The weighted perfect domination problem and its variants

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Abstract

A perfect dominating set of a graph $G = (V, E)$ is a subset $D$ of $V$ such that every vertex not in $D$ is adjacent to exactly one vertex in $D$. The perfect domination problem is to find the minimum size of a perfect dominating set of a graph. Suppose moreover that every vertex $v \in V$ has a cost $c(v)$ and every edge $e \in E$ has a cost $c(e)$. The weighted perfect domination problem is to find a perfect dominating set $D$ such that its total cost $c(D) = \sum_{v \in D} c(v) + \sum_{(u, v) \in E} c(u, v)$ is minimum. We also consider the following three variants of perfect domination. A perfect dominating set $D$ is called independent (resp. connected, total) if the subgraph $\langle D \rangle$ induced by $D$ has no edge (resp. is connected, has no isolated vertex). This paper first proves that the three variants of perfect domination are NP-complete for bipartite graphs and chordal graphs, except for the connected perfect domination in chordal graphs. We then present linear-time algorithms for the weighted perfect domination problem and its three variants in block graphs.

Keywords: Perfect domination; Independent perfect domination; Connected perfect domination; Total perfect domination; Block; Cut-vertex; Block graph; Dynamic programming

1. Introduction

The concept of domination in graph theory arises naturally from the facility location problem in operations research. Consider a geographical area that is partitioned into regions. Facilities are going to be placed in some of the regions. The problem is to choose a minimum number of regions at which to place these service facilities, so that each region is served by a facility in it or at least one facility adjacent to it when there is no facility in this region. We can model this location problem by

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considering a graph \( G = (V, E) \) whose vertices represent the regions and edges represent pairs of regions that are adjacent. The problem is then to find a minimum sized dominating set of \( G \), i.e., a subset \( D \) of \( V \) such that every vertex not in \( D \) is adjacent to at least one vertex in \( D \).

Depending on different requirements in various location problems, domination has many variants, e.g., independent domination, connected domination, total domination, dominating cycle, \( k \)-domination, edge domination, \( k \)-neighbor domination. Many studies \([4,5,9,11-13,15]\) had been made in finding polynomial-time algorithms for the domination problem and its variants in special classes of graphs and proving NP-completeness for these problems for general or special graphs.

The perfect domination problem is to find a minimum sized perfect dominating set of a graph \( G = (V, E) \), which is a subset \( D \) of \( V \) such that every vertex not in \( D \) is adjacent to exactly one vertex in \( D \). This minimum size is called the perfect domination number of \( G \) and is denoted by \( \delta(G) \).

Suppose moreover that in a graph \( G = (V, E) \) every vertex \( v \in V \) has a cost \( c(v) \) and every edge \( e \in E \) has a cost \( c(e) \). The weighted perfect domination problem, which was first introduced by Yen and Lee \([13]\), is to find a perfect dominating set \( D \) such that its total cost

\[
c(D) = \sum \{ c(v) : v \in D \} + \sum \{ c(u, v) : u \not\in D, v \in D, \text{ and } (u, v) \in E \}
\]

is minimum. This minimum cost is denoted by \( \delta(G, c) \). In the case when \( c(v) = 1 \) for each vertex \( v \) and \( c(e) = 0 \) for each edge \( e \), \( c(D) = |D| \). Thus so the perfect domination problem is a special case of the weighted perfect domination problem. In the facility location problem, \( c(v) \) can be viewed as the cost of a facility in vertex \( v \) and a region \( u \) not in \( D \) is served with cost \( c(u, v) \) by a unique facility \( v \) in \( D \), where \( c(u, v) \) can be viewed as the communication cost from region \( u \) to the facility \( v \). \( c(D) \) is then the total cost for the system.

The concept of perfect domination was introduced by Weichsel \([16]\), who studied perfect domination in hypercubes. Yen and Lee \([17]\) proved that the perfect domination problem is NP-complete for bipartite graphs and chordal graphs. They also gave linear-time algorithms for the weighted perfect domination problem on trees \([17]\) and series-parallel graphs \([18]\).

In this paper, we also study the following three variants of perfect domination. A perfect dominating set \( D \) is called independent, connected, or total if the subgraph \( \langle D \rangle \) induced by \( D \) is independent, connected, or without isolated vertices, respectively. The corresponding independent perfect domination number, connected perfect domination number, and total perfect domination number are denoted by \( \delta(G) \), \( \delta(G) \), and \( \delta(G) \), respectively. We also have the weighted versions of these variant perfect domination problems whose optimal values are denoted by \( \delta(G, c) \), \( \delta(G, c) \), and \( \delta(G, c) \), respectively.

Note that independent perfect domination is called efficient domination by Bange et al. \([2]\). They proved that the efficient domination problem is NP-complete for general graphs and gave a linear-time algorithm for the problem on trees. From the
coding theory point of view, Biggs [3] studied perfect d-codes. A perfect d-code of a graph \( G = (V, E) \) is a vertex set \( C \) such that every vertex \( v \in V \) is within distance \( d \) from exactly one vertex in \( D \). In conjunction with the study for the interconnection networks of parallel computers, Livingston and Stout [14] studied perfect d-dominating sets which are precisely the perfect d-codes. The independent perfect domination in this paper is the same as their perfect 1-domination. Recently, Chang et al. [6] gave a polynomial algorithm for the weighted independent perfect domination problem in cocomparability graphs. Chang and Liu [8] gave polynomial algorithms for the perfect domination problem and its three variants in interval graphs and circular-arc graphs.

This paper first proves that the three variants of perfect domination are NP-complete for bipartite graphs and chordal graphs, except the connected perfect domination in chordal graphs. Note that Chang and Liu [7] recently gave a linear-time algorithm for the weighted connected perfect domination problem in chordal graphs. We then give linear-time algorithms for the weighted perfect domination problem and its three variants in block graphs.

In a graph \( G = (V, E) \), a vertex \( x \) is a cut-vertex if deleting \( x \) and all edges incident to its increases the number of connected components. A block (or 2-connected component) of \( G \) is a maximal connected-induced subgraph without any cut-vertex. A block graph is a graph whose block are complete graphs. Note that trees and forests are block graphs. Block graphs were first studied by Harary [10].

2. NP-completeness in bipartite graphs and chordal graphs

This section shows that the independent, connected, and total perfect domination problems are NP-complete for bipartite graphs and chordal graphs, except the case of connected perfect domination in chordal graphs. We reduce the exact cover problem, which is NP-complete, to them.

Exact cover problem. Given a family \( F = \{S_1, S_2, \ldots, S_n\} \) of sets where each \( S_i \) is a subset of a set \( X = \{x_1, x_2, \ldots, x_m\} \), does there exist a subfamily of pairwise disjoint sets whose union is equal to \( X \)?

Theorem 1. The independent perfect domination problem is NP-complete for bipartite graphs.

Proof. Given an instance of the exact cover problem, construct a bipartite graph \( G = (V, E) \) as follows (see Fig. 1). Suppose \( F = \{S_1, S_2, \ldots, S_n\} \) is a family of sets where each \( S_i \) is a subset of \( X = \{x_1, x_2, \ldots, x_m\} \). First, each element \( x_j \) is a vertex and each set \( S_i \) is a vertex and \( x_j \) is adjacent to \( S_i \) if and only if \( x_j \in S_i \). For each \( S_i \), add
Fig. 1. The bipartite graph $G$ for the independent perfect domination.

a new vertex $a_i$ adjacent to vertex $S_i$. More precisely,

$$V = \{S_1, S_2, \ldots, S_n\} \cup \{x_1, x_2, \ldots, x_m\} \cup \{a_1, a_2, \ldots, a_n\}$$

and $E$ consists of

1. $(x_j, S_i)$ for all $x_j \in S_i$, $1 \leq j \leq m$, $1 \leq i \leq n$,
2. $(S_i, a_i)$ for all $1 \leq i \leq n$.

Suppose that $D$ is an independent perfect dominating set of $G$. $D \cap \{S_i, a_i\} \neq \emptyset$ for each $i$. Therefore, $i\delta(G) \geq n$. For the case in which $i\delta(G) = n$, an optimal solution $D^*$ contains exactly one vertex in $\{S_i, a_i\}$ for each $i$.

We claim that the exact cover problem has a positive answer if and only if $i\delta(G) = n$. Hence the NP-completeness of the exact cover problem implies the theorem.

Suppose the exact cover problem has a positive answer, i.e., $F$ has a subfamily $F^*$ of pairwise disjoint sets whose union is equal to $X$ or equivalently each element of $X$ is in exactly one set of $F^*$. Let $D^* = \{S_i: S_i \in F^*\} \cup \{a_i: S_i \in F - F^*\}$. It is easy to check that $D^*$ is an independent perfect dominating set of $G$ of size $n$. Hence $i\delta(G) = n$.

On the other hand, suppose $i\delta(G) = n$, i.e., there is an independent perfect dominating set $D^*$ of size $n$. By the previous discussion, $D^*$ contains exactly one vertex in $\{S_i, a_i\}$ for each $i$. Let $F^* = \{S_i: S_i \in D^*\}$. By the definition of independent perfect domination, $F^*$ is a subfamily of pairwise disjoint sets whose union is equal to $X$. 

If we add edges into the graph $G$ in the proof of Theorem 1 to make $X$ a clique, the resulting graph is a chordal graph. Using the same argument as in the proof of Theorem 1, we have the following theorem.

**Theorem 2.** The independent perfect domination problem is NP-complete for chordal graphs.

Note that the proofs in Theorems 1 and 2 may be easily modified to become proofs for the NP-completeness of perfect domination in bipartite graphs and chordal graphs.
Theorem 3. The connected perfect domination problem is NP-complete for bipartite graphs.

Proof. Given an instance of the exact cover problem, construct a bipartite graph $G = (V, E)$ as follows (see Fig. 2). Suppose $F = \{S_1, S_2, \ldots, S_n\}$ is a family of sets where each $S_i$ is a subset of $X = \{x_1, x_2, \ldots, x_m\}$. First, each element $x_j$ is a vertex and each set $S_i$ is a vertex and $x_j$ is adjacent to $S_i$ if and only if $x_j \in S_i$. Add vertices $y, z_1, z_2, \ldots, z_n, w_1, w_2, u_1, u_2, u_3, u_4$ and edges $(y, S_i), (w_1, w_2), (w_1, u_1), (w_1, u_2), (w_2, u_3), (w_2, u_4)$ to the graph such that $y$ is adjacent to all $x_j$ and all $z_i$, $w_1$ is adjacent to all $z_i$, and $w_2$ is adjacent to all $S_j$. More precisely,

$$V = \{S_1, S_2, \ldots, S_n\} \cup \{x_1, x_2, \ldots, x_m\} \cup \{y, z_1, z_2, \ldots, z_n, w_1, w_2, u_1, u_2, u_3, u_4\}$$

and $E$ consists of

1. $(x_j, S_i)$ for all $x_j \in S_i$, $1 \leq j \leq m$, $1 \leq i \leq n$,
2. $(x_j, y)$ for all $1 \leq j \leq m$,
3. $(y, z_i)$ for all $1 \leq i \leq n$,
4. $(z_i, w_i)$ for all $1 \leq i \leq n$,
5. $(w_2, S_i)$ for all $1 \leq i \leq n$,
6. $(y, w_2), (w_1, w_2), (w_1, u_1), (w_1, u_2), (w_2, u_3), (w_2, u_4)$.

We claim that the exact cover problem has a positive answer if and only if $c\delta(G) \leq n + 2$.

Suppose the exact cover problem has a positive answer, i.e., $F$ has a subfamily $F^*$ of pairwise disjoint sets whose union is equal to $X$ or equivalently each element in $X$ is in exactly one set in $F^*$. It is easy to check that $\{w_1, w_2\} \cup F^*$ is a connected perfect dominating set of size at most $n + 2$. Hence $c\delta(G) \leq n + 2$.

On the other hand, suppose $c\delta(G) \leq n + 2$, i.e., there is a connected perfect dominating set $D$ of size at most $n + 2$. First of all, $D$ must contain $w_1$ and $w_2$ to dominate $u_1, u_2, u_3, \text{and } u_4$. If $y \in D$, then $D$ should contain all $z_i$s since all $z_i$s are dominated by $w_1$ and $y$, and hence $|D| \geq n + 3$, which is impossible. If $D$ contains any $x_j$, then
D should contain y since y is dominated by w2 and xj, which is also impossible. Therefore, each xj is adjacent to exactly one Si in D. These Si's which dominate all xj's will form a subfamily of pairwise disjoint sets whose union is equal to X. □

There is no easy modification of the bipartite graph in the proof of Theorem 3 into a chordal graph. In fact, recently Chang and Liu [7] gave a linear-time algorithm for the connected perfect domination in chordal graphs. Although the connected perfect dominating set in the proof of Theorem 3 is total, we shall give another proof for the NP-completeness of total perfect domination in bipartite graphs. This new proof can be easily modified to produce results on chordal graphs.

**Theorem 4.** The total perfect domination problem is NP-complete for bipartite graphs.

**Proof.** Given an instance of the exact cover problem, construct a bipartite graph G = (V, E) as follows (see Fig. 3). Suppose \( F = \{S_1, \ldots, S_n\} \) is a family of sets where each \( S_i \) is a subset of \( X = \{x_1, x_2, \ldots, x_m\} \). First, each element \( x_j \) is a vertex and each set \( S_i \) is a vertex and \( x_j \) is adjacent to \( S_i \) if and only if \( x_j \in S_i \). For each \( S_i \), attach a path \( S_i - a_i - b_i \) of length two at vertex \( S_i \). More precisely,

\[
V = \{S_1, S_2, \ldots, S_n\} \cup \{x_1, x_2, \ldots, x_m\} \cup \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}
\]

and \( E \) consists of

1. \((x_j, S_i)\), for all \( x_j \in S_i \), \( 1 \leq j \leq m, 1 \leq i \leq n \),
2. \((S_i, a_i), (a_i, b_i)\) for all \( 1 \leq i \leq n \).

Suppose \( D \) is a total perfect dominating set of \( G \). First of all, \( D \) must contain all \( a_i \)'s and at least one vertex in \( \{S_i, b_i\} \) for each \( i \). So \( |D| \geq 2n \) and then \( t\delta(G) \geq 2n \). For the case of \( t\delta(G) = 2n \), an optimal solution \( D^* \) contains all \( a_i \)'s and exactly one vertex in \( \{S_i, b_i\} \) for each \( i \).

We claim that the exact cover problem has a positive answer if and only if \( t\delta(G) = 2n \).

Suppose the exact cover problem has a positive answer, i.e., \( F \) has a subfamily \( F^* \) of pairwise disjoint sets whose union is equal to \( X \) or equivalently each element in \( X \) is in

![Fig. 3. The bipartite graph G for the total perfect domination.](image)
exactly one set in \( F^* \). Let \( D^* = \{ S_i, a_i; S_i \in F^* \} \cup \{ a_i, b_i; S_i \in F - F^* \} \). It is easy to check that \( D^* \) is a total perfect dominating set of \( G \) of size \( 2n \). Hence \( t\delta(G) = 2n \).

On the other hand, suppose \( t\delta(G) = 2n \), i.e., there is a total perfect dominating set \( D^* \) of size \( 2n \). By the previous discussion, \( D^* \) contains all \( a_i \)'s and exactly one vertex in \( \{ S_i, b_i \} \) for each \( i \). Let \( F^* = \{ S_i; S_i \in D^* \} \). By the definition of total perfect domination, \( F^* \) is a subfamily of pairwise disjoint sets whose union is equal to \( X \).

If we add edges into the graph \( G \) in the proof of Theorem 4 to make \( X \) a clique, the resulting graph is a chordal graph. Using the same arguments as in the proof of Theorem 4, we have the following theorem.

**Theorem 5.** The total perfect domination problem is NP-complete for chordal graphs.

### 3. Weighted perfect domination in block graphs

This section gives a linear-time algorithm for the weighted perfect domination problem in block graphs by means of dynamic programming.

Suppose that \( G = (V, E) \) is a graph with costs \( c \) on vertices and edges. For technical reasons, we consider \( G \) as a graph rooted at a specified vertex \( u \). We define the following terms, which are the weighted perfect domination problem with boundary conditions.

\[
(PD_1) \quad \delta_1(G, u, c) = \min \{ c(D); u \in D \text{ and } D \text{ is a perfect dominating set of } G \}.
\]

A perfect dominating set \( D \) of \( G \) which contains \( u \) is called a \( \delta_1 \)-prefect dominating set of \( G \) (with respect to \( u \)).

\[
(PD_2) \quad \delta_0(G, u, c) = \min \{ c(D); u \notin D \text{ and } D \text{ is a perfect dominating set of } G \}.
\]

A perfect dominating set \( D \) of \( G \) which does not contain \( u \) is called a \( \delta_0 \)-perfect dominating set of \( G \) (with respect to \( u \)).

\[
(PD_3) \quad \delta_{fo}(G, u, c) = \min \{ c(D); D \subseteq V(G) - N[u] \text{ and } D \text{ is a perfect dominating set of } G - u \}.
\]

A perfect dominating set \( D \) of \( G - u \) which does not contain any vertex in \( N[u] \) is called a \( \delta_{fo} \)-perfect dominating set of \( G \) (with respect to \( u \)).

It is clear that for any graph \( G \), \( \delta(G, c) = \min \{ \delta_1(G, u, c), \delta_0(G, u, c) \} \).

Now consider the following composition of \( n \) graphs. Suppose \( G_1, G_2, \ldots, G_n \) are \( n \geq 2 \) graphs rooted at \( v_1, v_2, \ldots, v_n \), respectively. The composition of \( G_1, G_2, \ldots, G_n \) is the graph \( G \) rooted at \( v_1 \), which is obtained from the disjoint union of \( G_1, G_2, \ldots, G_n \) by adding edges to make the vertex set \( \{ v_1, v_2, \ldots, v_n \} \) a clique in \( G \) (see Fig. 4). Note that a connected block graph can be obtained from trivial graphs by a sequence of graph compositions.

The following theorem is the basis of the linear-time algorithm for the weighted perfect domination problem in block graphs. Note that it is true for general graphs.
Theorem 6. Suppose $G_1, G_2, \ldots, G_n$ are $n \geq 2$ graphs rooted at $v_1, v_2, \ldots, v_n$, respectively. $G$ is the graph rooted at $v_1$, which is obtained from the disjoint union of $G_1, G_2, \ldots, G_n$ by adding edges to make $U = \{v_1, v_2, \ldots, v_n\}$ a clique in $G$ (see Fig. 4). Then the following formulas hold:

(i) $\delta_1(G, v_1, c) = \min \left\{ \sum_{1 \leq i \leq n} \delta_1(G_i, v_i, c), \right.$

$\left. \delta_1(G_1, v_1, c) + \sum_{2 \leq i \leq n} \{c(v_1, v_i) + \delta_{f0}(G_i, v_i, c)\} \right\}$.

(ii) $\delta_0(G, v_1, c) = \min \left\{ \sum_{1 \leq i \leq n} \delta_0(G_i, v_i, c), \right.$

$\left. \min_{2 \leq j \leq n} \left\{ \delta_1(G_j, v_j, c) + \sum_{1 \leq i \leq n, i \neq j} \{c(v_j, v_i) + \delta_{f0}(G_i, v_i, c)\} \right\} \right\}$.

(iii) $\delta_{f0}(G, v_1, c) = \delta_{f0}(G_1, v_1, c) + \sum_{2 \leq i \leq n} \delta_0(G_i, v_i, c)$.

Proof. (i) Since $U$ is a clique in $G$, for any perfect dominating set $D$ of $G$, either $U \subseteq D$ or $D$ contains at most one vertex in $U$. Then, for any $\delta_1$-perfect dominating set $D$ of $G$, either $U \subseteq D$ or $D \cap U = \{v_1\}$. For the former case, $D = \bigcup_{1 \leq i \leq n} D_i$ where each $D_i$ is a $\delta_1$-perfect dominating set of $G_i$ and $c(D) = \sum_{1 \leq i \leq n} c(D_i)$. For the latter case, $D = \bigcup_{1 \leq i \leq n} D_i$, where $D_1$ is a $\delta_1$-perfect dominating set of $G_1$ and $D_i$ is a $\delta_{f0}$-perfect dominating set of $G_i$ for $2 \leq i \leq n$; and $c(D) = c(D_1) + \sum_{2 \leq i \leq n} \{c(v_1, v_i) + c(D_i)\}$. Thus Formula (i) holds.

(ii) Since $U$ is a clique in $G$, for any perfect dominating set $D$ which does not contain $v_1$, $D$ contains at most one vertex in $U$. Then, for any $\delta_0$-perfect dominating set $D$ of $G$, either $D \cap U = \emptyset$ or $D \cap U = \{v_j\}$ for some $2 \leq j \leq n$. For the former case, $D = \bigcup_{1 \leq i \leq n} D_i$ where each $D_i$ is a $\delta_0$-perfect dominating set of $G_i$; and $c(D) = \sum_{1 \leq i \leq n} c(D_i)$. For the latter case, $D = \bigcup_{1 \leq i \leq n} D_i$, where $D_j$ is a $\delta_1$-perfect dominating set of $G_j$ and each $D_i \neq D_j$ is a $\delta_{f0}$-perfect dominating set of $G_i$; and $c(D) = c(D_j) + \sum_{i \neq j} \{c(v_j, v_i) + c(D_i)\}$. Thus Formula (ii) holds.
(iii) Since $v_1$ is adjacent to all other $v_i$'s, for any $\delta_{f_0}$-perfect dominating set $D$ of $G$, $D \cap U = \emptyset$. Therefore, $D = \bigcup_{1 \leq i \leq n} D_i$, where $D_1$ is a $\delta_{f_0}$-perfect dominating set of $G_1$ and $D_i$ is a $\delta_0$-perfect dominating set of $G_i$ for $2 \leq i \leq n$, and $c(D) = \sum_{1 \leq i \leq n} c(D_i)$. Thus Formula (iii) holds. □

We now design an algorithm to solve the weighted perfect domination problem in block graphs based upon the dynamic programming approach. The algorithm starts from "outer" blocks of a block graph by repeatedly applying Theorem 6. In the algorithm, the problem can be solved in each block which will be rooted at a cut-vertex. The trouble is: we must find the blocks in a block graph. We overcome this difficulty by using the concept of block-cut-vertex structure of a block graph.

In a graph $G$, the intersection of two blocks is either empty or a cut-vertex. A vertex is a cut-vertex if and only if it is the intersection of two or more blocks. Moreover, suppose $G$ has $m$ blocks $B_1, B_2, \ldots, B_m$ and $n$ cut-vertices $v_1, v_2, \ldots, v_n$. Consider the block-cut-vertex structure $G^* = (V^*, E^*)$, where

$$V^* = \{B_1, B_2, \ldots, B_m, v_1, v_2, \ldots, v_n\}$$

and

$$E^* = \{(B_i, v_j): 1 \leq i \leq m, 1 \leq j \leq n, v_j \in B_i\}.$$ 

Then $G^*$ is a forest whose leaves are exactly those blocks containing just one cut-vertex in $G$ and whose isolated vertices are exactly those blocks without cut-vertices in $G$. A block which contains exactly one cut-vertex is called an end block of $G$. Note that $G^*$ is a tree if and only if $G$ is connected. The block-cut-vertex structure of a graph can be constructed in linear time by the depth first search, see Aho et al. [1]. In addition, it will be used as the traverse order for our algorithm to solve the weighted perfect domination problem in block graphs.

Fig. 5(a) is a block graph $G$. Fig. 5(b) shows its blocks and cut-vertices. Fig. 5(c) shows its block-cut-vertex structure $G^*$.

The block-cut-vertex structure of a block graph shows the relationship among blocks, where all the blocks interact at the cut-vertex only. Once the block-vertex-cut structure is found, then the dynamic programming approach can be easily applied. The calculation starts from the end blocks, and records the result in the cut-vertex (the root of the block) of the blocks. We then work inwardly in the block-cut-vertex structure $G^*$ of the graph, until all the vertices of $G^*$ are traversed. Then the solution of the whole graph will be obtained.

Algorithm WPDS-block-graph

**Input:** A connected block graph $G = (V, E)$ in which every vertex $v \in V$ has a cost $c(v)$ and every edge $e \in E$ has a cost $c(e)$.

**Output:** The value $\delta(G, c)$ of a minimum weighted perfect dominating set.
Fig. 5(a). A block graph $G$.

Fig. 5(b). The 8 blocks and 7 cut-vertices of $G$.

Fig. 5(c). The block-cut-vertex structure $G^*$ of $G$. 
Method

construct the block-cut-vertex structure \( G^* \) of \( G \);

for each vertex \( v \) of \( G \) do

\( \delta_1(v) \leftarrow c(v) \);

\( \delta_0(v) \leftarrow \infty \);

\( \delta_{fo}(v) \leftarrow 0 \);

end for;

\( G' \leftarrow G \);

while \( G' \) has more than one vertex do

choose a block \( B = (U, E_U) \) with at most one cut-vertex in \( G' \), where \( U = \{v_1, v_2, \ldots, v_n\} \) and \( v_1 \) is the only cut-vertex of \( G' \) when it has a cut-vertex;

\( \delta_1(v_1) = \min \{ \sum_{1 \leq i \leq n} \delta_1(v_i), \delta_1(v_1) + \sum_{1 \leq i \leq n} \{ c(v_1, v_i) + \delta_{fo}(v_i) \} \} \);

\( \delta_0(v_1) = \min \{ \sum_{1 \leq i \leq n} \delta_0(v_i), \min_{2 \leq j \leq n} \{ \delta_1(v_j) + \sum_{1 \leq j \neq i \leq n} \{ c(v_j, v_i) + \delta_{fo}(v_i) \} \} \} \);

\( \delta_{fo}(v_1) = \delta_{fo}(v_1) + \sum_{1 \leq i \leq n} \delta_0(v_i) \);

\( G' \leftarrow G' \setminus \{v_1, \ldots, v_n\} \);

end while;

\( \delta(G, c) \leftarrow \min \{ \delta_1(r), \delta_0(r) \} \), where \( r \) is the only vertex in \( G' \);

end WPDS-block-graph.

Theorem 7. Algorithm WPDS-block-graph solves the weighted perfect domination problem for block graphs in linear time.

Proof. The correctness of the theorem follows from Theorem 6. The algorithm is linear with respect to \(|V| + |E| \) since each block is considered once and in each block \( O(|N[v]|) \) operations are executed on each vertex \( v \).

With a slight modification by using pointers, we can find not only the value \( \delta(G, c) \) but also the corresponding optimal perfect dominating set \( D^* \).

4. Variants of perfect domination in block graphs

The three variants of the weighted perfect domination problem can be solved in similar ways. We give their derivation without detail proofs in this section.

For the weighted independent perfect domination problem, we consider the following different weighted independent perfect domination problems with boundary conditions.

\( (IPD_1) \) Finding \( i\delta_1(G, u, c) = \min \{ c(D) : u \in D \} \) and \( D \) is an independent perfect dominating set of \( G \). An independent perfect dominating set \( D \) of \( G \), which contains \( u \), is called an \( i\delta_1 \)-perfect dominating set of \( G \).

\( (IPD_0) \) Finding \( i\delta_0(G, u, c) = \min \{ c(D) : u \notin D \} \) and \( D \) is an independent perfect dominating set of \( G \). An independent perfect dominating set \( D \) of \( G \), which does not contain \( u \), is called an \( i\delta_0 \)-perfect dominating set of \( G \).
Finding $i\delta_f(G, u, c) = \min\{c(D): D \subset V(G) - N[u] \text{ and } D \text{ is an independent perfect dominating set of } G - u\}$. An independent perfect dominating set $D$ of $G - u$, which contains no vertex in $N[u]$, is called an $i\delta_f$-perfect dominating set of $G$.

It is clear that for any graph $G$, $i\delta(G, c) = \min\{i\delta_1(G, u, c), i\delta_0(G, u, c)\}$.

The following theorem is the basic theorem for the weighted independent perfect domination problem in block graphs.

**Theorem 8.** Suppose $G_1, G_2, \ldots, G_n$ are $n \geq 2$ graphs rooted at $v_1, v_2, \ldots, v_n$, respectively. $G$ is the graph rooted at $v_1$, which is obtained from the disjoint union of $G_1, G_2, \ldots, G_n$ by adding edges to make $U = \{v_1, v_2, \ldots, v_n\}$ a clique in $G$ (see Fig. 4). Then the following formulas hold:

1. $i\delta_1(G, v_1, c) = i\delta_1(G_1, v_1, c) + \sum_{2 \leq i \leq n} \{c(v_1, v_i) + i\delta_f(G_i, v_i, c)\}$,
2. $i\delta_0(G, v_1, c) = \min \left\{ \sum_{1 \leq i \leq n} i\delta_0(G_i, v_i, c), \min_{2 \leq j \leq n} \left\{ i\delta_1(G_j, v_j, c) + \sum_{1 \leq i \leq n} \{c(v_j, v_i) + i\delta_f(G_i, v_i, c)\} \right\} \right\}$,
3. $i\delta_f(G, v_1, c) = i\delta_f(G_1, v_1, c) + \sum_{2 \leq i \leq n} i\delta_0(G_i, v_i, c)$.

**Proof.** The proof is similar to Theorem 8 and omitted. □

Although we can use a similar approach as that for the weighted perfect domination problem to solve the weighted connected perfect domination problem in block graphs, a much simpler solution is presented here.

**Theorem 9.** Suppose $G = (V, E)$ is a connected block graph which contains at least one cut-vertex. Suppose $L$ is the set of non-cut-vertices in some end block. Then, $D$ is a connected dominating set of $G$ if and only if $V - L \subseteq D$. Furthermore, $V - \cup\{U - u: B = (U, E_B) \text{ is an end block with } u \text{ as its only cut-vertex such that } \sum_{x \in U - u} c(u, x) \leq \sum_{x \in U - u} c(x)\}$ is an optimal connected perfect dominating set of $G$.

**Proof.** The proof is trivial and omitted. □

To solve the weighted total perfect domination problem in block graphs, we consider the following four variant problems which are the original problem with boundary conditions.

**TPD$_1$** Finding $t\delta_1(G, u, c) = \min\{c(D): u \in D \text{ and } D \text{ is a total perfect dominating set of } G\}$. A total perfect dominating set $D$ of $G$, which contains $u$, is called an $t\delta_1$-perfect dominating set of $G$. 
(TPD₀) Finding \( tδ₀(G, u, c) = \min \{ c(D): u \notin D \text{ and } D \text{ is a total perfect dominating set of } G \} \). A total perfect dominating set \( D \) of \( G \), which does not contain \( u \), is called a \( tδ₀ \)-perfect dominating set of \( G \).

(\( TPD_{f₁} \)) Finding \( tδ₁(G, u, c) = \min \{ c(D): u \in D \text{ and } D \text{ is a perfect dominating set of } G \text{ such that } \langle D \rangle \text{ has no isolated vertex, except possibly } u \} \). A perfect dominating set \( D \) of \( G \) with \( u \in D \) and \( \langle D \rangle \) has no isolated vertices, except possibly \( u \), is called a \( tδ₁ \)-perfect dominating set of \( G \).

(\( TPD_{f₀} \)) Finding \( tδ₀(G, u, c) = \min \{ c(D): D \subset V(G) - N[u] \text{ and } D \text{ is a total perfect dominating set of } G - u \} \). A total perfect dominating set \( D \) of \( G - u \), which contains no vertex in \( N[u] \), is called a \( tδ₀ \)-perfect dominating set of \( G \).

It is clear that for any graph \( G \), \( tδ(G, c) = \min \{ tδ₁(G, u, c), tδ₀(G, u, c) \} \)

The following theorem is the basic theorem for the total weighted perfect domination problem in block graphs.

**Theorem 10.** Suppose \( G₁, G₂, \ldots, Gₙ \) are \( n \geq 2 \) graphs rooted at \( v₁, v₂, \ldots, vₙ \), respectively. \( G \) is the graph rooted at \( v₁ \), which is obtained from the disjoint union of \( G₁, G₂, \ldots, Gₙ \) by adding edges to make \( U = \{ v₁, v₂, \ldots, vₙ \} \) a clique in \( G \) (see Fig. 4). Then the following formulas hold:

(i) \( tδ₁(G, v₁, c) = \min \left\{ \sum_{1 \leq i \leq n} tδ₁(G_i, v_i, c) \right\} \)

\( tδ₁(G, v₁, c) + \sum_{2 \leq i \leq n} \{ c(v₁, v_i) + i tδ₀(G_i, v_i, c) \} \)\)

(ii) \( tδ₀(G, v₁, c) = \min \left\{ \sum_{1 \leq i \leq n} tδ₀(G_i, v_i, c), \right\} \)

\( \min_{2 \leq j \leq n} \left\{ tδ₁(G_j, v_j, c) + \sum_{1 \leq i \leq n} \{ c(v_j, v_i) + tδ₀(G_i, v_i, c) \} \right\} \}

(iii) \( tδ_{f₁}(G, v₁, c) = \min \left\{ \sum_{1 \leq i \leq n} tδ_{f₁}(G_i, v_i, c), \right\} \)

\( tδ_{f₁}(G_1, v₁, c) + \sum_{2 \leq i \leq n} \{ c(v₁, v_i) + tδ_{f₀}(G_i, v_i, c) \} \}

(iv) \( tδ_{f₀}(G, v₁, c) = tδ_{f₀}(G₁, v₁, c) + \sum_{2 \leq i \leq n} tδ₀(G_i, v_i, c) \)

**Proof.** The proof is similar to Theorem 8 and omitted. □

**References**

[7] M.S. Chang and Y.C. Liu, Polynomial algorithms for the weighted perfect domination problems on chordal graphs and split graphs, manuscript.