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Conservation Laws and Global Solutions of Linear First Order PDEs with Distributional Coefficients

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We treat linear partial differential equations of first order with distributional coefficients naturally related to physical conservation laws in the spirit of our preceding papers (which concern ordinary differential equations): the solutions are consistent with the classical ones. Under compatibility conditions we prove uniqueness and existence results. As an example we consider the problem $u_t + \delta_t u_x = 0$, $u(x, -1) = h(x)$ ($h \in C^2(\mathbb{R})$ is given); our theory grants that the unique solution in $C^2(\mathbb{R}^2) \oplus \mathscr{D}'_{\mathscr{I}}(\mathbb{R}^2)$ is $u(x,t) = h(x) - h'(0)\delta(x,t)$ and this has a physical meaning ($\mathscr{D}'_{\mathscr{I}}(\mathbb{R}^2)$ is the space of distributions with discrete support and δ is the Dirac measure at $(0,0)$). © 2001 Academic Press

0. INTRODUCTION: PHYSICAL MOTIVATION

Let us consider a physical system of spatial dimension one (coordinate x) and temporal dimension one (coordinate t). Let the state variable $u(x,t)$ be the density of matter at x at the instant t and $\phi(x,t)$ the flux of matter at x at the instant t. We may think of a fine tube placed along the x-axis with a circular cross-section of constant area S. To simplify, let us suppose that no matter has been created nor annihilated. Then the quantitative relation between u and ϕ is ruled by the equation

$$u_t + \phi_x = 0 \tag{0.1}$$

which is called the conservation law.

In this model the flux ϕ is a function of x and t. Of course, it may happen that ϕ, function of x and t, depends also on $u = u(x,t)$ or on its derivatives. For instance, letting $\phi = u^2/2$ then $\phi_x = uu_x$ and (0.1) reduces to $u_t + uu_x = 0$ which is Burger's equation. Letting $\phi = cu$ (c is a constant) in (0.1) we find $u_t + cu_x = 0$ which is a convection equation.

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Letting $\phi = -Du_x$ (D is a constant) in (0.1), we have $u_t - Du_{xx} = 0$ which is a diffusion equation.

Sometimes, when we try to describe certain physical situations by means of a differential equation, the conservation law (0.1) becomes naturally a partial differential equation of first order with distributional coefficients. Indeed, suppose that the physical setting forces us to consider a flux ϕ which is a $C^\infty(\mathbb{R}^2)$ function of x and t in all of the x, t-plane except on the positive part of the x-axis. There, at the instant $t = 0$ (the initial instant being at $t = -1$) a blow up is expected defined by a distribution with support on $\{(x,t) \in \mathbb{R}^2 \colon\ t = 0 \text{ and } x \geq 0\}$ which may depend on the density of matter $u(x,t)$ and on its derivatives. Let us exemplify this situation with the following simplified (so as to avoid technical complications) flux, written formally as

$$\phi = u_x(0,0)[H(x) \otimes \delta'(t)] - u_{xt}(0,0)[H(x) \otimes \delta(t)],$$

where H is the Heaviside function and δ is the Dirac measure at zero. The coefficients $u_x(0,0)$ and $u_t(0,0)$ were chosen so that the conservation law becomes an easy equation. We stress that this is only a formal expression, since we are treating functions and distributions on the same ground! Thus,

$$\begin{aligned} \phi_x &= u_x(0,0)[\delta(x) \otimes \delta'(t)] - u_{xt}(0,0)[\delta(x) \otimes \delta(t)] \\ &= u_x(0,0)\,\delta_t(x,t) - u_{xt}(0,0)\,\delta(x,t) \\ &= (\delta(x,t)u_x(x,t))_t - \delta(x,t)u_{xt}(x,t) = \delta_t(x,t)u_x(x,t), \end{aligned}$$

and (0.1) takes the form

$$u_t + \delta_t u_x = 0,$$

where $\delta_t = \delta_t(x,t) = \frac{\partial \delta}{\partial t}(x,t)$. We solve this highly non-classical equation in the final part of our paper under the "initial condition" $u(x,-1) = h(x)$ for a given $h \in C^2(\mathbb{R})$ and we give to the solution thus obtained a physical interpretation.

Such an equation and many others which express conservation laws are included in a general type of distributional equations to which our theory of distributional products affords a mathematical meaning and solutions.

1. THE PRODUCT OF DISTRIBUTIONS

Let $\mathscr{D}$ be the space of indefinitely differentiable complex functions defined on $\mathbb{R}^N$ with compact support, $\mathscr{D}'$ the space of distributions, and $L(\mathscr{D})$ the space of continuous linear maps $\mathscr{D} \to \mathscr{D}$. Among definitions of distribution products only two of them are general: Colombeau's construction [1] and our own. We will sketch the main ideas of this construction in the following. For details, the reader must see [4, 5].

First we define a product $T\phi \in \mathscr{D}'$ for $T \in \mathscr{D}'$ and $\phi \in L(\mathscr{D})$ by $\langle T\phi, x\rangle = \langle T, \phi(x)\rangle$ for all $x \in \mathscr{D}$. Next we define an epimorphism $\tilde{\zeta}$: $L(\mathscr{D}) \to \mathscr{D}'$ given by $\langle \tilde{\zeta}(\phi), x\rangle = \int\phi(x)$, for all $x \in \mathscr{D}$. Thus, given $T, S \in \mathscr{D}'$ we are tempted to define a natural product setting $TS := T\phi$, $\phi \in L(\mathscr{D})$ being such that $\tilde{\zeta}(\phi) = S$ (we say that $\phi \in L(\mathscr{D})$ is a representative operator of $S \in \mathscr{D}'$). Unfortunately, this product is not well defined because TS depends on the representative $\phi \in L(\mathscr{D})$ of $S \in \mathscr{D}'$.

This difficulty can be overcome if we fix $\alpha \in \mathscr{D}$ with $\int\alpha = 1$ and define s_α: $L(\mathscr{D}) \to L(\mathscr{D})$ by

$$[(s_\alpha \phi)(x)](y) = \int \phi[(\tau_y \check{\alpha})x]$$

for all $x \in \mathscr{D}$ and all $y \in \mathbb{R}^N$, where $\tau_y \check{\alpha}$: $\mathbb{R}^N \to \mathbb{C}$ is defined by $(\tau_y \check{\alpha})(t) = \check{\alpha}(t - y) = \alpha(y - t)$ for all $t \in \mathbb{R}^N$. It can be proved that, for each α in $\mathscr{D}$ with $\int\alpha = 1$, s_α is a linear operator and we have $s_\alpha \circ s_\alpha = s_\alpha$ (s_α is a projector in $L(\mathscr{D})$), $\operatorname{Ker} s_\alpha = \operatorname{Ker} \tilde{\zeta}$, and $\tilde{\zeta} \circ s_\alpha = \tilde{\zeta}$. Now, for each $\alpha \in \mathscr{D}$ with $\int\alpha = 1$ we define an α-product of $T \in \mathscr{D}'$ by $S \in \mathscr{D}'$ setting

$$T_{\dot{\alpha}} S := T(s_\alpha \phi) = (T * \check{\alpha})S, \tag{1.1}$$

where $\phi \in L(\mathscr{D})$ is a representative of $S \in \mathscr{D}'$. It is easy to prove that this α-product is independent of the representative of ϕ of S because $\operatorname{Ker} \tilde{\zeta} = \operatorname{Ker} s_\alpha$.

In general this α-product is neither commutative nor associative, but it is bilinear, has left unit element (the constant function with value 1 seen as a distribution), is distributive (to the right and to the left), and satisfies the usual rule for the derivative of the product,

$$D_k(T_{\dot{\alpha}} S) = (D_k T)_{\dot{\alpha}} S + T_{\dot{\alpha}}(D_k S),$$

where D_k is the usual k-partial derivative operator ($k = 1, 2, \ldots, N$). It is invariant for translations and also for any group G of unimodular transformations (linear transformations h: $\mathbb{R}^N \to \mathbb{R}^N$ with $|\det h| = 1$) if α is so invariant. It is not consistent with the classical Schwartz products [6] of distributions and functions.

In order to obtain consistency with the usual product of a distribution by a C^∞-function we are going to introduce some definitions and single out a certain subspace H_α of $L(\mathscr{D})$.

An operator $\phi \in L(\mathscr{D})$ is said to vanish in an open set Ω if $\phi(x) = 0$ for all $x \in \mathscr{D}$ with support contained in Ω. The support of an operator

$\phi \in L(\mathscr{D})$, supp ϕ, will be defined as the complement of the largest open set in which ϕ vanishes.

Let $\mathscr{N}$ be the set of operators of $L(\mathscr{D})$ with nowhere dense support and $\rho(C^\infty)$ the set of operators $\phi \in L(\mathscr{D})$ defined by $\phi(x) = \beta x$, for all $x \in \mathscr{D}$, with $\beta \in C^\infty$. For each $\alpha \in \mathscr{D}$ with $\int\alpha = 1$ let us consider the space $H_\alpha = \rho(C^\infty) \oplus s_\alpha(\mathscr{N}) \subset L(\mathscr{D})$. It can be proved that $\zeta_\alpha := \tilde{\zeta}|_{H_\alpha}: H_\alpha \to C^\infty \oplus \mathscr{D}'_m$ is an isomorphism. The space $\mathscr{D}'_m$ is denoted by $\mathscr{D}'_n$ in [4] and it is the space of nowhere dense supported distributions. Then if $T \in \mathscr{D}'$ and $S = \beta + f \in C^\infty \oplus \mathscr{D}'_m$, a new α-product can be defined by $T_{\dot{\alpha}} S := T\phi_\alpha$, where $\phi_\alpha \in H_\alpha$ is the representative of $S \in C^\infty \oplus \mathscr{D}'_m$. Now, this α-product is well defined because $\phi_\alpha = \zeta_\alpha^{-1}(S)$. Thus, we have

$$\begin{aligned} T_{\dot{\alpha}} S &= T_{\dot{\alpha}} \zeta_\alpha^{-1}(S) = T_{\dot{\alpha}} \zeta_\alpha^{-1}(\beta + f) = T_{\dot{\alpha}} \zeta_\alpha^{-1}(\beta) + T_{\dot{\alpha}} \zeta_\alpha^{-1}(f) \\ &= T\beta + (T * \check{\alpha}) f, \end{aligned} \tag{1.2}$$

and we get the consistency with the usual product of distributions with C^∞-functions, when these are placed at the right hand side. This is because if $S \in C^\infty$ then $f = 0$, $S = \beta$, and $T_{\dot{\alpha}} S = T\beta$. The remaining properties are the same of the α-product (1.1). Note also that if $\beta = 0$ the result of (1.2) is the same of (1.1).

Therefore, there exist lots of products. The test functions α are thus some kind of weights and it can physically be interesting for α to be invariant for the orthogonal group $\mathscr{O}(N)$ of $\mathbb{R}^N$ (in classical physics a product which is not invariant for this group clearly has no applications). In dimension $N = 1$, α will have even symmetry: $\check{\alpha} = \alpha$. We make this assumption in what follows. For instance, for the Dirac measure δ and the Heaviside function H, we have by (1.1) or (1.2)

$$\delta_{\dot{\alpha}} \delta = \delta_{\dot{\alpha}} (0 + \delta) = (\delta * \check{\alpha}) \delta = \check{\alpha} \delta = \alpha \delta = \alpha(0) \delta,$$

$$\begin{aligned} H_{\dot{\alpha}} \delta &= (H * \check{\alpha}) \delta = \left(\int_{-\infty}^{+\infty} \alpha(x - t) H(t) \, \mathrm{d}t \right) \delta = \left(\int_0^{+\infty} \alpha(-t) \, \mathrm{d}t \right) \delta \\ &= \tfrac{1}{2} \delta. \end{aligned}$$

In the setting of this theory, the α-products cannot be "completely" localized. This will be clear noting that for $T, S \in \mathscr{D}'$, supp$(T_{\dot{\alpha}} S) \subset$ supp(S) as for usual functions, but it doesn't happen that supp$(T_{\dot{\alpha}} S) \subset$ supp(T). In fact, if $a, b \in \mathbb{R}$,

$$\begin{aligned} (\tau_a \delta)_{\dot{\alpha}} (\tau_b \delta) &= [(\tau_a \delta) * \alpha](\tau_b \delta) = (\tau_a \alpha)(\tau_b \delta) = ((\tau_a \alpha)(b))(\tau_b \delta) \\ &= \alpha(b - a)(\tau_b \delta). \end{aligned}$$

Thus, the α-products are global products and when we apply them to differential equations, the solutions are naturally global solutions.

Finally, it is easy to see that formula (1.2) can be extended for $T \in \mathscr{D}'^p$ and $S \in C^p \oplus \mathscr{D}'_m$ ($p = 0, 1, 2, \dots$) where $\mathscr{D}'^p$ is the space of distributions of order $\le p$ in the sense of Schwartz [6]. So, assuming that $\mathscr{D}'^\infty$ means $\mathscr{D}'$, (1.2) is valid for $T \in \mathscr{D}'^p$ and $S \in C^p \oplus \mathscr{D}'_m$ with $p = 0, 1, 2, \dots, \infty$.

2. CLASSICAL SOLUTIONS AND W$_\alpha$-SOLUTIONS

Let us consider the linear Cauchy problem

$$P_h^v \equiv \begin{cases} au_x + bu_y + cu = v \\ u(r(t), s(t)) = h(t), \qquad \text{for all } t \in \mathbb{R} \end{cases} \tag{2.1}$$

in dimension $N = 2$ (the results extend immediately to $N \ge 2$). Here $a, b, c \in C^\infty(\mathbb{R}^2) \oplus \mathscr{D}'^p_\ell(\mathbb{R}^2)$, $\mathscr{D}'^p_\ell(\mathbb{R}^2) = \mathscr{D}'^p(\mathbb{R}^2) \cap \mathscr{D}'_\ell(\mathbb{R}^2)$, $p = 0, 1, 2, \dots, \infty$, $\mathscr{D}'_\ell(\mathbb{R}^2)$ is the space of distributions with discrete support in $\mathbb{R}^2$ (note that $\mathscr{D}'^p_\ell(\mathbb{R}^2) \subset \mathscr{D}'_m(\mathbb{R}^2)$), $v \in \mathscr{D}'(\mathbb{R}^2)$, $r, s, h \in C^{p+1}(\mathbb{R})$, and u is the unknown.

In the setting of classical Schwartz products, in order to solve the above problem we are forced to seek solutions in the narrow space $C^{p+1}(\mathbb{R}^2)$; we call such solutions *classical solutions*. Those solutions are clearly insufficient for applications in physical theories so that we must enlarge conveniently the concept of solution of the Cauchy problem. To do so, we associate to the problem P_h^v the problem Q_h^v defined by

$$Q_h^v \equiv \begin{cases} (u_x\gamma_1 + T_1 \dot{\alpha}\, u_x) + (u_y\gamma_2 + T_2 \dot{\alpha}\, u_y) + (u\gamma_3 + T_3 \dot{\alpha}\, u) = v \\ u(r(t), s(t)) = h(t), \qquad \text{for all } t \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $a = \gamma_1 + T_1$, $b = \gamma_2 + T_2$, $c = \gamma_3 + T_3$; $\gamma_1, \gamma_2, \gamma_3 \in C^\infty(\mathbb{R}^2)$; $T_1, T_2, T_3 \in \mathscr{D}'^p_\ell(\mathbb{R}^2)$ and the products $u_x\gamma_1, u_y\gamma_2, u\gamma_3$ are taken in the classical sense. The solutions of Q_h^v will be called *W_α-solutions of P_h^v with respect to the ruling group G*. They belong to the extended space $C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$ according to the following

DEFINITION 2.1. We say that $u \in C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$ is a *W_α-solution of P_h^v with respect to the ruling group G* when there exists an open set $\Omega \subset \mathbb{R}^2$ such that

(1) $(r(t), s(t)) \in \Omega$ for all $t \in \mathbb{R}$,

(2) the restriction u_Ω of u to Ω is a $C^{p+1}(\Omega)$-function,

(3) u satisfies Q_h^v.

Note that $u(r(t), s(t))$ makes sense. The W_α-solutions of P_h^v defined above are consistent with the classical solutions of P_h^v as shown by the following

THEOREM 2.2. *If $u \in C^{p+1}(\mathbb{R}^2)$ is a classical solution of P_h^v then, for all $\alpha \in \mathscr{D}(\mathbb{R}^2)$ G-invariant, with $\int \alpha = 1$, u is a W_α-solution of P_h^v with respect to G.*

Proof. It is sufficient to note that if $u = \beta + f \in C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$ is a classical solution of P_h^v then $f = 0$, $u = \beta + 0$, $u_x = \beta_x + 0$, $T_{1\dot{\alpha}} u_x = T_1 \beta_x + (T_1 * \check{\alpha}) \cdot 0 = T_1 \beta_x$, $u_x \gamma_1 = \beta_x \gamma_1, \ldots$, and Q_h^v is equivalent to

$$\begin{cases} (\beta_x \gamma_1 + T_1 \beta_x) + (\beta_y \gamma_2 + T_2 \beta_y) + (\beta \gamma_3 + T_3 \beta) = v \\ \beta(r(t), s(t)) = h(t), \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

which is the same as

$$\begin{cases} a\beta_x + b\beta_y + c\beta = v \\ \beta(r(t), s(t)) = h(t), \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

and so $u = \beta$ satisfies Q_h^v, which means that u is a W_α-solution of P_h^v. ∎

It should be stressed that, as we shall see, P_h^v may have no classical solutions in $C^{p+1}(\mathbb{R}^2)$ and still have a W_α-solution in $C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$. In some cases, this solution does not even depend on the G-invariant α-function. Unless otherwise specified, we suppose that $G = \mathscr{O}(2)$ is the orthogonal group in $\mathbb{R}^2$ as usual in nonrelativistic applications.

3. THE UNIQUENESS OF THE W_α-SOLUTION OF P_h^v IN $C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$

The main result depends on the following

LEMMA 3.1. *If $a, b, c \in C^\infty(\mathbb{R}^2)$, $f \in \mathscr{D}'_\ell(\mathbb{R}^2)$ is a solution of the equation $af_x + bf_y + cf = 0$ and moreover a and b do not vanish simultaneously in each point of $\mathbb{R}^2$, then $f = 0$.*

Proof. Suppose by contradiction that $f \neq 0$. Since f has discrete support, there exist an open set $\Omega \subset \mathbb{R}^2$ and $x_0 \in \Omega$ such that $\operatorname{supp}(f_\Omega) = \{x_0\}$. Thus f_Ω is a distribution with pointwise support and we can conclude that f_Ω is of finite order m. The restriction of the equation $af_x + bf_y = -cf$ to Ω leads immediately to a contradiction since the right hand side is a distribution in Ω of order m while the left hand side is a distribution in Ω of order $m + 1$. ∎

Now, if we adopt the same notations and assumptions we have made for P_h^v and Q_h^v in (2.1) and (2.2), we have the following uniqueness result.

THEOREM 3.2. *Given* $\alpha \in \mathcal{D}(\mathbb{R}^2)$, *G-invariant with* $\int \alpha = 1$, *suppose that*:

(1) *Problem*

$$\begin{cases} \gamma_1 \beta_x + \gamma_2 \beta_y + \gamma_3 \beta = 0 \\ \beta(r(t), s(t)) = 0, \qquad \text{for all } t \in \mathbb{R} \end{cases}$$

has a unique solution $\beta(x, y) \equiv 0$ *in* $C^{p+1}(\mathbb{R}^2)$.

(2) *At each point of* $\mathbb{R}^2$ *there exists* $i \in \{1, 2\}$ *such that* $\gamma_i + T_i * \alpha \neq 0$.

Then, if there exists a W_α*-solution of* P_h^v *in* $C^{p+1}(\mathbb{R}^2) \oplus \mathcal{D}'_\ell(\mathbb{R}^2)$ *with respect to G, this solution is unique.*

Proof. It is sufficient to prove that if $u = \beta + f \in C^{p+1}(\mathbb{R}^2) \oplus \mathcal{D}'_\ell(\mathbb{R}^2)$ is a W_α-solution of P_0^0 then $u = 0$. By Definition 2.1, there exists an open set $\Omega \subset \mathbb{R}^2$ such that $(\beta + f)_\Omega = \beta_\Omega + f_\Omega$ is a $C^{p+1}(\mathbb{R}^2)$-function (i.e., $f_\Omega = 0$) and

$$\begin{cases} (\beta_x + f_x)\gamma_1 + T_{1\dot\alpha}(\beta_x + f_x) + (\beta_y + f_y)\gamma_2 + T_{2\dot\alpha}(\beta_y + f_y) \\ \qquad + (\beta + f)\gamma_3 + T_{3\dot\alpha}(\beta + f) = 0 \\ (\beta + f)_\Omega(r(t), s(t)) = 0, \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

which means that

$$\begin{cases} \beta_x\gamma_1 + \beta_y\gamma_2 + \beta\gamma_3 \\ \qquad = -f_x\gamma_1 - f_y\gamma_2 - f\gamma_3 - T_1\beta_x - T_2\beta_y - T_3\beta \\ \qquad\quad - T_{1\dot\alpha}f_x - T_{2\dot\alpha}f_y - T_{3\dot\alpha}f & (3.1) \\ \beta(r(t), s(t)) = 0. & (3.2) \end{cases}$$

Noting that the left hand side of (3.1) is a $C^{p+1}(\mathbb{R}^2)$-function and that the right hand side belongs to $\mathcal{D}'_\ell(\mathbb{R}^2)$, we have

$$\beta_x\gamma_1 + \beta_y\gamma_2 + \beta\gamma_3 = 0, \tag{3.3}$$

$$f_x\gamma_1 + f_y\gamma_2 + f\gamma_3 + T_1\beta_x + T_2\beta_y + T_3\beta + T_{1\dot\alpha}f_x + T_{2\dot\alpha}f_y + T_{3\dot\alpha}f = 0. \tag{3.4}$$

By assumption, (3.3) and (3.2) imply $\beta = 0$ and so (3.4) implies

$$f_x\gamma_1 + f_y\gamma_2 + f\gamma_3 + (T_1 * \alpha)f_x + (T_2 * \alpha)f_y + (T_3 * \alpha)f = 0,$$

which is the same as

$$(\gamma_1 + T_1 * \alpha)f_x + (\gamma_2 + T_2 * \alpha)f_y + (\gamma_3 + T_3 * \alpha)f = 0.$$

Using (2) and Lemma 3.1 we conclude that $f = 0$. Thus, $u = \beta + f = 0$. ∎

4. THE EXISTENCE OF A W_α-SOLUTION OF P_h^v

With the same notations and assumptions we have made for P_h^v and Q_h^v in (2.1) and (2.2), we shall prove the following existence result.

THEOREM 4.1. *Given $\alpha \in \mathscr{D}(\mathbb{R}^2)$, G-invariant with $\int \alpha = 1$, there exists a W_α-solution of P_h^v in $C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$ with respect to the ruling group G if and only if the following conditions are satisfied*:

(a) $v \in C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$;

(b) *letting $v = \eta + g$ with $\eta \in C^{p+1}(\mathbb{R}^2)$ and $g \in \mathscr{D}'_\ell(\mathbb{R}^2)$, problem*

$$\begin{cases} \beta_x \gamma_1 + \beta_y \gamma_2 + \beta \gamma_3 = \eta & (4.1) \\ \beta(r(t), s(t)) = h(t) \qquad \text{for all } t \in \mathbb{R} & (4.2) \end{cases}$$

has a solution $\beta \in C^{p+1}(\mathbb{R}^2)$;

(c) *the differential equation*

$$(\gamma_1 + T_1 * \alpha) f_x + (\gamma_2 + T_2 * \alpha) f_y + (\gamma_3 + T_3 * \alpha) f$$
$$= -T_1 \beta_x - T_2 \beta_y - T_3 \beta + g \tag{4.3}$$

has a solution $f \in \mathscr{D}'_\ell(\mathbb{R}^2)$, such that $f_\Omega = 0$ for a certain open set $\Omega \subset \mathbb{R}^2$ and such that $(r(t), s(t)) \in \Omega$ for all $t \in \mathbb{R}$.

In this case, the W_α solution of P_h^v is $u = \beta + f$.

Proof. First, let us assume that $u = \beta + f \in C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$ is a W_α-solution of P_h^v with respect to the ruling group G. Then, by Definition 2.1 there exists an open set $\Omega \subset \mathbb{R}^2$ such that $(r(t), s(t)) \in \Omega$ for all $t \in \mathbb{R}$ and $u_\Omega = \beta_\Omega + f_\Omega \in C^{p+1}(\Omega)$. Then $f_\Omega = 0$ and $u_\Omega = \beta_\Omega$. Since u verifies Q_h^v we have

$$\begin{cases} (\beta_x + f_x)\gamma_1 + T_1 \beta_x + (T_1 * \alpha) f_x \\ \qquad + (\beta_y + f_y)\gamma_2 + T_2 \beta_y + (T_2 * \alpha) f_y \\ \qquad + (\beta + f)\gamma_3 + T_3 \beta + (T_3 * \alpha) f = v & (4.4) \\ \beta(r(t), s(t)) = h(t). & (4.5) \end{cases}$$

From (4.4) we conclude that $v \in C^{p+1}(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$. Letting $v = \eta + g$ with $\eta \in C^{p+1}(\mathbb{R}^2)$ and $g \in \mathscr{D}'_\ell(\mathbb{R}^2)$, we can write (4.4) as

$$\beta_x \gamma_1 + \beta_y \gamma_2 + \beta \gamma_3 - \eta = -f_x \gamma_1 - f_y \gamma_2 - f \gamma_3 - T_1 \beta_x - T_2 \beta_y - T_3 \beta$$
$$- (T_1 * \alpha) f_x - (T_2 * \alpha) f_y - (T_3 * \alpha) f + g.$$

The right hand side of this equality belongs to $\mathscr{D}'_{\ell}(\mathbb{R}^2)$ and the left hand side belongs to $C^{p+1}(\mathbb{R}^2)$. Hence they are both zero and (b) and (c) follow.

Now, suppose that (a), (b), and (c) are verified. Then $u = \beta + f$ is a W_α-solution of P_h^v with respect to the ruling group G, since (c) implies the existence of an open set Ω such that

(1) $(r(t), s(t)) \in \Omega$ for all $t \in \mathbb{R}$,

(2) $u_\Omega = \beta_\Omega + f_\Omega = \beta_\Omega \in C^{p+1}(\mathbb{R}^2)$.

Also, we have

(3) u satisfies Q_h^v,

because

$$\begin{aligned}
&(u_x\gamma_1 + T_{1\dot{\alpha}}u_x) + (u_y\gamma_2 + T_{2\dot{\alpha}}u_y) + (u\gamma_3 + T_{3\dot{\alpha}}u) \\
&\quad = (\beta_x + f_x)\gamma_1 + T_{1\dot{\alpha}}(\beta_x + f_x) \\
&\qquad + (\beta_y + f_y)\gamma_2 + T_{2\dot{\alpha}}(\beta_y + f_y) + (\beta + f)\gamma_3 + T_{3\dot{\alpha}}(\beta + f) \\
&\quad = \beta_x\gamma_1 + \beta_y\gamma_2 + \beta\gamma_3 + f_x\gamma_1 + f_y\gamma_2 + f\gamma_3 + T_1\beta_x + T_2\beta_y + T_3\beta \\
&\qquad + (T_1 * \alpha)f_x + (T_2 * \alpha)f_y + (T_3 * \alpha)f = \eta + g = v,
\end{aligned}$$

by (b) and (c). ∎

Thus, the proof of the existence of a W_α-solution for P_h^v is reduced to the proof of the existence of an ordinary solution $\beta \in C^{p+1}(\mathbb{R}^2)$ for the classical problem (4.1), (4.2) and to the proof of the existence of a solution $f \in \mathscr{D}'_{\ell}(\mathbb{R}^2)$ for the differential equation (4.3). In general, since f is a finite linear combination of derivatives of Dirac measures, it is not difficult to know whether there is a solution of (4.3) and even to determine this solution. This will be clear in our next example.

5. EXAMPLE

Given a function $h \in C^2(\mathbb{R})$, let us consider the problem

$$\begin{cases} u_t + \delta_t u_x = 0 & (5.1) \\ u(x, -1) = h(x) & (5.2) \end{cases}$$

that we have interpreted at the beginning. Since $\delta_t \in \mathscr{D}'^1_{\ell}(\mathbb{R}^2)$ we have $p = 1$ and $C^2(\mathbb{R}^2)$ is the space of the classical solutions u. Then $u_t \in C^1(\mathbb{R}^2)$ and from (5.1) we have $u_t = -\delta_t u_x$. Hence,

$$\begin{cases} u_t = 0 & (5.3) \\ \delta_t u_x = 0 & (5.4) \end{cases}$$

because $\operatorname{supp}(-\delta_t u_x) \subset \{(0,0)\}$ and $u_t \in C^1(\mathbb{R}^2)$. Thus $u(x,t) = \varphi(x)$ with $\varphi \in C^2(\mathbb{R}^2)$. From (5.2) it follows that $\varphi = h$ and so $u(x,t) = h(x)$. From (5.4) it also follows that $\delta_t h'(x) = 0$, $(\delta h'(x))_t = 0$, $h'(0)\delta_t = 0$, $h'(0) = 0$. Hence, *if* $h'(0) \neq 0$ *problem* (5.1), (5.2) *has no classical solutions*. Now, if we seek a W_α-solution $u = \beta + f \in C^2(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$ we must apply Theorem 4.1 and solve the problem

$$\begin{cases} \beta_t = 0 \\ \beta(x,-1) = h(x), \end{cases}$$

whose unique solution is $\beta(x,t) = h(x)$. Also, we must look for a solution $f \in \mathscr{D}'_\ell(\mathbb{R}^2)$ of (4.3), i.e., of equation

$$f_t + (\delta_t * \alpha) f_x = -\delta_t \beta_x. \tag{5.5}$$

Now, since

$$\delta_t \beta_x = \delta_t h'(x) = (\delta h'(x))_t = h'(0)\delta_t,$$

(5.5) is equivalent to

$$f_t + \alpha_t f_x = -h'(0)\delta_t.$$

Looking for a solution of the form $f = \text{constant} \cdot \delta$, it is easy to see that $f = -h'(0)\delta$ is indeed a solution (note that $\alpha_t(0,0) = 0$ and $\alpha_{tx}(0,0) = 0$ because α is $G = \mathscr{O}(2)$-invariant). Thus, by Theorem 4.1,

$$u(x,t) := h(x) - h'(0)\delta(x,t)$$

is a W_α-solution of (5.1), (5.2) for any $\alpha \in G$ with $\int\alpha = 1$. Applying Theorem 3.2 we conclude that this solution is the unique W_α-solution of this problem in $C^2(\mathbb{R}^2) \oplus \mathscr{D}'_\ell(\mathbb{R}^2)$ even if $h'(0) \neq 0$. Note that the W_α-solution does not depend on α and that if $h'(0) = 0$ then it reduces to the classical solution.

Physically, this means that if we consider an interval $[a,b]$ of $\mathbb{R}$, then the mass of matter $m(t)$ inside the interval $[a,b]$ is formally given by

$$m(t) = \int_a^b u(x,t) S \,\mathrm{d}x = S\int_a^b h(x)\,\mathrm{d}x - h'(0) S \int_a^b \delta(x,t)\,\mathrm{d}x,$$

and this mass is well defined for each t except for $t = 0$ if $0 \in [a,b]$, where the blow up occurs. We can better understand this phenomenon if we consider an interval of time $[t_0, t_1]$ and compute

$$\int_{t_0}^{t_1} m(t)\,\mathrm{d}t = (t_1 - t_0) S \int_a^b h(x)\,\mathrm{d}x - h'(0) S \int_{t_0}^{t_1} \left(\int_a^b \delta(x,t)\,\mathrm{d}x \right) \mathrm{d}t.$$

Thus

$$\int_{t_0}^{t_1} m(t)\,\mathrm{d}t = \begin{cases} (t_1 - t_0) S \int_a^b h(x)\,\mathrm{d}x \\ \qquad \text{if } (0,0) \notin [a,b] \times [t_0, t_1] \\ (t_1 - t_0) S \int_a^b h(x)\,\mathrm{d}x - h'(0) S \\ \qquad \text{if } (0,0) \in [a,b] \times [t_0, t_1]. \end{cases}$$

If $h'(0) = 0$ all this is very easy to interpret. If $h'(0) \neq 0$ this is a new result which cannot be obtained in the classical framework.

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