A generalized Weil representation for $SL_\ast(2, A_m)$, where $A_m = \mathbb{F}_q[x]/\langle x^m \rangle$

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1. Introduction

The Weil representation of symplectic groups is a very important and ubiquitous topic in mathematics for several reasons. For instance, it emerged in quantum mechanics but later enabled mathematicians to solve long standing problems in algebra, like the construction of all complex irreducible representations of the special linear group of rank 2 over a finite field [7], and later over a local field, except in residual characteristic two [3]. Originally these representations were constructed, in the more general case of the symplectic group over a local field, by A. Weil [9] taking advantage of
the representation theory of the related Heisenberg group, as described by the Stone–von Neumann theorem, in the real case [4]. Later, Soto-Andrade in [7] constructed Weil representations for the symplectic groups $Sp(2n, \mathbb{F}_q)$, by finding first a new presentation of these groups and by defining then linear operators on a suitable vector space which preserve the relations among the generators of the presentation. He obtained in this way a complete set of irreducible linear complex representations of the symplectic group $Sp(4, \mathbb{F}_q)$ by decomposing the two Weil representations associated to the two isomorphy types of quadratic forms of rank 4 over $\mathbb{F}_q$.

In [1], the authors extended the original method of construction of the Weil representation, via Heisenberg group and its Schrödinger representation to the case $Sp(2n, \mathbb{O}/\mathbb{P}^h)$, where $\mathbb{O}$ is the ring of integers of a local field, $\mathbb{P}$ is its unique maximal ideal, $\mathbb{O}/\mathbb{P}$ has odd characteristic and $l$ is a positive integer. The representation they constructed is a generalization of the classical $PGL$ presentation, and we compute the order of these groups; in Section 5, we construct a non-degenerate Heisenberg group and its Schrödinger representation to the case $det = 1$ irreducible components. They proved that their representations split multiplicity-freely into $l + 1$ irreducible components.

The groups $SL_* (2, A)$, where $A$ is a unitary ring with an involution $*$, are a non-commutative version of the classical group $SL(2, \mathbb{F})$, where $\mathbb{F}$ is a field. For example if $A$ is the ring of matrices $M_n(\mathbb{F})$ ( $\mathbb{F}$ is a field) and $a^* = a^t$, then $SL_* (2, A)$ is exactly the symplectic group $Sp(2n, \mathbb{F})$. When $A$ is a simple artinian ring, Pantoja, in [5], generalized the classical presentation of $SL(2, \mathbb{F})$ to $SL_* (2, A)$. In this work, we study the involutive structure of the truncated polynomial rings $A_m = \mathbb{F}_q[x]/(x^m)$ and the groups $SL_* (2, A_m)$. By using a presentation analogous to the one obtained in [5], we construct a Weil representation of $SL_* (2, A_m)$ as in [7]. If we consider the case $m = l = 1$, where $\mathbb{O}/\mathbb{P} = A_m = \mathbb{F}_q$, then both [1] and our work give back the aforementioned Weil representation of $SL(2, \mathbb{F}_q)$.

This paper is organized as follows: In Section 2, we recall the definitions of the group $SL_* (2, A)$ for an involutive ring $(A, *)$ (see also [6]); in Section 3, we obtain a full characterization of all involutions on the ring $A_m$, and we also prove the existence of just one kind of non-trivial involutions up to isomorphism; in Section 4, we provide a presentation of the groups $SL_* (2, A_m)$ (that we call Bruhat presentation), and we compute the order of these groups; in Section 5, we construct a non-degenerate quadratic $A_m$-module $(A_m, Q, B)$, and we compute the Gauss sum $S_{\varphi, Q}(1)$ associated to this non-degenerate quadratic form $Q$ and to a non-trivial character $\varphi$ of $\mathbb{F}_q^*$. Finally, in Section 6, we construct a Weil representation for $SL_* (2, A_m)$ by using the presentation obtained before. We will address the decomposition problem for this representation elsewhere.

2. The group $SL_* (2, A)$

Let $A$ be a unitary ring with an involution $a \mapsto a^*$, i.e., an anti-automorphism of $A$ of order two and let $Z(A)$ be the set of involutions of $A$. We denote by $A^\times$ the group of all invertible elements of $A$ and by $A^t$ the set of all elements $a \in A$ such that $a^* = a$, i.e., the symmetric elements with respect to $*$. If $T$ is a matrix with entries in $A$, we define $(T^*)_{ij} = (T_{ji})^*$. Then $(TS)^* = S^T T^*$ for all $T$ and $S$ in the ring of matrices $M(2, A)$, thus this defines an involution $*$ on $M(2, A)$. Let $M_* (2, A)$ be the set of matrices $g$ in $M(2, A)$ such that $\det g = \lambda g J$, where $\lambda g \in Z(A)$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M(2, A)$. We denote by $GL_* (2, A)$ the set of invertible elements in $M_* (2, A)$. Also, we define the $*$-determinant on $M_* (2, A)$ by $\det_*(g) = ad^* - bc^*$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We set $SL_* (2, A)$ for the subset of all $g$ such that $\det_*(g) = 1$. We have the following lemma.

**Lemma 1.** $GL_* (2, A)$ is a group under multiplication and $\det_*$ is an epimorphism of $GL_* (2, A)$ onto the group of all central symmetric invertible elements of $A$, such that $\ker \det_* = SL_* (2, A)$.

**Proof.** See [6]. □

We consider the matrices that are the $*$-analogous matrices of the classical generators of the group $SL_* (2, \mathbb{F}_q)$:

$$h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad t \in A^\times, \quad u(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in A^t, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
The matrices $h(t)$, $u(b)$ and $w$ satisfy the following universal relations

1. $h(t_1)h(t_2) = h(t_1t_2)$,
2. $u(b_1)u(b_2) = u(b_1 + b_2)$,
3. $h(t)u(b) = u(tbt^*)h(t)$,
4. $w^2 = h(-1)$,
5. $wh(t) = h(t^{* -1})w$,
6. $u(t)wu(t^{-1})w(1) = wh(-t^{-1})$.

3. The involutions of the ring $A_m$

Let $k = \mathbb{F}_q$ be the finite field with $q$ elements, $q$ odd. In this work, $m$ is always a positive integer. Set

$$A_m = k[x]/\langle x^m \rangle = \left\{ \sum_{i=0}^{m-1} a_i x^i : a_i \in k, \ x^m = 0 \right\}.$$ 

We denote by $\ast$ the $k$-linear involution on $A_m$ given by $x \mapsto -x$. In what follows involutions on $A_m$ are always $k$-linear mappings.

**Theorem 2.** Every non-trivial involution $\dagger$ of $A_m$ is isomorphic to the involution $\ast$ given by $\ast : x \mapsto x^* = -x$.

**Proof.** We set $x^\dagger = -x + a_1 x^{l_1} + h_{l+1}$, where $a_1 \in k$ and $h_{l+1} \in x^{l+1}A_m$. Then, $l$ is necessarily even. Therefore if we conjugate by the $k$-automorphism of $A_m$ given by $f(x) = x - \frac{a_1}{2} x^{l_1}$, we get $(f^{-1} \circ \dagger \circ f)(x) = -x + a_{l+2} x^{l+2} + h_{l+3}$, where $h_{l+3} \in x^{l+3}A_m$. By induction, the result follows because $x$ is nilpotent. \qed

In the following proposition, we prove that any non-trivial involution $\dagger$ on $A_m$ may be looked upon as a sort of deformation of the canonical involution $x \mapsto -x$.

**Proposition 3.** Let $\dagger$ be a non-trivial involution on $A_m$. Then

$$x^\dagger = \frac{-x}{1 + xq(x)}, \quad (1)$$

$$\left( \sum_{i=0}^{m-1} a_i x^i \right)^\dagger = \sum_{i=0}^{m-1} a_i x^{i*}, \quad (2)$$

where $q(x) \in A_m$ satisfies $x^2 q(x) = x^2 q(\frac{x}{1 + xq(x)})$ and conversely, that is, given $q(x) \in A_m$ such that $x^2 q(x) = x^2 q(\frac{x}{1 + xq(x)})$, if we define $\dagger : A_m \rightarrow A_m$ by (1) and (2), then $\dagger$ is an involution on $A_m$.

**Proof.** We write $x^\dagger = \sum_{i=1}^{m-1} a_i x^i$. Since $\dagger$ is an involution, $a_1$ is 1 or $-1$. If $a_1 = 1$, then $a_i = 0$ for all $i = 2, 3, \ldots, m-1$, i.e., $\dagger = \text{id}$. On the other hand, if $a_1 = -1$, then

$$x^\dagger = -x + \sum_{i=2}^{m-1} a_i x^i = -x + x^2 \sum_{i=2}^{m-1} a_i x^{i-2} = -x(1 - xh(x)),$$

where $h(x) = \sum_{i=2}^{m-1} a_i x^{i-2}$. Let $q(x) \in A_m$ such that

$$\left(1 + xq(x)\right)\left(1 - xh(x)\right) = 1.$$
Therefore

\[ x^\dagger = \frac{-x}{1 + xq(x)}. \]

Using the above expression of \( x^\dagger \) and the fact that \( x^{\dagger\dagger} = x \), we see that \( q(x) \) satisfies \( x^2 q(x) = x^2 q \left( \frac{-x}{1 + xq(x)} \right) \). The converse is straightforward. \( \square \)

**Remark 4.** We note that Theorem 2 still holds in the ring \( k[[x]] \) of formal power series. This suggests another approach to this matter: first, to prove Proposition 3 in the ring \( k[[x]] \). Second, to prove Theorem 2 noting that every involution \( \hat{\dagger} \) in \( k[[x]] \) of \( A_m \) can be lifted to an involution \( \hat{\dagger} \) in \( k[[x]] \) such that the following diagram commutes:

\[
\begin{array}{ccc}
  k[[x]] & \xrightarrow{\hat{\dagger}} & k[[x]] \\
  pr & \downarrow & pr \\
  A_m & \xrightarrow{\hat{\dagger}} & A_m.
\end{array}
\]

Theorem 2 in \( k[[x]] \) can be proved using \( q(x) \) from Proposition 3 and defining explicitly an automorphism \( f \) of \( k[[x]] \) such that \( f(h^*) = f(h)^* \) for all \( h \in k[[x]] \). We do not follow this point of view because our interest is focused on finite rings.

4. Presentation of the group \( SL_2^*(A_m) \)

From now on \( * \) will denote either the trivial involution or the non-trivial involution given by \( x^* = -x \) on \( A_m \). We are going to follow the ideas given in [5] in order to get a presentation of the group \( SL_2^*(A_m) \). A direct computation proves the following lemmas:

**Lemma 5.** Let \( a, c \) be two elements in \( A_m \) such that \( a \) or \( c \) is invertible and \( a^*c = c^*a \). Then there is a symmetric element \( s \) such that \( a + sc \) is an invertible element in \( A_m \).

**Lemma 6.** Let \( a, c \) be two non-invertible symmetric elements in \( A_m \). Then there is a symmetric invertible element \( x \) in \( A_m \) such that \( a - x^{-1}b \) and \( b + x \) are symmetric invertible elements in \( A_m \).

**Proposition 7.** The group \( SL_2^*(A_m) \) is generated by the set of matrices \( h(t), t \in A_m^\times, u(b), b \in A_m^\times \) and \( w \).

**Proof.** Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2^*(A_m) \). If \( c = 0 \), then \( g = h(a)u(a^{-1}b) \). Now, if \( c \) is invertible, then \( g = h(-c^{-1}a)u(c^*a)wu(c^{-1}d) \). Finally, if \( c \notin A_m^\times \cup \{0\} \), then \( a \) and \( c \) satisfy the conditions of Lemma 5, so we take \( s \in A_m^\times \) such that \( a + sc \in A_m^\times \). Therefore

\[ g = u(-s)h(-a - sc)wu(-a^n - c^ns)wu((a + sc)^{-1}(b + sd)), \]

and the proposition follows. \( \square \)

The set of generators \( h(t), t \in A_m^\times, u(b), b \in A_m^\times \) and \( w \) will be called Bruhat generators for \( SL_2^*(A_m) \). We put

\[ B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2^*(A_m) \right\}. \]

Then \( B \) is a subgroup of \( SL_2^*(A_m) \). By the above proposition, we have the following result.
Corollary 8. \( SL_n(2, A_m) = B \cup BwB \cup BwBwB \).

Lemma 9. In \( SL_n(2, A_m) \), we have:

(i) If \( h(t)u(a)wu(b)wu(c) = 1 \), then \( t = -1, a = -c \) and \( b = 0 \).

(ii) \( 1 \notin BwB \).

(iii) If \( h(t)u(a) = 1 \), then \( t = 1 \) and \( a = 0 \).

Proof. We suppose that \( h(t)u(a)wu(b)wu(c) = 1 \). Then \( h(t)u(a)w = -u(-c)wu(-b) \); writing this equality we have (i). Finally, a simple computation shows (ii) and (iii). \( \square \)

Let \( H \) be the abstract group generated by the objects \( h(t), t \in A_m^\times, u(b), b \in A_m^i \) and \( w \) with the following relations:

\[
\begin{align*}
(1) & \quad h(t_1)h(t_2) = h(t_1t_2), \\
(2) & \quad u(b_1)u(b_2) = u(b_1 + b_2), \\
(3) & \quad h(t)u(b) = u(tbt^*)h(t), \\
(4) & \quad w^2 = h(-1), \\
(5) & \quad wh(t) = h(t^{*-1})w, \\
(6) & \quad u(t)wu(t^{-1})wu(t) = wh(-t^{-1}).
\end{align*}
\]

From now on we will use the same symbols to denote the matrices in \( SL_n(2, A_m) \) and the abstract elements of \( H \). The \( w \)-length of an element \( g \in H \) is the minimal \( j \) such that \( g \in (BwB)^j \), where \( B \) is the subgroup of \( H \) generated by \( h(t), \) and \( u(b), t \in A_m^\times, b \in A_m^i \) and \( B^0 = B \). Similarly, we define the \( w \)-length of an element \( g \in SL_n(2, A_m) \).

Proposition 10. Every element in \( H \) has at most \( w \)-length 2.

Proof. Let \( g_1g_2t = g_3t' \), for \( g_i \in BwB, i = 1, 2, 3 \) and \( t, t' \) arbitrary in \( H \). This expression is equivalent to \( wu(a)wt'' = u(b)w \), where \( t'' \) in \( H \) using the relations defining \( H \). If either \( a \) or \( b \) is invertible, it is possible to reduce one \( w \) in above expression using the sixth relation that defines \( H \). If neither \( a \) nor \( b \) is invertible, there exists a symmetric invertible element \( x \in A_m^i \) such that \( a = x^{-1} \) and \( b + x \) are symmetric invertible elements. Multiplying the equation \( wu(a)wt'' = u(b)w \) by \( u(x) \) we get

\[ u(x)wu(a)wt'' = -wu(-x^{-1})hx^{-1})u(-(a - x^{-1})^{-1})wh(a - x^{-1})u(-(a - x^{-1})^{-1})t''. \]

This expression involves one \( w \) less, using the sixth relation. By induction, the lemma follows. For details, see [5]. \( \square \)

Now we can prove the following result.

Theorem 11. The set of matrices \( h(t), t \in A_m^\times, u(b), b \in A_m^i \) and \( w \) of \( SL_n(2, A_m) \) together with the relations:

\[
\begin{align*}
(1) & \quad h(t_1)h(t_2) = h(t_1t_2), \\
(2) & \quad u(b_1)u(b_2) = u(b_1 + b_2), \\
(3) & \quad h(t)u(b) = u(tbt^*)h(t), \\
(4) & \quad w^2 = h(-1), \\
(5) & \quad wh(t) = h(t^{*-1})w, \\
(6) & \quad u(t)wu(t^{-1})wu(t) = wh(-t^{-1}),
\end{align*}
\]

give a presentation of the group \( SL_n(2, A_m) \), which we call the Bruhat presentation of \( SL_n(2, A_m) \).
Finally by Lemma 9, we see that $\varphi(g) = 1$ implies $g = 1$. Thus the theorem follows. □

**Proposition 12.** The cardinality of $SL_2(2, A_m)$ is $(q - 1)q^{m+2(m-1)}(q + 1)$ or $(q - 1)q^{m+2(\frac{m+1}{2})-2}(q + 1)$ according to $*$ being the trivial involution or not, respectively.

**Proof.** Notice that $SL_2(2, A_m)$ acts on $M_{2 \times 1}(A_m)$ by left multiplication. Under this action, the cardinality of the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is either $(q - 1)q^{m+2(m-1)}(q + 1)$ or $(q - 1)q^{m+2(\frac{m+1}{2})-2}(q + 1)$ and the cardinality of the isotropy group of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $q^m$ or $q^{\frac{m+1}{2}}$ according to $*$ being the trivial involution or not. Hence, the theorem follows. □

5. A quadratic $A_m$-module

**Definition 13.** Let $(A, *)$ be an involutive ring and let $M$ be a right $A$-module. We say that $(M, Q, B)$ is a quadratic $A$-module if $Q : M \to A$ and $B : M \times M \to A$ is a bi-additive $A$-form such that:

\[ Q(ma) = a^*Q(m)a, \]
\[ B(m, n) = Q(m + n) - Q(m) - Q(n), \]
\[ B(ma, n) = B(m, na^*), \]
\[ B(m, n) = B(n, m)^*, \]
\[ B(ma, n) = a^*B(m, n), \]

for all $m, n \in M$ and for all $a \in A$.

The quadratic $A$-module $(M, Q, B)$ is non-degenerate if $B(m, n) = 0$ for all $m \in M$ implies $n = 0$.

We recall that we are writing $*$ for the trivial involution or the $k$-linear involution on $A_m$ given by $x^* = -x$. We now consider the non-degenerate quadratic $A_m$-module $(A_m, Q, B)$, where $Q : A_m \to A_m$ is given by $Q(t) = t^*t$ and $B : A_m \times A_m \to A_m$ is such that $B(t, s) = t^*s + ts^*$. Let $tr$ be the linear form on $A_m$ defined by $tr(\sum_{i=0}^{m-1} a_i x^i) = a_{m-1}$.

**Proposition 14.** Let $m$ be an odd number. Then

1. The form $tr$ on $A_m$ is $k$-linear and it is invariant under the involution $*$, i.e., $tr(t^*) = tr(t)$, for any $t \in A_m$.
2. If $*$ is the non-trivial involution, then the $k$-form $tr \circ B$ is a non-degenerate symmetric bilinear form.

**Proof.** (1) The case $* = \text{id}$ is trivial. From now on we will assume that $x^* = -x$. Let $t = \sum_{i=0}^{m-1} a_i x^i \in A_m$, then

\[ tr(t^*) = tr\left(\sum_{i=0}^{m-1} (-1)^i a_i x^i\right) = a_{m-1} = tr(t). \]

(2) It is clear that $tr \circ B$ is bilinear and symmetric. We now prove that it is non-degenerate. We note that

\[ tr \circ B(t, s) = tr(t^*s + ts^*). \]
Since $tr$ is invariant under the involution $\ast$, we see that $tr \circ B (t, s) = 2tr(t^\ast s)$. Then it suffices to show that the $k$-form $\tilde{B} : (t, s) \mapsto tr(t^\ast s)$ is non-degenerate. To this end, let $t \in A_m$ such that $\tilde{B}(t, s) = 0$, for any $s \in A_m$. We write $t^\ast = \sum_{i=0}^{m-1} a_i x^i \in A_m$. We consider $s_j = x^{m-j}, j = 1, 2, \ldots, m$. Then

$$0 = \tilde{B}(t, s_j) = tr \left( \sum_{i=0}^{m-1} a_i x^i s_j \right) = a_{j-1}.$$

Hence $a_{j-1} = 0$, for $j = 1, 2, \ldots, m$. Then $t^\ast = a_0 = t$. Finally, using that $\tilde{B}(t, 1) = 0$ we obtain $t = 0$ and $\tilde{B}$ is non-degenerate. Therefore the result follows.

A direct consequence of Proposition 14 is the following.

**Corollary 15.** If $\ast$ is the non-trivial involution and $m$ is an odd number, then the quadratic $A_m$-module $(A_m, Q, B)$ is non-degenerate.

**Remark 16.** If $\ast$ is the trivial involution, imitating the proof of Proposition 14, part (2), the $k$-form $tr \circ B$ is a non-degenerate symmetric bilinear. On the other hand, if $\ast$ is the non-trivial involution and $m$ is an even number, then the quadratic $A_m$-module $(A_m, Q, B)$ is degenerate. Indeed, we note that $x^{m-1}$ lies in the kernel of $B$ because, for any $s = \sum_{i=0}^{m-1} a_i x^i \in A_m$, we have

$$B(x^{m-1}, s) = (x^{m-1})^\ast s + x^{m-1}s^\ast = -a_0 x^{m-1} + a_0 x^{m-1} = 0.$$

### 5.1. Gauss sum for finite quadratic $A_m$-module

Let $(M, Q, B)$ be a finite non-degenerate quadratic $A_m$-module. Let $\theta$ be a non-trivial character of $A_m^\times$. The Gauss sum $S_{\theta \circ Q}$ associated to $(M, Q, B)$ and to $\theta$ is defined by

$$S_{\theta \circ Q} (a) = \sum_{m \in M} \theta(a Q(m)).$$

**Remark 17.** If $a$ is a symmetric element in $A_m$ and $(M, Q, B)$ is a non-degenerate quadratic $A_m$-module, then $(M, aQ, aB)$ is also a non-degenerate quadratic $A_m$-module.

**Lemma 18.** Let $(M, Q, B)$ be a non-degenerate quadratic $A_m$-module and $\theta$ be a non-trivial character of $A_m^\times$. Then:

1. If $a$ is an invertible element in $A_m$ and $t$ is a symmetric element in $A_m$, then

   $$S_{\theta \circ Q} (t) = S_{\theta \circ Q} (ata^\ast).$$

2. If $t$ is an invertible element in $A_m$, then

   $$S_{\theta \circ Q} (t) = S_{\theta \circ Q} (t^{\ast -1}).$$

**Proof.** (1) and (2) follow directly from the definition. \(\Box\)

**Lemma 19.** The group $A_m^\times$ acts on the set $(A_m^\times \cap A_m^\ast)$ of symmetric invertible elements by $(a, t) \mapsto ata^\ast$. This action has exactly two orbits.
Proof. The orbit of 1 is contained in the subset of $A_m$ consisting of all elements of the form $a_0 + \sum_{i=1}^{\lfloor m/2 \rfloor} a_{2i}x^{2i}$, where $a_0 \in (k^\times)^2$. On the other hand, if we take $a = a_0 + \sum_{i=1}^{\lfloor m/2 \rfloor} a_{2i}x^{2i}$, where $a_0 \in (k^\times)^2$, we can find a symmetric element $b$ such that $a = b^2 = b^* \cdot b$, i.e., $a$ belongs to the orbit of 1. Now, if $d \in k$ is a non-square, then $d$ is not in the orbit of 1. Finally, the sum of the cardinalities of these two orbits is exactly the cardinality of the set of symmetric invertible elements. Therefore the orbit of 1 and the orbit of $d$ exhaust the orbits of this action. □

Proposition 20. Let $\psi$ be a non-trivial character of the additive group $k^\times$. We put $\psi = \psi \circ \text{tr}$. Let $m$ be an odd number, then the map

$$\alpha : A_m^\times \cap A_m^s \to \mathbb{C}^\times,$$

$$t \mapsto \alpha(t) = \frac{S_{\psi \circ Q}(t)}{S_{\psi \circ Q}(1)}$$

is the sign character of the group $A_m^\times \cap A_m^s$.

Proof. Let $t$ be an element in $A_m^\times \cap A_m^s$. If $t$ belongs to the orbit of 1, then $ata^* = 1$, for some $a \in A_m^\times$. By Lemma 18, we see that $S_{\psi \circ Q}(t) = S_{\psi \circ Q}(ata^*) = S_{\psi \circ Q}(1)$ and therefore $\alpha(t) = 1$. On the other hand, if $t$ is in the orbit of $d$, where $d \in k$ is a non-square, then $ata^* = d$, for some $a \in A_m^\times$. By Lemma 18, we see that $S_{\psi \circ Q}(t) = S_{\psi \circ Q}(ata^*) = S_{\psi \circ Q}(d)$. Moreover, given that the quadratic $k$-form $\text{tr} \circ Q$ is non-degenerate, we have that $\text{tr} \circ Q$ is the orthogonal sum of the quadratic spaces $(U, Q_1)$ of rank 2s and $(k, tx^2)$, where $t \in k^\times$. Since the form $Q_1$ has rank 2s over the finite field $k$, the space $(U, Q_1)$ is isomorphic to $(U, dQ_1)$ (see [2]). On the other hand, we have

$$S_{\psi \circ Q_t}(1) + S_{\psi \circ Q_t}(d) = 2 \sum_{x \in k} \psi(x) = 0.$$

Then $S_{\psi \circ Q}(t) = S_{\psi \circ Q}(d) = S_{\psi \circ Q_1}(d)S_{\psi \circ Q_2}(d) = -S_{\psi \circ Q_1}(1)S_{\psi \circ Q_2}(1) = -S_{\psi \circ Q}(1)$. Therefore $\alpha(t) = -1$. Thus $\alpha$ is the sign character of $A_m^\times \cap A_m^s$. □

Let $\psi$ be a non-trivial character of $k^\times$. Set $\psi = \psi \circ \text{tr}$ and $m = 2s + 1$. Then we have the following.

Proposition 21. For the quadratic $A_m$-module $(A_m, Q, B)$, where $m = 2s + 1$, we have $S_{\psi \circ Q}(1) = q^{s} \sum_{a \in k} \psi((-1)^s(a^2))$.

Proof. Let $a = a_0 + a_1x + \cdots + a_{m-1}x^{m-1} \in A_m$, then

$$\text{tr}(aa^*) = \sum_{j=0}^{s-1} 2(-1)^ja_ja_{m-1-j} + (-1)^s a_s^2.$$

Therefore

$$S_{\psi \circ Q}(1) = \sum_{a \in A_m} \psi(\text{tr}(aa^*)) = \left(\prod_{i=1}^{s} \left(\sum_{a_{i-1}a_{m-i}} \psi(a_{i-1}a_{m-i-1})\right)\right) \sum_{a \in k} \psi((-1)^s(a^2))$$

$$= q^{s} \sum_{a \in k} \psi((-1)^s(a^2)).$$

□
Proposition 22. Let $k$ be the finite field of $q$ elements and let $\psi$ be a non-trivial character of $k^\times$. Then

$$\left(\sum_{a_i \in k} \psi(\alpha_i^2)\right)^2 = \left(\frac{-1}{k}\right)q,$$

where $\left(\frac{-1}{k}\right)$ is 1 or $-1$ according to $a$ being a square in $k^\times$ or not, respectively.

**Proof.** Recall that there are only two non-degenerate quadratic planes over $k$, up to isomorphism: the isotropic plane $(k^2, xy)$ and the anisotropic plane $(K, N)$, where $N$ denotes the norm of the unique quadratic extension $K$ of $k$. Moreover we can easily check that $S_{\psi \circ xy}(1) = q$ and $S_{\psi \circ N}(1) = -q$ (see [7]). Now, the sum $\left(\sum_{a_i \in k} \psi(\alpha_i^2)\right)^2$ is exactly the Gauss sum associated to the quadratic space $(k^2, x^2 + y^2)$. But this space is isomorphic either to $(k^2, xy)$ or to $(K, N)$ according to $-1$ being a square in $k$ or not, therefore the proposition follows. \qed

Lemma 23. For the quadratic $A_m$-module $(A_m, Q, B)$, we have $(S_{\psi \circ Q}(1))^2 = \alpha(-1)|A_m|$, where $|A_m|$ is the cardinality of the ring $A_m$.

**Proof.** The quadratic space $(A_m, Q, B)$ over $k$ has dimension $2s + 1$. By Proposition 21 we have

$$(S_{\psi \circ Q}(1))^2 = q^{2s} \left(\sum_{a_i \in k} \psi(-1)^i(\alpha_i^2)\right)^2.$$

We note that $\left(\frac{-1}{k}\right) = \alpha(a)$, for all $a \in k^\times$. Thus by Proposition 22 we have

$$(S_{\psi \circ Q}(1))^2 = q^{2s} \left(\sum_{a_i \in k} \psi(-1)^i(\alpha_i^2)\right)^2 = \alpha(-1)q^{2s+1} = \alpha(-1)|A_m|. \qed$$

Remark 24. Notice that Lemma 23 may be seen as an extension to the case of the nilpotent ring $A_m$ with an involution of Theorem 6.2(a) in [8].

6. Definition of the Weil representation of $SL_2(A_m)$

We recall that $*$ denotes the trivial involution or the $k$-linear involution given by $x^* = -x$ on $A_m$. Let $\psi$ be a non-trivial character of $k^\times$, set $\psi = \psi \circ \text{tr}$ and let $m$ be an odd number. We denote by $S_{\psi \circ Q}(1)$ the Gauss sum associated to $(A_m, Q, B)$ and to $\psi$. Then the function $\alpha$ from $(A_m^\times \cap A_m^s)$ to $\mathbb{C}^\times$ given by $\alpha : t \mapsto \alpha(t) = \frac{S_{\psi \circ Q}(t)}{S_{\psi \circ Q}(1)}$ is the sign character of the group $A_m^\times \cap A_m^s$. With these notations, we have:

Theorem 25. Let $W$ be the $\mathbb{C}$-vector space of all complex functions on $A_m$. Set

- $\rho(h(t))(f)(a) = \alpha(t)f(at)$, $f \in W$ and $a, t \in A_m$,
- $\rho(u(b))(f)(a) = \psi(bQ(a))f(a)$, $f \in W$ and $a, b \in A_m$,
- $\rho(w)(f)(a) = \frac{\alpha(-1)}{S_{\psi \circ Q}(1)} \sum_{c \in A_m} \psi(B(a, c))f(c)$, $f \in W$ and $a \in A_m$,

where $f \in W$ and $a, b, t$ in $A_m$. These formulas define a linear representation $(W, \rho)$ of $SL_2(A_m)$, which we call generalized Weil representation of $SL_2(A_m)$. 

Lemma 23 we have

The first three relations are straightforward. As for the fourth relation, we have:

$$\left( \rho(w) \circ \rho(w) \right)(f)(a) = \frac{1}{(A_m)^2} \sum_{b \in A_m} \sum_{c \in A_m} \psi(B(a, b)) \psi(B(b, c)) f(c)$$

$$= \frac{1}{(A_m)^2} \sum_{b \in A_m} \sum_{c \in A_m} \psi(B(a + c, b)) f(c)$$

$$= \frac{1}{(A_m)^2} \sum_{c \in A_m} f(c) \sum_{b \in A_m} \psi(B(a + c, b))$$

$$= \frac{1}{(A_m)^2} \sum_{b \in A_m} f(-a) \psi(B(0, b))$$

$$= \frac{|A_m|}{(A_m)^2} f(-a),$$

because $\psi \circ B(r, \cdot)$ is a non-trivial character of $A_m$ for $r \neq 0$, thus $\sum_{b \in A_m} \psi(B(r, b)) = 0$. Moreover, by Lemma 23 we have $\frac{|A_m|}{(A_m)^2} = \alpha(-1)$. Then

$$\left( \rho(w) \circ \rho(w) \right)(f)(a) = \frac{|A_m|}{(A_m)^2} f(-a) = \alpha(-1) f(-a) = \rho(h(-1)) f(a).$$

Hence the fourth relation follows. Now, we will prove the fifth relation:

$$\left( \rho(w) \circ \rho(h(t)) \right)(f)(a) = \frac{\alpha(-1) \alpha(t)}{\psi \circ Q(1)} \sum_{b \in A_m} \psi(B(a, b)) (f(bt))$$

$$= \frac{\alpha(-1) \alpha(t)}{\psi \circ Q(1)} \sum_{b' \in A_m} \psi(B(a, b't^{-1})) (f(b'))$$

$$= \frac{\alpha(-1) \alpha(t)}{\psi \circ Q(1)} \sum_{b' \in A_m} \psi(B(at^{-1}, b')) (f(b')).$$

By definition, we see that $\alpha(t) = \alpha(t^{-1})$. Therefore

$$\left( \rho(w) \circ \rho(h(t)) \right)(f)(a) = \frac{\alpha(-1) \alpha(t^{-1})}{\psi \circ Q(1)} \sum_{b' \in A_m} \psi(B(at^{-1}, b')) (f(b'))$$

$$= (\rho(h(t^{-1})) \circ \rho(w))(f)(a).$$

Hence the fifth relation is satisfied. Finally, we prove the sixth relation. We note this relation is equivalent to

$$\rho(w) \circ \rho(u(t)) \circ \rho(w) = \rho(h(t^{-1})) \circ \rho(u(-t)) \circ \rho(w) \circ \rho(u(-t^{-1})).$$
We compute first the left-hand side:

\[
(\rho(w) \circ \rho(u(t)) \circ \rho(w))(f)(a) = \frac{1}{(S_{\psi Q}(1))^2} \sum_{b \in A_m} \sum_{c \in A_m} \psi(B(a, b)) \psi(tQ(b)) \psi(B(b, c)) f(c) \\
= \frac{1}{(S_{\psi Q}(1))^2} \sum_{b \in A_m} \sum_{c \in A_m} \psi(B(a, b) + tQ(b) + B(b, c)) f(c).
\]

We put \(b = t^{-1} \tilde{b}\), so

\[
B(a, b) + tQ(b) + B(b, c) = t^{-1}(B(a, \tilde{b}) + Q(\tilde{b}) + B(\tilde{b}, c)) \\
= t^{-1}(B(a + c, \tilde{b}) + Q(\tilde{b})) \\
= t^{-1}(Q(a + c + \tilde{b}) - Q(a + c)).
\]

Therefore

\[
(\rho(w) \circ \rho(u(t)) \circ \rho(w))(f)(a) \\
= \frac{1}{(S_{\psi Q}(1))^2} \sum_{c \in A_m} \sum_{b \in A_m} \psi(t^{-1}(Q(a + c + \tilde{b}) - Q(a + c))) f(c) \\
= \frac{1}{(S_{\psi Q}(1))^2} \sum_{c \in A_m} \psi(-t^{-1}(Q(a + c))) f(c) \sum_{b \in A_m} \psi(t^{-1}(Q(a + c + \tilde{b}))).
\]

Since \(S_{\psi Q}(t^{-1}) = \sum_{b \in A_m} \psi(t^{-1}(Q(a + c + \tilde{b})))\), for any \(c\), we have

\[
(\rho(w) \circ \rho(u(t)) \circ \rho(w))(f)(a) = \frac{S_{\psi Q}(t^{-1})}{(S_{\psi Q}(1))^2} \sum_{c \in A_m} \psi(-t^{-1}(Q(a + c))) f(c).
\]

On the other hand, we have

\[
\rho(h(-t^{-1})) \circ \rho(u(-t)) \circ \rho(w) \circ \rho(u(-t^{-1}))(f)(a) \\
= \frac{\alpha(t^{-1})}{S_{\psi Q}(1)} \sum_{b \in A_m} \psi(B(-at^{-1}, b))(\rho(u(-t^{-1}))) f(b) \\
= \frac{\alpha(t^{-1})}{S_{\psi Q}(1)} \sum_{b \in A_m} \psi(B(-at^{-1}, b)) \psi(-t^{-1}Q(b)) f(b) \\
= \frac{\alpha(t^{-1})}{S_{\psi Q}(1)} \sum_{b \in A_m} \psi(-t^{-1}(B(a, b) - Q(b) - Q(a))) f(b) \\
= \frac{\alpha(t^{-1})}{S_{\psi Q}(1)} \sum_{b \in A_m} \psi(-t^{-1}(Q(a + b))) f(b).
\]
Now, from the proof of Lemma 19, the element $t^{-1}$ is in the orbit of 1 or $t^{-1}$ is in the orbit of $d$, where $d \in k$ is a non-square, according to $t^{-1}$ being a square or not, respectively. Then there exists an invertible element $a \in A_m$ such that $at^{-1}a^* = 1$ or $at^{-1}a^* = d$. Therefore, by Lemma 18 we have

$$\frac{S_{\psi \circ Q} (t^{-1})}{(S_{\psi \circ Q} (1))^2} = \frac{S_{\psi \circ Q} (at^{-1}a^*)}{(S_{\psi \circ Q} (1))^2} = \frac{\alpha (t^{-1})}{S_{\psi \circ Q} (1)},$$

for all $t \in A_m^s \cap A_m^s$. From here the sixth relation follows and therefore $(W, \rho)$ is a linear representation of $SL_2(A_m)$ on $W$.  

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References