt{-\det\bigl(\eta_{\mu\nu}-2\sqrt{2}\kappa|S|^{-1}(\tilde{F}_{\mu\nu}-gb_{\mu\nu})\bigr)}\\&\quad+\frac{1}{8}\mathrm{Im}\,\frac{1}{S}\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu}-gb_{\mu\nu})(\tilde{F}_{\rho\sigma}-gb_{\rho\sigma})+\frac{g}{24\kappa}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}+\mathcal{L}_{ST,bos.}.\end{aligned} \tag{4.43}$$

This is to be compared with the electric theory (4.14):

$$\begin{aligned}\mathcal{L}_{DBI,bos.}&=\frac{2g\,\mathrm{Re}\,\Phi-\xi_1}{8\kappa}-\frac{1}{8\kappa^2}\mathrm{Re}\,S\sqrt{-\det(\eta_{\mu\nu}-2\sqrt{2}\kappa F_{\mu\nu})}\\&\quad+\frac{1}{8}\mathrm{Im}\,S\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}-\frac{g}{4}\epsilon^{\mu\nu\rho\sigma}b_{\mu\nu}F_{\rho\sigma}+\frac{g}{24\kappa}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}+\mathcal{L}_{ST,bos.}.\end{aligned} \tag{4.44}$$

Hence, the duality from the electric to the magnetic theory corresponds to the transformations

$$b_{\mu\nu}\to 0,\qquad F_{\mu\nu}\to\tilde{F}_{\mu\nu}-gb_{\mu\nu},\qquad S\to S^{-1},\qquad \eta_{\mu\nu}\to|S|\eta_{\mu\nu}, \tag{4.45}$$

which can be also derived from electric–magnetic duality applied on the bosonic DBI theory only. The inversion of S combined with its imaginary shift, which is a symmetry of (4.44), generate $SL(2,\mathbb{R})$.

4.5. Double-tensor formulation and connection with the string fields

As mentioned in the introduction, in IIB superstrings compactified to four dimensions with eight residual supercharges, the dilaton belongs to a double-tensor supermultiplet. This representation of $N=2$ supersymmetry includes two Majorana spinors, two antisymmetric tensors $B_{\mu\nu}$ (NS–NS) and $C_{\mu\nu}$ (R–R) with gauge symmetries

$$\delta_{gauge}B_{\mu\nu}=2\partial_{[\mu}\Lambda_{\nu]},\qquad \delta'_{gauge}C_{\mu\nu}=2\partial_{[\mu}\Lambda'_{\nu]} \tag{4.46}$$

and two (real) scalar fields, the NS–NS dilaton and the R–R scalar, for a total of 4_B+4_F physical states. In principle, both antisymmetric tensors can be dualized to pseudoscalar fields with axionic shift symmetry, in a version of the effective field theory where the dilaton belongs to a hypermultiplet with four scalars in a quaternion-Kähler[23] manifold possessing three perturbative shift isometries, since the R–R scalar has its own shift symmetry. It is easy to see that only two shift isometries, related to the two antisymmetric tensors, commute, while all three together form the Heisenberg algebra. Indeed, in the double-tensor basis, the R–R field strength is modified [25] due to its anomalous Bianchi identity to $3\partial_{[\lambda}C_{\mu\nu]} - 3C^{(0)}\partial_{[\lambda}B_{\mu\nu]}$. Thus, a shift of the R–R scalar $C^{(0)}$ by a constant λ is accompanied by an appropriate transformation of $C_{\mu\nu}$ to leave its modified field-strength invariant:

$$\delta_H C^{(0)} = \lambda, \qquad \delta_H C_{\mu\nu} = \lambda B_{\mu\nu}. \tag{4.47}$$

It follows that δ_{gauge}, δ'_{gauge} and δ_H verify the Heisenberg algebra, with a single non-vanishing commutator

$$[\delta_{gauge}, \delta_H] = \delta'_{gauge}. \tag{4.48}$$

To establish the connection of the general formalism described in the previous subsections with string theory, we would like to identify the double-tensor multiplet with the universal dilaton hypermultiplet and study its coupling to the Maxwell goldstino multiplet of a single D-brane, in the rigid (globally-supersymmetric) limit. To this end, we transform the $N = 2$ double-tensor into a single-tensor representation by dualizing one of its two $N = 1$ linear multiplet components L', containing the R–R fields $C_{\mu\nu}$ and $C^{(0)}$, into a chiral basis $\Phi + \bar{\Phi}$. In this basis, the two R–R isometries correspond to constant complex shifts of the $N = 1$ superfield Φ. Imposing this symmetry to the kinetic function of Eqs. (2.7)–(2.8), one obtains (up to total derivatives, after superspace integration):

$$\mathcal{H}(L, \Phi, \bar{\Phi}) = \alpha\left(-\frac{1}{3}L^3 + \frac{1}{2}L(\Phi + \bar{\Phi})^2\right) + \beta\left(-L^2 + \frac{1}{2}(\Phi + \bar{\Phi})^2\right), \tag{4.49}$$

where α and β are constants. Note that the second term proportional to β can be obtained from the first by shifting $L + \beta/\alpha$. For $\alpha = 0$ however, it corresponds to the free case of quadratic kinetic terms for all fields of the single-tensor multiplet. The coupling to the Maxwell goldstino multiplet is easily obtained using Eqs. (4.12), (4.22) and (2.18). Up to total derivatives, the action is:

$$\begin{aligned}\mathcal{L} = \int d^2\theta\, d^2\bar{\theta}\Bigg[&\alpha\left(-\frac{1}{3}L^3 + \frac{1}{2}L(\Phi + \bar{\Phi})^2\right) + \beta\left(-L^2 + \frac{1}{2}(\Phi + \bar{\Phi})^2\right) \\ &- g(\Phi + \bar{\Phi})V_1(WW)\Bigg] + g\int d^2\theta\left[\chi^\alpha W_\alpha - \frac{i}{2\kappa}Y - \frac{\xi_1}{4g}X(WW)\right] + \text{c.c.}\end{aligned} \tag{4.50}$$

In general, the four-form field is not inert under the variation δ_H of Eq. (4.47) [26]. In our single-tensor formalism, $\delta_H L = 0$ and $\delta_H \Phi = c$ where c is complex when combined with the axionic shift δ'_{gauge} of $\mathrm{Im}\,\Phi$ dual to $C_{\mu\nu}$ of Eq. (4.46); in addition

$$\delta_H Y = -ic\kappa X(WW). \tag{4.51}$$

With this variation, the Lagrangian, including the Chern–Simons interaction, is invariant under the Heisenberg symmetry.

We can now dualize back $\Phi + \bar{\Phi}$ to a second linear multiplet L' by first replacing it with a real superfield U:

[23] For supergravity. The limit of global supersymmetry is a hyperkähler manifold, which is Ricci-flat.

$$\mathcal{L} = \int d^2\theta\, d^2\bar{\theta} \left[\alpha\left(-\frac{1}{3}L^3 + \frac{1}{2}LU^2\right) + \beta\left(-L^2 + \frac{1}{2}U^2\right) - U\left(mL' + gV_1\right)\right]$$
$$+ g\int d^2\theta \left[\chi^\alpha W_\alpha - \frac{i}{2\kappa}Y - \frac{\xi_1}{4g}X\right] + \text{c.c.}, \tag{4.52}$$

where the constant m corresponds to a rescaling of L'. Solving for U,

$$U = \frac{mL' + gV_1}{\alpha L + \beta}, \tag{4.53}$$

delivers the double-tensor Lagrangian

$$\tilde{\mathcal{L}} = \int d^2\theta\, d^2\bar{\theta} \left[-\frac{\alpha}{3}L^3 - \beta L^2 - \frac{1}{2}\frac{(mL' + gV_1)^2}{\alpha L + \beta}\right]$$
$$+ g\int d^2\theta \left[\chi^\alpha W_\alpha - \frac{i}{2\kappa}Y - \frac{\xi_1}{4g}X\right] + \text{c.c.}, \tag{4.54}$$

where as before $V_1 = V_1(WW)$ and $X = X(WW) = \frac{1}{2}\overline{DD}V_1(WW)$. It is invariant under variation (4.51) of the four-form superfield combined with $\delta_H L' = 2c(\alpha L + \beta)/m$.

After elimination of the Maxwell auxiliary field (choosing $m = \sqrt{2}$)

$$d_{2,bos.} = \frac{gC}{2\kappa}\sqrt{\frac{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}{(\frac{\sqrt{2}gC'}{\alpha C+\beta} - \xi_1)^2 + 2g^2C^2}}, \tag{4.55}$$

the component expansion of the bosonic Lagrangian is

$$\tilde{\mathcal{L}}_{bos.} = (\alpha C + \beta)\left[\frac{1}{2}(\partial_\mu C)^2 + \frac{1}{2}\partial_\mu\left(\frac{C'}{\alpha C + \beta}\right)^2 + \frac{1}{12}(3\partial_{[\mu}b_{\nu\rho]})^2\right]$$
$$+ \frac{1}{12(\alpha C + \beta)}\left(3\partial_{[\mu}b'_{\nu\rho]} + \frac{g\kappa}{\sqrt{2}}\omega_{\mu\nu\rho} - \frac{C'}{\alpha C + \beta}3\partial_{[\mu}b_{\nu\rho]}\right)^2$$
$$- \frac{g}{4\kappa\sqrt{2}}\left(\frac{C'}{\alpha C + \beta} + \frac{\xi_1}{\sqrt{2}g}\right) + \frac{g}{4\kappa\sqrt{2}}\sqrt{\left(\frac{C'}{\alpha C + \beta} + \frac{\xi_1}{\sqrt{2}g}\right)^2 + C^2}$$
$$\times\sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})} - \frac{g}{4}\epsilon^{\mu\nu\rho\sigma}b_{\mu\nu}F_{\rho\sigma} + \frac{g}{24\kappa}\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} \tag{4.56}$$

in terms of the Maxwell Chern–Simons form $\omega_{\nu\rho\sigma} = 3A_{[\nu}F_{\rho\sigma]}$.

We expect that this action describes the globally-supersymmetric limit of the effective four-dimensional action of a D-brane coupled to the universal dilaton hypermultiplet of the perturbative type II string. As mentioned previously, its general form in the local case depends also on two constant parameters, upon imposing the perturbative Heisenberg isometries, that correspond to the tree and one-loop contributions [15]. It is tempting to identify these two parameters with α and β of our action. Moreover, by identifying the two antisymmetric tensors $b_{\mu\nu}$ and $b'_{\mu\nu}$ with the respective NS–NS $B_{\mu\nu}$ and R–R $C_{\mu\nu}$ and the combination $C'/(\alpha C + \beta)$ with the R–R scalar $C^{(0)}$, as the Heisenberg transformations indicate, one finds that the two actions match up to normalization factors depending on the NS–NS dilaton that should correspond to the scalar C. Finding the precise identifications, which certainly depend on the way one should take the rigid limit that decouples gravity, is an interesting question beyond our present analysis restricted to global supersymmetry.

5. Nonlinear $N = 2$ QED

We will now show that the effective theory presented above describing a super-Higgs phenomenon of partial (global) supersymmetry breaking can be identified with the Higgs phase of nonlinear $N = 2$ QED, up to an appropriate choice of the single-tensor multiplet kinetic terms. We will then analyze its vacuum structure in the generally allowed parameter space.

In linear $N = 2$ quantum electrodynamics (QED), the Lagrangian couples a hypermultiplet with two chiral superfields (Q_1, Q_2) to the vector multiplet (V_1, V_2) or (X, W_α). The $U(1)$ gauge transformations of the hypermultiplet are linear, and Q_1 and Q_2 have opposite $U(1)$ charges:

$$\begin{aligned}\mathcal{L}_{QED} = &\int d^2\theta\, d^2\bar\theta\,\big[\bar Q_1 Q_1 e^{V_2} + \bar Q_2 Q_2 e^{-V_2}\big] + \int d^2\theta\, \frac{i}{\sqrt2} X Q_1 Q_2 + \text{c.c.}\\ &+ \mathcal{L}_{Max.} + \Delta\mathcal{L},\end{aligned}\tag{5.1}$$

where $\mathcal{L}_{Max.}$ includes (canonical) gauge kinetic terms and $\Delta\mathcal{L}$ contains three parameters:

$$\Delta\mathcal{L} = m\int d^2\theta\, Q_1 Q_2 + \text{c.c.} + \int d^2\theta\, d^2\bar\theta\,[\xi_1 V_1 + \xi_2 V_2].\tag{5.2}$$

The hypermultiplet mass term with coefficient m can be eliminated by a shift of X and $\xi_{1,2}$ are the two Fayet–Iliopoulos coefficients. Since $\xi_1 \int d^2\theta\, d^2\bar\theta\, V_1 = -\frac14 \int d^2\theta\, \xi_1 X + \text{c.c.}$, the complete superpotential w is

$$w = \left(\frac{i}{\sqrt2} X + m\right) Q_1 Q_2 - \frac14 \xi_1 X.$$

There are six real auxiliary fields, f_{Q_1}, f_{Q_2}, d_1 and d_2 but only four are actually independent[24]: $Q_1 \bar f_{Q_1} = Q_2 \bar f_{Q_2}$. Since the metric is canonical, $\det K_{i\bar j} = 1$ and trivially hyperkähler. If $\xi_1 = \xi_2 = 0$, the gauge symmetry is not broken and the hypermultiplet mass $m + i\langle X\rangle/\sqrt2$ is arbitrary. Any nonzero ξ_1 or ξ_2 induces $U(1)$ symmetry breaking with all fields having the same mass. In any case, $N = 2$ supersymmetry remains unbroken at the global minimum.

In order to first bring the theory to a form allowing dualization to our single-tensor formulation, we use the holomorphic field redefinition[25]

$$\begin{aligned}&Q_1 = a\sqrt{\Phi} e^{\Phi'}, \qquad Q_2 = ia\sqrt{\Phi} e^{-\Phi'},\\ &Q_1 Q_2 = ia^2 \Phi, \qquad Q_1/Q_2 = -ie^{2\Phi'},\end{aligned}\tag{5.3}$$

with $a^2 = 1/\sqrt2$. The QED Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{QED} = &\frac{1}{\sqrt2}\int d^2\theta\, d^2\bar\theta\,\sqrt{\Phi\bar\Phi}\big[e^{\Phi'+\bar\Phi'+V_2} + e^{-\Phi'-\bar\Phi'-V_2}\big] + \mathcal{L}_{Max.}\\ &+ \int d^2\theta\left[-\frac12 \Phi(X - \sqrt2 im) - \frac14 \xi_1 X\right] + \text{c.c.} + \xi_2 \int d^2\theta\, d^2\bar\theta\, V_2.\end{aligned}\tag{5.4}$$

While the gauge transformation of Φ' is $\delta_{U(1)}\Phi' = \Lambda_c$, Φ is gauge invariant. Since the Kähler potential is now a function of $\Phi' + \bar\Phi'$, with a Stückelberg gauging of the axionic shift of Φ', the chiral Φ' can be dualized to a linear L using an $N = 1$ Legendre transformation. The result is

[24] We use the same notation for a chiral superfield Φ, Q_1, Q_2, ... and for its lowest complex scalar component field.

[25] This field redefinition has constant Jacobian.

$$\mathcal{L}_{QED} = \int d^2\theta\, d^2\bar\theta\, \big[\sqrt{2\Phi\bar\Phi + L^2} - L\ln\big(\sqrt{2\Phi\bar\Phi+L^2}+L\big)\big] + \mathcal{L}_{Max.} - \int d^2\theta \left[\frac{1}{2}X\Phi + \chi^\alpha W_\alpha - \frac{i}{\sqrt{2}}m\Phi + \frac{1}{4}\xi_1 X\right] + \text{c.c.} + \xi_2 \int d^2\theta\, d^2\bar\theta\, V_2. \quad (5.5)$$

The dual single-tensor QED theory has off-shell $N=2$ invariance (the Laplace equation (2.7) is verified) and the two multiplets are now coupled by an $N=2$ Chern–Simons interaction (2.19). Notice that the free quadratic kinetic terms of the charged hypermultiplet lead to a highly non-trivial kinetic function in the single-tensor representation. Moreover, there are only four auxiliary fields, f_Φ, d_1 and d_2. The Legendre transformation defines the scalar field C in L as

$$e^{2\,\mathrm{Re}\,\Phi'} = \frac{1}{\sqrt{2\Phi\bar\Phi}}\big(\sqrt{2\Phi\bar\Phi + C^2} + C\big), \qquad e^{-2\,\mathrm{Re}\,\Phi'} = \frac{1}{\sqrt{2\Phi\bar\Phi}}\big(\sqrt{2\Phi\bar\Phi + C^2} - C\big) \quad (5.6)$$

and Eqs. (5.3) relate then C and Φ with Q_1 and Q_2:

$$C = |Q_1|^2 - |Q_2|^2, \qquad \Phi = -\sqrt{2}i\,Q_1Q_2. \quad (5.7)$$

According to Eq. (4.11), the nonlinear DBI version of $N=2$ QED is obtained by replacing in Lagrangian (5.5) X by $X(WW)$, which includes DBI gauge kinetic terms, by omitting $\mathcal{L}_{Max.}$ which is removed by the third constraint (4.2) and by adding the four-form term $\frac{i}{2\kappa}\int d^2\theta\, Y + \text{c.c.}$:

$$\mathcal{L}_{QED,DBI} = \int d^2\theta\, d^2\bar\theta\, \big[\sqrt{2\Phi\bar\Phi + L^2} - L\ln\big(\sqrt{2\Phi\bar\Phi+L^2}+L\big) + \xi_2 V_2\big] - \int d^2\theta \left[\left(\frac{1}{2}\Phi + \frac{1}{4}\xi_1\right)X(WW) - \frac{i}{\sqrt{2}}m\Phi + \chi^\alpha W_\alpha - \frac{i}{2\kappa}Y\right] + \text{c.c.} \quad (5.8)$$

Notice that two additional terms appear compared to the action studied in Section 4: an Fayet–Iliopoulos term proportional to ξ_2 and a term linear in Φ which is also invariant under the second (nonlinear) supersymmetry (2.4); they generate, together with ξ_1 the general parameter space of nonlinear QED coupled to a charged hypermultiplet. Without loss of generality, we choose m to be real, while the choice $\xi_1 = -1/\kappa$ would canonically normalize gauge kinetic terms for a background where Φ vanishes. We may return to chiral superfields (Φ, Φ') or (Q_1, Q_2) to write the DBI theory as[26]

$$\mathcal{L}_{QED} = \int d^2\theta\, d^2\bar\theta\, \big[\bar Q_1 Q_1 e^{V_2} + \bar Q_2 Q_2 e^{-V_2} + \xi_2 V_2\big] + \int d^2\theta \left[\left(\frac{i}{\sqrt{2}}Q_1Q_2 - \frac{1}{4}\xi_1\right)X(WW) + mQ_1Q_2 + \frac{i}{2\kappa}Y\right] + \text{c.c.} \quad (5.9)$$

Since $X(WW)|_{\theta=0}$ only depends on fermion fields, the auxiliary fields f_1 and f_2 only contribute to the bosonic Lagrangian by a hypermultiplet mass term

$$\big(|f_1|^2 + |f_2|^2\big)_{bos.} = m^2\big(|Q_1|^2 + |Q_2|^2\big)$$

to be added to the scalar potential obtained from Eq. (4.16) with the substitutions

$$2g\,\mathrm{Re}\,\Phi - \xi_1 \;\to\; 2\sqrt{2}\,\mathrm{Im}(Q_1Q_2) - \xi_1, \qquad gC \;\to\; C + \xi_2 = \xi_2 + |Q_1|^2 - |Q_2|^2$$

[26] See Eq. (4.17).

(since we have chosen $g = 1$). The complete potential is then[27]

$$V_{QED,DBI} = \frac{1}{8\kappa}\big(2\sqrt{2}\,\mathrm{Im}(Q_1Q_2) - \xi_1\big)\left[\sqrt{1 + \frac{2[\xi_2 + |Q_1|^2 - |Q_2|^2]^2}{[2\sqrt{2}\,\mathrm{Im}(Q_1Q_2) - \xi_1]^2}} - 1\right] + m^2\big(|Q_1|^2 + |Q_2|^2\big). \tag{5.10}$$

The analysis is then very simple. The first line vanishes only for

$$\big\langle \xi_2 + |Q_1|^2 - |Q_2|^2 \big\rangle = 0, \qquad \big\langle 2\sqrt{2}\,\mathrm{Im}(Q_1Q_2) - \xi_1 \big\rangle > 0. \tag{5.11}$$

The first condition is the usual D-term equation $\langle d_2 \rangle = 0$ for the Maxwell superfield. The second condition is necessary to have a well-defined DBI gauge kinetic term at the minimum. Hence, if $m = 0$, conditions (5.11), which can always be solved, define the vacuum of the theory. Choosing $\langle Q_1 \rangle = v$ and $\langle Q_2 \rangle = \sqrt{v^2 + \xi_2}$, with v real (and arbitrary), we find a massive vector boson which, along with a real scalar and the two Majorana fermions

$$\frac{1}{\sqrt{2v^2 + \xi_2}}\big[v\psi_{Q_1} - \sqrt{v^2 + \xi_2}\,\psi_{Q_2}\big] \pm i\lambda,$$

makes a massive $N = 1$ vector multiplet of mass $\sqrt{v^2 + \xi_2/2}$. Hence the potentially massless gaugino λ, with its goldstino-like second supersymmetry variation $\delta^*\lambda_\alpha = -\frac{1}{\sqrt{2}\kappa}\eta_\alpha + \cdots$, has been absorbed in the massive $U(1)$ gauge boson multiplet. This is possible only because the second supersymmetry transformation of the four-form field compensates the gaugino nonlinear variation. The fermion

$$\sqrt{v^2 + \xi_2}\,\psi_{Q_1} + v\psi_{Q_2}$$

is massless and corresponds to the fermion of the chiral superfield Φ in the single-tensor formalism, in agreement with our analysis in Section 4.2 [see below Eq. (4.16)]. With two real scalars, it belongs to a massless $N = 1$ chiral multiplet.

If $m \neq 0$, a supersymmetric vacuum has $\langle Q_1 \rangle = \langle Q_2 \rangle = 0$. It only exists if $\xi_2 = 0$ and $\xi_1 \neq 0$. The second condition is again to have DBI gauge kinetic terms on this vacuum. In this case, the $U(1)$ gauge symmetry is not broken, the goldstino vector multiplet remains massless and the hypermultiplet has mass m. If $m \neq 0$, a nonzero Fayet–Iliopoulos coefficient ξ_2 breaks then $N = 1$ linear supersymmetry. Note that the single-tensor formalism is appropriate for the description of the Higgs phase of nonlinear QED in a manifest $N = 1$ superfield basis (with respect to the linear supersymmetry), while the charged hypermultiplet representation is obviously convenient for describing the Coulomb phase.

One can finally expand the action (5.9) in powers of κ in order to find the lowest-dimensional operators that couple the goldstino multiplet of partial supersymmetry breaking to the $N = 2$ hypermultiplet. Besides the dimension-four operators corresponding to the gauge factors $e^{\pm V_2}$, one obtains a dimension-six superpotential interaction $\sim \kappa Q_1 Q_2 W^2$ coming from the solution of the nonlinear constraint $X = \kappa W^2 + \mathcal{O}(\kappa^3)$; it amounts to a field-dependent correction to the $U(1)$ gauge coupling.

[27] The auxiliary d_2 is given in Eq. (4.13).

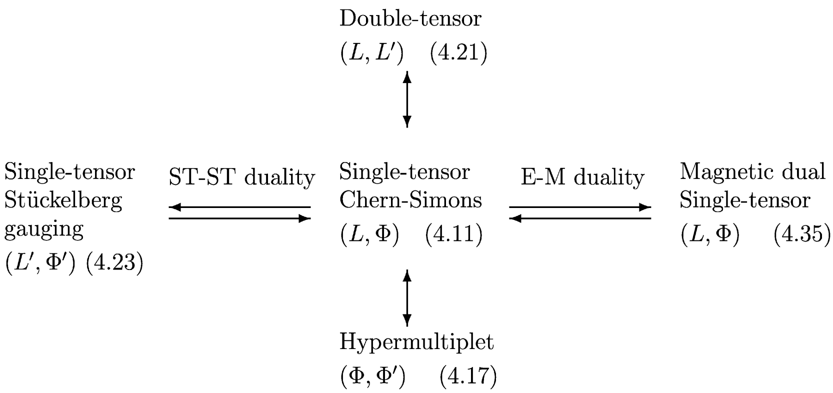

Fig. 1. Web of dualities: double arrows indicate duality transformations preserving off-shell $N = 2$ supersymmetry, simple arrows are $N = 1$ off-shell dualities only, leading to theories with on-shell $N = 2$ supersymmetry. The $N = 1$ superfields and the related equations are indicated.

6. Conclusions

In this work we have studied the interaction of the Maxwell goldstino multiplet of $N = 2$ nonlinear supersymmetry to a hypermultiplet with at least one isometry. The starting point was to describe the hypermultiplet in terms of a single-tensor multiplet, which admits an off-shell $N = 2$ formulation, and introduce a coupling using a Chern–Simons interaction. This system describes the coupling of a D-brane to bulk fields of $N = 2$ compactifications of type II strings, in the rigid limit of decoupled gravity. Using $N = 1$ and $N = 2$ dualities, we have also obtained equivalent formulations of the nonlinear Maxwell theory coupled to a matter $N = 2$ supermultiplet. This web of theories is summarized in Fig. 1.

Specializing to the case of the universal dilaton hypermultiplet, we determined the action completely in the rigid limit, using the Heisenberg symmetry of perturbative string theory, up to an arbitrary constant parameter which, in the quaternion-Kähler case of $N = 2$ supergravity, corresponds to the string one-loop correction [15]. An interesting open question is to realize this decoupling limit directly from the supergravity-coupled system.

We have shown how the above system applies to the Higgs phase of $N = 2$ nonlinear QED coupled to a charged hypermultiplet. Allowing a hypermultiplet mass scale and a Fayet–Iliopoulos term in the two-dimensional parameter space, the vacuum structure includes phases with broken and unbroken linear $N = 1$ supersymmetry and/or $U(1)$ gauge symmetry.

It is interesting to note that in the Higgs phase the goldstino vector multiplet combines with the hypermultiplet to form an $N = 1$ massive vector and a massless chiral superfield. This novel super-Higgs mechanism is possible without gravity because the hypermultiplet is charged under the $U(1)$ partner of the goldstino. In the $N = 1$ case, the goldstino multiplet can be gauged only by gravity and is absorbed by the gravitino that acquires a mass.

In principle, it is straightforward to introduce additional hypermultiplets. Obviously only one of them will ‘absorb’ the goldstino providing mass to the $U(1)$ vector. This action describes also the low-energy limit of spontaneous partial supersymmetry breaking $N = 2 \to N = 1$, when the breaking is ‘small’ in the matter (hypermultiplet) sector. This is analogous, in the case of a single $N = 1$ nonlinear supersymmetry, to the effective action of the goldstino coupled to $N = 1$ multiplets at energies higher than their soft breaking masses. It is then known that this action is obtained by simply identifying the constrained goldstino multiplet with the so-called spurion [23]. One may try to develop the analogy in the $N=2$ nonlinear case and derive the structure of possible 'soft' terms associated to the partial $N=2 \rightarrow N=1$ breaking. As a step further, one could try to integrate out the $N=2$ superpartners and obtain the effective action at much lower energies, describing the interactions of the $N=2$ goldstino multiplet to $N=1$ superfields. This would be directly relevant for constructing brane effective theories involving non-Abelian gauge groups and charged matter. It could also be used for studying a supersymmetric extension of the Standard Model in the presence of a second supersymmetry nonlinearly realized due to its breaking at a high scale.

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Appendix A. Conventions for $N = 1$ superspace

The $N=1$ supersymmetry variation of a superfield V is $\delta V = (\epsilon Q + \bar{\epsilon}\bar{Q})V$, with supercharges verifying the algebra

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\partial_\mu. \tag{A.1}$$

On V, the supersymmetry algebra is then

$$[\delta_1, \delta_2]V = -2i\left(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1\right)\partial_\mu V. \tag{A.2}$$

The covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\left(\sigma^\mu\bar{\theta}\right)_\alpha\partial_\mu, \qquad \bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\left(\theta\sigma^\mu\right)_{\dot{\alpha}}\partial_\mu \tag{A.3}$$

anticommute with supercharges and verify

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\partial_\mu \tag{A.4}$$

as well. The identities

$$DD\theta\theta = \overline{DD}\,\overline{\theta\theta} = -4, \qquad \int d^2\theta\, d^2\bar{\theta} = -\frac{1}{4}\int d^2\theta\,\overline{DD} = -\frac{1}{4}\int d^2\bar{\theta}\, DD, \tag{A.5}$$

only valid under a spacetime integral $\int d^4x$, are commonly used.

The $N=1$ supersymmetry variations of the components (z, ψ, f) of a chiral superfield Φ, $\bar{D}_{\dot{\alpha}}\Phi = 0$, are

$$\begin{aligned}
\delta z &= \sqrt{2}\epsilon\psi, \\
\delta\psi_\alpha &= -\sqrt{2}\left[f\epsilon_\alpha + i\left(\sigma^\mu\bar{\epsilon}\right)_\alpha\partial_\mu z\right], \\
\delta f &= -\sqrt{2}i\,\partial_\mu\psi\sigma^\mu\bar{\epsilon}.
\end{aligned} \tag{A.6}$$

The bosonic expansions of the chiral superfields used in the text are:

$$W_\alpha(y,\theta) = \theta_\alpha d(y) + \frac{i}{2}\left(\theta\sigma^\mu\bar\sigma^\nu\right)_\alpha F_{\mu\nu}(y),$$
$$\chi_\alpha(y,\theta) = -\frac{1}{4}\theta_\alpha C(y) + \frac{1}{4}\left(\theta\sigma^\mu\bar\sigma^\nu\right)_\alpha b_{\mu\nu}(y),$$
$$\Phi(y,\theta) = \phi(y) - \theta\theta f_\phi(y), \tag{A.7}$$

and any other chiral superfield has an expansion similar to Φ. In this notation $\bar\chi_{\dot\alpha} = (\chi_\alpha)^*$ but $\bar W_{\dot\alpha} = -(W_\alpha)^*$. Since $L = D^\alpha\chi_\alpha - \bar D_{\dot\alpha}\bar\chi^{\dot\alpha}$, the linear superfield has bosonic expansion

$$L(x,\theta,\bar\theta) = C + \theta\sigma^\mu\bar\theta v_\mu + \frac{1}{4}\theta\theta\overline{\theta\theta}\Box C,$$
$$v_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu b^{\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^{[\nu}b^{\rho\sigma]} = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma}. \tag{A.8}$$

With these expansions,

$$\int d^2\theta\, d^2\bar\theta \left[-L^2 + \frac{1}{2}(\Phi + \bar\Phi)^2\right]$$

is the Lagrangian of a free, canonically-normalized, single-tensor $N = 2$ multiplet. Its bosonic content is

$$\frac{1}{2}(\partial_\mu C)\left(\partial^\mu C\right) + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}, \qquad H_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}.$$

These identities are useful:

$$D_\alpha D_\beta = \frac{1}{2}\epsilon_{\alpha\beta}DD, \qquad \bar D_{\dot\alpha}\bar D_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\overline{DD},$$
$$[D_\alpha, \overline{DD}] = -4i\left(\sigma^\mu\bar D\right)_\alpha\partial_\mu, \qquad [\bar D_{\dot\alpha}, DD] = +4i\left(D\sigma^\mu\right)_{\dot\alpha}\partial_\mu,$$
$$DDW_\alpha = 4i\left(\sigma^\mu\partial_\mu\bar W\right)_\alpha, \qquad \overline{DD}\bar W_{\dot\alpha} = -4i\left(\partial_\mu W\sigma^\mu\right)_{\dot\alpha}.$$

Further identities (with identical conventions) can be found in an appendix of Ref. [9].

Appendix B. Solving the quadratic constraint

The quadratic constraint $\mathcal{Z}^2 = 0$ must be solved to obtain the magnetic DBI theory coupled to a single-tensor multiplet. Using the expansion

$$\mathcal{Z}(y,\theta,\tilde\theta) = Z(y,\theta) + \sqrt{2}\tilde\theta\omega(y,\theta) - \tilde\theta\tilde\theta\left[\frac{i}{2}\Phi_{\mathcal{Z}} + \frac{1}{4}\overline{DD}\bar Z(y,\theta)\right],$$

in terms of the $N = 1$ chiral superfields Z, ω_α and $\Phi_{\mathcal{Z}}$, the constraint is equivalent to the single equation

$$Z = -\frac{\omega\omega}{i\Phi_{\mathcal{Z}} + \frac{1}{2}\overline{DD}\bar Z}. \tag{B.1}$$

The electric constraint equation (4.3), which was solved by Bagger and Galperin [7] using a method which applies to Eq. (B.1) as well, corresponds to the particular case $\omega_\alpha = iW_\alpha$, $\Phi_{\mathcal{Z}} = -i/\kappa$ and $Z = X$. Following then Ref. [7], the solution of Eq. (B.1) is

$$Z(\omega\omega, \Phi_{\mathcal{Z}}) = \frac{i}{\Phi_{\mathcal{Z}}}\left(\omega\omega + \overline{DD}\left[\frac{\omega\omega\overline{\omega\omega}}{|\Phi_{\mathcal{Z}}|^2 + A + \sqrt{|\Phi_{\mathcal{Z}}|^4 + 2A|\Phi_{\mathcal{Z}}|^2 + B^2}}\right]\right), \tag{B.2}$$

where

$$A = -\frac{1}{2}(DD\omega\omega + \overline{DD}\,\overline{\omega\omega}) = A^*,$$
$$B = -\frac{1}{2}(DD\omega\omega - \overline{DD}\,\overline{\omega\omega}) = -B^*.$$

Another useful expression is

$$Z(\omega\omega, \Phi_{\mathcal{Z}}) = \frac{i}{\Phi_{\mathcal{Z}}}\left(\omega\omega + \overline{DD}\left[\frac{\omega\omega\overline{\omega\omega}}{(DD\omega\omega)(\overline{DD}\,\overline{\omega\omega})}\left\{|\Phi_{\mathcal{Z}}|^2 + A - \sqrt{|\Phi_{\mathcal{Z}}|^4 + 2A|\Phi_{\mathcal{Z}}|^2 + B^2}\right\}\right]\right). \tag{B.3}$$

In the text, we need the bosonic content of $Z(\omega\omega, \Phi_{\mathcal{Z}})$. We write:

$$\omega_\alpha(y, \theta) = \theta_\alpha \rho + \frac{1}{2}\left(\theta\sigma^\mu\bar{\sigma}^\nu\right)_\alpha P_{\mu\nu} + \cdots, \tag{B.4}$$

where ρ is a complex scalar (2 bosons), $P_{\mu\nu}$ a real antisymmetric tensor (6 bosons) and dots indicate omitted fermionic terms. Hence,

$$\omega\omega = \theta\theta\left[\rho^2 + \frac{1}{2}P^{\mu\nu}P_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}P_{\mu\nu}P_{\rho\sigma}\right] + \cdots,$$
$$A = 2\left(\rho^2 + \bar{\rho}^2\right) + 2P^{\mu\nu}P_{\mu\nu} + \cdots,$$
$$B = 2\left(\rho^2 - \bar{\rho}^2\right) + i\epsilon^{\mu\nu\rho\sigma}P_{\mu\nu}P_{\rho\sigma} + \cdots.$$

Since the bosonic expansion of ω_α carries one θ_α, it follows from solution (B.2) that the bosonic $Z(\omega\omega, \Phi_{\mathcal{Z}})$ has a $\theta\theta$ component only, and that this component only depends on ρ, $P_{\mu\nu}$ and the lowest scalar component of $\Phi_{\mathcal{Z}}$ (which we also denote by $\Phi_{\mathcal{Z}}$). As a consequence, the bosonic $Z(\omega\omega, \Phi_{\mathcal{Z}})$ does not depend on the auxiliary scalar $f_{\Phi_{\mathcal{Z}}}$ of $\Phi_{\mathcal{Z}}$. We then find:

$$Z(\Phi_{\mathcal{Z}}, \omega\omega)_{bos.} = \frac{i\bar{\Phi}_{\mathcal{Z}}}{|\Phi_{\mathcal{Z}}|^2}\omega\omega - \frac{i\bar{\Phi}_{\mathcal{Z}}}{4|\Phi_{\mathcal{Z}}|^2}\theta\theta\left(|\Phi_{\mathcal{Z}}|^2 + A - \sqrt{|\Phi_{\mathcal{Z}}|^4 + 2A|\Phi_{\mathcal{Z}}|^2 + B^2}\right)_{\theta=0}. \tag{B.5}$$

The parenthesis is real. In terms of component fields:

$$\begin{aligned} Z = {} & -\frac{i\bar{\Phi}_{\mathcal{Z}}}{4|\Phi_{\mathcal{Z}}|^2}\theta\theta\left[|\Phi_{\mathcal{Z}}|^2 - i\epsilon^{\mu\nu\rho\sigma}P_{\mu\nu}P_{\rho\sigma} - 2\left(\rho^2 - \bar{\rho}^2\right)\right] \\ & + \frac{i\bar{\Phi}_{\mathcal{Z}}}{4|\Phi_{\mathcal{Z}}|^2}\theta\theta\left[\left(|\Phi_{\mathcal{Z}}|^2 + 2\left(\rho^2 + \bar{\rho}^2\right)\right)^2 - 16\rho^2\bar{\rho}^2 + 4\left(\rho^2 - \bar{\rho}^2\right)i\epsilon^{\mu\nu\rho\sigma}P_{\mu\nu}P_{\rho\sigma} \right. \\ & \left. + 4|\Phi_{\mathcal{Z}}|^2 P^{\mu\nu}P_{\mu\nu} - \left(\epsilon^{\mu\nu\rho\sigma}P_{\mu\nu}P_{\rho\sigma}\right)^2\right]^{1/2} + \cdots. \end{aligned} \tag{B.6}$$

The decomposition (4.27), $\mathcal{Z} = \tilde{\mathcal{W}} + 2g\mathcal{Y}$, indicates that

$$\rho = -\frac{g}{2}C + i\tilde{d}_2, \qquad P_{\mu\nu} = gb_{\mu\nu} - \tilde{F}_{\mu\nu}, \qquad \Phi_{\mathcal{Z}} = 2g\Phi. \tag{B.7}$$

In Lagrangian (4.35), we need the imaginary part of the $\theta\theta$ component of $Z(\omega\omega, \Phi_{\mathcal{Z}})$:

$$\begin{aligned}\operatorname{Im} Z(\omega\omega,\Phi_{\mathcal{Z}})|_{\theta\theta} = {} & -\frac{g\operatorname{Re}\Phi}{2} + \frac{\operatorname{Re}\Phi}{8g|\Phi|^2}\big\{16g^4|\Phi|^4 + 8g^2|\Phi|^2\big(g^2C^2 - 4\tilde{d}_2^2\big) \\ & - 16g^2C^2\tilde{d}_2^2 + 16g^2|\Phi|^2(\tilde{F}_{\mu\nu} - gb_{\mu\nu})\big(\tilde{F}^{\mu\nu} - gb^{\mu\nu}\big) \\ & + 8gC\tilde{d}_2\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) \\ & - \big[\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma})\big]^2\big\}^{1/2} \\ & + \frac{\operatorname{Im}\Phi}{8g|\Phi|^2}\big[\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) - 4gC\tilde{d}_2\big]. \end{aligned} \tag{B.8}$$

We now use

$$\begin{aligned} -\det\left(|\Phi|\eta_{\mu\nu} + \frac{\sqrt{2}}{g}P_{\mu\nu}\right) &= -|\Phi|^4\det\left(\eta_{\mu\nu} + \frac{\sqrt{2}}{g|\Phi|}P_{\mu\nu}\right) \\ &= |\Phi|^4 + \frac{|\Phi|^2}{g^2}P^{\mu\nu}P_{\mu\nu} - \frac{1}{16g^4}\big(\epsilon^{\mu\nu\rho\sigma}P_{\mu\nu}P_{\rho\sigma}\big)^2 \end{aligned} \tag{B.9}$$

to rewrite

$$\begin{aligned}\operatorname{Im} Z(\omega\omega,\Phi_{\mathcal{Z}})|_{\theta\theta} = {} & -\frac{g\operatorname{Re}\Phi}{2} + \frac{\operatorname{Re}\Phi}{4g|\Phi|^2}\left\{-4g^4|\Phi|^4\det\left[\eta_{\mu\nu} - \frac{\sqrt{2}}{g|\Phi|}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})\right]\right. \\ & - 4g^2\tilde{d}_2^2\big(2|\Phi|^2 + C^2\big) + 2g^4C^2|\Phi|^2 \\ & \left. + 2gC\tilde{d}_2\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma})\right\}^{1/2} \\ & + \frac{\operatorname{Im}\Phi}{8g|\Phi|^2}\big[\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) - 4gC\tilde{d}_2\big]. \end{aligned} \tag{B.10}$$

As a check, choosing $\Phi = -1/(2g\kappa)$ and $g = 0$ to decouple the single-tensor multiplet leads back to theory (4.4) since in that case $\tilde{d}_2 = 0$.

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