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Improved Error Bounds for Newton-Like Iterations under Chen-Yamamoto Conditions

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Abstract—New *a posteriori* error bounds are provided for Newton-like iterations under Chen-Yamamoto conditions, which improve earlier ones.

Keywords—Banach space, Newton-like iteration.

1. INTRODUCTION

In this note we are concerned with the problems of approximating a locally unique zero x^* of the equation

$$F(x) + G(x) = 0, \tag{1}$$

in a Banach space E_1, where F, G are nonlinear operators defined on $U(x_0, R) = \{x \in E_1 \mid \|x - x_0\| \le R\} \subseteq E_1$ with values in a Banach space E_2.

Yamamoto in [1] provided a comparison list for almost all known results until approximately 1986 for *a posteriori* error bounds for Newton's method. We note that this list will be incomplete today. See [2–5] for example and the references there after 1986. Since then, he attempted (sometimes with Chen) to extend some of the results (but not all) obtained in [1] to be valid for Newton-like methods, by applying the same unifying principle. However, Yamamoto or others have not found results of the type we present here for Newton-like methods (see in particular Remark 1 that follows). We note that they have for Newton's method only [1].

Chen and Yamamoto in [6] established a convergence theorem for the Newton-like iteration

$$x_{n+1} = x_n - A(x_n)^{-1}(F(x_n) + G(x_n)), \qquad \text{for} \quad n \ge 0. \tag{2}$$

Here $A(x_n)$ denotes a linear operator which is a conscious approximation of the Frechet derivate $F'(x_n)$ of F evaluated at $x = x_n$ for $n \ge 0$. They also provided error bounds for the same iteration.

Here we show how to improve their error bounds.

2. CONVERGENCE ANALYSIS

The following conditions were considered in [6] for $n = 0$.

We assume that $A(x_n)^{-1}$ exists and for all $x_n, x, y \in U(x_0, r) \subseteq U(x_0, R)$, $t \in [0, 1]$

$$\|A(x_n)^{-1}(A(x) - A(x_n))\| \le v_n(r) + b_n, \tag{3}$$

$$\|A(x_n)^{-1}(F'(x + t(y - x)) - A(x))\| \le w_n(r + t\,\|y - x\|) - v_n(r) + c_n, \qquad \text{and} \tag{4}$$

$$\|A(x_n)^{-1}(G(x) - G(y))\| \le e_n(r)\|x - y\|, \tag{5}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX where $w_n(r+t) - v_n(r)$, $t \geq 0$ and $e_n(r)$ are nondecreasing nonnegative functions with $w_n(0) = v_n(0) = e_n(0) = 0$ for all $n \geq 0$, $v_n(r)$ are differentiable, $v_n'(r) > 0$ for all $r \in [0, R]$, and the constants b_n, c_n satisfy $b_n \geq 0$, $c_n \geq 0$, and $b_n + c_n < 1$ for all $n \geq 0$. It is convenient to introduce for all $n, i \geq 0$

$$a_n = \|A(x_n)^{-1}(F(x_n) + G(x_n))\|, \qquad \varphi_{n,i}(r) = a_i - r + c_{n,i} \int_0^r w_n(t)\,dt,$$

$$z_n = z_n(r) = 1 - v_n(r) - b_n, \qquad \psi_n(r) = c_{n,i} \int_0^r e_n(t)\,dt, \qquad c_{n,i} = z_n(r_i)^{-1},$$

$$h_{n,i} = \varphi_{n,i}(r) + \psi_{n,i}(r), \qquad r_n = \|x_n - x_0\|, \qquad a_n = \|x_{n+1} - x_n\|,$$

the equations

$$r = a_n + c_{0,n} \left(\int_0^r (w_0(r_n + t) + e_n(r_n + t))\,dt + (b_n + c_n - 1) r \right), \tag{6}$$

$$r = a_n + c_{n,n} \left(\int_0^r (w_n(r_n + t) + e_n(r_n + t))\,dt + (b_n + c_n - 1) r \right), \tag{7}$$

$$a_n = r + c_{0,n} \left(\int_0^r (w_0(r_n + t) + e_n(r_n + t))\,dt + (b_n + c_n - 1) r \right), \tag{8}$$

$$a_n = r + c_{n,n} \left(\int_0^r (w_n(r + t) + e_n(r_n + t))\,dt + (b_n + c_n - 1) r \right), \tag{9}$$

and the scalar iterations $\{s^0_{k,n}\}$, $\{s_{k,n}\}$ (for each fixed $n \geq 0$), given by

$$s_{0,n} = s^0_{n,n} = 0, \quad s^0_{k+1,n} = s^0_{k,n} + \frac{h_{0,n}(s^0_{k,n} + r_n)}{c_{0,n} z_0(s^0_{k,n} + r_n)}, \qquad \text{for} \quad k \geq 0, \tag{10}$$

$$s_{k+1,n} = s_{k,n} + \frac{h_{n,n}(s_{k,n} + r_n)}{c_{n,n} z_{n,n}(s_{k,n} + r_n)}, \qquad \text{for} \quad k \geq n. \tag{11}$$

Then, as in [6, Theorems 1 and 2] and the remark in [5, p. 993] we can show the following theorem.

THEOREM. *Let $F, G : U(x_0, R) \subseteq E_1 \to E_2$ be nonlinear operators. Assume:*

(i) *the function $h_{0,0}(r)$ has a unique zero s^*_0 in the interval $[0, R]$ and $h_{0,0}(R) \leq 0$;*
(ii) *the following estimates are true:*

$$\frac{h_{n,n}(r + t_n)}{c_{n,n} z_{n,n}(r + r_n)} \leq \frac{h_{0,n}(r + r_n)}{c_{0,n} z_n(r + r_n)}, \qquad \text{for all } r \in [0, R - r_n] \text{ and for each fixed } n \geq 0. \tag{12}$$

Then we have the following.

(a) *The scalar iterations $\{s^0_{k+1,n}\}$ and $\{s_{k+1,n}\}$ for $k \geq 0$, given by (10) and (11) are monotonically increasing and converge to s^*_n and s^{**}_n for each fixed $n \geq 0$, which are the unique solutions of equations (6) and (7) in $[0, R - s_n]$, respectively with $s^{**}_n \leq s^*_n$ for all $n \geq 0$.*
(b) *The Newton-like iteration $\{x_n\}$ $n \geq 0$, generated by (2) is well defined, remains in $U(x_0, s^*)$ for all $n \geq 0$ and converges to a solution x^* of the equation $F(x) + G(x) = 0$, which is unique in $U(x_0, R)$.*
(c) *The following estimates are true for all $n \geq 0$:*

$$\|x_{n+1} - x_n\| \leq s_{n+1,n+1} - s_{n,n} \leq s^0_{n+1,n+1} - s^0_{n,n}, \tag{13}$$

$$\|x^* - x_n\| \leq s^{**}_n - s_{n,n} \leq s^*_n - s^0_{n,n} \leq s^*_0 - s^0_{n,0}, \tag{14}$$

$$\|x^* - x_n\| \geq I^*_n, \tag{15}$$

$$\|x^* - x_n\| \geq I^{**}_n, \qquad \text{and} \tag{16}$$

$$I^{**}_n \leq I^*_n, \tag{17}$$

where I^*_n and I^{**}_n are the solutions of the equation (8) and (9), respectively.

REMARK 1. We observe that our iteration (10) is really iteration (7) in [3]. Hence, the estimates (13) and (14) improve the corresponding ones in [6, p. 40,45]. Estimates of the form (15), (16), and (17) were not given in [3], but they were given in [5, p. 989; 4, p. 673; 7, p. 134] (for $G = 0$), when $A(x) = F'(x)$ and under special cases of the conditions (3)–(5).

REMARK 2. The direction of the inequality (17) can be reversed if inequalities (12) are reversed.

REMARK 3. Note, that the above results remain true if conditions (3)–(5) are satisfied for every $r \in [0, R - r_n] \subseteq [0, R]$.

REMARK 4. If conditions (3)–(5) are satisfied only for $n = 0$, then we can choose $v_n(r) = c_{n,0}v_0(r)$, $w_n(r) = c_{n,0}w_0(r)$, $e_n(r) = c_{n,0}e_0(r)$, $b_n = c_{n,0}b_0$, and $c_n = c_{n,0}c_0$ for all $n \geq 1$. Then conditions (3)–(5) will be satisfied. Otherwise, if the same conditions are satisfied only for a fixed $m > 0$, then the first m terms of (2) can be dropped. Conditions (3)–(5) will then be satisfied with the above choices of function and parameters. Moreover, we can then set $m = 0$.

REMARK 5. Similar results can easily follow if we consider a more general Newton-like iteration of the form $y_{n+1} = y_n - A(y_n)^{-1}(F(y_n) + G(y_n))$, for all $y_0 \in U(z, R)$ and $n \geq 0$ (see, also [6, p. 39]).

REMARK 6. The conditions (3)–(5) and (12) are not difficult to realize. For simplicity let

$$\alpha_n = \frac{\|A(x_n)^{-1}(F(x) - F(y))\|}{\|x - y\|}, \qquad b_n = \sup_{x,y\in U(x_0,R)} \alpha_n, \qquad \gamma_n = \sup_{x,y\in U(x_0,R-r_n)} \alpha_n,$$

$$e_0(r) = \delta_0, \qquad e_0(r + r_n) = \delta_{0,n},$$

$$v_0(r) = \varepsilon_0, \qquad v_0(r + r_n) = \varepsilon_{0,n},$$

$$w_0(r) = \eta_0, \qquad w_0(r + r_n) = n_{0,n},$$

$$v_n(r) = \theta_n = \sup_{x,y\in U(x_0,R)} (\|A(x_n)^{-1}(A(x) - A(x_n))\| - b_n),$$

$$v_n(r + r_n) = \theta_{n,n} = \sup_{x,y\in U(x_0,R-r_n)} (\|A(x_n)^{-1}(A(x) - A(x_n))\| - b_n),$$

$$\lambda_n = w_n(r) = \sup_{x,y\in U(x_0,R)} (\|A(x_n)^{-1}(F'(x + t(y - x)) - A(x)\| + v_n(r) - c_n), \qquad \text{and}$$

$$\lambda_{n,n} = w_n(r + c_n) = \sup_{x,y\in U(x_0,R-r_n)} (\|A(x_n)^{-1}(F'(x + t(y - x)) - A(x)\| + v_n(r) - c_n),$$

for all $n \geq 0$. It can now easily be seen that with the above choices many natural sufficient conditions can be given, so that inequalities (12) are satisfied for all $n \geq 0$. (See, also [7]). One can refer to [8] for some applications of these ideas to the solution of integral equations.

REMARK 7. We can define the sequence $\{s_n^1\}$, $n \geq 0$ by $s_0^1 = 0$, $s_{n+1}^1 = s_n^1 + (h_{n,n}(s_n^1 + r_n))/c_{n,n}$ for all $n \geq 0$.

Then, under the hypotheses of the theorem, we can easily show that

$$\|x_{n+1} - x_n\| \leq s_{n+1}^1 - s_n^1 \leq s_{n+1,n+1} - s_{n,n} \leq s_{n+1,n+1}^0 - s_{n,n}^0, \qquad \text{and}$$
$$\|x^* - x_n\| \leq t^* - s_n^1 \leq s_n^{**} - s_{n,n} \leq s_n^* - s_{n,n}^0 \leq s_0^* - s_{n,0}^0,$$

for all $n \geq 0$, where $t^* = \lim_{n\to\infty} s_n^1$.

As in Remark 1, we note the above two error estimates improve further the corresponding results in [6, p. 40,45].

Relevant work on the subject but following a completely different approach can be found in [3] and the references there.

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