FUNDAMENTALS OF PLANAR ORDERED SETS*

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A finite ordered set is planar when it can be represented in the plane with the point \( a \) lower than \( b \) in the plane whenever \( a \) is less than \( b \) in the ordered set, and with a straight-line segment from \( a \) to \( b \) whenever \( a \) is covered by \( b \). We prove it is equivalent to allow instead an arc from \( a \) to \( b \) as long as the arc does not go below the level of \( a \) or above the level of \( b \) (whenever \( a \) is covered by \( b \)). Our result is analogous to the well-known graph-theoretic result of K. Wagner and I. Fáry.

All graphs and digraphs (directed graphs) are finite, without loops of multiple edges. In addition, all ordered sets are understood to be finite. The covering digraph of an ordered set \( \langle X; \leq \rangle \) has vertex set \( X \) and a directed edge \( ab \) from \( a \) to \( b \) exactly when \( a \) is covered by \( b \). (The corresponding graph is a covering graph.) As the title of this paper indicates, our original motivation was covering digraphs. However, it seems natural to deal with arbitrary digraphs. The definition of planar digraphs that we give in Section 1 is motivated by the way that covering digraphs are usually drawn. An ordered set is planar exactly when its covering digraph is acyclic. A planar digraph is necessarily acyclic. A digraph can be nonplanar even though the corresponding graph is planar.

Whenever possible, we extend the usual terminology for ordered sets to acyclic digraphs. For example, the vertex \( a \) of an acyclic digraph is called a zero (or least element) whenever there is a directed path from \( a \) to every other vertex. Also, an acyclic digraph is bounded if it has both a zero and one. It is known that a planar bounded ordered set must be a lattice (see, for example, Kelly and Rival [6]). An edge \( ab \) of an acyclic digraph \( G \) is called extraordinary when there is a directed path from \( a \) to \( b \) that does not use the edge \( ab \). An acyclic digraph is an extraordinary digraph exactly when it has no extraordinary edges. Moreover, any acyclic digraph can be turned into a covering digraph by placing a new vertex (of degree two) on each extraordinary edge.

Our main result (Theorem 4) shows that the arcs representing the edges of planar digraphs do not need to be (straight-line) segments. Our main result is an analog of the result proved by Wagner [10] and, later, by Fáry [4] that every planar graph has straight planar representation (each edge is represented by a

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segment). In both these papers, edges were added to a plane graph to create a new plane graph that has the same vertices and is triangulated; i.e., each face, including the infinite one, is a triangle. (We wrote “infinite” because we always use “bounded” in the order-theoretic sense.) We use an analogous triangulation result (Theorem 3) in the proof of our main result.

In Section 1, we define planar digraphs (allowing arcs as edges) and present some results for planar graphs that are needed later. All subsequent sections are concerned with plane digraphs and their planar representations. In Section 2, we define when two planar representations of the same digraph are similar (intuitively, they do not differ significantly), and apply a result from Section 1 to obtain a similar polygonal planar representation (each edge is a polygonal arc). Theorem 1 produces a similar polygonal plane representation in which all the edges are monotonic (loosely speaking, they “go upwards”). In Section 4, we embed any planar digraph into a planar bounded digraph on the same set of vertices and prove Theorem 3 on triangulation. Finally, Theorem 4 shows that every plane digraph has a similar straight planar representation. Our definition of planar ordered sets is thus shown to be a conservative extension of the usual one that requires each edge to be a segment.

1. Plane digraphs and graphs

A digraph is plane if all its vertices and edges are in the plane such that: each edge $ab$ is an arc (homeomorph of a segment) between the vertices $a$ and $b$; two edges can only intersect at their endpoints; for each (directed) edge $ab$, $a$ is below $b$ (i.e., $\pi_2 a < \pi_2 b$) and

$$\pi_2(ab) \subseteq [\pi_2 a, \pi_2 b]. \quad (*)$$

(The second projection function is denoted by $\pi_2$.) We call (*) the betweenness condition. A digraph $G$ is planar if it is isomorphic to a plane digraph (which is called a planar representation of $G$). An ordered set is planar when its covering digraph is. A plane digraph is straight if each edge is a segment.

Our definition of plane digraphs will be justified in Section 5 when we complete the proof that any planar digraph has a straight planar representation. Let us call an arc monotonic if no two distinct points on it have the same second coordinate. We call a plane digraph monotonic if each edge is a monotonic arc. A straight plane digraph is certainly monotonic. By the Intermediate Value Theorem, the betweenness condition is superfluous for any monotonic plane digraph. The necessity of the betweenness condition is illustrated by Fig. 1. (The lattice, $2^3$, shown there does not have a straight planar representation.)

Our graph-theoretic terminology usually agrees with Bollobás [2]. A plane graph has its vertices and edges in the plane, with arcs for its edges; its edges cross only at vertices. We call a plane graph or digraph polygonal if all its edges
are polygonal arcs. The cyclic order of the neighbours of each vertex is the counterclockwise order in which the edges to the neighbors first touch a circle that is centered at the vertex but contains no other vertex. In addition, each cycle in a plane graph has either a counterclockwise or clockwise orientation.

We shall define an equivalence relation (called similarity) between planar representations of the same planar graph, such that two similar representations have the same cyclic order about each vertex and have the same orientation of each cycle. In fact, for a connected planar graph, these two conditions are equivalent to similarity.

In subsequent sections, most arguments are combinatorial, although some topology is used in the proofs of Theorems 1 and 2. In this section, we shall use the topology of the plane to justify our transition to a more combinatorial setting. The reader can omit the proofs of Lemmas 1 and 2 without significantly affecting his understanding of the rest of the paper.

There are three natural ways to specify that two planar representations of a graph \( G \) are similar. One way is to have an orientation-preserving homeomorphism \( \varphi \) of the plane (to itself) that maps corresponding vertices and edges to each other. Secondly, the homeomorphism \( \varphi \) can be defined only on some rectangle (or disk) containing both plane representations and the orientation-preserving assumption is replaced by the condition that \( \varphi \) is the identity on the boundary. Thirdly, we can require that the mapping \( \varphi \) of the second case is actually a deformation that preserves the boundary values; i.e., there is a isotopy from the identity to \( \varphi \) in which each intermediate homeomorphism leaves the boundary pointwise invariant. Each condition is clearly stronger than the preceding one. In fact, all three conditions are equivalent. The Jordan Curve Theorem and the Schoenflies Extension Theorem (see, for example, [8, Theorem 10.4]) can be used to show that the first implies the second. (The second condition's homeomorphism agrees with the first condition's on every finite component of the complement of the graph in the plane.) Tietze's Deformation Theorem ([8, Theorem 11.1]) says that the second condition implies the third. The third condition can be visualized in terms of a rubber sheet that is clamped at the edges.

We now give combinatorial conditions for similarity. Let \( G \) be a connected plane graph. The main result of Adkisson and MacLane [1] shows that an
orientation-preserving topological embedding of $G$ into the sphere can be extended to a homeomorphism of the sphere iff $G$ and its image have the same cyclic order about each vertex. Consequently, two planar representations of a connected planar graph are similar iff they have the same cyclic order about each vertex and each cycle has the same orientation. Now suppose that $G$ is a disconnected planar graph and consider two planar representations of it. Add additional edges to the first planar representation of $G$ to give a planar representation of a connected graph $H$; the two planar representations of $G$ are similar iff the second one can be extended to a similar planar representation of $H$.

In general, the order in which the edges from a vertex $v$ last intersect a circle centered at the vertex does not agree with the cyclic order about $v$. However, the following lemma shows that these two orders do agree when the circle is sufficiently small.

**Lemma 1.** For every vertex $v$ of a plane graph, there exists $\rho > 0$ such that, for every circle $C$ with center $v$ and radius less than $\rho$, the edges to the neighbors of $v$ last touch $C$ in the cyclic order about $v$.

**Proof.** Let $v$ be a vertex of the plane graph $G$, and let $D$ be a circle centered at $v$ that does not contain any other vertex in its interior. Let $v_1, v_2, \ldots, v_n$ be the neighbors of $v$ in the cyclic order about $v$, and let $y_i$ be the first intersection (starting from $v$) of the arc $vv_i$ with $D$ for $1 \leq i \leq n$. Clearly, $y_1, y_2, \ldots, y_n$ are counterclockwise around $D$. Let $\rho$ be the minimum distance from $v$ to the subarc $y_i v_i$ for $1 \leq i \leq n$. Let $C$ be a circle with center $v$ and radius less than $\rho$, and let $x_i$
be the last intersection (starting from \( v \)) of \( vu_i \) with \( C \) for \( 1 \leq i \leq n \). After observing that \( x_i \) and \( y_i \) are the only places where the subarc \( x_i y_i \) of \( vu_i \) intersects \( C \) or \( D \), it is obvious that \( x_1, x_2, \ldots, x_n \) are counterclockwise around \( C \).

Let \( C \) be an arc from \( a \) to \( b \) in the plane. We call a polygonal arc from \( a \) to \( b \) consistent with \( C \) if it consists of segments from \( x_i \) to \( x_{i+1} \) for \( 0 \leq i < n \), where \( x_0 = a, x_1, x_2, \ldots, x_n = b \) are points of \( C \) that are in this order along \( C \).

**Lemma 2** (Polygonal Approximation). For any planar representation of a planar graph, there is a similar polygonal planar representation in which the vertices are unmoved and each new polygonal arc is consistent with its original edge.

**Proof.** Let \( G \) be a plane graph with vertices \( v_1, v_2, \ldots, v_n \) and let \( e_1, e_2, \ldots, e_m \) be the edges of \( G \). By adding edges, we can assume that \( G \) is connected. Choose \( s > 0 \) so that \( \sqrt{2}s \) is less than the value \( \rho \) of Lemma 1. We also assume that \( s \) is small enough so that the closed squares of edge-length \( 2s \) that are centered at each vertex of \( G \) are pairwise disjoint. Let \( S_i \) denote the square centered at \( v_i \) with edge-length \( s \). If the vertex \( v_i \) is incident to the edge \( e_k \), then \( v_{ik} \) will denote the last intersection (starting from \( v_i \)) of \( e_k \) with \( S_i \).

Choose \( \delta > 0 \) such that \( 2\delta \) is less than both \( s \) and the minimum distance between two distinct arcs that are each expressible as the subarc of \( e_k \) from \( v_{ik} \) to \( v_{jk} \) for some \( k \). Divide the plane into closed squares of edge-length \( \delta \) by drawing all the lines that have equations of the form \( x = l\delta \) or \( y = l\delta \) for an integer \( l \). (Henceforth, we write “square” without qualification whenever we mean such a square.) For each \( i \), \( S'_i \) will denote the smallest rectangle that contains \( S_i \) and is bounded by such lines. If the vertex \( v_i \) is incident to the edge \( e_k \), then \( v'_ik \) will denote the last intersection (starting from \( v_i \)) of \( e_k \) with \( S'_i \). Since the distance of each corner of \( S'_i \) from \( v_i \) is less than \( \sqrt{2}s \), the points of the form \( v'_ik \) are counterclockwise around \( S'_i \) when the edges \( e_k \) incident to \( v_i \) are taken in cyclic order.

If the edge \( e_k \) joins the vertices \( v_i \) and \( v_j \), then \( e'_k \) will denote the subarc of \( e_k \) from \( v'_ik \) to \( v'_jk \). We shall define a polygonal arc from \( v'_ik \) to \( v'_jk \) that is consistent with \( e'_k \). As indicated in Fig. 2, this new polygonal curve consists of segments that join some points of \( e'_k \) that are on the boundaries of squares. In each part of Fig. 2, the previous segment is shown inside the square(s) with a solid boundary, and the other segment is the new one that replaces all of the arc that is shown except for the small terminal part. The arc \( e'_k \) never returns to any of the squares shown in Fig. 2.

We now define the replacement of an arc by a polygonal one that is indicated in Fig. 2. Let \( P \) and \( Q \) be the points on the original arc that define the segment \( PQ \) which replaces the part of the original arc from \( P \) to \( Q \). We shall define the point \( R \) on the arc that gives the next segment \( QR \). As shown in Fig. 2, \( Q \) is on the
boundary of the one or two squares that the segment $PQ$ crosses. The *neighboring region* consists of those squares that contain $Q$ but do not contain $P$. (In Fig. 2, these are the squares that do not have a solid boundary.) We define $R$ to be the last point on the original arc that is in the neighboring region. Observe that $R$ comes after $Q$ on the arc and that the segment $QR$ lies in the neighboring region.

We now define the polygonal arc that replaces the edge $e_k$. Choose the smallest rectangular region of two or four squares that contains $V$; in its interior. We start with the segment that joins $v'_{i_k}$ to the last point of $e'_k$ that is in this region. We obtain the polygonal arc $\alpha$ by repeating the procedure of Fig. 2 until we reach $v'_{j_k}$. (Since $v'_{i_k}$ is on the boundary of $S'_i$, it is on the boundary of a square.) Obviously, only the endpoints of $\alpha$ are contained in $S'_i$ or $S'_j$. Thus, we obtain a polygonal arc $\beta$ from $v_i$ to $v_j$ by adding a segment to each end of $\alpha$. Clearly, $\beta$ is compatible with $e_k$.

Since none of the new polygonal arcs cross, we have defined a planar representation of $G$ that is polygonal. Moreover, the original and new planar representations have the same cyclic order about each vertex. However, additional conditions must be imposed to ensure that all cycles have the same orientation. The concept of *winding number* is needed (as presented, for example, in Chinn and Steenrod [3]). We indicate the argument. For each cycle $C$ of the graph $G$, fix a point $P_C$ in the plane that lies inside $C$ in the first planar representation. Add additional vertices to each edge of $G$ in the original planar representation until, for each new edge that lies on a cycle $C$, there is a ray that starts at $P_C$ and misses the edge. The parameters $s$ and $\delta$ must be chosen small enough so that the polygonal approximation (as defined above) of each edge does not intersect any of the rays defined for that edge. Let $R$ be the ray from $P_C$ that misses the edge $e$. Firstly, $s$ is chosen so that $R$ does not intersect either square of edge-length $2s$ that is centered at an endvertex of $e$. Secondly, $\delta$ is chosen so that $R$ can be covered by squares of edge-length $\delta$ (as used above) that do not intersect $e$. This is done for each edge $e$. Since, in both planar representations, each cycle $C$ has the same winding number (equal to plus or minus one) with respect to $P_C$, the proof of the lemma is complete. 

We shall apply the Polygonal Approximation Lemma in the next section to make the study of planar digraphs more combinatorial. For background, let us mention a few results about planar ordered sets. (These are not used in the sequel.) The planarity of lattices has been studied extensively. We have already mentioned that a bounded ordered set is planar exactly when it is a planar lattice. We refer the reader to Section 3 of Kelly and Trotter [7] for a detailed discussion of planar lattices. For lattices, planarity is equivalent to dimension at most two. (For the definition of dimension, see for example, [7].) Dimension theory is of no use in the general theory of planar ordered sets because Kelly [5] has shown that there are planar ordered sets of any finite dimension. A subposet of a planar
ordered set need not be planar. For example, form \( P \) from the ordinal sum of two copies of \( \mathbb{2} \) (every element of one copy is less than every element of the other) by identifying the one of the bottom copy with the zero of the top copy. \( P \) is certainly planar, but the subposet obtained by deleting the middle element is nonplanar. C.R. Platt [9] has shown that a lattice is planar iff a related graph (the covering graph of the lattice with a new edge added from zero to one) is planar.

2. Similarity and polygonal approximations for digraphs

There are three different conditions for planar representations of a digraph to be considered similar; each condition is more restrictive than the corresponding condition for graphs given in Section 1. In the third condition, we require, in addition to the third condition for graphs, that each intermediate homeomorphism map the original plane digraph onto a planar digraph representation. Clearly, the two representations of Fig. 3 are not similar in this third sense although they are similar as planar representations of graphs. The augmented digraph of a digraph \( G \), denoted by \( G^* \), is formed from \( G \) by adding, to each vertex of \( G \) which has at least two neighbors and for which all neighbors are in the same direction, a single pendent edge in the opposite direction (with its new endvertex). Any planar representation of a digraph can obviously be extended to a planar representation of its augmented digraph. We say that two planar representations of a digraph are similar (in the first or second sense) when the two corresponding planar representations of the augmented digraph are similar as planar representations of graphs (in the same sense). By the corresponding results for graphs, the first two senses are equivalent and follow from the third sense. We are unable to show that the third sense is equivalent to the first two. Henceforth, similarity of planar representations of digraphs is understood to be in the first or second sense.

In a plane digraph, there is a linear ordering (understood to be from left to right) of the lower neighbors of each vertex and a similar linear ordering of the upper neighbors. (These two linear orderings obviously determine the cyclic order about a vertex of the corresponding plane graph.) The corresponding result for graphs (see Section 1) shows that two planar representations of a connected digraph \( G \) are similar exactly when the lower and upper neighbors of each vertex

![Fig. 3. Two planar representations.](image-url)
have the same ordering and each cycle has the same orientation. (As we shall show in a subsequent paper, the condition that each cycle have the same orientation is redundant.)

Let $G$ be a plane digraph. By Lemma 2, there is a polygonal approximation of the plane graph corresponding to the augmented digraph $G^*$. Since each segment on each new edge joins points on the original edge, this polygonal approximation is also a planar representation of $G^*$ as a digraph. Thus, any planar representation of a digraph has a polygonal plane representation that is similar to it. In addition, observe that this polygonal approximation of the plane digraph $G$ will be monotonic whenever $G$ is.

Before continuing, let us explain our interest in similarity and its various definitions for digraphs. When a person initially draws a planar ordered set, he often uses curved monotonic arcs. While doing so, he tries to arrange the lower and upper covers of each element to achieve a pleasing picture. Intuitively, our main result (Theorem 4) says that the original drawing can be deformed into one with segments as edges in which all the orderings of lower and upper covers are preserved. We wrote "intuitively" because the word "deformed" indicates the third sense of similarity for planar representations of digraphs. Recall that we cannot show that the third sense of similarity is a consequence of our definition of similarity for digraphs (although we believe that it is). In the proof of Theorem 1, we shall use the topological definition of similarity for graphs that we gave in Section 1.

For a straight plane graph, there is $\varepsilon > 0$ such that if every vertex is moved by at most $\varepsilon$, then every vertex can still be joined by segments (that do not cross) to all its neighbors. (Take $\varepsilon$ to be one-third of the minimum value which is the distance between two vertices or between a vertex and an edge that are not incident.) By the third definition of similarity, all of these planar representations will be similar. Given a polygonal plane digraph, we apply this result to obtain a similar polygonal planar representation of the digraph in which no two corner points are at the same level. (The corner points of a polygonal arc are the points where the slope of the polygonal arc changes, including the endpoints.) First we move each of the digraph’s vertices vertically by at most half the allowed amount so that the lower vertex of each edge remains below its upper vertex. We then move the remaining corner points vertically so that each polygonal arc representing an edge satisfies the betweenness condition.

3. Monotonic planar representations

**Theorem 1.** For any polygonal planar representation of a digraph, there is a polygonal planar representation that is monotonic and similar to the original one.
Since the proof of Theorem 1 is complicated, we discuss it first. If an edge of a polygonal plane digraph is not monotonic, then there are two distinct points on it that are at the same level. (A level is a horizontal line.) For example, we can have a bump as shown in Fig. 4. That figure shows part of a single edge starting at its lower vertex; the solid circles are corner points. The corner point $i$ could be the upper vertex of the edge, but only if it were higher than shown in Fig. 4. The proof will make each edge monotonic by removing all its bumps.

We give two examples that illustrate how the proof of Theorem 1 removes bumps. In Fig. 5, the left-hand diagram shows a plane digraph in which the edge $cd$ returns to the level of $p$. (As usual, open circles represent vertices.) We chose the point $q$ on the edge $cd$ so that the subarc from $p$ to $q$ of the current edge can be replaced by the segment from $p$ to $q$ in the final planar representation (shown on the right in Fig. 5). The point $q$ was chosen low enough so that the segment $pq$ did not touch the edge $fe$. To obtain a similar planar representation, the edges incident to $b$ could not be left where they were. The vertex $b$ was moved downward to a position below $pq$. Observe that the edges $ab$ and $cb$ are no longer straight after the bump is removed.

Figure 6 presents a more complex situation. We do not describe the final monotonic planar representation. (It is very easy to obtain in two steps from the proof.) Rather, we use this example to raise some problems that the proof must overcome. The point $q$ is chosen on the edge $ae$. As before, the subarc from $p$ to $q$ of the edge $ae$ will be replaced by the segment from $p$ to $q$ in a new planar representation.
representation. The new location for the edge $ab$ must be below the segment $pq$. Thus, there must be horizontal as well as vertical movement applied to the edge $ab$. Clearly, the final position of $c$ must also be below the segment $pq$. However, this means that the point $r$ must be lowered so that the new edge $dc$ will satisfy the betweenness condition.

We now precisely define a bump on one edge of a polygonal plane digraph (see Fig. 4). Let $p$, $s$ and $t$ be three distinct points that are in this order on the edge (starting at the lower vertex). The points $s$ and $t$ are consecutive corner points. The points $p$ and $s$ are the same level, and $t$ is above this level. We also require that the subarc $a$ from $p$ to $s$ is above the level of $s$ except at its endpoints. Finally, we require that $t$ is outside the region bounded by $a$ and the segment $ps$. The corner point $s$ is “where the bump occurs.”

We say that a new polygonal planar representation removes the bump
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described above if, for some point \( q \) after \( s \) on the segment \( st \), the new subarc from \( p \) to \( q \) in the new planar representation is monotonic, and the parts of the edge before \( p \) and after \( q \) are unchanged. Since a bump is uniquely determined by the corner point \( s \), there are only a finite number of bumps on each edge. There may be more than one bump at the same level. The proof of Theorem 1 shows how to remove a bump at the lowest possible level (chosen earliest along the edge if more than one bump is at this level). After this removal, all the bumps that occurred on the subarc from \( p \) to \( t \) are gone. (The bump at \( s \) was the last one on this subarc.) All bumps with a later value of \( s \) remain and retain their old value of \( s \), although their value of \( p \) may change.

Next, we show that an edge can be made monotonic by removing its bumps. The new planar representation in the proof of Theorem 1 that removes a bump on an edge does not change the number of bumps on any other edge. A vee on a polygonal arc consists of three consecutive corner points \( r, s, t \) such that \( r \) and \( t \) are both above the level of \( s \). The vee is said to “occur” at \( s \). Clearly, a vee occurs at each bump.

**Lemma 3.** Let \( e \) be a polygonal directed edge in the plane that does not contain a horizontal segment. If a vee occurs at the corner point \( s \) of \( e \), then a bump occurs at \( s \) or below the level of \( s \).

**Proof.** Let \( r, s, t \) be a vee on \( e \), where \( r \) precedes \( s \) on \( e \). Let \( p \) be the last point on \( e \) before \( r \) that is at the level of \( s \). Clearly, the subarc \( a \) from \( p \) to \( s \) is above the level of \( s \) except at its endpoints. If a vee occurred below the level of \( s \), then this vee could replace the original one. Therefore, we can assume that no vee occurs below the level of \( s \).

Suppose that \( t \) is inside the region bounded by \( a \) and the segment \( ps \). Since the upper vertex \( w \) cannot be in this region by the betweenness condition, the subarc from \( t \) to \( w \) goes below the level of \( s \). Let \( u \) be the first corner point after \( t \) that is below the level of \( s \). We start at \( u \) and continue along the edge \( e \) as long as we are going downward in the plane. Since there are no horizontal segments, we reach a corner point \( b \) such that the next corner point \( c \) is above the level of \( b \). If \( a \) denotes the corner point preceding \( b \) on the subarc from \( t \) to \( b \), then \( a, b, c \) is obviously a vee that is at a lower level than \( s \). This contradiction shows that a bump occurs at \( s \), completing the proof. \( \square \)

**Lemma 4.** Let \( e \) be a polygonal directed edge in the plane that does not contain a horizontal segment. Let \( c \) be a corner point of \( e \), and assume that the corner point \( d \) that follows \( c \) on \( e \) is above the level of \( c \). If no bump occurs at \( c \) or below the level of \( c \), then the subarc from the lower vertex \( v \) to \( d \) is monotonic. Consequently, if a polygonal edge of a plane digraph has no bumps or horizontal segments, then it is monotonic.
Proof. Assume all hypotheses of the lemma are satisfied. Starting from \( d \), we continue backward along the edge \( e \) as long as we are going downward in the plane. If we do not reach \( v \), then a vee occurs at the last point we reach, and this point is at or below \( c \). This would mean, by Lemma 3, that a bump occurs at or below \( c \), contrary to assumption. Since we reach \( v \), the subarc from \( v \) to \( d \) is monotonic. □

**Proof of Theorem 1.** We start with a polygonal planar representation of a fixed digraph. By the last observation of Section 2, we can assume that no edge contains a horizontal segment in the initial polygonal plane digraph. Although we add many more corner points in one step of our construction, this condition will be preserved. We select a non-monotonic edge and call it the “current edge.”

By Lemma 4, it suffices to define a similar new planar representation that removes the bump on the current edge that is at the lowest possible level (chosen earliest along the edge if more than one bump is at this level), and to show that the number of bumps on any other edge is unchanged. The starting polygonal planar representation of the digraph is just called “the plane digraph.” The level of this bump will be called the “reference” level.

Let the bump be as in Fig. 4 with \( p, r, s \) and \( t \) be as shown there (and as given in the definition of a bump). We shall assume that \( p \) is to the left of \( s \) as in Fig. 4. There are no bumps on the current edge that are below the reference level, and no bump occurs at the reference level and before \( c \) on the current edge. By Lemma 4, the subarc from the lower vertex to \( p \) is monotonic, and so lies below the reference level except at \( p \). (Note that \( p \) could be the lower vertex of the current edge.) The main polygon is the closed set bounded by the subarc \( \alpha \) from \( p \) to \( s \) and the segment \( ps \). (The interior of the main polygon is shaded in Fig. 4.)

We shall choose a point \( q \) on the segment \( st \) so that, at the very end of our construction, the subarc of the current edge from \( p \) to \( q \) can be replaced with the segment \( pq \). Thus, the subarc of the current edge up to \( q \) will be monotonic at the end of our construction. We require that \( q \) is distinct from \( s \) and low enough so that \( rs \) is the only edge of \( \alpha \) that the segment \( pq \) intersects after it leaves \( p \). Let \( u \) denote the intersection of \( pq \) and \( rs \). (The segment \( uq \) is shown dotted in Fig. 4.) Finally, we require that the closed set bounded by \( us, sq, qu \) contains no point of the plane digraph that is not on \( us \) or \( sq \).

Let the height of any compact set be the maximum distance between one of its points and the reference level. Let \( \beta \) be a subarc of any edge \( e \) (not necessarily the current edge) of the plane digraph such that \( \beta \) is above or at the reference level and its endpoints are distinct and at the reference level (Note that \( \beta \) can intersect the reference level at more than two points.) Also, let \( a \) and \( b \) be the leftmost and rightmost intersection points of \( \beta \) with the reference level. If the height of \( \beta \) is less than the height of the main polygon, and each part of \( e \) adjacent to \( \beta \) immediately goes below the reference level, then we call the closed set bounded by \( \beta \) and the segment \( ab \) an extra polygon. (Note that interior of an
extra polygon is nonempty but does not have to be connected.) An extra polygon is maximal if it is not contained in any other extra polygon or in the main polygon. There may be no extra polygons. Unless indicated otherwise, “polygon” means the main polygon or a maximal extra polygon.

By the betweenness condition, the upper vertex of the current edge is not in the main polygon. When an edge leaves an extra polygon, it goes below the reference level. By Lemma 3, there is no vee on the current edge below the reference level. Consequently, the current edge does not intersect any extra polygon. By similar reasoning, the subarc α from p to s is the only part of the current edge that is in the main polygon.

We are about to define a new planar representation of the original digraph. It is the “penultimate” one because it does not remove the bump at s on the current edge. We shall shrink the main polygon until it is below the line through p and q except at p. In doing so, the height of every polygon will be multiplied by a positive number which is less than 1. Adding at most one short vertical segment to each vertex of the original plane digraph produces a polygonal plane representation of the augmented digraph. These new segments can be chosen so they do not cross the segment uq. Thus, we can assume that the original digraph is augmented.

For convenience, we henceforth assume that p is (0,0). Thus, the reference level is the x-axis. We first map every point (x, y) in the main polygon to (x + δy, y) for some sufficiently large δ in order to give each point that is above the x-axis a positive x-coordinate. (Choose δ so that each corner point of the main polygon except p and q is given a positive x-coordinate.) Clearly, there are positive reals m and ε', with ε' < 1, such that, whenever 0 < ε ≤ ε', sending each point (x, y) in the transformed main polygon to (mx, εy) maps the main polygon into the original main polygon, and maps every point of the main polygon except p below the line through p and q. Similarly, for each maximal extra polygon, there are real numbers a, b, ε with 0 < a, ε' < 1 such that the map (x, y) → (ax + b, εy) with 0 < ε ≤ ε' maps the original extra polygon into itself. We now let ε be the minimum of the values of ε' for all polygons. Henceforth, we shall assume that all the above maps are defined with this common value of ε < 1.

Let R be a closed rectangle, with horizontal and vertical sides, whose interior contains the plane digraph. Also, let R+ consist of those points in R whose second coordinates are non-negative. Let ψ be the composite map defined for the main polygon. We shall specify a polygonal arc γ that starts at p, ends at a point s' slightly to the right of s on the x-axis, and is above the x-axis except at its endpoints. Each remaining corner point of γ is outside the main polygon and near a corner point of α. Clearly, such an arc γ can be found so the interior of the simple closed curve formed by γ and the segment ps' contains all of the main polygon that is above the x-axis. We can assume that γ intersects the plane digraph at exactly two points: at p and at the intersection point t' of the final segment of γ with st. We can also assume that any point of the plane digraph that is in the closed
set bounded by $\gamma$ and $p's$ is in the main polygon or is on the segment $st'$. Finally, we assume that the segment from $\psi s$ to $t'$ only intersects the image of the main polygon at $\psi s$. By the Schoenflies Extension Theorem (see Section 1), $\psi$ can be extended to a homeomorphism of $R^+$ that fixes each point of the plane digraph—except on $st'$—that is outside the main polygon. This extension fixes each point on $\gamma$ and on the boundary of $R$. We can also require that the extension of $\psi$ maps $st'$ to the segment from $\psi s$ to $t'$. By a similar argument, each map defined for a maximal extra polygon can be extended to a homeomorphism of $R^+$ that fixes each point on the augmented digraph that is outside the corresponding polygon and fixes each point on the boundary of $R$.

We shall define a homeomorphism $\varphi$ from $R$ to itself that is the identity on the boundary. On $R^+$, this homeomorphism is the composite, in some order, of the homeomorphisms that extend the maps associated with the polygons. Observe that $\varphi$ fixes both $p$ and $q$. Let $L$ be a horizontal line that is slightly below the $x$-axis so that there are no corner points of the plane digraph on $L$, or between $L$ and the $x$-axis. Below and on $L$, $\varphi$ is the identity function. Each segment $uw$ that was part of the digraph in the original planar representation (with $u$ on $L$ and $w$ on the reference level) should be mapped by $\varphi$ to the segment joining $v$ and $\varphi w$. Since $\varphi$, as defined on $R^+$, preserves the order of points on the $x$-axis, it is easy to extend $\varphi$ to $R$ so that the previous conditions are satisfied and $\varphi$ preserves $y$-coordinates below the $x$-axis. The penultimate planar representation is the image of the original plane digraph under $\varphi$. By the second definition of similarity for graphs we have, as a graph, a similar planar representation.

If a point $a$ on an edge of the plane digraph is in some polygon, then every point before $a$ on this edge is in a polygon or is below the reference level. Moreover, if $a$ is in some polygon, but some point before $a$ is not in the same polygon, then the lower vertex of the edge is below the reference level. (We omit the easy verifications. For $a$ on the subarc $\alpha$ from $p$ to $s$, recall that the subarc of the current edge up to $p$ is below the $x$-axis except at $p$.) Consequently, each new edge satisfies the betweenness condition so that the penultimate plane graph is a planar representation of the digraph. Since the digraph is augmented, the original and penultimate planar representations of the digraph are similar.

Observe that $\varphi$ and its inverse preserve $y$-coordinates on and below the $x$-axis. Let $ab$ be a segment of the plane digraph with $0<\pi_2 a < \pi_2 b$. By the previous paragraph, $ab$ is a subset of some polygon or it is disjoint from every polygon. Consequently, there are no horizontal segments in the penultimate plane digraph.

We show that a bump occurs at $b$ in the original plane digraph if and only if a bump occurs at $\varphi b$ in the penultimate plane digraph. This is obvious if $b$ is on or below the $x$-axis or if $b$ is on the current edge. Let a bump on a different edge occur at $b$ with a positive $y$-coordinate, let $a$ be the earlier point on the bump that is at the same level as $b$, let $c$ be the corner point after $b$, and let $\beta$ be the subarc from $a$ to $c$. Since $\beta$ does not intersect the $x$-axis, it is completely outside every polygon or
it remains inside the same polygon. Thus, \( q(\beta) \) clearly defines a bump. Since \( q \)
maps the set of points above the x-axis to itself, the opposite implication is now immediate.

In the penultimate plane digraph, we now replace the subarc from \( p \) to \( q \) of the
current edge by the segment \( pq \) to define the final planar representation. Since
this change can be effected by a suitable homeomorphism, this final planar
representation is similar to the penultimate one, and therefore to the original
one. (The argument is much the same as the one used to prove similarity. A
simple closed curve is defined that passes through \( p \) and \( q \), and approximates the
simple closed curve composed of \( q(\alpha) \) and the segments \( (qs)t', \ t'q \) and \( pq \).
Moreover, the interior of the new simple closed curve contains every point on the
latter curve except \( p \) and \( q \). No other point of the penultimate plane digraph is
inside or on the new simply closed curve.) Since the subarc from \( p \) to \( t' \) was the
only part of the current edge that was changed in going to the penultimate planar
digraph, it is clear that the final planar representation has removed the bump at \( s \).
Since the number of bumps on any other edge is the same as in the original planar
representation, the proof is complete. \( \square \)

Using the polygonal approximation of Section 2, we have the

**Corollary.** For any planar representation of a digraph, there is a similar planar
representation that is polygonal and monotonic.

### 4. Embedding and triangulation

Each edge \( ab \) of a monotonic plane digraph \( G \) uniquely determines a
continuous function \( f \) with domain \([\pi_2a, \pi_2b]\) such that the arc is parameterized
as \( \langle f(y), y \rangle \); we call \( f \) the edge function of the edge \( ab \). A point \( \langle x, y \rangle \) in the
plane is to the left of a monotonic arc from \( p \) (the lower point) to \( q \) if
\( \pi_2p \leq y < \pi_2q \) and \( x \leq x' \), where \( \langle x', y \rangle \) is on the arc.

**Theorem 2.** If \( G \) is a monotonic plane digraph with only one vertex at each of the
lowest and highest levels, then monotonic edges can be added to \( G \) to form a
bounded plane digraph on the same set of vertices. Moreover, if \( G \) is polygonal,
then the bounded plane digraph can also be chosen to be polygonal.

**Proof.** Let \( \alpha \) and \( \beta \) be the vertices of \( G \) at the lowest and highest level
respectively. By duality, it suffices to add edges until every vertex except \( \alpha \) is the
second vertex of some edge. Let \( c \) be a vertex distinct from \( \alpha \) that has indegree
zero. Draw a vertical line downward from \( c \). If such a line never hits any other
part of \( G \), then it is easy to define a monotonic arc from \( \alpha \) to \( c \) that does not
intersect any other part of \( G \). (If \( G \) is polygonal or \( \alpha \) is an isolated vertex, the
vertical line is stopped at a point a little above the level of \( \alpha \), and then the
segment from this point to \( \alpha \) is added. In general, a simple case of our subsequent argument is needed to define the start of the arc from \( \alpha \). If the vertical line first hits a vertex of \( G \), then the new arc is obvious. We can therefore assume that the vertical line first hits the edge \( ab \) at the point \( p \) as in Fig. 7. However, we do not assume that the plane digraph \( G \) is straight as indicated there or even that it is polygonal. Let \( u = \pi_2 a \), let \( v = \pi_2 b \), and let \( d \) be the point on the vertical line that is at the level of \( b \). We shall define an arc from \( u \) to \( d \). The edge \( ac \) will then be represented by this arc together with the segment from \( d \) to \( c \). The arc \( M = ap \cup pd \) (where \( ap \) is the first part of the edge \( ab \) and \( pd \) is a vertical segment) is useful as a construction device because no edge of \( G \) except \( ab \) intersects \( M - \{a\} \). If some point on an edge of \( G \) except \( ab \) is to the left of \( M \), and is at a level in the interval \((u, v]\), then every point on that edge is strictly to the left of \( M \). We assume that \( b \) is to the left of \( M \) as in Fig. 7.

Let \( \mathcal{F} \) (respectively, \( \mathcal{G} \)) consist of the edge functions of all edges except \( ab \) that contain a point whose second coordinate is in \((u, v]\) and is to the left (respectively, right) of \( M \). We add one additional function to each of \( \mathcal{F} \) and \( \mathcal{G} \). To \( \mathcal{F} \), we add the edge function of \( ab \), and to \( \mathcal{G} \), we add a function \( g \) with domain \([u, v]\) that has the form

\[
g(y) = f(y) + \delta(y - u),
\]

where \( f \) is the edge function of \( ab \) and the positive constant \( \delta \) is large enough so that every point on the arc \( (g(y), y) \) is to the right of \( M \).

Fig. 7. Adding a new edge.
Let \( l \) be the (pointwise) supremum of \( \mathcal{F} \), and let \( r \) be the infimum of \( \mathcal{G} \), where the domain of each of these functions is \([u,v]\). Each of these functions is continuous except for a finite number of jump discontinuities, and each one is continuous at \( u \). There is no point of the digraph \( G \) that lies strictly between these two curves. We can retain this property when \( l \) and \( r \) are suitably redefined as linear functions in a neighborhood of each discontinuity so that the new functions are continuous (and even piecewise linear if the plane digraph \( G \) is polygonal). Assume that this has been done. Let \( \lambda \) be the unique real number such that

\[
\pi_2 d = \lambda l(v) + (1 - \lambda) r(v).
\]

The new arc from \( a \) to \( d \) is given by \( \langle \lambda l(y) + (1 - \lambda) r(y), y \rangle \) for \( y \in [u,v] \). If all the original edges are polygonal, then so is the new edge \( ac \) formed by this new arc and the segment \( dc \).

**Corollary 1.** Any planar digraph \( G \) is a subdigraph of a bounded planar digraph on the same vertex set. In fact, for any planar representation of \( G \), there is a bounded plane digraph on the same vertex set that is polygonal and monotonic, in which the induced planar representation of \( G \) is similar to the original one.

**Proof.** By the corollary to Theorem 1, there is a polygonal monotonic planar representation of \( G \) that is similar to the original one. By the final observation of Section 2, one vertex at each of the lowest and highest levels can be moved to produce a similar polygonal planar representation to which Theorem 2 can be applied. \( \square \)

As in graph theory, we introduce a combinatorial meaning for *homeomorphic*. A digraph is *homeomorphic* to any digraph obtained by adding vertices to its edges.

**Corollary 2.** A digraph on \( n \geq 3 \) vertices is planar iff it is homeomorphic to a subdigraph of the covering digraph of a planar lattice on \( 3n - 5 \) elements.

**Proof.** By Corollary 1, any planar digraph is a subdigraph of a bounded planar digraph with the same number of vertices. We turn the bounded planar digraph into the covering digraph of a planar lattice by adding a vertex to each extraordinary edge. The number of extraordinary edges obviously cannot be more than the number of finite faces. A triangulated plane graph on \( n \geq 3 \) vertices has \( 2n - 5 \) finite faces, the maximum possible number for a graph with this many vertices. A planar lattice can be enlarged by adding a new zero. \( \square \)

Let \( P \) be the planar ordered set consisting of the ordinal sum of a 2-element antichain \( A \) and an \((n - 2)\)-element antichain \( B \). (Every element of \( A \) is less than every element of \( B \).) This example shows that the size of the lattice in Corollary 2 is best possible, even for covering digraphs. Since any digraph on at most three vertices is planar, Corollary 2 provides a characterization of planar digraphs in
terms of planar lattices. As we noted in Section 1, Platt [9] has characterized planar lattices in terms of planar graphs.

Let $G$ be monotonic bounded plane digraph. The left boundary (of $G$) consists of the vertices and edges lying on the leftmost arc from 0 to 1. The left and right boundary of each finite face of $G$ consists of edges that form a monotonic arc, and these two arcs only intersect at their lowest and highest levels. A triangle in a digraph consists of three vertices $a$, $b$, $c$ and edges $ab$, $bc$ and $ac$. A face of a plane digraph is triangular if its boundary is a triangle. We now consider the analog of graph triangulation.

**Theorem 3.** If $G$ is a monotonic bounded plane digraph with at least three vertices, with no two vertices at the same level, then monotonic edges can be added to $G$ to form a bounded plane digraph on the same vertex set in which every face is triangular—except, possibly, the infinite one when $01$ is an edge of $G$. In addition, each boundary of the new plane digraph has at most two edges. The new plane digraph can be chosen to be polygonal if $G$ is.

**Proof.** In each face, we first decide which edge can be added. We then use a simple version of the argument in Theorem 2 to draw the edge. If $01$ is not an edge of $G$, then we first add it as the new left boundary. For this one edge, we indicate how to specify the arc that represents it. Let $(g(y), y)$ be the original left boundary arc. Start with a point $p$ between the levels of 0 and 1. Going downward from $p$, 'interpolate' as in Theorem 2 between the functions $f$ and $g$, where $f(y) = g(y) - \delta(y - \pi_2(0))$. The rest of the arc is obtained by a dual procedure.

If $a$, $b$, $c$, $d$ are consecutive vertices (going upward) along the left boundary $C$ of a face $F$, then the edges $ac$ and $bd$ cannot both be to the left of $C$. If either of these edges is not to the left of $C$, then it is not an edge of $G$. Therefore, we can add one of these two edges to the face $F$. We iterate this procedure until the left and right boundary of each face (including the infinite face) has at most three vertices. Each finite face that is not triangular has middle vertices $a$ and $b$ on its two boundaries; suppose that $a$ is lower than $b$ in the plane. Clearly, $ab$ is not an edge of the plane digraph, and we can add this edge across the face. The proof is now complete. \[\Box\]

**Corollary.** If $G$ is a plane digraph with at least three vertices, then $G$ is similar to a subdigraph of a monotonic bounded plane digraph that has the same vertex set and contains the edge $01$, in which no two vertices are at the same level and every face is triangular—except, possibly, the infinite one. In addition, each boundary of the new plane digraph has at most two edges.

**Proof.** By the corollary to Theorem 1, there is a polygonal monotonic planar representation of $G$. By the final observation of Section 2, we can assume that no two vertices of $G$ are at the same level. Now apply Theorems 2 and 3. \[\Box\]
5. Straight planar representations

In Theorem 4, we obtain a straight planar representation of any planar digraph. For planar lattices, the transition from a monotonic planar representation to a straight one is given in Theorem 2.5 of Kelly and Rival [6]. (Recall that a plane lattice is the same thing as a plane bounded ordered set.)

**Theorem 4.** Any plane digraph has a similar straight planar representation.

**Proof.** Let $G$ be a plane digraph. By the corollary to Theorem 3, we can assume that $G$ is a monotonic bounded plane digraph that contains the edge 01, in which no two vertices are at the same level and every face is triangular—except, possibly, the infinite one. In addition, each boundary of $G$ has at most two edges.

We prove, by induction on the number of vertices, that $G$ has a similar straight planar representation. We can certainly assume that $G$ has more than three vertices. If 01 is not on either boundary, then the inductive assumption applies to the part of $G$ to the left of 01 (including the edge 01). Thus, the digraphs to the left and right of 01 have similar straight planar representations, and it is easy to transform one so it matches the other along the segment from 0 to 1.

We have thus reduced to the case that the infinite face of $G$ is triangular. The rest of our proof follows Fáry [4]. If there is a triangle of $G$ that is not the boundary of the infinite face and includes a vertex in its interior, then induction applies to the inside and outside of this triangle, and their two straight planar representations can be patched together. We now assume that this case does not hold. Let $a$ be a vertex of $G$ that is not on either boundary. Let $b_1, b_2, \ldots, b_k$ be the lower neighbors of $a$ and let $c_1, c_2, \ldots, c_l$ be the upper neighbors of $a$ (both from left to right). Clearly, both $k$ and $l$ are nonzero. Let $b = b_h$ be the lower neighbor of $a$ that is at the highest level.

By Lemma 4 of [4], $C = (b_1, \ldots, b_k, c_1, \ldots, c_l, b_1)$ is a cycle of $G$ and $a$ is the only vertex of $G$ in the interior of $C$. Consequently, $k + l \geq 4$. We form a new plane digraph $H$ from $G$ by deleting the interior of $C$ and using the interpolation technique of Theorems 2 and 3 to add the following $(k + l - 3)$ edges as monotonic arcs inside $C$:

- $b_i b$ when $1 \leq i \leq k$, $i \neq h$, $i \neq h - 1$ if $h > 1$, and $i \neq h + 1$ if $h < k$.
- $bc_i$ when $1 \leq j \leq l$, $j \neq 1$ if $h = 1$, and $j \neq l$ if $h = k$.

Since no vertex is inside a triangle in $G$, none of these edges are in $G$, so that there are no multiple edges in $H$. By induction, there is a straight planar representation $\sigma(H)$ that is similar to the plane digraph $H$. Observe that, in $\sigma(H)$, $b$ is higher than $b_i$ for all $i \neq h$, and $b$ is lower than $c_j$ for all $j$. For $\sigma(H)$, there is a disk $D$ centered at $b$ such that every point in $D$ that is inside $C$ can be joined with a segment inside $C$ to all the vertices of $C$. We delete the interior of $C$ in $\sigma(H)$, and place $a$ in $D$ and in the interior of $C$. Since there exists at least one
segment of the form \( bc_j \) that lies inside or on \( C \), we can assume that \( a \) is above \( b \). We can also require \( a \) to be below the level of every \( c_j \). We complete the definition of the straight planar representation of \( G \) by joining \( a \) with a segment to each vertex of \( C \). Since we have preserved both the cyclic order about \( a \) and the orientation of \( C \), the final straight planar representation is similar to the original plane digraph \( G \).

**References**