The Fuglede-Putnam Theorem
and Normal Products of Matrices

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ABSTRACT

The rectangular matrix version of the Fuglede-Putnam theorem is used to prove that, for rectangular complex matrices A and B, both AB and BA are normal if and only if \( A^*AB = BAA^* \) and \( B^*BA = ABB^* \). We deduce some results relating the rank of A and the factors in a polar decomposition of A to the normality of AB and BA.

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Under the assumption that A and B are normal \( n \times n \) complex matrices, N. A. Wiegmann [12] proved that AB and BA are normal if and only if \( A^*AB = BAA^* \) and \( B^*BA = ABB^* \). In [13], Wiegmann improved this by omitting the requirement that B be normal. In this note, we show that the

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assumption on the normality of $A$ can also be removed. Moreover, we shall assume that $A$ and $B$ are rectangular matrices of appropriate dimensions.

Let $A$ be a nonsingular normal matrix, and let $A = UH$, where $U$ is unitary and $H$ is positive definite Hermitian. Clarifying a result of Wiegmann's [13], Gibson [4] remarks that $AB$ and $BA$ are normal if and only if $BU$ is normal and $HBU = BUH$. In Theorems 3, 4, and 5, we remove the restriction that $A$ be normal, and we use Theorem 2 to investigate to what extent this result can be generalized to singular $A$.

For the sake of completeness, we give an elementary proof of the rectangular matrix version of the Fuglede-Putnam theorem [2; 8; 9; 10, p.300; 11], which is essentially to be found in [6, p. 65]. Our principal result (Theorem 2) will follow immediately from this theorem. For a related application of the Fuglede-Putnam theorem see [5; 6, p. 68].

Denote by $\mathbb{C}^{m \times n}$ the set of all $m \times n$ complex matrices.

**Theorem 1 (Fuglede-Putnam).** Let $P \in \mathbb{C}^{m \times n}$, $Q \in \mathbb{C}^{n \times m}$, $T \in \mathbb{C}^{n \times n}$. If $P$ and $Q$ are normal and $PT = TQ$, then $P^*T = TQ^*$.

**Proof.** Since the matrix $P \oplus Q$ is normal, there exists a scalar polynomial $g$ such that $(P \oplus Q)^* = g(P \oplus Q)$. This implies that $P^* = g(P)$ and $Q^* = g(Q)$. Hence, $P^*T = g(P)T = Tg(Q) = TQ^*$.

**Remark 1.** Let $f$ be a function defined on the spectra of $P \in \mathbb{C}^{m \times n}$ and $Q \in \mathbb{C}^{n \times m}$, in the sense of Gantmacher [3, p. 96]. Then there exists a polynomial $g$ such that $f(P) = g(P)$ and $f(Q) = g(Q)$. Hence, if $T \in \mathbb{C}^{n \times n}$, it follows that $PT = TQ$ implies that

$$f(P)T = g(P)T = Tg(Q) = Tf(Q).$$

Letting $f(\lambda) = \bar{\lambda}$ for the normal matrices $P$ and $Q$ of Theorem 1, we obtain our proof of that theorem. In the proof of Theorem 3 we use another application of this result. Let $f(\lambda) = \lambda^{1/2} > 0$ for $\lambda > 0$. If $H \in \mathbb{C}^{m \times m}$ and $K \in \mathbb{C}^{n \times n}$ are positive semidefinite Hermitian, then $H = f(H^2)$ and $K = f(K^2)$. Hence, if $T \in \mathbb{C}^{n \times n}$ with $H^2T = TK^2$, then $HT = TK$.

**Theorem 2.** Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then $AB$ and $BA$ are normal if and only if $A^*AB = BAA^*$ and $ABB^* = B^*BA$. 
Proof. Assume that $AB$ and $BA$ are normal. Then $(AB)^*$ and $(BA)^*$ are normal. Hence, since

$$A^*(AB)^* = A^*B^*A^* = (BA)^*A^*,$$

by Theorem 1, $A^*AB = BAA^*$. Similarly, from $(AB)^*B^* = B^*(BA)^*$, we obtain $ABB^* = B^*BA$. Conversely, if $A^*AB = BAA^*$ and $ABB^* = B^*BA$, then multiplying the first equation by $B^*$ and the second one by $A^*$ we see that $AB$ and $BA$ are normal. 

Remark 2. A result of J. Williamson [14] can be used instead of Theorem 1 to obtain a proof of Theorem 2. Assume that $AB$ and $BA$ are normal. It follows from Williamson's Theorem 2 that there exist unitary matrices $U \in \mathbb{C}^{mn}$, $V \in \mathbb{C}^{nn}$ and rectangular diagonal matrices $F, G \in \mathbb{C}^{mn}$ such that $A = UFV$ and $B^* = UGV$. Then

$$A^*AB = V^*F^*FG^*U^* = V^*G^*FF^*U^* = BAA^*,$$

$$ABB^* = UFG^*GV = UGG^*FV = B^*BA.$$

Remark 3. The result that Theorem 1 implies Theorem 2 may be put into a more general context. Let $\mathfrak{A}$ be an algebra over the complex numbers with an involution $^*$ (see [1]). An element $P \in \mathfrak{A}$ is called normal if $PP^* = P^*P$. We define $\mathfrak{A}$ to be a Fuglede-Putnam algebra if, for all normal $P, Q \in \mathfrak{A}$ and $T \in \mathfrak{A}$, the relation $PT = TQ$ implies $P^*T = TQ^*$. Let $\mathfrak{A}$ be a Fuglede-Putnam algebra and let $A, B \in \mathfrak{A}$. We have shown that $AB$ and $BA$ are normal if and only if $A^*AB = BAA^*$ and $ABB^* = B^*BA$. An example of a Fuglede-Putnam algebra is the algebra of all bounded operators on a Hilbert space, with involution the usual adjoint. Other examples may be found in [7].

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It is well known that every $A \in \mathbb{C}^{nn}$ has a polar decomposition as $A = UH$ where $H \in \mathbb{C}^{nn}$ is positive semidefinite Hermitian and $U \in \mathbb{C}^{nn}$ is unitary. If $A$ is singular, $U$ is not unique. We have the following theorem.

Theorem 3. Let $A = UH$, where $H \in \mathbb{C}^{mn}$ is positive semidefinite Hermitian and $U \in \mathbb{C}^{nn}$ is unitary, and let $B \in \mathbb{C}^{nm}$.

(a) If $BU$ is normal and $HBU = BUH$, then $AB$ and $BA$ are normal.
(b) If $AB$ and $BA$ are normal, then $HBU = BUH$. 
Proof. Suppose that $BU$ is normal and $HBU = BUH$. Then

$$BAA^* = BUH(UH)^* = BUH^2U^* = H^2BUU^* = H^2B = (UH)^*UHB = A^*AB.$$  

(1)

Since $BU$ is normal and $HBU = BUH$, from Theorem 1, we also have $H(BU)^* = (BU)^*H$. Hence,

$$ABB^* = UHBU(BU)^* = UBU(BU)^*H = U(BU)^*BUH = UU*B*BUH = B*BA.$$  

(2)

Therefore, by Theorem 2, $AB$ and $BA$ are normal. This proves (a). To prove (b), let $AB$ and $BA$ be normal and note that there exists a positive semidefinite Hermitian $K \in \mathbb{C}^{n \times n}$ such that $A = KU$. Using Theorem 2, we obtain

$$H^2B = A^*AB = BAA^* = BK^2.$$  

Hence, since $H$ and $K$ are positive semidefinite Hermitian, $HB = BK$ (see Remark 1). Then $HBU = BKU = BUH$.

Theorem 4. Let $A = UH$, where $H \in \mathbb{C}^{n \times n}$ is positive semidefinite Hermitian and $U \in \mathbb{C}^{n \times n}$ is unitary. The following are equivalent:

(a) $\text{rank}(A) > n - 1$;

(b) if $B \in \mathbb{C}^{n \times n}$ such that $AB$ and $BA$ are normal, then $BU$ is normal.

Proof. Let $\text{rank}(A) > n - 1$, and let $B \in \mathbb{C}^{n \times n}$ be such that $AB$ and $BA$ are normal. From Theorem 2 and part (b) of Theorem 3, we see that

$$(BU)^*BUH = U*B*BA = U*ABB^* = HBU(BU)^* = BU(BU)^*H.$$  

(3)

Hence, if $\text{rank}(H) = \text{rank}(A) = n$, then $BU$ is normal. Suppose that $\text{rank}(A) = n - 1$. Then there exist a unitary $V \in \mathbb{C}^{n \times n}$ and a positive definite Hermitian matrix $L$ of order $n - 1$ such that

$$VHV^* = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}.$$  

(4)
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Let

\[ VBUV^* = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \]

where \( G_{22} \in \mathbb{C} \). Since \( L \) is nonsingular, from part (b) of Theorem 3 we see that \( G_{12} = 0 \) and \( G_{21} = 0 \). Then Eq. (3) implies that \( G_{11} \) is normal. Moreover, since \( G_{22} \in \mathbb{C} \), we see that \( BU \) is normal. Hence (a) \( \Rightarrow \) (b).

Let \( \text{rank}(A) = k < n - 1 \). There exist \( L \in \mathbb{C}^{kk} \) and unitary \( V \in \mathbb{C}^{nn} \) such that \( VHV^* \) has the form (4). Since \( m = n - k > 2 \), there exists \( R \in \mathbb{C}^{mm} \) such that \( R \) is not normal. Let

\[ B = V^* \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} VU^*. \]

Then

\[ BUH = HBU, \quad BU(BU)^*H = (BU)^*BUH. \]

These equations imply \( A^*AB = BAA^* \) and \( B^*BA = ABB^* \) by an argument similar to that at the beginning of the proof of Theorem 3 [see (1) and (2)]. Hence, by Theorem 2, \( AB \) and \( BA \) are normal. However, \( BU \) is not normal. Let

Clearly, Theorems 3 and 4 imply the following theorem.

**Theorem 5.** Let \( A = UH \), where \( H \in \mathbb{C}^{nn} \) is positive semidefinite Hermitian and \( U \in \mathbb{C}^{nn} \) is unitary. The following are equivalent:

(a) \( \text{rank}(A) > n - 1 \);

(b) if \( B \in \mathbb{C}^{nn} \), then \( AB \) and \( BA \) are normal if and only if \( BU \) is normal and \( HBU = BUH \).

**REFERENCES**


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