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Semantics for finite delay[1]

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Abstract

We produce a fully abstract model for a notion of process equivalence taking into account issues of fairness, called by Milner *fair bisimilarity*. The model uses Aczel's anti-foundation axiom and it is constructed along the lines of the anti-founded model for SCCS given by Aczel. We revisit Aczel's semantics for SCCS where we prove a unique fixpoint theorem under the assumption of guarded recursion. Then we consider Milner's extension of SCCS to include a finite delay operator ε. Working with fair bisimilarity we construct a fully abstract model, which is also fully abstract for *fortification*. We discuss the solution of recursive equations in the model. The paper is concluded with an investigation of the algebraic theory of fair bisimilarity.

1. Fairness and finite delay

A typical *fairness* notion ensures that a process that is infinitely often enabled must be taken infinitely often. Fairness often leads to the failure of continuity of semantic operations, when the semantic domain is a DCPO (directed complete partial order), see for example [15], and hence to the need for transfinite induction. At the same time, certain properties of programs, such as *liveness*, cannot be proven unless fairness is assumed. In addition, fairness is a significant issue in hardware and software systems such as communication protocols, distributed databases and asynchronous circuits [7].

In this report we assume a simple notion of fairness: unbounded but finite delay of subprocesses in concurrent computation. Consider a programming language with

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parallel constructs $P|Q$. A synchronous parallel operator forces both components to proceed at the same speed with lock-step synchronization at the ticking of a universal clock. Effectively then the speed of the system is that of the slowest component. Construing $|$ as an asynchronous parallel raises issues of fairness. A move of the compound process $P|Q$ is either a move of P or one of Q. We may then say that P moves while Q *delays* (or vice versa). This can be expressed more succintly by introducing a special action 1 to indicate the passage of time. Delaying for one unit of time is then regarded as an idle transition $Q \xrightarrow{1} Q$. In a language with recursion, nondeterministic choice and prefixing, such as SCCS, delay δ is a derived operator definable by the pointwise recursion $\delta P \equiv \mu x(1.x + P)$ (x not free in P). Intuitively, δP may either perform the actions of P or idle for one unit of time and then get to a state where it may either perform the actions of P or else idle for another unit of time, and so on. Given a synchronous parallel operator $\|$, asynchrony is captured by defining $P|Q = P\|\delta Q + \delta P\|Q$, as discussed in Milner [20]. However δ allows for perpetual delay and so $P|Q$ can exhibit *unfair* behaviour. For example, if $P \equiv \mu x(a.x)$ and $Q \equiv \mu x(b.x)$, then $P|Q$ can perform either of a^ω or b^ω, which is unfair as it precludes the other process from proceeding. This creates the need for a delay operator that only allows for arbitrarily long but *finite* delay. Adding such an operator ε, *fair asynchrony* can be defined as $P\|\varepsilon Q + \varepsilon P\|Q$.

A finite delay operator ε was first introduced in Milner's technical report [19] and subsequently studied by Hennessy in [12, 13]. We first review the basics from Milner's report.

1.1. SCCS with finite delay (SCCS + ε)

The language $\mathscr{L}$ of SCCS + ε is that of the synchronous calculus of [20] with the addition of an operation symbol ε (the finite delay operator). Process terms are defined by the following schema, where A is a fixed abelian group of basic actions.

$$P ::= 0 \mid x, x \in Var \mid a.P,\ a \in A \mid \textstyle\sum_{i\in I} P_i \mid P\|P \mid P|_L, 1 \in L \subseteq A \mid \varepsilon P \mid \mu_i \vec{x}\vec{P}.$$

The operational semantics is the usual for SCCS with the addition of the wait and fulfill rules for ε

$$\text{(Delay)}\ \frac{}{\varepsilon E \xrightarrow{1} \varepsilon E}\text{(wait)} \qquad \frac{E \xrightarrow{a} F}{\varepsilon E \xrightarrow{a} F}\text{(fulfill)}$$

Since the actions of εP are exactly those of δP the two processes will be identified by bisimilarity: $\varepsilon P \simeq \delta P$. To make the distinction the operational semantics needs to be extended to include information about the infinite behaviour of processes. Certain infinite strings of actions must be deemed inadmissible for a process as they may involve infinite delay.

Where $u = a_1 a_2 \cdots \in A^+$ is a (finite or infinite) sequence of actions a *u-computation* of P is a sequence

$$P = P_0 \xrightarrow{a_1} P_1 \xrightarrow{a_2} P_2 \to \cdots$$

where we leave implicit the proof information (justification by rules of the individual actions a_i). The part of it which begins with P_i, for some $i \geqslant 0$ is called a *sequel* of the computation. A computation of the form

$$\varepsilon P \xrightarrow{1} \varepsilon P \xrightarrow{1} \dots$$

in which every instance of the silent action is justified by the wait rule is called a *waiting*.

A *context* (with n "holes") is an expression of the form $\mathscr{C}[X_1,\dots,X_n]$ built from product and restriction, for example $X_1 \| (X_2|_L)$. If P is the agent $P \equiv \mathscr{C}[P_1,\dots,P_n]$, for some agents $P_1,\dots,P_n$, then each agent P_i is a *subagent* of P. Given the rules of action, every u-computation of P

$$P \equiv \mathscr{C}[P_1,\dots,P_n] \xrightarrow{a_1} \mathscr{C}_1[P_{11},\dots,P_{n1}] \xrightarrow{a_2} \cdots$$

is inferred from u_i-computations, $1 \leqslant i \leqslant n$, of the subagents of P

$$P_i \xrightarrow{a_{i1}} P_{i1} \xrightarrow{a_{i2}} P_{i2} \xrightarrow{a_{i3}} \cdots$$

where the jth action in P's computation is the product (taken in the abelian group A) of the actions $a_{1j},\dots,a_{nj}$. Each of these computations is called a *subcomputation* of P's computation.

Definition 1.1. 1. A computation is *admissible* iff either it is finite or else it has no sequel with a waiting subcomputation. Otherwise it is *inadmissible*.

2. If $u \in A^\omega$, then P admits u iff P has some admissible u-computation. Otherwise P prevents u.

Some simple examples follow.

Example 1.2. $\mu x(a.x)$ admits a^ω, δP admits 1^ω but εP may prevent 1^ω. If $P \equiv a.x \| \varepsilon Q$, $Q \equiv \mu y(b.y)$ and $ab \neq a$, then μxP prevents the sequence a^ω since the only possible a^ω-computation of μxP involves a waiting subcomputation of εQ. Finally, if $P \equiv a.(b.0 + \varepsilon x)$, then the only possible computation of $a1^\omega$ from μxP is the computation

$$\mu xP \xrightarrow{a} b.0 + \varepsilon(\mu xP) \xrightarrow{1} \varepsilon(\mu xP) \xrightarrow{1} \cdots$$

Hence μxP prevents the sequence $a1^\omega$. The only admissible infinite sequences for μxP are sequences of the form $a1^{m_2}a1^{m_4}a\dots a1^{m_{2k}}a\dots$, for some natural numbers m_{2k}, $k \in \omega$.

What needs to be determined now is an appropriate concept of identity on processes. Obviously, this cannot be bisimilarity since the distinction between δP and εP cannot be made.

Milner [19] proposed *fortification equivalence* as the individuation principle for processes.

Definition 1.3. A binary relation $\mathscr{R}$ on processes is a *fortification relation* if $P\mathscr{R}Q$ implies that for all $a \subset A$ and $u \in A^{\omega}$

1. $P \xrightarrow{a} P' \Rightarrow \exists Q'\ Q \xrightarrow{a} Q'$ and $P'\mathscr{R}Q'$
2. $Q \xrightarrow{a} Q' \Rightarrow \exists P'\ P \xrightarrow{a} P'$ and $P'\mathscr{R}Q'$
3. P prevents u implies Q prevents u.

Fortification equivalence, denoted by $\sim$, is the symmetrization of the largest fortification relation, which we shall henceforth call simply *fortification* and denote by $\prec$. In other words we define $P \sim Q$ iff $P \prec Q \prec P$. Thus, for example, $\varepsilon P \sim \varepsilon\varepsilon P$ and $\varepsilon\delta P \sim \delta P$, for any process term P.

In [19] some possible alternatives are briefly mentioned. *Fair bisimulation*, among them, is defined by making clause 3 in the definition of fortification symmetric. We can then define *fair bisimilarity*, denoted by $\approx$, as the largest fair bisimulation. Bisimilarity will be denoted by $\simeq$ throughout this report. Clearly, $\approx \subseteq \prec \subseteq \simeq$ and, since $\approx$ is a symmetric fortification, $\approx \subseteq \sim$. It should be clear also that restricting to the fragment of SCCS $+ \varepsilon$ without the finite delay operator ε all four relations coincide.

Allowing ε in the signature, define a term P as *finite* if it has no subterm of the form $\mu_i \bar{x}\bar{P}$. Then it is not hard to see that the restrictions to finite terms of all the above relations again coincide. The proof relies on the observation that a finite term admits no infinite sequences, hence clause 3 in the above definition is vacuous. We make this official in the following.

Lemma 1.4. *For finite terms P and Q, $P \approx Q$ iff $P \sim Q$ (iff $P \simeq Q$).*

Incidentally, this precludes giving any interesting characterization of either fair bisimilarity or fortification equivalence with respect to some set of axioms for *finite* terms.

The reasons presented in [19] for favouring fortification against fair bisimilarity are that we seem to lose the interesting law $\delta P \prec \varepsilon P$ and the least (with respect to fortification) fixpoint theorem: If $E\{Q/x\} \prec Q$, then $\mu x E \prec Q$. This is not necessarily so. We think of the process algebra for SCCS $+ \varepsilon$ as a pre-ordered algebra $\mathscr{L} = \langle L, =, <, \Sigma \rangle$ where $<$ is a pre-order and Σ is the signature of operators of SCCS $+ \varepsilon$. In the algebra arising in the natural way from the operational semantics, however, the identity $=$ on processes is interpreted as *fair bisimilarity* $\approx$ (and not as *fortification equivalence*) and the pre-order $<$ is interpreted as fortification $\prec$. The set of (in)equations to be satisfied is discussed in Section 3.3. Since fair bisimilarity is a fortification relation the least fixpoint theorem is still available, namely if $P\{Q/x\} \approx Q$, then $\mu x P \prec Q$ (assuming guarded recursion).

The drawback with using *fortification equivalence* as identity of processes is that we lose connection with bisimilarity. It is desirable that, since bisimilarity proves to be too coarse an equivalence relation, then the improved identity criterion $\sim'$ should be a *refinement* of bisimilarity. This condition is obviously satisfied by fair bisimilarity and it is then worth investigating this relation on its own right.

In this report we construct a final coalgebra for an appropriate endofunctor on the category of classes which is fully abstract for fair bisimilarity, by the universal properties of final coalgebras. We verify that our semantics is also fully abstract for fortification (hence also for fortification equivalence) and investigate the solution of recursive equations in the model. In addition, we show in Section 3.3 that all the interesting equations for fortification equivalence that are singled out in [19] also hold for fair bisimilarity (which is a refinement of fortification equivalence since fair bisimilarity is clearly a fortification relation).

1.2. Structure of this paper

In Section 2 we review from Aczel [1, 2] the semantics he proposes for SCCS. Our treatment of recursive terms differs from that in [2]. We prove a unique fixpoint theorem under the assumption of guarded recursion, used later in the proof of Theorem 3.20 that any two fixpoints of the functional induced in our model by a guarded open term must have the same finitary behaviour.

In Section 3 we introduce a natural notion of *extended transition system* (ETS) as a transition system equipped with an environment map V assigning to every "process" in the system a set of infinite sequences u of actions (an *admissibility set*), intuitively those sequences along which the "process" is allowed to evolve. In the general case V is just an arbitrary assignment. ETS's can be conveniently regarded as coalgebras for the class functor

$$\Phi = Pow(A \times -) \times Pow(A^{\omega})$$

(where A is the set of actions and for a class X, $PowX$ is the class of sub*sets* of X).

We verify that Φ satisfies the conditions of the Special Final Coalgebra Theorem of [1], hence that the class

$$\mathscr{P} = \bigcup \{x \mid x \text{ is a set and } x \subseteq \Phi x\}$$

is a final coalgebra for Φ. We derive from finality of full-abstractness theorem for fair bisimilarity: $P \approx Q$ iff $[\![P]\!]_{\Phi} = [\![Q]\!]_{\Phi}$. Furthermore, we show that full-abstractness for fortification also holds: $P \prec Q$ iff $[\![P]\!]_{\Phi} \prec [\![Q]\!]_{\Phi}$, Theorem 3.16.

In Section 3.2.2 we turn to studying the interpretation of recursive terms. We observe that a unique fixpoint theorem fails but that any two fixpoints can be identified on grounds of finitary only behaviour, Theorem 3.20. Furthermore, we verify that any two fixpoints p, q with the same admissibility sets ($Vp = Vq$) must be identical and that if $Vq \subseteq Vp$ then $p \prec q$, Theorem 3.18. We also derive a least fixpoint theorem for *definable* fixpoints (Definition 3.21 and Theorem 3.23) using Milner's least fixpoint theorem in [19] and full abstractness for fortification, Theorem 3.16.

The paper is concluded with Section 3.3 where we verify that the basic equational theory of fortification equivalence outlined in [19] is sound in the structure $\mathscr{P}$, hence sound for fair bisimilarity by full abstractness of the model.

Incidentally, the question of a fully-abstract semantics for fortification equivalence is still open. Since our model is itself an extended transition system we do get that

for any process terms P, Q, $P \sim Q$ iff $[\![P]\!]_\Phi \sim [\![Q]\!]_\Phi$ (see Theorem 3.16). However, $\sim$ is not the identity relation on our semantic structure. One direction to follow is to take a quotient of our semantic structure $\mathscr{P}$ with respect to fortification equivalence and appropriately characterize representatives of the equivalence classes. We leave this question open.

Hennessy [12] provided a fully abstract model for SCCS $+ \varepsilon$ for a notion of *testing equivalence* in the tradition initiated by Hennessy and de Nicola [8] and further explored by Hennesy in [14–16]. In [10] the present author has made an attempt, with M.Z. Kwiatkowska, to semantically capture the notion of admissible sequence and of fortification equivalence. We produced a model of generalized synchronization trees over the SCCS synchronization algebra, extending the framework developed by Winskel [26] and motivated by Hennessy [15]. We showed that the model can capture the notion of admissibility but full-abstractness must fail in this framework.

In producing here a fully abstract model for fair bisimilarity we extend the framework developed by Aczel [1, 3] and then also investigated by Rutten in [22, 23] and Rutten and Turi in [24, 25].

Since we work within a nonstandard set-theory, we have collected our set-theoretic assumptions in Appendix A, where we also review some of the technical aspects of Aczel's approach to modelling processes as hypersets.

2. Transition systems as coalgebras

A transition system over A is a structure $(X, (\xrightarrow{a})_{a\in A})$, where X is a class and $\xrightarrow{a}$ is a binary relation on X for each $a \in A$. We call the system *set-based* if for each $a \in A$ and $x \in X$ the class $\{x' \mid x \xrightarrow{a} x'\}$ is a *set*. In the sequel, by a transition system we always mean a set-based system. A (set-based, as agreed) transition system can be regarded as a coalgebra for the functor $\Theta \equiv Pow(A \times -)$, where for a class X, $Pow X$ is the class of subsets of X and for a function $f : X \to Y$ and a subset $U \subseteq X$, $Pow(f)(U) = \{fx \mid x \in U\}$. A *coalgebra* for an endofunctor on C (where C is some category) is a pair (X, α), where X is an object of C and α is a morphism $\alpha : X \to \Theta X$. In the particular case where C is the category of classes and $\Theta = Pow(A \times -)$ the structure map α and the transition relations $\xrightarrow{a}$ are interdefinable by

$$(a, y) \in \alpha x \text{ iff } x \xrightarrow{a} y.$$

Coalgebras for a functor Θ form a category C_Θ with morphisms $f : (X, \alpha) \to (Y, \beta)$ the maps $f : X \to Y$ such that the square below commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \Theta X & \xrightarrow[\Theta f]{} & \Theta Y \end{array}.$$

In other words, for any $x \in X$, $y \in Y$,

$$fx \xrightarrow{a} y \text{ iff } \exists x'\ x \xrightarrow{a} x' \text{ and } fx' = y.$$

Remark 2.1. It is not hard to show in standard ZFC set-theory that the functor Θ has a final coalgebra. Indeed, we may take the class of all rooted transition systems, turn it to a transition system and verify that it forms a weakly final coalgebra, that is to say a coalgebra (F, ϕ) such that for any coalgebra (X, α) there is at least one morphism $f : (X, \alpha) \to (F, \phi)$. We can then take the quotient of (F, ϕ) by bisimilarity and verify that the resulting system is a final coalgebra.

Here we take the alternative approach of [1]. What we aim at is a direct set-theoretic modeling of the abstract behaviour of processes. Modeling processes as sets requires that we drop the foundation axiom because of recursively defined processes. For example if $P \equiv \mu x(a.x)$ then we wish to model P as the set of pairs (b, Q) such that $P \xrightarrow{b} Q$. More precisely, if $[\![.]\!]$ is the semantic map, we should have $[\![P]\!] = \{(b, [\![Q]\!]) \mid P \xrightarrow{b} Q\}$. For the particular example this would give a set satisfying the equation $x = \{(a, x)\}$. Such a set does not belong, of course, in the well-founded universe. For that reason we turn to Aczel's anti-founded set-theory ZFA + GC (we use global choice because we prefer to work with classes rather than sets). Our set-theoretic assumptions are spelled out in Appendix A.

2.1. *Final coalgebra semantics for SCCS*

In [1] and later in [2], Aczel worked out the final coalgebra semantics for SCCS (and for CSP, in [2]). We briefly review the basics for two reasons: First, because this will give us a useful warm-up before we address the question of final coalgebra semantics for SCCS + ε (SCCS with finite delay), and second because we will add a unique fixpoint theorem under an assumption of guarded recursion, used in the proof of Theorem 3.20.

If we let $J = \bigcup \{x \mid x \subseteq \Theta x\}$, then J can be shown to be the largest fixpoint for Θ. Furthermore, the special final coalgebra theorem (see Appendix A) applies from which we can conclude that $J = Pow(A \times J)$ is a final coalgebra for $\Theta = Pow(A \times -)$, where the structure of SCCS map $\alpha : J \to Pow(A \times J)$ is the identity on J. Denote the signature of SCCS operators by Σ and let T_Σ be the SCCS terms, finite and infinite (where some subterm is a recursive term $\mu \bar{x} \bar{E}$). The operational semantics for SCCS is a transition system, hence a Θ-coalgebra. By finality of J, let $[\![.]\!]_\Theta : T_\Sigma \to J$ be the unique coalgebra map. Based on finality of J we can also show that

Proposition 2.2. *J is closed under the SCCS operations.*

Furthermore, the semantic map is a Σ-homomorphism in the sense of the following:

Proposition 2.3. *For any process terms*

1. $[\![a.P]\!]_\Theta = a.[\![P]\!]_\Theta := \{(a, [\![P]\!]_\Theta)\}$,

2. $[\![P+Q]\!]_\Theta = [\![P]\!]_\Theta + [\![Q]\!]_\Theta := [\![P]\!]_\Theta \cup [\![Q]\!]_\Theta$,
3. $[\![P|_L]\!]_\Theta = [\![P]\!]_\Theta|_L := \{(a,[\![Q]\!]_\Theta) \mid a \in L \text{ and } (a,[\![Q]\!]_\Theta) \in [\![P]\!]_\Theta\}$,
4. $[\![P\|Q]\!]_\Theta = [\![P]\!]_\Theta\|[\![Q]\!]_\Theta := \{(ab,[\![P']\!]_\Theta\|[\![Q']\!]_\Theta) | P \xrightarrow{a} P' \text{ and } Q \xrightarrow{b} Q'\}$.

The proofs can be found in [1, 2].

To have a treatment of recursion we extend the semantic map $[\![.]\!]_\Theta$ to an interpretation $[\![.]\!]^e_\Theta$ of open terms, where e is a variable assignment (an environment) $e : Var \to J$ (Var is the set of variables of the SCCS language). Then if P is open, $Fv(P) = \{x_1,\ldots,x_n\}$, $[\![P]\!]_\Theta$ can be regarded as a function $[\![P]\!]_\Theta : J^n \to J$. If $s_1,\ldots,s_n \in J$, then $[\![P]\!]_\Theta(s_1,\ldots,s_n) = [\![P]\!]^{e[x_1:=s_1,\ldots,x_n:=s_n]}_\Theta$.

To prove a unique fixpoint theorem we will assume that the variables are *guarded* where we say that $x \in Fv(P)$ is guarded in P iff every free occurrence of x is within a subterm $a.Q$ of P.

Theorem 2.4. *Let $Fv(P) = \{x\}$, x guarded in P, and $[\![.]\!]_\Theta : T_\Sigma \to J$ the semantic map. If $[\![\mu xP]\!]_\Theta = s$ then $[\![P]\!]_\Theta(s) = s$. Furthermore, if $s' \in J$ is such that $[\![P]\!]_\Theta(s') = s'$, then $s = s'$.*

Proof. The proof is similar to that of Theorem 3.18, in fact simpler as we need not be concerned here with the environment sets Vp of processes as we do in that proof. Otherwise, the proof proceeds by first verifying that Lemmas 3.24 and 3.29 hold and then by ordinal induction. Within that a subinduction is needed, all as in the proof of Theorem 3.18. The definition of the maps $[\![P]\!]_\Theta$, for open P, is detailed in Lemma 3.17. □

As a corollary we obtain Milner's unique fixpoint theorem in [20], namely

Theorem 2.5 (Milner [20]). *Assume x is guarded in P. If $P\{Q/x\} \simeq Q$, then $\mu xP \simeq Q$.*

Proof. From the hypothesis both $[\![\mu xP]\!]_\Theta$ and $[\![Q]\!]_\Theta$ are fixpoints of the functional $[\![P]\!]_\Theta$, hence $[\![\mu xP]\!]_\Theta = [\![Q]\!]_\Theta$. The kernel of the semantic map $[\![.]\!]_\Theta$ is a bisimulation, hence $\mu xP \simeq Q$. □

3. Extended TSs and final coalgebras

Every process term P of SCCS+ε comes with a set V_P of infinite sequences of actions that it admits. The possible infinitary behaviour of P is constrained by the environment V_P since P may be able to perform a string u of actions that the environment V_P forbids ($u \notin V_P$). The typical example of course is with terms of the form εP. εP can perform an infinite sequence of wait actions $\varepsilon P \xrightarrow{1} \varepsilon P$, but $1^\omega \notin V_{\varepsilon P}$ unless P can perform an infinite sequence of internal moves as a result of synchronization.

The operational semantics of a process language like SCCS + ε is an *extended transition system* (ETS). An ETS is a TS with extra structure, to account for infinitary

behaviour. Thus an ETS over a set A of basic actions is a structure $(X,(\xrightarrow{a})_{a\in A},V)$ where $V:X \to Pow(A^\omega)$ is a function assigning a set Vx of infinite sequences of actions to every $x \in X$. Intuitively Vx prescribes what infinite strings of actions x is allowed to perform. We first note that taking V to be the map assigning the admissible sequences to each process term P of SCCS $+\varepsilon$ the operational semantics for SCCS $+\varepsilon$ is an ETS. More precisely we have the following, where T is the set of closed finite or infinite terms:

Proposition 3.1. *The structure map $V:T \to Pow(A^\omega)$ satisfies the following*:

1. $V(0)=\emptyset$,
2. $V(a.P)=\{a^\frown v \mid v \in V(P)\} := a.V(P)$,
3. $V(P+Q)=V(P)\cup V(Q) := V(P)+V(Q)$,
4. $V(P|_L)=V(P)\cap L^+ := V(P)|_L$,
5. $V(P\|Q)=\{u\cdot v \mid u \in V(P)$ *and* $u \in V(Q)\} := V(P)\|V(Q)$ ($u\cdot v$ *is the pointwise product of the sequences* u *and* v),
6. $V(\varepsilon P)=\bigcup_{n\in\omega}\{(1^n)^\frown v \mid v \in V(P)\} := \varepsilon V(P)$,
7. $V(\mu xP)=V(P\{\mu xP/x\})$.

Proof. The proof follows from Definition 1.1 of admissible sequence. □

Note that by item 7 in the above proposition the admission set of a recursive process μxP is a fixpoint of the related functional on $Pow(A^\omega)$. For many simple examples it turns out to be the largest fixpoint but we do not know if this is true in general.

ETSs can be turned to a category with morphisms the transition system maps $f:(X,(\xrightarrow{a})_{a\in A},V) \to (Y,(\xrightarrow{a})_{a\in A},U)$ subject to the additional requirement that environment constraints are preserved, namely $Vx = Ufx$. Extended systems then form a subcategory of the category of transition systems and we need to investigate the question of the existence of a final object.

3.1. Final extended transition systems

We discuss in this section the questions of existence and of basic properties for a final ETS.

As for plain transition systems, it is not hard to see within standard set-theory that a final object exists in the category of ETSs, specializing a general categorical construction of a final coalgebra from a weakly final one in our particular context. We may take the class of all rooted extended systems and turn it to an extended transition system in the natural way. This system $\mathscr{V}_0$ is weakly final in the sense that a morphism $f:\mathscr{X} \to \mathscr{V}_0$ always exists, for any ETS $\mathscr{X}$. To produce a final object we need to factor out by an appropriate equivalence relation $\approx$. The only critical point in the choice of $\approx$ (which we will take to be fair bisimilarity) is that it should be possible to show that if

$$\mathscr{X} \underset{\xrightarrow{g}}{\xrightarrow{f}} \mathscr{Y}$$

are two morphisms, then the relation $\mathscr{R} \subseteq Y \times Y$ defined by $y \mathscr{R} y'$ iff there exists $x \in X$ such that $y = fx$ and $y' = gx$ is a subrelation of $\approx$. Indeed, suppose a suitable notion of identity $\approx$ is given satisfying the above condition. Let $\mathscr{V}$ be the quotient $(\mathscr{V}_0)_{/\approx}$. $\mathscr{V}$ is of course weakly final. Suppose now

$$\mathscr{X} \underset{\underset{g}{\longrightarrow}}{\overset{f}{\longrightarrow}} \mathscr{V}$$

are two morphisms and let $\mathscr{R}$ be the relation described above. Since we assume $\mathscr{R} \subseteq \approx$ to conclude that $f = g$ we only need to observe that in the quotient $\mathscr{V} = (\mathscr{V}_0)_{/\approx}$ the identity relation is exactly the relation $\approx$.

The appropriate individuation principle on extended systems that we work with is that of fair bisimilarity. A relation $\mathscr{R}$ on an extended transition system $\mathscr{X} = (X, (\overset{a}{\rightarrow})_{a \in A}, V)$ is a *fair bisimulation* if $x \mathscr{R} y$ implies

$Vx = Vy$ and for all $a \in A$

- $x \overset{a}{\rightarrow} x' \Rightarrow \exists y' \ (y \overset{a}{\rightarrow} y'$ and $x' \mathscr{R} y')$,
- $y \overset{a}{\rightarrow} y' \Rightarrow \exists x' \ (x \overset{a}{\rightarrow} x'$ and $x' \mathscr{R} y')$.

Extending Milner's definition [19] to any ETS, $\mathscr{R}$ is a *fortification relation* if $x \mathscr{R} y$ implies

$Vy \subseteq Vx$ and for all $a \in A$

- $x \overset{a}{\rightarrow} x' \Rightarrow \exists y' \ (y \overset{a}{\rightarrow} y'$ and $x' \mathscr{R} y')$,
- $y \overset{a}{\rightarrow} y' \Rightarrow \exists x' \ (x \overset{a}{\rightarrow} x'$ and $x' \mathscr{R} y')$.

Fair bisimilarity is the largest fair bisimulation

$$\approx = \bigcup \{\mathscr{R} \mid \mathscr{R} \text{ is a fair bisimulation}\}.$$

Similarly, *fortification* is defined as the largest fortification relation

$$\prec = \bigcup \{\mathscr{R} \mid \mathscr{R} \text{ is a fortification relation}\}$$

A fundamental property of a final ETS is that it is *strongly extensional* in the sense of Theorem 3.2. The proof of the theorem can be given along the lines of similar results in Aczel [2] and Rutten and Turi [25].

Theorem 3.2. *If $\mathscr{F} = (F, (\overset{a}{\rightarrow})_{a \in A}, V)$ is a final ETS and $p, q \in F$, then $p \approx q$ iff $p = q$.*

Proof. The proof is an immediate consequence of the following Proposition, taking f_X to be identity on $\mathscr{F}$. □

Proposition 3.3. *Assume $\mathscr{F} = (F, (\overset{a}{\rightarrow})_{a \in A}, V)$ is a final ETS and let $\mathscr{X} = (X, (\overset{a}{\rightarrow})_{a \in A}, U)$ be any ETS and $f_X : \mathscr{X} \rightarrow \mathscr{F}$ the unique morphism. Then for any $x, y \in X$, $x \approx y$ iff $f_X x = f_X y$.*

Proof. In its turn, the proof of this proposition follows from the next two lemmas.

Lemma 3.4. *If* $f : (X, (\xrightarrow{a})_{a\in A}, V) \to (Y, (\xrightarrow{a})_{a\in A}, U)$ *is a morphism, then its kernel* $K = ker(f) = \{(x,x') \mid fx = fx'\}$ *is a fair bisimulation on* X.

Proof. Assume $fx = fz$. Then $Vx = U(fx) = U(fz) = Vz$. Next assume $x \xrightarrow{a} x'$. Then $fx \xrightarrow{a} fx'$ and hence $fz \xrightarrow{a} fx'$. By definition of morphisms there must be a z' such that $z \xrightarrow{a} z'$ and $fz' = fx'$. □

We will regard ETSs as coalgebras for the class functor

$$\Phi \equiv Pow(A \times -) \times Pow(A^{\omega}).$$

An ETS is then a triple $\mathscr{X} = (X, \eta, V_X)$, where $\eta \times V_X : X \to \Phi X$ is the structure map. For $x \in X$, $(\eta \times V_X)x = (\eta x, V_X x)$, where $(a, y) \in \eta x$ is understood as $x \xrightarrow{a} y$ and $V_X x \subseteq Pow(A^{\omega})$ is the environment of x. We phrase our second lemma needed for the proof of Proposition 3.3 in these terms.

Lemma 3.5. *Let* $\mathscr{X} = (X, (\xrightarrow{a})_{a\in A}, V)$ *be an ETS. A binary relation* R *on* X *is a fair bisimulation iff there exists a structure map* $\eta : R \to Pow(A \times R)$ *and a map* $V_R : R \to Pow(A^{\omega})$ *such that the natural projection maps* $\pi_i : R \to X$ *are morphisms of extended transition systems.*

Proof. If R is a fair bisimulation define η by $(x, y) \xrightarrow{a} (x', y')$ iff $x \xrightarrow{a} x'$ and $y \xrightarrow{a} y'$ and let $V_R(x, y) = Vx$ $(= Vy)$. It is immediate that the projections are morphisms.

For the converse, given (R, η, V_R) we assume that $\pi_i : R \to X$ are ETS morphisms. Given $(x, y) \in R$ we have

$$Vx = V\pi_1(x, y) = V_R(x, y) = V\pi_2(x, y) = Vy.$$

If $x \xrightarrow{a} x'$, then $\pi_1(x, y) \xrightarrow{a} x'$. Hence there must be $(x'', y') \in R$ such that $(x, y) \xrightarrow{a} (x'', y')$ and $\pi_1(x'', y') = x'$, that is $x'' = x'$. Then $y \xrightarrow{a} y'$ and $(x', y') \in R$. We may thus conclude that R is a fair bisimulation. □

3.2. Abstract processes as hypersets

As with plain SCCS, however, we are not interested in the final system constructed as a quotient by fair bisimilarity of the weakly final system of all rooted ETSs. Rather, we shall use the special final coalgebra theorem to obtain an extended transition system whose objects are hypersets. By uniqueness of final objects, up to isomorphism, we can think of the hypersets modelling processes as representatives of the fair-bisimilarity equivalence classes.

We regard an ETS as a coalgebra for the functor

$$\Phi \equiv Pow(A \times -) \times Pow(A^{\omega}) \equiv \Theta \times Pow(A^{\omega}).$$

What we are interested in is a solution to the recursive equation

$$X = Pow(A \times X) \times Pow(A^{\omega})$$

in the category of classes which, we also need to verify, is a final coalgebra for this functor (where the structure map is the identity).

By set-continuity of Φ (which is immediate)[1] the class

$$\mathscr{P} = \bigcup\{x \mid x \subseteq Pow(A \times x) \times Pow(A^{\omega})\}$$

is the largest fixpoint of Φ. To apply the special final coalgebra theorem of [1] (see Appendix A) we need to verify the following:

Proposition 3.6. *The functors $\Phi_K = Pow(A \times -) \times K$, where K is a constant class functor, are standard and uniform on maps.*

The proof is rather straightforward. Yet, for reader's convenience it is given at the end of Appendix A as it makes use of notational conventions and definitions detailed in that Appendix. When $K := Pow(A^{\omega})$ we simply write Φ.

Given now that Φ is also uniform on maps, by the special final coalgebra theorem the class $\mathscr{P}$ is a final coalgebra with structure map the identity on $\mathscr{P}$. Every $p \in \mathscr{P}$ is then a pair $p = (\alpha p, Vp)$, where $\alpha p \subseteq Pow(A \times \mathscr{P})$ and $Vp \subseteq A^{\omega}$. The transition system structure is determined by the map α by letting $p \xrightarrow{a} q$ iff $(a, q) \in \alpha p$.

By Theorem 3.2 and finality, $\mathscr{P}$ is strongly extensional, that is to say identity on $\mathscr{P}$ coincides with fair bisimilarity: $p \approx q$ iff $p = q$. A consequence of finality of $\mathscr{P}$ is the following:

Proposition 3.7. *$\mathscr{P}$ is closed under all the operations of SCCS + ε.*

Proof. Explicitly, the operations on $\mathscr{P}$ are defined as follows:

1. $0 = (\emptyset, \emptyset)$,
2. $a.p = (\{(a, p)\}, a.Vp)$, where $a.Vp := \{a^\frown v \mid v \in Vp\}$,
3. $p + q = (\alpha p \cup \alpha q, Vp \cup Vq)$,
4. $p|_L = (\{(a, p'|_L) \mid a \in L, (a, p') \in \alpha p\}, Vp \cap L^+)$,
5. $\alpha(p \| q) = \{ab, p' \| q') \mid (a, p') \in \alpha p, (b, q') \in \alpha q\}$,
   $V(p \| q) = \{u \cdot v \mid u \in Vp \text{ and } v \in Vq\}$,
6. $\varepsilon p = (\{(1, \varepsilon p)\} \cup \alpha p, \bigcup_{n \in \omega}\{(1^n)^\frown v \mid v \in Vp\})$.

The proof that these are well-defined operations on $\mathscr{P}$ relies on finality of $\mathscr{P}$ or merely from the fact that $\mathscr{P} = Pow(A \times \mathscr{P}) \times Pow(A^{\omega})$. For example, for the parallel operator $\|$ we can turn $\mathscr{P} \times \mathscr{P}$ to an ETS by letting $(p, q) \xrightarrow{c} (p', q')$ iff there exist a, b such that $c = ab$, $p \xrightarrow{a} p'$ and $q \xrightarrow{b} q'$. The environment $V(p, q)$ is defined as we defined $V(p \| q)$. By finality of $\mathscr{P}$ there is a unique coalgebra morphism $\| : \mathscr{P} \times \mathscr{P} \to \mathscr{P}$, which shows that $\mathscr{P}$ is closed under the product operator defined above.

[1] For definitions see Appendix A.

For εp, given $p \in \mathscr{P}$, the equation

$$x = \left(\{(1,x)\} \cup \alpha p,\ \bigcup_{n\in\omega} \{(1^n)^\frown v \mid v \in Vp\} \right)$$

must have a unique solution, by the Solution Lemma (see Appendix A). If $\mathscr{P}'$ is the ETS containing $\mathscr{P}$ and all the solutions p' (one for each $p \in \mathscr{P}$), then again let $\varepsilon : \mathscr{P}' \to \mathscr{P}$ be the unique coalgebra morphism. This shows that $\mathscr{P}$ is closed under the operator ε as we defined it above. □

Remark 3.8. The delay operator δ is defined in a similar way. For each $p \in \mathscr{P}$, δp is the unique solution to the equation

$$x = \left(\{(1,x)\} \cup \alpha p,\ \{1^\omega\} \cup \bigcup_{n\in\omega} \{(1^n)^\frown v \mid v \in Vp\} \right).$$

That $\mathscr{P}$ is closed under δ follows by the same argument showing that it is closed under ε.

Now if T is the set of closed SCCS+ε terms we have a semantic map $[\![.]\!]_\Phi : T \to \mathscr{P}$, by finality of $\mathscr{P}$. We can easily verify the following:

Proposition 3.9. *The semantic map respects the operators, that is* $[\![a.P]\!]_\Phi = a.[\![P]\!]_\Phi$, $[\![P+Q]\!]_\Phi = [\![P]\!]_\Phi + [\![Q]\!]_\Phi$ *etc. In particular, if* $p = [\![P]\!]_\Phi$, *then* $\delta p = [\![\delta P]\!]_\Phi$.

The theorem below is a consequence of Theorem 3.2.

Theorem 3.10 (Full Abstractness). *The model* $\mathscr{P}$ *is fully abstract for fair bisimilarity. In other words, for any closed process terms* P, Q *of SCCS* $+ \varepsilon$ *we have* $P \approx Q$ *iff* $[\![P]\!]_\Phi = [\![Q]\!]_\Phi$.

The model is in fact also fully abstract for fortification. The proof is given at the end of the next subsection as it makes use of the approximation of the relation $\prec$ introduced there.

3.2.1. Approximations

For a number of proofs we need to have approximations $\approx^\alpha$, $\alpha \in Ord$, of the relation $\approx$. We also define approximations $\prec^\alpha$ of the fortification relation, used in the proof that the model is fully abstract for fortification. Given a transition system T and $s, t \in T$, to prove that $s \prec t$ (or $s \approx t$) we must exhibit a fortification (a fair bisimulation) and this is not always convenient. The approximations will provide us with a suitable proof technique just like the approximations $\simeq^\alpha$ of bisimilarity given in [20] serve the same purpose.

As in the case of bisimulation we approximate $\prec$ from above by a decreasing sequence of relations $\prec^\alpha$, where $\alpha \in Ord$, defined as follows:

$$\prec^0 = S \times S,$$

$$\prec^\lambda = \bigcap_{\alpha\in\lambda} \prec^\alpha,$$

$s \prec^{\alpha+1} t$ iff $Vt \subseteq Vs$ and for all $a \in A$

$$s \xrightarrow{a} s' \Rightarrow \exists t'\ t \xrightarrow{a} t' \text{ and } s' \prec^\alpha t'$$

$$t \xrightarrow{a} t' \Rightarrow \exists s'\ s \xrightarrow{a} s' \text{ and } s' \prec^\alpha t'.$$

Lemma 3.11. *For any (not necessarily set-based) transition system, if* $\alpha \in \beta$ *then* $\prec^\beta \subseteq \prec^\alpha$.

Proof. The proof is by induction on β. □

We let $\prec' = \bigcap_{\alpha\in Ord} \prec^\alpha$. Recall that we say that the TS is *set-based* iff for all $a \in A$, the class $\{t' \mid t \xrightarrow{a} t'\}$ is a set, for each $t \in S$.

Proposition 3.12. *If the transition system is set-based, then*

$$\prec = \prec' \left(= \bigcap_{\alpha\in Ord} \prec^\alpha \right).$$

Proof. First, let α be the least ordinal (if it exists) for which we can find a pair s, t such that $s \prec t$ but $s \not\prec^\alpha t$. Then argue that α cannot be 0 or a limit. Remains to show that α is not a successor either. If $\alpha = \delta + 1$ use definitions to derive a contradiction. This shows that $\prec \subseteq \bigcap_{\alpha\in Ord} \prec^\alpha$.

To show the converse it is enough to verify that the relation $\prec' = \bigcap_{\alpha\in Ord} \prec^\alpha$ is a fortification since then it must be contained in the largest fortification $\prec$. Suppose that $s \prec' t$ and let $s \xrightarrow{a} s'$. Then for each ordinal α we have $s \prec^{\alpha+1} t$ and so we can find t'_α such that $t \xrightarrow{a} t'_\alpha$ and $s'' \prec^\alpha t'$. Then there are proper-class many ordinals α, β such that $t'_\alpha = t'_\beta$. In fact the class *On* of ordinals with this property is a nonempty initial segment of *Ord* by Lemma 3.11, hence *On* must be all of *Ord*. Hence there is a t' such that $t \xrightarrow{a} t'$ and for all ordinals α, $s' \prec^\alpha t'$. Since in particular $s \prec^1 t$ we also have $Vt \subseteq Vs$ and so $\prec'$ defined as the intersection $\bigcap_{\alpha\in Ord} \prec^\alpha$ is a fortification. □

In the particular case where the system is *image-finite*, that is the set $\{t' \mid t \xrightarrow{a} t'\}$ is a finite set for each a and t there is no need for ascending to transfinite ordinals.

Proposition 3.13. *If the ETS is image finite, then* $\prec = \prec^\omega$ $(= \bigcap_{n\in\omega} \prec^n)$.

By completely analogous arguments we can prove that

Theorem 3.14. *For a set-based system* $\approx = \bigcap_{\alpha \in Ord} \approx_\alpha$. *If, in particular, the system is image-finite, then* $\approx = \bigcap_{n \in \omega} \approx_n$.

The definition of the approximations $\approx_\alpha$ is analogous to that $\prec^\alpha$. The only difference is that at successor stages we require that $Vs = Vt$ rather than $Vt \subseteq Vs$.

Example 3.15. The following example is taken from Milner [19, Section 6], where he shows that the two terms are fortification equivalent (we verify they are fairly bisimilar).

To show that $(P \| Q)|_{\{1,a\}} \approx R$, where

$$P \equiv \mu x.\varepsilon(ab.x), \qquad Q \equiv \mu x.\delta(\bar{b}.x), \qquad R \equiv \mu x.\varepsilon(a.x)$$

we proceed by induction on α. The cases $\alpha = 0$ or a limit ordinal are trivial. Assuming $(P \| Q)|_{\{1,a\}} \approx_\alpha R$ we need to show $(P \| Q)|_{\{1,a\}} \approx_{\alpha+1} R$. It is fairly easy to verify that the two terms admit exactly the same sequences. Now if $(P \| Q)|_{\{1,a\}} \xrightarrow{c} E$, then either $c = 1$ and the move follows from $P \xrightarrow{1} P$ and $Q \xrightarrow{1} Q$, or else $c = a$ and the move follows from $P \xrightarrow{ab} P$ and $Q \xrightarrow{\bar{b}} Q$. In either case $E \equiv (P \| Q)|_{\{1,a\}}$ and $R \xrightarrow{c} R$. By induction $(P \| Q)|_{\{1,a\}} \approx_\alpha R$. Hence by definition $(P \| Q)|_{\{1,a\}} \approx_{\alpha+1} R$. Thus we may conclude $(P \| Q)|_{\{1,a\}} \approx R$.

As an application of the approximations defined we now prove full abstractness of the model $\mathscr{P}$ with respect to fortification.

Theorem 3.16 (Full abstractness for fortification). *For any closed process terms* P, Q, $P \prec Q$ *iff* $[\![P]\!]_\Phi \prec [\![Q]\!]_\Phi$. *Hence also* $P \sim Q$ *iff* $[\![P]\!]_\Phi \sim [\![Q]\!]_\Phi$.

Proof. Note first that $\prec$ on the right-hand-side is fortification on $\mathscr{P}$, defined in the natural way since $\mathscr{P}$ itself is an extended transition system.

Assuming $P \prec Q$ we verify that $[\![P]\!]_\Phi \prec^\alpha [\![Q]\!]_\Phi$ for all ordinals α. The cases $\alpha = 0$ or a limit ordinal are trivial. For the successor case first notice that since $[\![.]\!]_\Phi$ is a coalgebra map and $P \prec Q$ we must have

$$V([\![Q]\!]_\Phi) = V(Q) \subseteq V(P) = V([\![P]\!]_\Phi).$$

Next suppose $[\![P]\!]_\Phi \xrightarrow{a} p'$. Then $p' = [\![P']\!]_\Phi$ for some process term P' such that $P \xrightarrow{a} P'$. From $P \prec Q$ let Q' be such that $Q \xrightarrow{a} Q'$ and $P' \prec Q'$. By induction $[\![P']\!]_\Phi \prec^\alpha [\![Q']\!]_\Phi$. Hence we may conclude that $[\![P]\!]_\Phi \prec^{\alpha+1} [\![Q]\!]_\Phi$. Thus the semantic map is monotone with respect to fortification. The converse is similar. □

3.2.2. Recursion

The semantic map $[\![.]\!]_\Phi$ picks out some object $p \in \mathscr{P}$ as the interpretation of $\mu_i \bar{x} \bar{P}$. We verify below the $[\![\mu_i \bar{x} \bar{P}]\!]_\Phi$ is indeed a fixpoint. We also point out that, unlike the case of SCCS and the semantic structure J, a unique fixpoint theorem now fails

(Theorem 3.18 and Example 3.19). However $[\![\mu_i \tilde{x}\tilde{P}]\!]_\Phi$ is least in the fortification preorder among the *definable* fixpoints (Definition 3.21 and Proposition 3.23). As it should be expected, any two fixpoints are shown to be indistinguishable on grounds of finitary behaviour alone (Theorem 3.20).

In order to state our main theorem we need to be able to regard open terms as functions on our model. This can be done in a straightforward way in most cases except for terms of the form $\mu y.P(x,y)$ where $x \neq y$ occurs free.

Lemma 3.17. *For each assignment $e : Var \to \mathscr{P}$ there is an extension $[\![-]\!]^e_\Phi$ of the semantic map $[\![-]\!]_\Phi$ from open terms to elements of $\mathscr{P}$ such that*

1. $[\![P]\!]^e_\Phi = [\![P]\!]_\Phi$ *for a closed term P, and* $[\![x]\!]^e_\Phi = ex$,
2. $[\![a.Q]\!]^e_\Phi = a.[\![Q]\!]^e_\Phi$,
3. $[\![P+Q]\!]^e_\Phi = [\![P]\!]^e_\Phi + [\![Q]\!]^e_\Phi$,
4. $[\![P|_L]\!]^e_\Phi = ([\![P]\!]^e_\Phi)|_L$,
5. $[\![P\|Q]\!]^e_\Phi = [\![P]\!]^e_\Phi \| [\![Q]\!]^e_\Phi$,
6. $[\![\varepsilon P]\!]^e_\Phi = \varepsilon([\![P]\!]^e_\Phi)$,
7. $[\![\mu y.P]\!]^e_\Phi = [\![P\{\mu y.P/y\}]\!]^e_\Phi$.

Proof. Enrich the language of SCCS $+\varepsilon$ by adding a name $\hat{p}$ for each element $p \in \mathscr{P}$ and close under the recursive clauses for term formation. The extended transition system that results is specified by $V(\hat{p}) = V(p)$ and $\hat{p} \xrightarrow{a} \hat{q}$ provided $p \xrightarrow{a} q$ in $\mathscr{P}$. By finality of $\mathscr{P}$ let $[\![-]\!]'_\Phi$ be the unique morphism into $\mathscr{P}$. Given an open term P, $Fv(P) = \{x_1,\ldots,x_n\}$, define $[\![P]\!]^e_\Phi := [\![P\{\widehat{ex_i}/x_i,\ i=1,\ldots,n\}]\!]'_\Phi$. Properties 1–7 follow immediately. □

An open term P with the free variable x can be thus regarded as inducing a function on $\mathscr{P}$, defined by $[\![P]\!]_\Phi(r) := [\![P]\!]^{e[x:=r]}_\Phi$. Similarly for terms with more than one free variables.

Theorem 3.18. *Let $Fv(P) = \{x\}$, x guarded in P, and let $p = [\![\mu xP]\!]_\Phi$. Then $[\![P]\!]_\Phi(p) = p$, that is μxP is interpreted as a fixpoint. Furthermore, let $q,r \in \mathscr{P}$ and assume $[\![P]\!]_\Phi(q) = q$ and $[\![P]\!]_\Phi(r) = r$. If in addition $Vq = Vr$ (respectively $Vq \supseteq Vr$) then $q = r$ (respectively, $q \prec r$).*

The proof will be given at the end of this section. Interestingly, comparison of admitted sequences only at the top level (that is, for q and r and not recursively for successors of them) is sufficient. However, the assumption that $Vq = Vr$ cannot be dropped.

Example 3.19. If Q is any process term, then both $[\![\delta Q]\!]_\Phi$ and $[\![\varepsilon Q]\!]_\Phi$ are fixpoints of $[\![1.x + Q]\!]_\Phi$. The first is a fixpoint by Theorem 3.18 since $\delta Q \equiv \mu x(1.x + Q)$. That $[\![\varepsilon Q]\!]_\Phi$ is also a fixpoint is shown in Section 3.3, where we verify that for any $p \in \mathscr{P}$ the equation $\varepsilon p = 1.\varepsilon p + p$ holds. Hence a unique fixpoint theorem does not hold in $\mathscr{P}$.

For a second counterexample to unique fixpoints in $\mathscr{P}$ consider the term $P \equiv a.x$. By the Solution Lemma (see Appendix) each of the equations

$$x = (\{(a,x)\}, \emptyset), \tag{1}$$

$$x = (\{(a,x)\}, \{a^\omega\}) \tag{2}$$

is guaranteed to have a (unique) solution. Call them p and q, respectively. It is obvious that both $p, q \in \mathscr{P} = Pow(A \times \mathscr{P}) \times Pow(A^\omega)$. Futhermore, since $a.\emptyset = \emptyset$ and $a.\{a^\omega\} = \{a^\omega\}$ it is clear that $a.p = p$ and $a.q = q$. In fact p is the interpretation of the term $\mu x(a.x)\|\varepsilon 0$, which prevents a^ω. Applying Theorem 3.18 we can immediately conclude that $\mu x(a.x) \prec \mu x(a.x)\|\varepsilon 0$.

Any two fixpoints are however indistinguishable on grounds of finitary only behaviour, that is to say they must be bisimilar. Uniqueness up to bisimilarity can be established using the unique fixpoint Theorem 2.4 for the sematics J of SCCS.

Theorem 3.20. *Let $Fv(P) = \{x\}$, x guarded in P, and $! : \mathscr{P} \to J$ the unique Θ-coalgebra morphism. Assume $p, q \in \mathscr{P}$ are fixpoints of $[\![P]\!]_\Phi$. Then $!p = !q$.*

The proof is given at the end of this section.

Definition 3.21. An object $p \in \mathscr{P}$ is *definable* in the language $\mathscr{L}$ of $SCCS + \varepsilon$ iff there is a process term P such that $p = [\![P]\!]_\Phi$.

Of course not every $p \in \mathscr{P}$ is $\mathscr{L}$-definable.

Example 3.22. If $p = (\emptyset, A^\omega)$, then clearly p is not definable in $\mathscr{L}$, since the set of admissible sequences of a term with no transitions must be empty. The unique solution p to the equation $x = (\{(a,x)\}, \emptyset)$ is definable in $\mathscr{L}$, as we have pointed out in Example 3.19, since $p = [\![\mu x(a,x)\|\varepsilon 0]\!]_\Phi$.

Restricting to the $\mathscr{L}$-definable processes in $\mathscr{P}$ a least fixpoint theorem follows from Theorem 3.16 and from Milner's least fixpoint theorem in [19]: If $P\{Q/x\} \prec Q$, then $\mu xP \prec Q$.

Proposition 3.23. *Let $Fv(P) = \{x\}$, x guarded in P. If q is a definable fixpoint of $[\![P]\!]_\Phi$, then $[\![\mu xP]\!]_\Phi \prec q$.*

Proof. Recall that $\mathscr{P}$ is strongly extensional ($p \approx q$ iff $p = q$) and that $\approx$ is itself a fortification relation. The proof then follows from the fact that for all process terms P, Q, $P \prec Q$ implies $[\![P]\!]_\Phi \prec [\![Q]\!]_\Phi$, Theorem 3.16, and the least fixpoint theorem of [19]. □

It did not seem possible to drop the restriction that the fixpoint q is definable and still get a least fixpoint theorem with respect to the fortification pre-order on $\mathscr{P}$. The rest of this subsection is taken up with the proofs of Theorems 3.18 and 3.20.

Proof of Theorem 3.18. That $[\![\mu xP]\!]_\Phi$ is a fixpoint of the functional $[\![P]\!]_\Phi$ follows from the observation that if q is a process definable from Q, then $[\![P\{Q/x\}]\!]_\Phi = [\![P]\!]_\Phi(q)$ which can be shown by structural induction on P. For the rest of the proof we will need two lemmas which we state and prove below.

Lemma 3.24. *For all ordinals α, all $p,q,r \in \mathscr{P}$ and sets $L \subseteq A$, $1 \in L$, if $p \approx_\alpha q$, then $p|_L \approx_\alpha q|_L$ and $p\|r \approx_\alpha q\|r$. Similarly if we replace $\approx_\alpha$ with $\prec^\alpha$.*

Proof. We give the proof for $\approx_\alpha$ as the variant with $\prec^\alpha$ is very similar.

The proof is by induction on α and it is immediate for the cases $\alpha = 0$ or a limit ordinal. Assume now $p \approx_{\beta+1} q$. Then $Vp = Vq$, by definition of the approximation relations $\approx_\delta$. So we have $V(p|_L) := L^\omega \cap Vp = L^\omega \cap Vq := V(q|_L)$.

Next suppose $p|_L \xrightarrow{a} p_1$. Then $a \in L$, $p \xrightarrow{a} p'$ for some p' such that $p_1 = p'|_L$. By $p \approx_{\beta+1} q$, $q \xrightarrow{a} q'$ for some $q' \approx_\beta p'$. Letting $q_1 = q'|_L$ we get $q|_L \xrightarrow{a} q_1$ and, by induction, $p'|_L \approx_\beta q'|_L$, that is $p_1 \approx_\beta q_1$. By a symmetric argument if $q|_L \xrightarrow{a} q_1$ we can find p_1 such that $p|_L \xrightarrow{a} p_1$ and $p_1 \approx_\beta q_1$. Hence $p|_L \approx_{\beta+1} q|_L$.

For the parallel operator and since $Vp = Vq$ it follows $V(p\|r) = V(q\|r)$. If $p\|r \xrightarrow{c} s_1$, then there exist a,b,p',r' such that $c = ab$, $p_1 = p'\|r'$ and $p \xrightarrow{a} p', r \xrightarrow{a} r'$. By hypothesis $p \approx_{\beta+1} q$, hence $q \xrightarrow{a} q'$ for some q' such that $p' \approx_\beta q'$. By induction $p'\|r' \approx_\beta q'\|r'$. Similarly if we start by assuming $q\|r \xrightarrow{c} t_1$. Hence $p\|r \approx_{\beta+1} q\|r$. □

If C is any one hole context and $p,q \in \mathscr{P}$ are such that $Vp = Vq$ then it is apparently true that $V([\![C]\!]_\Phi(p)) = V([\![C]\!]_\Phi(q))$. Technically the proof is by induction on *prevention ordinals*, defined in [19]. We cite below the main fact about these ordinals.

Lemma 3.25 (Milner [19]). *There exists a partial assignment ord of ordinal numbers to pairs (P,u), where P is a process term of the finite delay calculus and $u \in A^\omega$ such that*

1. *ord (P,u) is defined iff P prevents u,*
2. *ord $(P,u) <$ ord $(a.P,au)$,*
3. *ord (P,u), ord $(Q,u) <$ ord $(P+Q,u)$,*
4. *Either ord $(P,u) <$ ord $(P\|Q,u\cdot v)$ or ord $(Q,v) <$ ord $(P\|Q,u\cdot v)$,*
5. *ord $(P,u) <$ ord $(P|_L,u)$, assuming $u \in L^\omega$,*
6. *ord $(P,u) <$ ord $(\varepsilon P, 1^n u)$, for all $n \geqslant 0$,*
7. *ord $(Q\{P/x\},u) <$ ord $(\mu x.Q,u)$, where $P \equiv \mu x.Q$, x guarded in Q.*

Lemma 3.26. *Let C be a one-hole context and $p,q \in \mathscr{P}$ such that $Vp = Vq$. Then $V([\![C]\!]_\Phi(p)) = V([\![C]\!]_\Phi(q))$.*

Proof. A detailed proof is rather tedious and so we only give a sketch. The proof is by induction on prevention ordinals. To be more precise, the argument proceeds in the extended language containing a constant $\hat{p}$ for each $p \in \mathscr{P}$ (and closed under the

recursive clauses for term formation). The definition of prevention ordinals is slightly modified by letting $ord(\hat{p},u)=0$ if $u \not\in Vp$ and undefined otherwise. The rest of the argument is straightforward, proceeding by examining the structure of C. The case $C \equiv x$, a single variable, is taken care of by the hypothesis that $Vp = Vq$. □

For the proof of Lemma 3.29 below we will need a subinduction on the *guard-depth* of expressions.

Definition 3.27. The *guard depth* $gd(E)$ of an expression E is defined by structural induction as follows:

1. $gd(x) = gd(a.E) = 0$,
2. $gd(\sum_{i\in I} E_i) = \sup\{gd(E_i)+1 \mid i\in I\}$,
3. $gd(E\|F) = \max\{gd(E)+1, gd(F)+1\}$,
4. $gd(E|_L) = gd(\varepsilon E) = gd(E)+1$,
5. $gd(\mu_i \bar{x}\bar{E}) = gd(E_i)+1$.

This ordinal is defined in Milner [19] from where we also quote below the main fact about it.

Lemma 3.28. *If the $\bar{x}$ are guarded in E, then $gd(E\{\bar{F}/\bar{x}\}) = gd(E)$.*

Lemma 3.29. *For all ordinals α, all $p,q\in\mathscr{P}$ and any one-hole context C, if $p \approx_\alpha q$, then $[\![C]\!]_\Phi(p) \approx_\alpha [\![C]\!]_\Phi(q)$. Similarly if we replace $\approx_\alpha$ with $\prec^\alpha$.*

Proof. By a one-hole context we simply mean an open term C with just one free variable, for example $a.x + b.0, \mu y.(a.x + b.y)$. We only give the proof for $\approx_\alpha$.

The proof is by induction on α. The ordinal induction hypothesis is that for every $\beta < \alpha$ and every context Q in one hole if $p \approx_\beta q$, then $[\![Q]\!]_\Phi(p) \approx_\beta [\![Q]\!]_\Phi(q)$. If α is either 0 or a limit ordinal then the claim is trivially true. Suppose now $\alpha = \beta + 1$, let C be any one-hole context and assume $p \approx_{\beta+1} q$.

We will show that $p \approx_{\beta+1} q$ implies $[\![C]\!]_\Phi(p) \approx_{\beta+1} [\![C]\!]_\Phi(q)$. The hypothesis implies $Vp = Vq$ and then also $V([\![C]\!]_\Phi(p)) = V([\![C]\!]_\Phi(q))$, by Lemma 3.26. To show that $[\![C]\!]_\Phi(p) \approx_{\beta+1} [\![C]\!]_\Phi(q)$ we proceed by induction on the guard depth of C and verify that the bisimulation clauses in the definition of $\approx_{\beta+1}$ hold. We examine the cases for C.

$C \equiv x$: Immediate.

$C \equiv a.C'$: The guard depth of C is 0 in this case. The only possible moves are $[\![C]\!]_\Phi(p) \xrightarrow{a} [\![C']\!]_\Phi(p)$ and, similarly, $[\![C]\!]_\Phi(q) \xrightarrow{a} [\![C']\!]_\Phi(q)$. Since $p \approx_{\beta+1} q$ implies $p \approx_\beta q$, the ordinal induction hypothesis implies that $[\![C']\!]_\Phi(p) \approx_\beta [\![C']\!]_\Phi(q)$. Given that the admissibility sets coincide we may conclude $[\![C]\!]_\Phi(p) \approx_{\beta+1} [\![C]\!]_\Phi(q)$.

$C \equiv C' + C''$: Similar.

$C \equiv C'|_L$: Since $gd(C')$ is strictly below $gd(C)$ we have $[\![C']\!]_\Phi(p) \approx_{\beta+1} [\![C']\!]_\Phi(q)$. Suppose now $[\![C]\!]_\Phi(p) \xrightarrow{a} p'$. Then $a \in L$ and $[\![C']\!]_\Phi(p) \xrightarrow{a} p_1$ for some p_1 such

that $p' = p_1|_L$. Then there is some q_1 such that $[\![C']\!]_\Phi(q) \xrightarrow{a} q_1$ with $p_1 \approx_\beta q_1$. By Lemma 3.24 we get $p_1|_L \approx_\beta q_1|_L$, i.e. $p' \approx_\beta q'$. We may then conclude that $[\![C]\!]_\Phi(p) \approx_{\beta+1} [\![C]\!]_\Phi(q)$.

$C \equiv C'\|C''$: The argument is similar and uses again Lemma 3.24.

$C \equiv \varepsilon C'$: By Proposition 3.9 we have $[\![\varepsilon C']\!]_\Phi(p) = \varepsilon([\![C']\!]_\Phi(p))$ and similarly for q. If the move is a wait move then use the fact that by the ordinal induction hypothesis we have $[\![\varepsilon C']\!]_\Phi(p) \approx_\beta [\![\varepsilon C']\!]_\Phi(q)$. Otherwise the move follows from $[\![C']\!]_\Phi(p) \xrightarrow{a} p'$. Since $gd(C')$ is strictly below $gd(C)$ we may use induction.

$C \equiv \mu y.K$: The claim $[\![\mu y.K]\!]_\Phi(p) \approx_{\beta+1} [\![\mu y.K]\!]_\Phi(q)$ is equivalent to the claim $[\![K\{\mu y.K/y\}]\!]_\Phi(p) \approx_{\beta+1} [\![K\{\mu y.K/y\}]\!]_\Phi(q)$. Since the guard depth of $K\{\mu y.K/y\}$ is strictly below the guard depth of $\mu y.K$ we may appeal to induction. □

We can now turn to the proof of Theorem 3.18. We assume that $Fv(P) = \{x\}, x$ guarded in P, and that p, q are fixpoints of $[\![P]\!]_\Phi$ such that $Vp = Vq$ (the case $Vp \subseteq Vq$ is very similar). We want to show that $p = q$, in other words that for all α, $p \approx_\alpha q$. We prove the following claim: For any ordinal α and any guarded one-hole context P' the hypotheses above imply that $[\![P']\!]_\Phi(p) \approx_\alpha [\![P']\!]_\Phi(q)$. Instantiating this to the case of the context P we obtain $[\![P]\!]_\Phi(p) \approx [\![P]\!]_\Phi(q)$, hence $p = [\![P]\!]_\Phi(p) = [\![P]\!]_\Phi(q) = q$.

The proof is by induction on α where we assume that for all $\beta < \alpha$ and any guarded one-hole context K we have $[\![K]\!]_\Phi(p) \approx_\beta [\![K]\!]_\Phi(q)$. Now if α is either 0 or a limit ordinal the claim is trivial and so we may assume $\alpha = \beta+1$. If P' is a guarded one-hole context we need to prove that $[\![P']\!]_\Phi(p) \approx_{\beta+1} [\![P']\!]_\Phi(q)$. Given Lemma 3.26 we only need to worry about the bisimulation clauses. The proof is by a subinduction on the guard depth of P' where we assume as subinduction hypothesis that for any guarded one-hole contex C with $gd(C) < gd(P')$ the claim $[\![C]\!]_\Phi(p) \approx_{\beta+1} [\![C]\!]_\Phi(q)$ holds. We examine the cases for P' (which cannot be a variable, by the guardedness assumption).

$P' \equiv a.P_1$: Then $[\![P']\!]_\Phi(p) = a.[\![P_1]\!]_\Phi(p)$ (similarly for q) and the only possible moves are $[\![P']\!]_\Phi(p) \xrightarrow{a} [\![P_1]\!]_\Phi(p)$ and $[\![P']\!]_\Phi(q) \xrightarrow{a} [\![P_1]\!]_\Phi(q)$. Note that the base induction hypothesis (on ordinals) implies that $p = [\![P]\!]_\Phi(p) \approx_\beta [\![P]\!]_\Phi(q) = q$. But $p \approx_\beta q$ implies, by Lemma 3.29, $[\![P_1]\!]_\Phi(p) \approx_\beta [\![P_1]\!]_\Phi(q)$. Note that x may not be guarded in P_1, which is why we needed to prove Lemma 3.29. It then follows that $[\![P']\!]_\Phi(p) \approx_{\beta+1} [\![P']\!]_\Phi(q)$.

$P' \equiv P_1 + P_2$: Since the guard depths of P_1, P_2 are strictly below that of P' we have $[\![P_i]\!]_\Phi(p) \approx_{\beta+1} [\![P_i]\!]_\Phi(q)$. This implies $[\![P']\!]_\Phi(p) \approx_{\beta+1} [\![P']\!]_\Phi(q)$.

$P' \equiv P_1|_L$: Similar.

$P' \equiv P_1\|P_2$: Similar.

$P' \equiv \varepsilon P_1$: Suppose $[\![\varepsilon P_1]\!]_\Phi(p) \xrightarrow{a} p'$. Since by Proposition 3.9 $[\![\varepsilon P_1]\!]_\Phi(p) = \varepsilon([\![P_1]\!]_\Phi(p))$ the a-move to p' may be either a wait move $a = 1$ or one of $[\![P_1]\!]_\Phi(p)$. In the first case use the fact that $[\![\varepsilon P_1]\!]_\Phi(p) \approx_\beta [\![\varepsilon P_1]\!]_\Phi(q)$. In the latter case and since the guard depth of P_1 is strictly below that of εP_1 we have, by induction, $[\![P_1]\!]_\Phi(p) \approx_{\beta+1} [\![P_1]\!]_\Phi(q)$. We may then conclude that $[\![\varepsilon P_1]\!]_\Phi(p) \approx_{\beta+1} [\![P_1]\!]_\Phi(q)$.

$P' \equiv \mu y.Q$: Immediate, by considering $Q\{P'/y\}$ which has strictly smaller guard depth.

We have then shown that $[\![P']\!]_\Phi(p) \approx [\![P']\!]_\Phi(q)$ for any guarded one-hole context P'. In particular, $[\![P]\!]_\Phi(p) \approx [\![P]\!]_\Phi(q)$ and since p and q are fixpoints it follows that $p \approx q$. By strong extensionality of $\mathscr{P}$ with respect to fair bisimilarity, Theorem 3.10, the conclusion $p = q$ follows and this completes the proof of Theorem 3.18 □

Proof of Theorem 3.20. The proof of this theorem follows from the following proposition. Recall that Θ is the class functor $Pow(A \times -)$ and that J is a fixpoint for Θ and a final Θ-coalgebra.

Proposition 3.30. *Let* $! : \mathscr{P} \to J$ *be the unique* Θ*-coalgebra morphism. Then* $!$ *is a* Σ*-homomorphism, where* Σ *is the SCCS* $+ \varepsilon$ *signature. In other words* $!(a.p) = a.!p$, $!(p+q) = !p + !q$, $!(p|_L) = (!p)|_L$, $!(p\|q) = !p\|!q$ *and* $!(\varepsilon p) = \varepsilon(!p)$.

Proof. For prefixing, summation and restriction the claim is immediate from definitions given also that for any abstract process $p \in \mathscr{P}$ we have $!p = \{(a, !q) \mid p \xrightarrow{a} q\}$.

For delay, the operator ε on J is defined as in [1] by $\varepsilon j = j \cup \{(1, j)\}$, for all $j \in J$. We show $!(\varepsilon p) \simeq_\alpha \varepsilon(!p)$ for all ordinals α. By definition of the approximations for bisimilarity (see [20]) the cases $\alpha = 0$ or a limit are trivial. For the successor case $\alpha = \beta + 1$ suppose $!(\varepsilon p) \xrightarrow{a} k$. Then $k = !q$ for some $q \in \mathscr{P}$ such that $\varepsilon p \xrightarrow{a} q$. If q is εp and $a = 1$, then we have $\varepsilon(!p) \xrightarrow{1} \varepsilon(!p)$ and by induction $!(\varepsilon p) \simeq_\beta \varepsilon(!p)$. Otherwise, $p \xrightarrow{a} q$ and thereby $!p \xrightarrow{a} !q$. Since $\varepsilon(!p) = !p \cup \{(1, !p)\}$ it follows that $\varepsilon(!p) \xrightarrow{a} !q$. Hence $!(\varepsilon p) \simeq_{\beta+1} \varepsilon(!p)$. By induction $!(\varepsilon p) \simeq (!p)$ and thereby the two processes are equal (by strong extensionally of J with respect to bisimilarity).

For the case of the parallel operator $\|$ we define the map $\hat{(.)} : \mathscr{P} \to J$ by

$$\hat{r} = \begin{cases} !p\|!q & \text{if } r = p\|q \text{ for some } p, q, \\ !r & \text{otherwise.} \end{cases}$$

We first need to verify that this map is well-defined. We claim that for all p_1, p_2, q_1, q_2 if $p_1\|q_1 = p_2\|q_2$, then $!p_1\|!q_1 = !p_2\|!q_2$. If this fails, let α be the least ordinal for which we can find p_i's and q_i's such that $p_1\|q_1 = p_2\|q_2$ but $!p_1\|!q_1 \not\simeq_{\alpha+1} !p_2\|!q_2$. We will derive a contradiction.

Suppose first that $!p_1\|!q_1 \xrightarrow{c} k$, some $k \in J$. Then there exist a, b, p_{11}, q_{11} such that $c = ab$, $k = !p_{11}\|!q_{11}$ and $p_1 \xrightarrow{a} p_{11}, q_1 \xrightarrow{b} q_{11}$. If we set $r = p_{11}\|q_{11}$, then by $p_1\|q_1 = p_2\|q_2$ we must have $p_2\|q_2 \xrightarrow{c} r$. Hence there exist a', b', p_{22}, q_{22} such that $c = a'b'$, $r = p_{22}\|q_{22}$ and $p_2 \xrightarrow{a'} p_{22}$, $q_2 \xrightarrow{b'} q_{22}$. Given $p_{11}\|q_{11} = r = p_{22}\|q_{22}$ we get by induction $!p_{11}\|!q_{11} \simeq_\alpha !p_{22}\|!q_{22}$. Setting $k' = !p_{22}\|!q_{22}$ we have then obtained $!p_2\|!q_2 \xrightarrow{c} k'$ and $k \simeq_\alpha k'$.

Next suppose $!p_2\|!q_2 \xrightarrow{c} k$. By a symmetric argument we can find k' such that $!p_1\|!q_1 \xrightarrow{c} k'$ and $k \simeq_\alpha k'$. By definition of the approximations it follows that $!p_1\|!q_1 \simeq_{\alpha+1} !p_2\|!q_2$, contrary to hypothesis.

Therefore, the map $\hat{(.)} : \mathscr{P} \to J$ is well-defined. Next we verify that it is a Θ-coalgebra morphism, from which it follows that for any $p \in \mathscr{P}$, $\hat{p} = !p$, by uniqueness of !. If $\hat{p} \xrightarrow{c} k$ for some $k \in J$ and p is not of the form $q \| r$ then there is nothing to prove since $\hat{p} = !p$ and ! is a Θ-coalgebra morphism. Suppose now $p = q \| r$ and then we assume $!q \| !r \xrightarrow{c} k$. Then there exists a, b, i, j such that $k = i \| j$, $c = ab$ and $!q \xrightarrow{a} i$, $!r \xrightarrow{b} j$. Hence for some q', r' we have $q \xrightarrow{a} q'$, $r \xrightarrow{b} r'$ and $!q' = i, !r' = j$. Then $k = !q' \| !r'$, where $q \| r \xrightarrow{ab} q' \| r'$. This shows that the map $\hat{(.)} : \mathscr{P} \to J$ is a Θ-coalgebra morphism and thereby it must coincide with the final map !. Hence for the case of parallel, too, we have $!(p \| q) = !p \| !q$. □

Corollary 3.31. *Let $Fv(P) = \{x\}$. Then the square below commutes*

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{[\![P]\!]_\Phi} & \mathscr{P} \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle !} \\ J & \xrightarrow[[\![P]\!]_\Theta]{} & J \end{array}.$$

In other words, for any $p \in \mathscr{P}$ we must have $!([\![P]\!]_\Phi(p)) = [\![P]\!]_\Theta(!p)$.

Proof. The proof is by structural induction on P using the previous proposition. □

The proof of Theorem 3.20 now easily follows. If x is guarded in P, where $Fv(P) = \{x\}$ and $[\![P]\!]_\Phi(p) = p$, $[\![P]\!]_\Phi(q) = q$, then $!p = !([\![P]\!]_\Phi(p)) = [\![P]\!]_\Phi(!p)$ and similarly $!q = !([\![P]\!]_\Theta(q)) = [\![P]\!]_\Theta(!q)$. Since the functional $[\![P]\!]_\Theta$ on J must have a unique fixpoint, by Theorem 2.4, it follows that $!p = !q$. Hence any two fixpoints are indistinguishable on grounds of finitary behavior. □

3.3. On the algebraic theory of fair bisimilarity

In Section 1 we described the language $\mathscr{L}$ of the SCCS + ε process algebra. The basic (in)equational theory E for SCCS + ε is the extension of the first-order calculus of (in)equality generated by the axioms of Table 1. All (in)equations in Table 1 have been considered in [19] as the basic algebraic theory of fortification equivalence. Our soundness theorem implies that these (in)equations hold for fair bisimilarity, which is a refinement of both bisimilarity and fortification equivalence.

By an *$\mathscr{L}$-structure* we mean a pre-ordered set $\mathscr{M} = \langle M, =, <, \Sigma \rangle$ closed under all Σ-operations (Σ is the signature of SCCS + ε). An *interpretation* in an $\mathscr{L}$-structure $\mathscr{M}$ is a map $[\![.]\!]^{\mathscr{M}} : T \to \mathscr{M}$ from closed terms[2] to elements of $\mathscr{M}$ such that $[\![.]\!]$ is a

[2] Any interpretation $[\![.]\!]^{\mathscr{M}}$ can be extended to an interpretation $[\![.]\!]^{\mathscr{M},e}$ of open terms by structural induction and given an assignment $e : Var \to \mathscr{M}$.

Table 1
Equations for fair bisimilarity

Summation
1. $x + 0 = x$
2. $x + y = y + x$
3. $x + (y + z) = (x + y) + z$
4. $x + x = x$

Product
5. $x \| 0 = 0$
6. $x \| y = y \| x$
7. $x \| (y \| z) = (x \| y) \| z$
8. $(a.x) \| (b.y) = (ab).(x \| y)$
9. $x \| (y + z) = (x \| y) + (x \| z)$

Restriction
10. $(a.x)|_L = \begin{cases} a.(x|_L) & \text{if } a \in L \\ 0 & \text{otherwise} \end{cases}$
11. $(x + y)|_L = (x|_L) + (y|_L)$
12. $(x|_L)|M) = x|_{L \cap M}$

Delay
13. $\varepsilon\varepsilon x = \varepsilon x$
14. $\varepsilon x = x + 1.\varepsilon x = x + \varepsilon x$
15. $\varepsilon x \| \varepsilon y = \varepsilon(x \| \varepsilon y + \varepsilon x \| y)$
16. $(\varepsilon x)|_L = \varepsilon(x|_L)$
17. $\varepsilon\delta x = \delta x$
18. $\varepsilon x \| \delta y = \varepsilon(x \| \delta y + \varepsilon x \| y)$
19. $\delta x < \varepsilon x$

Σ-homomorphism (operators are respected). An *$\mathscr{L}$-model* is a pair $(\mathscr{M}, [\![.]\!]^{\mathscr{M}})$ where $[\![.]\!]^{\mathscr{M}}$ is an interpretation and $\mathscr{M}$ is a *Σ-algebra*, that is to say an $\mathscr{L}$-structure in which all (in)equations of Table 1 are valid.

We write $\vdash_E P = Q$ (and similarly $\vdash P < Q$, usually omitting the subscript E in both cases) if the equation $P = Q$ (or the inequation $P < Q$) is derivable from the axioms of the theory E by (in)equational reasoning. As usual, if P and Q are open terms then $\vdash P = Q$ means that the closure by universal quantification on the free variables of P and Q is provable from E. For example, $\vdash x + x = x$ means that $\vdash \forall x(x + x = x)$.

For closed terms P, Q and an $\mathscr{L}$-model $\mathscr{M}$ we say that $\mathscr{M} \models P = Q$ iff $[\![P]\!]^{\mathscr{M}} = [\![Q]\!]^{\mathscr{M}}$. If P, Q are open terms and $e: Var \to \mathscr{M}$ is an assignment of elements of $\mathscr{M}$ to the variables of the language, then we say that $\mathscr{M} \models (P = Q)[e]$ iff $[\![P]\!]^{\mathscr{M},e} = [\![Q]\!]^{\mathscr{M},e}$ (the equation $P = Q$ is *satisfiable* in the model by the assignment e). Finally, we say that the equation $P = Q$ is *valid* (or *sound*) in $\mathscr{M}$, in notation $\mathscr{M} \models P = Q$, iff $\mathscr{M} \models (P = Q)[e]$ for any assignment e. Similarly for inequations.

In the previous sections we verified that $\mathscr{P}$ is closed under the SCCS + ε operations and that a Σ-homomorphism $[\![.]\!]_\Phi : T \to \mathscr{P}$ exists. We now verify that $\mathscr{P}$ is a Σ-algebra.

Theorem 3.32 (Soundness). *The theory E is sound in the structure $\mathscr{P}$. That is to say $\vdash_E P = Q$ implies $\mathscr{P} \models P = Q$ and similarly $\vdash P < Q$ implies $\mathscr{P} \models P < Q$.*

Proof. By strong extensionality of $\mathscr{P}$ it suffices to show that $[\![P]\!]^e_\Phi \approx [\![Q]\!]^e_\Phi$ rather than $[\![P]\!]^e_\Phi = [\![Q]\!]^e_\Phi$, which we do in some cases by using the approximation of the relation $\approx$ via the sequence $(\approx^\alpha)_{\alpha \in Ord}$. We first deal with soundness of the delay axioms.

If $\varepsilon\varepsilon p \neq \varepsilon p$ for some p, let α be the least ordinal for which $\varepsilon\varepsilon p \not\approx^\alpha \varepsilon p$. Then α cannot be 0 or a limit. Suppose there is an ordinal δ such that $\alpha = \delta + 1$. By minimality of α we have $\varepsilon\varepsilon p \approx^\delta \varepsilon p$. Note first that $V(\varepsilon\varepsilon p) = V(\varepsilon p)$ follows immediately from the definition of the operation ε.

Now suppose $\varepsilon\varepsilon p \xrightarrow{a} q$. If this follows from the fact that $\varepsilon p \xrightarrow{a} q$ nothing to prove. Otherwise the move is $\varepsilon\varepsilon p \xrightarrow{1} \varepsilon\varepsilon p$. But then $\varepsilon p \xrightarrow{1} \varepsilon p$ and $\varepsilon\varepsilon p \approx^\delta \varepsilon p$, by induction. Similarly if we start by assuming that $\varepsilon p \xrightarrow{a} q$. Hence by definition $\varepsilon\varepsilon p \approx^{\delta+1} \varepsilon p$, contrary to hypothesis. Thus $\varepsilon\varepsilon p \approx \varepsilon p$ and by strong extensionality of J it follows that $\varepsilon\varepsilon p = \varepsilon p$.

The identities $\varepsilon p = p + \varepsilon p = p + 1 : \varepsilon p$ follow directly from the definition of the operator ε on J.

For the law involving restriction, notice that $V(\varepsilon(p|_L)) = V((\varepsilon p)|_L)$ follows by definitions and the fact that the restriction set L must contain the silent move 1. Again, if identity fails let α be the least ordinal for which $(\varepsilon p)|_L \not\approx^\alpha \varepsilon(p|_L)$. Then α must be a successor $\alpha = \delta + 1$. If $(\varepsilon p)|_L \xrightarrow{a} q$ makes a move $a \neq 1$, then $a \in L$ and $\varepsilon p \xrightarrow{a} q$. But this move must follow from $p \xrightarrow{a} q$, from which (since $a \in L$) $p|_L \xrightarrow{a} q$ and thereby $\varepsilon(p|_L) \xrightarrow{a} q$ follows. Otherwise the move is a silent move following from $\varepsilon p \xrightarrow{1} \varepsilon p$. But also $\varepsilon(p|_L) \xrightarrow{1} \varepsilon(p|_L)$ and by induction $(\varepsilon p)|_L \approx^\delta \varepsilon(p|_L)$. Similarly if we start with the assumption that $(\varepsilon p|_L) \xrightarrow{a} q$. Hence by definition of the approximating relations it follows that $(\varepsilon p)|_L \approx^{\delta+1} \varepsilon(p|_L)$, contrary to hypothesis. Thus by induction $(\varepsilon p)|_L = \varepsilon(p|_L)$.

Next we verify the law involving the synchronous product. The identity of the environment assignments follows from definitions. We proceed again by induction on the approximation. So suppose $\alpha = \delta + 1$ is the least ordinal such that $\approx^\alpha$ distinguishes the two.

Suppose $\varepsilon p \| \varepsilon q \xrightarrow{ab} p' \| q'$, following from $\varepsilon p \xrightarrow{a} p'$ and $\varepsilon q \xrightarrow{a} q'$. We distinguish the cases according to whether the actions a, b are silent moves or not.

Suppose first that $a = 1 = b$, so that the move is $\varepsilon p \| \varepsilon q \xrightarrow{1} \varepsilon p \| \varepsilon q$. The process $\varepsilon(p \| \varepsilon q + \varepsilon p \| q)$ can also make a 1 move to itself and by induction $\varepsilon p \| \varepsilon q \approx^\delta \varepsilon(p \| \varepsilon q + \varepsilon p \| q)$. The other cases are similar, with $\varepsilon(p \| \varepsilon q + \varepsilon p \| q)$ always matching an ab move of $\varepsilon p \| \varepsilon q$.

Hence by induction it follows that $\varepsilon p \| \varepsilon q = \varepsilon(p \| \varepsilon q + \varepsilon p \| q)$.

Soundness of the identities $\varepsilon\delta p = \delta p$ and $\varepsilon p \| \delta q = \varepsilon(p \| \delta q + \varepsilon p \| q)$ follows by similar arguments.

Soundness of the axioms for choice is immediate from the corresponding properties of union. Soundness of the axioms for the synchronous product and for restriction follows from definitions alone.

Finally, soundness of the inequation $\delta x < \varepsilon x$ follows by definition of δ on $\mathscr{P}$, Remark 3.8. □

4. Conclusions

Considering the infinitary behaviour leads to a more intensional view of processes. Aczel suggests [2] that using *coloured* transition systems one can perhaps capture the particular aspect of intensionality of interest. A coloured system is a coalgebra for the functor $\Psi := Pow(A\times -)\times Col$, where Col is a set of *colours*. In [2] a particular colouring is chozen leading to a model for CSP. Colours, in our own case, have been taken to be sets of infinite sequences of actions. In intuitive, but not quite accurate terms, one can think of our model as being like Aczel's model for SCCS except for decorating every node with a set (of admissible sequences). This is not really an accurate picture, however, for the simple reason that the SCCS model is bound to identify (being fully abstract for bisimilarity) the delay operators δ and ε and no subsequent "decorating" will distinguish the two. The intensional distinctions we have sought to make are finer than those made by bisimilarity and they required carrying out a fresh construction.

The formal study of finite delay is still at a basic level. We point out below some of the questions left open.

1. We have shown validity of the basic (in)equational theory in our model (hence validity for fair bisimilarity). Completeness of the theory in $\mathscr{P}$ is the statement that if $p = q$ (or $p \prec q$) holds in $\mathscr{P}$ for definable p and q, $p = [\![P]\!]_\Phi$, $q = [\![Q]\!]_\Phi$, then the equation $P = Q$ (respectively, the inequality $P < Q$) is derivable in the theory E. Given full abstractness of $\mathscr{P}$ this is equivalent to the statement that $\approx$ is contained in the smallest Σ-congruence generated by the equational axioms of the theory (similarly for $\prec$). In yet different words, it is equivalent to the statement that $P \approx Q$ implies $\vdash P = Q$. We cannot hope to get a completeness theorem for the basic theory E, however, unless it is extended to include some form of an induction principle. This is one of the questions that was also left open in [19]. Hennessy [13] has initiated a study of the axiomatization of finite delay in cases simpler than the full SCCS + ε. Furthermore, his approach is based on a notion of *testing* rather than on fair bisimilarity or fortification equivalence.

2. Another question left open is that of a logical characterization of fair bisimilarity and fortification. A simple way to proceed is as follows. Let $\mathscr{L}$ be the language built on the atomic sentences Vp, where Vp is the set of admissible sequences for the process p, and closed under infinitary conjunction $\bigwedge_{i\in I}$, negation $\neg$ and modal operators $\langle a\rangle$ indexed by atomic actions $a\in A$. Define satisfaction in the usual way for compound sentences and let $p \models Vq$ iff $Vq \subseteq Vp$. Assume the usual notion of modal depth except for decreeing that $md(Vp) = 1$ for any atomic sentence of the form Vp, and stratify $\mathscr{L}$ by letting $\mathscr{L}_\alpha = \{\phi\in\mathscr{L} \mid md(\phi)\leqslant\alpha\}$. Let also $\mathscr{L}(p), \mathscr{L}_\alpha(p)$ have their obvious meaning and $\mathscr{L}^+$ be the fragment without negation. Essentially by the same argument as in [17] it can be shown that

Proposition 4.1. *For all ordinals* α, $p \approx_\alpha q$ *iff* $\mathscr{L}_\alpha(p) = \mathscr{L}_\alpha(q)$. *Similarly,* $p \prec^\alpha q$ *iff* $\mathscr{L}^+_\alpha(p) \subseteq \mathscr{L}^+_\alpha(q)$. *Consequently,* $p \approx q$ *iff* $\mathscr{L}(p) = \mathscr{L}(q)$ *and* $p \prec q$ *iff* $\mathscr{L}^+(p) \subseteq \mathscr{L}^+(q)$. *Hence also* $p \sim q$ *iff* $\mathscr{L}^+(p) = \mathscr{L}^+(q)$.

What would be more interesting, however, is to build the modal language on atomic sentences that are not the admissibility sets Vp but rather the sentences of a temporal language capturing facts as the property of a process p to admit a sequence u of actions. As pointed out in [19] the connection between the finite delay operator ε and the *eventually* operator of temporal logic needs to be investigated and clarified.

We hope to take up these issues in another report.

Appendix A. Set-theoretic assumptions

We let V be the class of all pure sets. We assume a proper class V' of *atoms* and work in the universe $V[V']$ of sets that may involve atoms in their build-up.[3] An axiomatization of set theory with atoms can be found in Barwise [5] (where atoms are called *urelements*). Assuming a global form of choice any two proper classes are equinumerous, hence we will typically write x_a for the atom labelled by the pure set a and assume that $a \neq b$ iff $x_a \neq x_b$ (which is to say that $x_- : V \simeq V'$ is a bijection). For a class B of pure sets (atoms) we let B' be the class $\{x_b \mid b \in B\}$ (respectively, $\{b \mid x_b \in B\}$) and extend the priming notation to functions between classes in the obvious way. We use the term "set" to refer to both *pure* sets and sets that may involve atoms in their build-up. For a class X of atoms, $V[X]$ will denote the class of sets with atoms from X possibly occurring in their build-up. These sets will be also referred to as X-sets.

From Aczel [1] we recall the following, where C is the (large) category of classes.

Definition A.1. A class functor $\Theta : C \to C$ is *set-continuous* iff

1. Θ is monotone, i.e. $X \subseteq Y$ implies $\Theta X \subseteq \Theta Y$ and
2. set-based, that is to say that for any class X and $a \in \Theta X$ there is a subset $x \subseteq X$ such that $a \in \Theta x$.

It is not hard to see that, as it is pointed out in [1], Θ is set-continuous iff for any class X

$$\Theta X = \bigcup \{\Theta x \mid x \in V \text{ and } x \subseteq X\}$$

which is what justifies the terminology "set-continuous".

[3] If postulating a proper class of atoms is an offending assumption to the reader let us reassure them that we could equivalently consider two disjoint copies of V, $\{0\} \times V$ and $\{1\} \times V$, call "atoms" the sets of the form $(0,x)$ where $x \in V$ and "pure sets" the pairs of the form $(1,x)$ with $x \in V$.

It is shown in [1] that if Θ is a set-continuous class-operator then the class

$$J_\Theta = \bigcup \{x \in V \mid x \subseteq \Theta x\}$$

is the largest fixed point of Θ (the proof uses dependent choice). The special final coalgebra theorem (which uses AFA) also proven in [1] asserts that, under some additional assumptions on the class functor Θ, (J_Θ, Id) is a final Θ-coalgebra. These additional assumptions (to be detailed in a minute) hold for the functor $\Theta = Pow(A \times -)$, from which it follows that the semantic map from a transition system to the final system can be taken to be the map

$$[\![x]\!] = \{(a, [\![y]\!]) \mid x \xrightarrow{a} y\}.$$

Use of AFA is essential here since if the transition system is presented as the operational semantics of a process language with recursion the sets above may well fail to be well-founded. Once again, consider the term $\mu x(a.x)$. Its interpretation yields the singleton set $[\![P]\!] = \{(a, [\![P]\!])\}$. The anti-foundation axiom implies that there is a unique set satisfying the equation $x = \{(a,x)\}$, which of course is a non-wellfounded set.

The assumptions hinted at above are described in the following definitions.

Definition A.2. A class functor $\Theta : C \to C$ is standard if it is set-continuous and preserves inclusion maps: if $i_{X,Y} : X \hookrightarrow Y$ is the inclusion map (for any $x \in X$, $i_{X,Y} x = x \in Y$), then $\Theta i_{X,Y} : \Theta X \to \Theta Y$ is also the inclusion map $i_{\Theta X, \Theta Y} : \Theta X \hookrightarrow \Theta Y$.

For the next definition we need to first recall (from [1]) the following:

Lemma A.3 (Substitution lemma, AFA). *Fix a class X of atoms. For each function $\pi : X \to V$, assigning a pure set πx to each atom $x \in X$, there exists a unique function $\hat{\pi} : V[X] \to V$ that assigns a pure set $\hat{\pi} a$ to each X-set a such that*

$$\hat{\pi} a = \{\hat{\pi} b \mid b \in a \cap V[X]\} \cup \{\pi x \mid x \in a \cap X\}. \tag{3}$$

We also recall

Lemma A.4 (Solution lemma, AFA). *Fix a class X of atoms. If u_x is an X-set for each $x \in X$, then the system of equations (soe)*

$$x = u_x \quad (x \in X,\ u_x \in V[X])$$

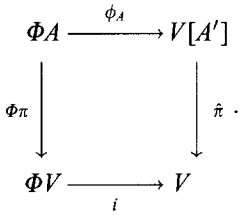
has a unique solution in V (the class of all pure sets): There is a map $\pi : X \to V$ such that $\pi x = \hat{\pi} u_x$.

The following definition describes the final condition needed for the statement of the special final coalgebra theorem.

Definition A.5. A standard class functor $\Phi: C \to C$ is uniform on maps iff for every class $A \subseteq V$ there is a map $\phi_A : \Phi A \to V[A']$ (assigning an A'-set to each pure set in ΦA) with the following universal property:

Given any assignment of pure sets $\pi' : A' \to V$ to the class A' of atoms and where $\hat{\pi}$ is the substitution map and $\pi : A \to V$ the obvious map, for any $u \in \Phi A$ we have $(\Phi\pi)u = \hat{\pi}(\phi_A u)$.

In a diagram, the equation above means that the square below commutes:



It is then shown in [1] that

Theorem A.6 (Special final coalgebra theorem, AFA). *If Φ is a standard class functor that is uniform on maps then (J_Φ, Id) is a final coalgebra for Φ, where J_Φ is the largest fixed point of Φ.*

We give below details of the promised proof of Proposition 3.6.

Proof of Proposition 3.6. Set-continuity and preservation of inclusion maps is rather straightforward, hence Φ_K is a standard functor. For the uniformity part, notice first that the functor Pow is (standard and) uniform on maps, where for a class B we define ϕ_B on $u \in Pow(B)$ by $\phi_B u = \{x_b \mid b \in u \subseteq B\}$. Also, the functor $Pow(A \times -)$ is uniform on maps where we now define θ_B on $u \subseteq A \times B$ by

$$\theta_B u = \{(a, x_b) \mid (a, b) \in u \subseteq A \times B\}$$

For the functor $Pow(A \times -) \times K$ we simply define a map $\phi_B = \theta_B \times K$. To be explicit, given $w \in (Pow(A \times B) \times K)$, $w = (u, k)$ with $u \in Pow(A \times B)$ and $k \in K = Pow(A^\omega)$. Then

$$\phi_B(u, k) = (\{(a, x_b) \mid (a, b) \in u \subseteq A \times B\}, k).$$

The rest follows by the observation that if Θ is (standard and) uniform on maps then so is $\Phi_K = \Theta \times K$, where K is a constant functor with value some fixed class K. If θ_A is the map satisfying the condition of Definition A.5, we let $\phi_A = \theta_A \times K$. Assume a standard representation of pairing: $(x, y) = \{\{x\}, \{x, y\}\}$. Let now $\pi' : A' \to V$ be any map. Then observe that since K is a pure class $\hat{\pi}$ is identity on members of K, that is

$\hat{\pi}k = k$ for all $k \in K$. By the following small computation

$$\begin{aligned}\hat{\pi}\phi_A(u,k) &= \hat{\pi} \circ (\theta_A \times K)(u,k)) = \hat{\pi}(\theta_A u, k) \\ &= \hat{\pi}(\{\{\theta_A u\}, \{\theta_A u, k\}\}) = \{\{\hat{\pi}\theta_A u\}, \{\hat{\pi}\theta_A u, k\}\} \\ &= \{\{(\Theta\pi)u\}, \{(\Theta\pi)u, k\}\} = ((\Theta\pi)u, k) \\ &= ((\Theta\pi) \times K)(u,k) = (\Theta \times K)\pi(u,k) \\ &= \Phi\pi(u,k)\end{aligned}$$

the proof that Φ_K is uniform on maps is complete. □

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