Stieltjes Type Theorems for Orthogonal Polynomials of Two Variables

Zhong-xuan Luo and Ren-hong Wang

Institute of Mathematical Sciences, Dalian University of Technology,
Dalian 116024, China

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In this paper, some important properties of orthogonal polynomials of two variables are investigated. The concepts of invariant factor for orthogonal polynomials of two variables are introduced. The presented results include Stieltjes type theorems for multivariate orthogonal polynomials and the corresponding asymptotic expansion formulas.

Key Words: orthogonal polynomials of two variables; invariant factor; Stieltjes type theorems.

1. INTRODUCTION

It is well known that one of the most important characteristics of orthogonal polynomials in one variable is the three-term recurrence relation. Let \( \{\omega_n(x)\}_{n=0}^{\infty} \) be a sequence of orthogonal polynomials with unit coefficient of the highest degree term. Then

\[
\omega_{n+2}(x) = (x - \alpha_{n+2})\omega_{n+1} - \lambda_{n+1}\omega_n(x).
\]  

Stieltjes [4] proved the following interesting result from the three-term recurrence relation (1):

\textit{Stieltjes Theorem.} Let \( x < a \) or \( x > b \). Then the rational function

\[
\frac{\psi_n(x)}{\omega_n(x)}
\]  

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converges to the integral \( \int_a^b \frac{\rho(t)}{x-t} \, dt \), and (2) is a unique rational function such that the following limit
\[
\lim_{x \to \infty} x^{2n+1} \left( \int_a^b \frac{\rho(t)}{x-t} \, dt - \frac{\psi_n(x)}{\omega_n(x)} \right)
\]
exists.

This theorem reveals the relation between orthogonal polynomials and the theory of continued fractions. In other words, any orthogonal polynomial on a finite interval could be obtained from the continued fraction expansion of the function \( \int_a^b \frac{\rho(t)}{x-t} \, dt \). Indeed, it is well known that the orthogonal polynomial of degree \( n \) on \( [a, b] \) could also be obtained from the \( [n-1]/[n] \) type Padé approximation of the function \( \int_a^b \frac{\rho(t)}{x-t} \, dt \).

The aim of this paper is to study the corresponding theory and formulas for orthogonal polynomials of two variables. One of the difficulties is that the space of polynomials of degree \( n \) that are orthogonal to all polynomials of lower degree is of a dimension that is greater than one. It follows that an invariant factor of multivariate orthogonal polynomials will appear in the Stieltjes type theorems. We shall state some definitions and notations, introduce the invariant factor of orthogonal polynomials of two variables, and study its properties in Section 2. In Section 3, we shall present the main results of the paper—Stieltjes type theorems and the corresponding asymptotic expansion properties.

2. DEFINITIONS, NOTATIONS, AND PRELIMINARIES

Let \( \Pi_2 \) be the ring of all polynomials in two variables and let \( P_n \) be the space of all polynomials of total degree \( n \). Let \( D \subset \mathbb{R}^2 \) be a given bounded region and let \( \rho(x, y) \) be a weight function on \( D \).

Two polynomials \( f(x, y), g(x, y) \in \Pi_2 \) are said to be orthogonal polynomials related to the weight function \( \rho(x, y) \) on \( D \), if
\[
\iint_D \rho(x, y) f(x, y) g(x, y) \, dx \, dy = 0.
\]

Let \( V_n \) be the space of polynomials of total degree \( n \) that are orthogonal to all polynomials of lower degree; then \( V_n \) is a vector space of dimension \( n + 1 \), and
\[
P_n = \bigoplus_{k=0}^n V_k \quad \text{and} \quad \Pi_2 = \bigoplus_{k=0}^{+\infty} V_k.
\]

The two polynomials \( p(x, y) \) and \( q(x, y) \) are said to be equivalent, if there is a nonzero real number \( a \) such that
\[
p(x, y) = a \cdot q(x, y).
\]
Let \( \{ \omega_0^k(x, y), \omega_1^k(x, y), \ldots, \omega_k^k(x, y) \} \) be an orthogonal polynomial basis of \( \Pi_2 \) related to \( \rho(x, y) \) on \( D \). For a \( \omega^k = \{ \omega_0^k(x, y), \omega_1^k(x, y), \ldots, \omega_k^k(x, y) \} \), we define

\[
\Delta_1^k(x, y) = \begin{vmatrix}
\omega_0^k(x, y) & \frac{\partial}{\partial y} \omega_0^k(x, y) & \ldots & \frac{\partial^n}{\partial y^n} \omega_0^k(x, y) \\
\omega_1^k(x, y) & \frac{\partial}{\partial y} \omega_1^k(x, y) & \ldots & \frac{\partial^n}{\partial y^n} \omega_1^k(x, y) \\
\vdots & \vdots & \ddots & \vdots \\
\omega_k^k(x, y) & \frac{\partial}{\partial y} \omega_k^k(x, y) & \ldots & \frac{\partial^n}{\partial y^n} \omega_k^k(x, y)
\end{vmatrix} = \det[\delta_1^k(x, y)],
\]

(3)

and

\[
\Delta_2^k(x, y) = \begin{vmatrix}
\omega_0^k(x, y) & \frac{\partial}{\partial x} \omega_0^k(x, y) & \ldots & \frac{\partial^n}{\partial x^n} \omega_0^k(x, y) \\
\omega_1^k(x, y) & \frac{\partial}{\partial x} \omega_1^k(x, y) & \ldots & \frac{\partial^n}{\partial x^n} \omega_1^k(x, y) \\
\vdots & \vdots & \ddots & \vdots \\
\omega_k^k(x, y) & \frac{\partial}{\partial x} \omega_k^k(x, y) & \ldots & \frac{\partial^n}{\partial x^n} \omega_k^k(x, y)
\end{vmatrix} = \det[\delta_2^k(x, y)],
\]

(4)

The polynomials \( \Delta_1^k(x, y) \), \( \Delta_2^k(x, y) \), and \( \Delta(x, y) \) are said to be an invariant factor in \( y \), invariant factor in \( x \), and invariant factor of \( \omega^k \), respectively. They have the following properties:

**Proposition 1.**

(a) The functions \( \Delta_1^k(x, y) \) and \( \Delta_2^k(x, y) \) are one variable polynomials of degree \( \frac{k(k+1)}{2} \) in \( x \) and of degree \( \frac{k(k+1)}{2} \) in \( y \), resp.; \( \Delta_1(x, y) \) is a polynomial of degree \( \frac{(k-1)k}{2} \) in \( x \) and \( y \), resp.

(b) Up to a constant multiple, \( \Delta_1^k(x, y), \Delta_2^k(x, y), \) and \( \Delta(x, y) \) only depend on the region \( D \) and the weight function \( \rho(x, y) \) and are independent of collection of orthogonal polynomial vector \( \omega^k \).
Proof. It is easy to see that $\Delta^{(1)}_k(x, y), \Delta^{(2)}_k(x, y)$, and $\Delta_k(x, y)$ are polynomials in $x, y$. Since the total degree of $\omega^j_k(x, y)$ is $k$, it follows from the properties of the determinant that $\frac{\partial}{\partial y}\Delta^{(1)}_k(x, y) = 0$ and $\frac{\partial}{\partial y}\Delta^{(2)}_k(x, y) = 0$. It follows from the principle of the matrix function that $\frac{\partial}{\partial y}\Delta^{(1)}_k(x, y) = 0$, $\frac{\partial}{\partial y}\Delta^{(2)}_k(x, y) = 0$, and $\frac{\partial}{\partial y}\Delta_k(x, y) = 0$, which completes the proof of (a).

Let $\tilde{o}^k = \{\omega^0_k(x, y), \omega^1_k(x, y), \ldots, \omega^k_k(x, y)\} \in V_k$ and $\tilde{U}_k = \{u^0_k(x, y), u^1_k(x, y), \ldots, u^k_k(x, y)\} \in V_k$ be two different orthogonal polynomial vectors of total degree $k$. It follows from the basic theory of multivariate orthogonal polynomials that there are two nonsingular matrices $A_{(k+1)\times(k+1)}$ and $B_{k\times k}$ such that

$$
\begin{pmatrix}
\omega^0_k(x, y) \\
\omega^1_k(x, y) \\
\vdots \\
\omega^k_k(x, y)
\end{pmatrix} = A_{(k+1)\times(k+1)} \begin{pmatrix}
\omega^0_k(x, y) \\
\omega^1_k(x, y) \\
\vdots \\
\omega^k_k(x, y)
\end{pmatrix},
$$

$$
\begin{pmatrix}
u^0_k(x, y) \\
\nu^1_k(x, y) \\
\vdots \\
\nu^k_k(x, y)
\end{pmatrix} = B_{k\times k} \begin{pmatrix}
u^0_k(x, y) \\
\nu^1_k(x, y) \\
\vdots \\
\nu^k_k(x, y)
\end{pmatrix}.
$$

Hence

$$
\begin{align*}
\delta^{(1)}_k(x, y)_o &= A_{(k+1)\times(k+1)} \cdot \delta^{(1)}_k(x, y)_U, \\
\delta^{(2)}_k(x, y)_o &= A_{(k+1)\times(k+1)} \cdot \delta^{(2)}_k(x, y)_U, \\
\delta_k(x, y)_o &= \begin{pmatrix} A_{(k+1)\times(k+1)} & 0 \\ 0 & B_{k\times k} \end{pmatrix} \cdot \delta_k(x, y)_U,
\end{align*}
$$

and

$$
\begin{align*}
\Delta^{(1)}_k(x, y)_o &= |A_{(k+1)\times(k+1)}| \Delta^{(1)}_k(x, y)_U, \\
\Delta^{(2)}_k(x, y)_o &= |A_{(k+1)\times(k+1)}| \Delta^{(2)}_k(x, y)_U, \\
\Delta_k(x, y)_o &= |A_{(k+1)\times(k+1)}| |B_{k\times k}| \Delta_k(x, y)_U,
\end{align*}
$$

which completes the proof of the proposition.

In what follows, we use the notation $\Delta^{(1)}_k(x, y) := \Delta^y_k(x)$ and $\Delta^{(2)}_k(x, y) = \Delta^x_k(y)$. Denote the algebraic cofactor of $\frac{\partial}{\partial y}\omega^k_k$ in (3) by $A^{(1)}_j(x, y)$, the algebraic cofactor of $\frac{\partial}{\partial x}\omega^k_k$ in (4) by $A^{(2)}_j(x, y)$ $i, j = 0, 1, \ldots, k$, and the algebraic cofactor of the element which lies in the $i$th row and $j$th column in (5) by $A_{ij}(x, y)$, respectively.
For a given integer \( l (0 \leq l \leq k) \), let

\[
\Phi^{(1)}_l(u, v; x, y) = \frac{1}{l!} (v - y)^l \Delta^l_k(x) - \sum_{i=0}^{k-l} A_{ki}^{(1)}(x, y) \omega^{k}_i(u, v)
\]

(6)

\[
\Phi^{(2)}_l(u, v; x, y) = \frac{1}{l!} (u - x)^l \Delta^l_k(y) - \sum_{i=0}^{k-l} A_{ki}^{(2)}(x, y) \omega^{k}_i(u, v)
\]

(7)

\[
\Phi^{(l+k)}(u, v; x, y) = \frac{1}{l!} (u - x)^l \Delta^l_k(x, y) - \sum_{i=0}^{k-l} A_{ki}^{l+k}(x, y) \omega^{k}_i(u, v) - \sum_{i=0}^{k-l} A_{ki}^{l+k+1}(x, y) \omega^{k-1}_i(u, v)
\]

(8)

(9)

\[
\Phi^{(l+k+1)}(u, v; x, y) = \frac{1}{l!} (u - x)^l \Delta^l_k(x, y) - \sum_{i=0}^{k-l} A_{ki}^{l+k+1}(x, y) \omega^{k}_i(u, v) - \sum_{i=0}^{k-l} A_{ki}^{l+k+2}(x, y) \omega^{k-1}_i(u, v)
\]

Lemma 2. For \( l = 0, 1, \ldots, k \), the functions \( \Phi^{(1)}_l(u, v; x, y), \Phi^{(2)}_l(u, v; x, y), \Phi^{(l+k+1)}(u, v; x, y) \) are polynomials in \( u, v, x, y \).

Proof. Without loss of generality, we shall only give the proof for the case \( l = 0 \). Since

\[ \Delta^k_k(x) = \sum_{i=0}^{k} A_{ki}^{(1)}(x, y) \omega^{k}_i(x, y), \]

hence

\[ (u - x)\Phi^{(0)}_l(u, v; x, y) = \sum_{i=0}^{k} A_{ki}^{(1)}(x, y) (\omega^{k}_i(u, v) - \omega^{k}_i(x, y)). \]

Using Taylor expansion formula to \( \omega^{k}_i(u, v) \), we have

\[
\omega^{k}_i(u, v) = \sum_{j=0}^{k} \frac{1}{j!} \left( (u - x) \frac{\partial}{\partial x} + (v - y) \frac{\partial}{\partial y} \right)^j \omega^{k}_i(x, y)
\]

\[
= \sum_{j=0}^{k} \frac{1}{j!} \frac{\partial^j}{\partial y^j} \omega^{k}_i(x, y)(v - y)^j + (u - x) \cdot Q_j(u, v; x, y),
\]

where \( Q_j(u, v; x, y) \) is a polynomial in \( u, v, x, y \). We have

\[
(u - x)\Phi^{(0)}_l(u, v; x, y)
\]

\[
= \sum_{i=0}^{k} A_{ki}^{(1)}(x, y) \left[ \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j}{\partial y^j} \omega^{k}_i(x, y)(v - y)^j + (u - x) \cdot Q_j(u, v; x, y) \right]
\]
\[= \sum_{j=1}^{k} \frac{1}{j!} (v - y)^j \sum_{i=0}^{k} A_{i0}^{(1)}(x, y) \frac{\partial^j}{\partial y^j} \omega_i(x, y)\]
\[\quad + (u - x) \cdot \sum_{i=0}^{k} A_{i0}^{(1)}(x, y)Q_i(u, v; x, y).\]

Because
\[\sum_{i=0}^{k} A_{i0}^{(1)}(x, y) \frac{\partial^j}{\partial y^j} \omega_i(x, y) = 0, \quad j = 1, 2, \ldots, k,\]
we have
\[\Phi_{(1)}^0(u, v; x, y) = \sum_{i=0}^{k} A_{i0}^{(1)}(x, y) \cdot Q_i(u, v; x, y).\]

A similar argument applies to \(\Phi_{(2)}^j(u, v; x, y), \Phi^j(u, v; x, y), \) and \(\Phi^{j+k+1}(u, v; x, y)\).

3. STIELTJES TYPE THEOREMS IN ORTHOGONAL POLYNOMIALS OF TWO VARIABLES

In this section, we shall give Stieltjes type theorems for multivariate orthogonal polynomials and some rational asymptotic expansion formulas. Let

\[a = \min\{x|(x, y) \in \mathbf{D}\}, \quad b = \max\{x|(x, y) \in \mathbf{D}\},\]
\[c = \min\{y|(x, y) \in \mathbf{D}\}, \quad d = \max\{y|(x, y) \in \mathbf{D}\},\]
and \(\mathbf{D} = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}.

For a given weight function \(\rho(x, y)\) on \(\mathbf{D}\), a nonnegative integer \(0 \leq l \leq k\) and \(1 \leq l' \leq k\), let
\[\Psi_{(1)}^l(x, y) = \int_{\mathbf{D}} \rho(u, v)\Phi_{(1)}^l(u, v; x, y)(v - y)^l \, du \, dv \quad (10)\]
\[\Psi_{(2)}^l(x, y) = \int_{\mathbf{D}} \rho(u, v)\Phi_{(2)}^l(u, v; x, y)(u - x)^l \, du \, dv \quad (11)\]
\[\Psi^l(x, y) = \int_{\mathbf{D}} \rho(u, v)\Phi^l(u, v; x, y)(v - y)^l \, du \, dv \quad (12)\]
\[\Psi^{l+k}(x, y) = \int_{\mathbf{D}} \rho(u, v)\Phi^{l+k}(u, v; x, y)(u - x)^l \, du \, dv. \quad (13)\]

It follows from Lemma 2 that \(\Psi_{(1)}^l(x, y), \Psi_{(2)}^l(x, y), \Psi^l(x, y), \) and \(\Psi^{l+k}(x, y)\) are polynomials in \(x, y\). In particular, if \(l = 0\), then \(\Psi_{(1)}^0(x, y)\) and \(\Psi_{(2)}^0(x, y)\) are one variable polynomials in \(x\) and \(y\), respectively.
THEOREM 3 (Stieltjes type 1). Let \((x, y) \in \mathbb{R}^2 - \overline{D}\). Then the rational function
\[
\frac{\Psi_1'(x, y)}{\Delta_k^1(x)} \quad \text{and} \quad \frac{\Psi_2'(x, y)}{\Delta_k^2(y)}
\]
converges to the following integrals \((k \to +\infty)\)
\[
\frac{1}{n!} \iint_{\overline{D}} \frac{\rho(u, v)(y-v)^{2j}}{x-u} \, du \, dv \quad \text{and} \quad \frac{1}{n!} \iint_{\overline{D}} \frac{\rho(u, v)(x-u)^{2j}}{y-v} \, du \, dv,
\]
respectively.

THEOREM 4 (Stieltjes type 2). Let \((x, y) \in \mathbb{R}^2 - \overline{D}\). Then the rational function
\[
\frac{\Psi^n(x, y)}{\Delta_k(x, y)}
\]
converges to the following integrals \((k \to +\infty)\):
\[
\frac{1}{n!} \iint_{\overline{D}} \frac{\rho(u, v)(y-v)^{2n-1}}{x-u} \, du \, dv \quad \text{if} \quad 0 \leq n \leq k
\]
\[
\frac{1}{(n-k)!} \iint_{\overline{D}} \frac{\rho(u, v)(x-u)^{2(n-k)-1}}{y-v} \, du \, dv \quad \text{if} \quad k+1 \leq n \leq 2k.
\]

Proof of Theorem 4. Without loss of generality, we assume that \(x > b\) \(y > d\). For given \(x, y\) and \(l = 0, 1, \ldots, k\), let
\[
\Lambda_k^l(z_{00}, z_{10}, \ldots, z_{0k-2})
\]
\[
= \iint_{\overline{D}} \rho(u, v) \left[ \frac{1}{n!} (v-y)^{l} + (u-x)(v-y) \sum_{i+j=0}^{k-2} z_{ij}(u-x)(v-y)^{l} \right] \, du \, dv, \quad (14)
\]
\[
\Lambda_{k}^l(z_{00}, z_{10}, \ldots, z_{0k-2})
\]
\[
= \iint_{\overline{D}} \rho(u, v) \left[ \frac{1}{n!} (u-x)^{l} + (u-x)(v-y) \sum_{i+j=0}^{k-2} z_{ij}(u-x)(v-y)^{l} \right] \, du \, dv, \quad (15)
\]
Consider the minimum problem
\[
M_k^l = \Lambda_k^l(z_{00}, z_{10}, \ldots, z_{0k-2}) = \min_{(z_{00}, z_{10}, \ldots, z_{0k-2})} \Lambda_k^l(z_{00}, z_{10}, \ldots, z_{0k-2}).
\]
It is easy to see that there exists a unique solution for this minimum problem, and
\[
\frac{\partial \Lambda_k^l}{\partial z_{ij}} = 0, \quad 0 \leq i + j \leq k - 2, \quad 0 \leq l \leq 2k.
\]
Hence, for the solution \( \{ \bar{z}_{ij} \}_{0 \leq i + j \leq k-2} \), we have

\[
\int_D \rho(u, v) \left[ \frac{1}{l!} (v - y)^l + (u - x)(v - y) \sum_{i+j=0}^{k-2} \bar{z}_{ij} (u - x)^i (v - y)^j \right] \\
\times (u - x)^s (v - y)^t \, du \, dv = 0, \quad 0 \leq l \leq k
\]

and

\[
\int_D \rho(u, v) \left[ \frac{1}{(l-k)!} (u - x)^{(l-k)} + (u - x)(v - y) \sum_{i+j=0}^{k-2} \bar{z}_{ij} (u - x)^i (v - y)^j \right] (u - x)^t (v - y)^s \, du \, dv = 0, \\
k < l \leq 2k, \quad 0 \leq s + t \leq k-2. \quad (16)
\]

Let

\[
H^l(u, v) = \frac{1}{l!} (v - y)^l + (u - x)(v - y) \sum_{i+j=0}^{k-2} \bar{z}_{ij} (u - x)^i (v - y)^j, \quad 0 \leq l \leq k,
\]

\[
H^l(u, v) = \frac{1}{(l-k)!} (u - x)^{(l-k)} + (u - x)(v - y) \sum_{i+j=0}^{k-2} \bar{z}_{ij} (u - x)^i (v - y)^j,
\]

\[
k < l \leq 2k.
\]

The formula (16) implies that

\[
\int_D \rho(u, v) H^l(u, v) u^s v^t \, du \, dv = 0, \quad 0 \leq s + t \leq k-2, \quad 0 \leq l \leq 2k, \quad (17)
\]

and \( H^l(u, v) \) is a polynomial of total degree \( k \), which is orthogonal to all polynomials of degree less than \( k - 1 \). It follows from the properties of the orthogonality theory that there are real numbers \( \{ M_{lj} \}_{j=0}^{2k} \) such that

\[
H^l(u, v) = \sum_{j=0}^{k} M_{l,j} \omega_j^k(u, v) + \sum_{j=0}^{k-1} M_{l,j+k+1} \omega_{j+1}^k(u, v). \quad (18)
\]

Hence, we have

\[
\begin{pmatrix}
H^0(u, v) \\
H^1(u, v) \\
\vdots \\
H^k(u, v) \\
H^{k+1}(u, v) \\
\vdots \\
H^{2k}(u, v)
\end{pmatrix} = \begin{pmatrix}
M_{00} & M_{01} & \cdots & M_{02k} \\
M_{10} & M_{11} & \cdots & M_{12k} \\
\vdots & \vdots & \ddots & \vdots \\
M_{2k0} & M_{2k1} & \cdots & M_{2k2k}
\end{pmatrix} \begin{pmatrix}
\omega_0^k(u, v) \\
\omega_1^k(u, v) \\
\vdots \\
\omega_{k-1}^k(u, v)
\end{pmatrix} := M \cdot \vec{\omega}.
\]
For an $l$, by differentiating formula (18) and taking $u = x, v = y$, we have
\[
\sum_{j=0}^{k} M_{lj} \frac{\partial^j}{\partial y^j} \omega_j^i + \sum_{j=0}^{k-1} M_{lj+k+1} \frac{\partial^j}{\partial y^j} \omega_j^{i-1} = \delta_{il}, \quad i, l = 0, 1, \ldots, k,
\]
\[
\sum_{j=0}^{k} M_{lj+k} \frac{\partial^j}{\partial x^j} \omega_j^k + \sum_{j=0}^{k-1} M_{lj+k+1} \frac{\partial^j}{\partial x^j} \omega_j^{k-1} = \delta_{il}, \quad i, l = 1, 2, \ldots, k,
\]
where $\delta_{il}$ is Kronecker’s symbol.
This implies that
\[
M = \delta_k(x, y)^{-1} = \frac{1}{\Delta_k(x, y)} \begin{pmatrix} A_{00}(x, y) & A_{10}(x, y) & \cdots & A_{2k,0}(x, y) \\ A_{01}(x, y) & A_{11}(x, y) & \cdots & A_{2k,1}(x, y) \\ \vdots & \vdots & \ddots & \vdots \\ A_{0,2k}(x, y) & A_{1,2k}(x, y) & \cdots & A_{2k,2k}(x, y) \end{pmatrix}.
\]
Hence, we get
\[
H^l(u, v) = \frac{1}{\Delta_k(x, y)} \left( \sum_{i=0}^{n} A_{ji}(x, y) \omega_j^i(u, v) + \sum_{i=0}^{k-1} A_{j+i+1,i}(x, y) \omega_j^{i-1}(u, v) \right), \quad (19)
\]
and for $0 \leq l \leq k$,
\[
M'_l := \int_{B} \rho(u, v) [H^l(u, v)]^2 \frac{du \, dv}{(x-u)(y-v)}
\]
\[
= \int_{B} \rho(u, v) H^l(u, v) \left[ \frac{1}{\pi} (v-y)^{2l} + (u-x)(v-y) \sum_{i=0}^{k} z_i(u-x)(v-y) \right] \frac{du \, dv}{(x-u)(y-v)}
\]
\[
= \frac{1}{\pi} \int_{B} \rho(u, v) H^l(u, v) \left[ \frac{(v-y)^l}{(x-u)(y-v)} \right] du \, dv
\]
\[
= \frac{1}{\pi} \int_{B} \rho(u, v) \sum_{i=0}^{k} A_{ji}(x, y) \omega_j^{l}(u, v) + \sum_{i=0}^{k-1} A_{j+i+1,i}(x, y) \omega_j^{l-1}(u, v) - \frac{1}{\pi} (v-y)^l \Delta_k(x, y)
\]
\[
\times (v-y)^d du \, dv + \left( \frac{1}{\pi} \right)^2 \int_{B} \rho(u, v) \frac{(v-y)^{2l-1}}{x-u} du \, dv
\]
\[
= \left( \frac{1}{\pi} \right)^2 \int_{B} \rho(u, v) \frac{(v-y)^{2l-1}}{x-u} du \, dv - \frac{1}{\pi} \Psi(x, y) \Delta_k(x, y).
\]
It remains to prove $\lim_{k \to \infty} M'_l = 0$. By an argument similar to Stieltjes theorem in one variable [4], it is not hard to show that there exists a real number $0 < q < 1$ such that
\[
0 \leq M'_k \leq q^2 M'_{k-1} \leq \cdots \leq q^{2k-2} M'_1,
\]
which implies that the statement holds for $0 \leq l \leq k$. A similar argument applies to the case $l' = 1, \ldots, k$, which completes the proof of Theorem 4.
Proof of Theorem 3. Without loss of generality, we assume that \( x > b \).
For a nonnegative integer \( l \leq k \), let
\[
H^l(u, v) = \frac{1}{l!} (v - y)^l + (u - x) \sum_{i+j=0}^{k-1} z_{ij}(u - x)^i(v - y)^j
\]
(20)
\[
\overline{H}^l(u, v) = \frac{1}{l!} (u - x)^l + (v - y) \sum_{i+j=0}^{k-1} z_{ij}(u - x)^i(v - y)^j.
\]
(21)

Analogous to the proof of Theorem 4, one uses a similar argument to \( H^l(u, v) \) and \( \overline{H}^l(u, v) \) to prove the theorem.

Let
\[
h^l_{(1)}(u,v;x,y) = \begin{vmatrix}
\omega^k_0(x,y) & \ldots & \frac{\partial^k-1}{\partial x^{k-1}} \omega^k_0(x,y) & \omega^k_0(u,v) & \frac{\partial^k}{\partial y^k} \omega^k_0(x,y) & \ldots & \frac{\partial^k}{\partial x^k} \omega^k_0(x,y) \\
\omega^k_1(x,y) & \ldots & \frac{\partial^k-1}{\partial x^{k-1}} \omega^k_1(x,y) & \omega^k_1(u,v) & \frac{\partial^k}{\partial y^k} \omega^k_1(x,y) & \ldots & \frac{\partial^k}{\partial x^k} \omega^k_1(x,y) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\omega^k_k(x,y) & \ldots & \frac{\partial^k-1}{\partial x^{k-1}} \omega^k_k(x,y) & \omega^k_k(u,v) & \frac{\partial^k}{\partial y^k} \omega^k_k(x,y) & \ldots & \frac{\partial^k}{\partial x^k} \omega^k_k(x,y)
\end{vmatrix}
\]
\[
h^l_{(2)}(u,v;x,y) = \begin{vmatrix}
\omega^k_0(x,y) & \ldots & \frac{\partial^k-1}{\partial x^{k-1}} \omega^k_0(x,y) & \omega^k_0(u,v) & \frac{\partial^k}{\partial y^k} \omega^k_0(x,y) & \ldots & \frac{\partial^k}{\partial x^k} \omega^k_0(x,y) \\
\omega^k_1(x,y) & \ldots & \frac{\partial^k-1}{\partial x^{k-1}} \omega^k_1(x,y) & \omega^k_1(u,v) & \frac{\partial^k}{\partial y^k} \omega^k_1(x,y) & \ldots & \frac{\partial^k}{\partial x^k} \omega^k_1(x,y) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\omega^k_k(x,y) & \ldots & \frac{\partial^k-1}{\partial x^{k-1}} \omega^k_k(x,y) & \omega^k_k(u,v) & \frac{\partial^k}{\partial y^k} \omega^k_k(x,y) & \ldots & \frac{\partial^k}{\partial x^k} \omega^k_k(x,y)
\end{vmatrix}
\]
and for \( l' = 0, 1, \ldots, 2k \), \( h^{l'}(u,v;x,y) \) is the determinant that the \( l' \)th column is replaced by \((\omega^k_0(u,v), \ldots, \omega^k_k(u,v), \omega^{k-1}_0(u,v), \ldots, \omega^{k-1}_k(u,v))^T\) in (5).

Let
\[
\overline{C}_{ij}(x,y) = \int_D \rho(u,v) h^{l}_{(1)}(u,v;x,y) u^{i'} v^{j'} \, du \, dv
\]
\[
\overline{C}^l_{ij}(x,y) = \int_D \rho(u,v) h^{l}_{(2)}(u,v;x,y) u^{i'} v^{j'} \, du \, dv
\]
\[
\overline{C}_{ij}(x,y) = \int_D \rho(u,v) h^{l'}(u,v;x,y) u^{i'} v^{j'} \, du \, dv.
\]

It follows from the properties of orthogonal polynomials that for \( 0 \leq l \leq k \),
\[
\overline{C}_{ij}(x,y) = 0, \quad \text{for } 0 \leq i + j \leq k - 1;
\]
\[
\overline{C}_{ij}(x,y) = 0, \quad \text{for } 0 \leq i + j \leq k - 1;
\]
for \( l' = 0, 1, \ldots, 2k \),
\[
\hat{C}_{ij}^l(x, y) = 0, \quad \text{for } 0 \leq i + j \leq k - 2.
\]

Similar to the processing of Proposition 1, it is easy to see that if \( j = 0 \), every algebraic cofactor \( A_{ij}^{(1)}(x, y) \) is a one variable polynomial in \( x \); if \( j > 0 \), every algebraic cofactor \( A_{ij}(x, y) \) is a polynomial in \( x, y \) but is of degree 1 in \( y \). So we have

**Lemma 5.** For \( i + j \geq k \), \( \hat{C}_{ij}^0(x, y) \) and \( \hat{C}_{ij}^0(x, y) \) are one variable polynomials in \( x \) and \( y \), respectively; \( \hat{C}_{ij}(x, y) \) and \( \hat{C}_{ij}^l(x, y) (l = 1, 2, \ldots, k) \) are polynomials in \( x, y \) but are of degree 1 in \( y \).

It is easy to verify that

**Lemma 6.** None of \( \hat{C}_{ij}(x, y), \hat{C}_{ij}^0(x, y), i + j = k \), and \( \hat{C}_{ij}^l(x, y), i + j = k - 1 \) are zero.

Because for \( |x| > \max\{|a|, |b|\} \) and \( |y| > \max\{|c|, |d|\} \) there are uniformly convergent series
\[
\frac{1}{x - u} = \frac{1}{x} \sum_{i=0}^{\infty} u^i \quad \text{and} \quad \frac{1}{y - v} = \frac{1}{y} \sum_{i=0}^{\infty} v^i,
\]
we have

**Theorem 7.** The fractions \( \frac{\Psi_{ij}^l(x, y)}{\Delta^l_k(x)} \) and \( \frac{\Psi_{ij}^l(x, y)}{\Delta^l_k(y)} \) have the following asymptotic expansion properties: for sufficiently large \( |x| \) and \( |y| \), \( 0 \leq l \leq k \)
\[
\begin{align*}
\frac{1}{l!} \int_D \frac{\rho(u, v)(y - u)^{2l}}{x - u} \, du \, dv - \frac{\Psi_{ij}^l(x, y)}{\Delta^l_k(x)} &= \frac{1}{x \Delta^l(x)} \left( \frac{\hat{C}_{k-2l}^l}{x^{k-2l}} + \cdots \right), \\
\frac{1}{l!} \int_D \frac{\rho(u, v)(x - u)^{2l}}{y - v} \, du \, dv - \frac{\Psi_{ij}^l(x, y)}{\Delta^l_k(y)} &= \frac{1}{y \Delta^l(y)} \left( \frac{\hat{C}_{k-2l}^l}{y^{k-2l}} + \cdots \right),
\end{align*}
\]
respectively.

For \( \frac{\Psi(x, y)}{\Delta_k(x, y)} \), we shall only give out its asymptotic expansion property for the case of \( l = 0 \).

**Theorem 8.** For sufficient large \( |x| \) and \( |y| \),
\[
\int_D \frac{\rho(u, v)}{(x - u)(y - v)} \, du \, dv - \frac{\Psi^0(x, y)}{\Delta_k(x, y)} = \frac{1}{xy \Delta_k(x, y)} \left( \frac{\hat{C}_{k-10}^0}{x^{k-1}} + \frac{\hat{C}_{k-21}^0}{x^{k-2}y} + \cdots \right).
\]
Finally, we give the invariant factors corresponding to some given planar regions and weight function \( \rho(x, y) = 1 \):

1. \( T_2 : \{(x, y) | x \geq 0, y \geq 0, x + y \leq 1\}, \rho(x, y) = 1 \).

\[
\Delta_1'(x) = 3x - 1 \\
\Delta_2'(x) = 4(-1 + 5x)(1 - 8x + 10x^2) \\
\Delta_3'(x) = 108(1 - 7x)(1 - 12x + 21x^2)(1 - 15x + 45x^2 - 35x^3) \\
\Delta_4'(x) = 27648(1 - 9x)(1 - 16x + 36x^2)(1 - 21x + 84x^2 - 84x^3) \\
\times (1 - 24x + 126x^2 - 224x^3 + 126x^4) \\
\cdots
\]

\[
\Delta_1'(y) = \Delta_1'(y), \quad \Delta_2'(y) = \Delta_2'(y), \quad \Delta_3'(y) = \Delta_3'(y), \quad \Delta_4'(y) = \Delta_4'(y) \\
\cdots
\]

and

\[
\Delta_1(x, y) = 1 \\
\Delta_2(x, y) = -100(1 - 4x - 4y + 12xy) \\
\Delta_3(x, y) = -148176(1 - 16x - 16y + 75x^2 + 240xy + 75y^2) \\
- 90x^3 - 1050x^2y - 1050xy^2 - 90y^3 + 1170x^3y \\
+ 4305x^2y^2 + 1170xy^3 - 4500x^3y^2 \\
- 4500x^2y^3 + 4500x^3y^3) \\
\cdots
\]

2. \( S_2 : \{(x, y) | x^2 + y^2 \leq 1\}, and \rho(x, y) = 1 \).

\[
\Delta_1'(x) = x \\
\Delta_2'(x) = -4(1 - 2x)x(1 - 2x) \\
\Delta_3'(x) = 144x^2(1 - 6x^2)(1 - 2x^2) \\
\Delta_4'(x) = 55296x^2(1 - 8x^2)(3 - 8x^2)(1 - 2x - 4x^2)(1 + 2x - 4x^2) \\
\cdots
\]

\[
\Delta_1'(y) = \Delta_1'(y), \quad \Delta_2'(y) = \Delta_2'(y), \quad \Delta_3'(y) = \Delta_3'(y), \quad \Delta_4'(y) = \Delta_4'(y) \\
\cdots
\]
and

\[
\begin{align*}
\Delta_1(x, y) &= 1 \\
\Delta_2(x, y) &= -32xy \\
\Delta_3(x, y) &= -4608xy(2 - 9x^2 - 9y^2 + 36x^2y^2) \\
\Delta_4(x, y) &= 6115295232x^2y^2(2 - 19x^2 - 19y^2 + 32x^4 + 164x^2y^2 + 32y^4 \\
&\quad - 256x^4y^2 - 256x^2y^4 + 384x^4y^4) \\
&\quad - 256x^4y^2 - 256x^2y^4 + 384x^4y^4
\end{align*}
\]

\[\cdots\]

REFERENCES