A note on an unusual type of generalized polar decomposition

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ABSTRACT

Inspired by the paper of Faßbender and Ikramov [H. Faßbender, Kh.D. Ikramov, A note on an unusual type of polar decomposition, Linear algebra appl. 429 (2008) 42–49], in this note we introduce an unusual type of generalized polar decomposition for a rectangular matrix \( A \) of the form \( A = GE \), where \( G \) is a complex symmetric matrix and \( E \) is a partial isometric matrix. Following the pattern used in the paper mentioned above, we call this decomposition a symmetric-partial-isometric generalized polar decomposition or an SPIGPD for short. Some properties of this decomposition are presented and results of SPIGPD related to conjugate-normal matrices are also obtained.

1. Introduction and preliminaries

In this note, let \( C_{m \times n} \) be the set of \( m \times n \) complex matrices and \( C_{r \times r}^{m \times n} \) be the subset of \( C_{m \times n} \) consisting of matrices with rank \( r \). Let \( I_r \) be the identity matrix of order \( r \). Given \( A \in C_{r \times n}^{m \times n} \), the symbols \( A^T, A^*, r(A), \) and \( R(A) \) stand for the transpose, conjugate transpose, rank, and range of \( A \), respectively. Furthermore, without specification, we always assume that \( m \geq n > r \).

For a complex rectangular matrix \( A \in C_{r \times n}^{m \times n} \), there exist a Hermitian positive semidefinite matrix \( G \in C_{m \times m}^{m \times n} \) and a partial isometric matrix \([1]\) \( E \in C_{m \times n}^{m \times n} \) such that

\[
A = GE.
\]

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This decomposition is called the generalized polar decomposition [1] of \( A \). Usually, when \( r = n \), (1.1) is called polar decomposition [5]. The decomposition (1.1) can be calculated from the singular value decomposition (SVD)
\[
A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* = U_1 \Sigma V_1^*
\]
by
\[
G = U_1 \Sigma U_1^*, \quad E = U_1 V_1^*,
\]
where \( U = (U_1, U_2) \in C_{m \times m} \) and \( V = (V_1, V_2) \in C_{n \times n} \) are unitary matrices, \( U_1 \in C_{m \times r} \), \( V_1 \in C_{n \times r} \), \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r) \), and \( \sigma_1 \geq \cdots \geq \sigma_r > 0 \) are the non-zero singular values of \( A \). Though there always exists a unique polar decomposition of \( A \in C_{m \times n} \) when \( r = n \), it is not true in general for generalized polar decomposition of \( A \in C_{m \times n} \). However, it has been proved that it is unique if the decomposition (1.1) satisfies
\[
R(E) = R(G).
\]
This condition can be found in [1].

Like the polar decomposition [3,5], decomposition (1.1) should be termed a left generalized polar decomposition, since matrix \( A \in C_{m \times n} \) can also be rewritten as
\[
A = EH,
\]
where \( E \in C_{m \times n} \) is a partial isometric matrix and \( H \in C_{n \times n} \) is a Hermitian positive semidefinite matrix, and they can be computed from SVD by
\[
H = V_1 \Sigma V_1^*, \quad E = U_1 V_1^*.
\]
We call decomposition (1.5) a right generalized polar decomposition of \( A \). Similar to (1.4), when the following condition (1.7) [1] holds, decomposition (1.5) is unique
\[
R(E^*) = R(H).
\]
In addition, it is worth to note that the partial isometric polar factor \( E \) in the decompositions (1.1) and (1.5) are the same. However, it is generally untrue for the new type of generalized polar decomposition introduced in the following.

Similar to the generalization from polar decomposition to the symmetric-unitary polar decomposition (SUPD) [3], in this note, we present a new type of generalized polar decomposition of \( A \in C_{m \times n} \), namely,
\[
A = GE
\]
or
\[
A = QH,
\]
where \( E, Q \in C_{m \times n} \) are partial isometric matrices, and \( G \in C_{m \times m}, H \in C_{n \times n} \) are complex symmetric matrices.

Similar to the names of the left and right generalized polar decomposition and following the pattern used in [3], we call (1.8) a left symmetric-partial-isometric generalized polar decomposition (SPIGPD) and (1.9) a right SPIGPD. Like SUPD, we will present some properties of the new decomposition and results on the SPIGPD related to conjugate-normal matrices later in this note. In the following, we summarize some results on the partial isometric matrices and generalized polar decomposition.

**Lemma 1.1.** Let \( E \in C_{m \times n} \). Then the following statements are equivalent:

(a) \( E \) is a partial isometric matrix;

(a*) \( E^* \) is a partial isometric matrix;

(b) \( E^*E \) is an orthogonal projector on \( R(E^*) \);
(b*) $EE^* \text{ is an orthogonal projector on } R(E)$;
(c) $EE^*E = E$;
(c*) $E^*EE^* = E^*$;
(d) $E^* = E^\dagger$;
(e) $E^\dagger$ is a partial isometric matrix.

This lemma can be found in [1] (i.e., Theorem 5 in Chapter 6 in [1]). Moreover, throughout this note $E^\dagger$ denotes the Moore–Penrose inverse [1] of $E \in \mathbb{C}^{m \times n}$.

**Lemma 1.2.** Let (1.1) be the generalized polar decomposition of $A \in \mathbb{C}^{r \times n}_{\mathbb{C}}$ and the condition (1.4) be satisfied. Then

(a) $G = EA^* = AE^*$ and $G = \frac{1}{2}(EA^* + AE^*)$;
(b) $EE^*$ is an orthogonal projector on $R(E)$ and $R(A)$, i.e., $EE^* = P_{R(E)} = P_{R(A)}$;
(c) $E^*E$ is an orthogonal projector on $R(E^*)$ and $R(A^*)$, i.e., $E^*E = P_{R(E^*)} = P_{R(A^*)}$.

**Proof.** The results of (a) can be verified easily by using (1.2) and (1.3). For (b) and (c), they can be derived by combining with Lemma 1.1 and Theorem 6 in Chapter 6 of [1].

**Lemma 1.3.** Let (1.1) be the generalized polar decomposition of a square matrix $A \in \mathbb{C}^{r \times m}_{\mathbb{C}}$ and the condition (1.4) be satisfied. Then, $A$ is a normal matrix if and only if any of the following conditions is fulfilled:

(a) $A^*$ admits a generalized polar decomposition $A^* = GQ$ satisfying $R(G) = R(Q)$, where $Q$ is a partial isometric matrix;
(b) $GE = EG$;
(c) $AE^* = E^*A$;
(d) $AG = GA$.

**Proof.** The fact that condition (b) is equivalent to $A$ being a normal matrix can be found in [1], i.e., Example 52 in Chapter 6 in [1]. Therefore, it is enough to prove condition (a) and (b) $\iff$ (c) $\iff$ (d). We first give the proof of condition (a).

**Sufficiency.** Suppose that $A$ and $A^*$ have their generalized polar decompositions as those described in this lemma. Considering the conditions (1.4) and $R(G) = R(Q)$, and (b) in Lemma 1.2, we have

$$AA^* = GEE^*G = G^2, \quad A^*A = GQQ^*G = G^2.$$ 

Hence, $A$ is a normal matrix.

**Necessity.** Observe that if $A$ is a normal matrix with the generalized polar decomposition as in (1.1) satisfying condition (1.4), then (b) holds, i.e., $GE = EG$. It follows that

$$A^* = E^*G = GE^* = G, \quad R(G) = R(E^*) = R(Q).$$

Thus, we obtained the desired result.

Now we show that (b) $\iff$ (c) $\iff$ (d). The implications (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) are easy to derive by noting the fact that $GE = EG$ implies $R(E) = R(G) = R(E^*)$, and (b), (c) ofLemma 1.2. The implication (c) $\Rightarrow$ (b) is also not difficult to obtain. In fact, from (c) and considering the condition (1.4), and (b) of Lemma 1.2, we have

$$GEE^* = E^*GE \iff (EE^*G)^* = E^*GE \iff GE = EE^*GE \iff EG = GE.$$ 

Next, we mainly prove (d) $\Rightarrow$ (b). From (d), and using (1.3) and the fact $U_1^*U_1 = I_r$, we have
\[ GEG = GGE \iff U_1 \Sigma U_1^T U_1 V_1^* U_1 \Sigma U_1^T V_1^* = U_1 \Sigma U_1^T U_1 \Sigma U_1^T U_1 V_1^* \]
\[ \iff U_1 \Sigma V_1^* U_1 \Sigma U_1^T = U_1 \Sigma \Sigma V_1^* \Rightarrow V_1^* U_1 \Sigma = \Sigma V_1^* U_1. \]  
(1.10)

Since
\[ GE = U_1 \Sigma V_1^*, \quad EG = U_1 V_1^* U_1 \Sigma U_1^*, \]
which combined with (1.10) gives \( EG = U_1 \Sigma V_1^* U_1 U_1^* \). Then, it suffices to show that \( V_1^* U_1 U_1^* = V_1^* \).

From (d) again, we have \( R(GE^* G) = R(E^* G G) \). Note that
\[ R(GE^* G) \subseteq R(G), \quad r(G) = r \]
and
\[ r(GE^* G) = r(U_1 \Sigma U_1^* V_1 U_1^* U_1 \Sigma U_1^*) = r(U_1^* V_1) = r. \]

We have
\[ R(GE^* G) = R(G). \]
Similarly, it follows from
\[ R(E^* G G) \subseteq R(E^*), \quad r(E^*) = r \]
and
\[ r(E^* G G) = r(V_1 U_1^* U_1 \Sigma U_1^* U_1 \Sigma U_1^*) = r(V_1 \Sigma \Sigma U_1^*) = r \]
that
\[ R(E^* G G) = R(E^*). \]
As a result,
\[ R(E) = R(G) = R(E^*). \]

Observe that
\[ R(E) = R(U_1) = R(E^*) = R(V_1) \quad \text{and} \quad EE^* = U_1 V_1^* V_1 U_1^* = U_1 U_1^*. \]

Note that in the above derivation, the fact \( V_1^* V_1 = I \) is used. Thus, from (b) of Lemma 1.2, we have
\[ U_1 U_1^* V_1 = V_1 \iff V_1^* U_1 U_1^* = V_1^*. \] Therefore, the whole proof is completed. \( \square \)

2. Symmetric-partial-isometric generalized polar decomposition

In the following, we introduce the SPIGPD of \( A \in C_r^{m \times n} \) by using its SVD (1.2). Rewrite (1.2) as
\[ A = \left( U_1 \Sigma U_1^T \right) \left( U_1 V_1^* \right) = GE. \]  
(2.1)

It is easy to find that
\[ G = U_1 \Sigma U_1^T \in C_r^{m \times m} \]  
(2.2)

is a complex symmetric matrix, and
\[ E = U_1 V_1^* \in C_r^{m \times n} \]  
(2.3)

is a partial isometric matrix. In fact, it can be verified by using Lemma 1.1 and the following fact
\[ EE^* E = U_1 V_1^* U_1 U_1^* U_1 V_1^* = U_1 V_1^* = E. \]

The above consideration establishes the existence of a left SPIGPD for every matrix \( A \in C_r^{m \times n} \). Similarly, we can establish the existence of a right SPIGPD by rewriting (1.2) as
Lemma 1.1. It follows from (2.1) that right SPIGPD is analogous. Corresponding ones for SUPD [3], we omit the detail here.

Similarly to the SUPD, each left (right) SPIGPD of $A \in \mathbb{C}_r^{m \times n}$ can be obtained through an appropriate eigenvector matrix $U$. In addition, when matrix $A \in \mathbb{C}_r^{m \times n}$ undergoes a unitary congruence transformation, the symmetric polar factor $G$ ($H$) transforms a unitary congruence one and the partial isometric polar factor $E$ ($Q$) transforms a unitary similarity one. Since all arguments are similar to the corresponding ones for SUPD [3], we omit the detail here.

Next, we present the counterparts of results in Lemma 1.2 for the left SPIGPD. The argument for the right SPIGPD is analogous.

Theorem 1.2. Let (2.1) be the SPIGPD of $A \in \mathbb{C}_r^{m \times n}$ and the condition (2.5) be satisfied. Then

(a) $G = \overline{EA^*} = AE^*$ and $G = \frac{1}{2}(EA^* + AE^*)$;
(b) $EE^*$ is an orthogonal projector on $R(E)$ and $R(A^*)$, i.e., $EE^* = P_{R(E)} = P_{R(A^*)}$;
(c) $E^*E$ is an orthogonal projector on $R(E^*)$ and $R(A^*)$, i.e., $E^*E = P_{R(E^*)} = P_{R(A^*)}$.

Proof. Considering the condition (2.5), the results of (a) can be obtained from the following facts

$$EA^* = \overline{EE^*G} = G, \quad AE^* = GEE^* = (EE^*G)^* = G, \quad \frac{1}{2}(EA^* + AE^*) = G.$$ 

For (b) and (c), we only need to show that $R(E) = R(\overline{A})$ and $R(E^*) = R(A^*)$, respectively, by considering Lemma 1.1. It follows from (2.1) that

$$R(\overline{A}) = R(GE) \subseteq R(G), \quad R(A^*) = R(E^*G) \subseteq R(E^*).$$
Furthermore, from condition (2.5), (2.7) and the fact $\mathcal{G} = \mathcal{G}^\ast$, we have
\[
\begin{align*}
\rho(\mathcal{G}) &= \rho(\mathcal{A}^\dagger) \iff \rho(\mathcal{G}^\ast \mathcal{G}) = \rho(\mathcal{A}^\dagger) \iff \rho(\mathcal{G}) = \rho(\mathcal{A}), \\
\rho(E^\ast) &= \rho(E) = \rho(\mathcal{G}) = \rho(\mathcal{A}) = \rho(\mathcal{A}^\ast).
\end{align*}
\]
Thus the desired results are obtained. Therefore, we complete the whole proof. □

From Theorem 2.1, it is not difficult to find that $E E^\ast$ and $E^\ast E$ are invariant though $E$ may have infinite choices in general, since the orthogonal projectors on $R(\mathcal{A})$ and $R(\mathcal{A}^\ast)$ are unique. Note that
\[
\begin{align*}
\mathcal{P}_R(\mathcal{A}) &= \mathcal{A}^\dagger = \mathcal{U}_1 \mathcal{U}_1^\dagger, \quad \mathcal{P}_R(\mathcal{A}^\ast) = \mathcal{A}^\dagger = \mathcal{V}_1 \mathcal{V}_1^\dagger.
\end{align*}
\]
Thus, $E E^\ast$ and $E^\ast E$ can be calculated by $\mathcal{U}_1 \mathcal{U}_1^\dagger$ and $\mathcal{V}_1 \mathcal{V}_1^\dagger$, respectively. Moreover, from these facts we can also find that the ranks of partial isometric polar factor $E$ and symmetric polar factor $G$ of $A \in \mathbb{C}^{m \times n}$ are invariant and are the same as that of $A \in \mathbb{C}^{m \times n}$.

3. Results related to conjugate-normal matrices

We call a square matrix $A$ to be conjugate-normal if
\[
AA^\ast = A^\ast A.
\]
Conjugate-normal matrices play a significant role in the theory of unitary congruence. More results on such matrices can be found in [4]. In this section, we present some results of the SPIGPD related to conjugate-normal matrices. They are the same as the corresponding ones of the SUPD in form [3,4]. The following is the main theorem.

**Theorem 3.1.** A square matrix $A \in \mathbb{C}^{m \times m}$ is conjugate-normal if and only if any of the following conditions is fulfilled:

(a) $A$ and $A^T$ admit SPIGPDs $A = GE$ and $A^T = GQ$ satisfying condition (2.5) and $R(Q) = R(G)$, respectively, with the same symmetric polar factor $G$;

(b) There exists an SPIGPD of $A, A = GE$, satisfying condition (2.5) such that
\[
E(\mathcal{G}G) = (\mathcal{G}G)E; \tag{3.1}
\]

(c) There exists an SPIGPD of $A, A = GE$, satisfying condition (2.5) such that
\[
E(A^\ast A) = (A^\ast A)E; \tag{3.2}
\]

(d) There exists an SPIGPD of $A, A = GE$, satisfying condition (2.5) such that
\[
A^\ast \mathcal{G} = \mathcal{G} A. \tag{3.3}
\]

**Proof.** Noting the conditions (2.5) and $R(Q) = R(G)$, condition (a) can be proved by using a method similar to the one given in [4]. The Necessities of conditions (b), (c), and (d) can be also proved by using a method similar to the one given in [3] by considering the condition (2.5) and Theorem 2.1. However, the method of proving the sufficiencies of conditions (b), (c), and (d) in [3] is disabled here. Because, in this case, $E$ is not a unitary matrix any more but a partial isometric one instead. We need to introduce an alternative method. Our trick is first proving the sufficiency of condition (b), then proving the facts that (c) and (d) imply (b), respectively. Moreover, note that the equalities in (3.1)–(3.3) are always satisfied no matter what values of $G$ and $E$ are given. Therefore, without loss of generality, we may assume that $G$ and $E$ are given in (2.2) and (2.3), respectively, in the following proof.
We first prove the sufficiency of condition (b). From (3.1), we have
\[ R(E^*(GG)) = R((GG)E^*). \]
Note that
\[ R(E^*(GG)) \subseteq R(E^*), \quad r(E^*) = r \]
and
\[ r(E^*(GG)) = r \left( V_1 U_1^T \Sigma U_1^T \Sigma U_1^T \right) = r(V_1 \Sigma^2 U_1^T) = r. \]
Then
\[ R(E^*(GG)) = R(E^*). \]
Similarly, we can get
\[ R((GG)E^*) = R(G). \]
Therefore,
\[ R(E) = R(G) = R(E^*). \tag{3.4} \]
Since
\[ A^*A = E^*GG, \quad AA^* = GEE^*G, \]
using conditions (3.1) and (3.4), we have
\[ A^*A = E^*E(GG) = GG, \quad AA^* = GG. \]
Thus, we obtain the desired result, i.e., \( A^*A = GG = AA^* \).

Now, we show that (c) and (d) imply (b), respectively. From (3.2), we get
\[ R(E(A^*A)) = R((A^*A)E). \]
Note that
\[ R(E(A^*A)) \subseteq R(E), \quad r(E) = r \]
and
\[ r(E(A^*A)) = r \left( \overline{U}_1 V_1^* V_1 \Sigma U_1^* U_1 \Sigma V_1^* \right) = r \left( \overline{U}_1 \Sigma^2 V_1^* \right) = r. \]
Then
\[ R(E(A^*A)) = R(E). \]
Similarly, note that
\[ R((A^*A)E) = R(E^*GGEE) \subseteq R(E^*), \quad r(E^*) = r \]
and
\[ r((A^*A)E) = r \left( V_1 \Sigma U_1^* U_1 \Sigma V_1^* \overline{U}_1 V_1^* \right) = r \left( \overline{U}_1 \Sigma^2 V_1^* \right) = r. \]
Thus,
\[ R((A^*A)E) = R(E^*). \]
As a result, (3.4) holds. From (3.2) again, we have
\[ EE^*GG = E^*GG, \]
According to (3.4), (3.5), and the comments at the end of the preceding section, we can get

Thus, the proof of the implication (c) $\Rightarrow$ (b) is completed.

It follows from (3.3) that

which implies

Note that

and

Then

Similarly, note that

and

Then

Therefore, (3.4) holds. In addition, it is easy to induce that

From (3.3) again, we have

According to (3.4), (3.5), and the comments at the end of the preceding section, we can get

Thus, in view of (3.6) and (3.7), it is seen that

Hence,

Thus the desired result is obtained. Therefore, we complete the whole proof. □

In this theorem, we only consider the left SPIGPD. A similar assertion can be stated for the right SPIGPD.
4. Concluding remarks

In this note, we study a new type of generalized polar decomposition. Some results related to the new decomposition are presented. It is interesting to extend such type decomposition to the weighted polar decomposition [6]. This is our future topic.

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