NOTE

**M-CHAIN GRAPHS OF POSETS**

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Received October 21, 1986
Revised October 26, 1987

The $m$-chain graph of a finite poset is defined as a generalization of the covering graph. 2-chain graphs of posets whose covering graphs are trees are characterized.

1. Introduction

Covering graphs of various types of finite partially ordered sets have been extensively studied since Ore in [2] asked for a characterization of such graphs in general. In this note we introduce a graph, called the $m$-chain graph of a poset, that reduces to the covering graph when $m = 1$. Since we will show how to construct the $m$-chain graph for $m > 2$ as the 2-chain graph of another poset, we are concerned primarily with 2-chain graphs. Utilizing the line diagraph of the oriented covering graph of a poset, we characterize the 2-chain graphs of posets whose covering graphs are trees.

2. Definitions and observations

In order to minimize definition overload, we assume familiarity with standard graph theory and poset terminology. All sets are finite. For a poset $P$ and $x, y \in P$ we let $x < y$ denote the fact that $y$ covers $x$. For $m$ a positive integer, an $m$-chain in the poset $P$ is a chain $x_1x_2 \cdots x_m$ such that $x_i < x_{i+1}$ for all $i = 1, \ldots, m-1$. Clearly $m$-chain exist only when $P$ has height at least $m - 1$. The $m$-chain graph of the poset $P$, denoted $G_m(P)$, is the graph whose vertices are the $m$-chains of $P$.

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and the \( m \)-chain \( x_1x_2 \cdots x_m \) is adjacent to the \( m \)-chain \( y_1y_2 \cdots y_m \) if and only if

(i) \( x_i = y_{i+1} \) for \( i = 1, 2, \ldots, m-1 \) and \( y_1 < x_1 \),

or

(ii) \( x_{i+1} = y_i \) for \( i = 1, 2, \ldots, m-1 \) and \( x_m < y_m \).

Note that \( G_m(P) \) is only defined if \( P \) has height at least \( m-1 \), \( G_m(P) \) has no edges if the height of \( P \) is \( m-1 \), and \( G_1(P) \) is the usual covering graph of \( P \).

The most interesting problem concerning \( G_m(P) \) is to characterize those graphs that are the \( m \)-chain graphs of some poset. Of course, for \( m=1 \) this is still unknown except for certain restricted classes of posets. (See [3] for a review and references.) We now make a few observations about \( G_m(P) \) in general.

Since two \( m \)-chains \( x = x_1 \cdots x_m \) and \( y = y_1 \cdots y_m \) of \( P \) are adjacent in \( G_m(P) \) if and only if \( \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_m\} \) is an \((m+1)\)-chain, an edge in \( G_m(P) \) really can be identified simply as an \((m+1)\)-chain in \( P \). Thus for the graph \( G = (V, E) \) to be the \( m \)-chain graph of a poset \( P \), there must be exactly \( |V| \) number of \( m \)-chains and \( |E| \) number of \((m+1)\)-chains in \( P \). Also, if we define \( x < y \) for case (ii) above and \( y \prec x \) for case (i), then the transitive closure of \( < \) partially orders the set of \( m \)-chains of \( P \). Denote this partially ordered set by \( P_m \).

Then it is easy to see that the covering graph of \( P_m \) is just \( G_m(P) \), so that \( G_m(P) \) must, for example, always be triangle-free. In addition we have the key fact that \( G_2(P_m) = G_{m+1}(P) \), which in effect reduces the study of \( m \)-chain graphs to that of \( 2 \)-chain graphs. Another useful observation is the following. Let \( D \) be the digraph that is the naturally oriented covering graph of \( P \) (i.e. \( D \) is what now is sometimes called the ‘diagram’ of \( P \)). Then the underlying graph of the line digraph of \( D \) is just \( G_2(P) \).

We conclude this section with some examples. First we show that if \( G \) is a tree, then \( G \) is an \( m \)-chain graph for all \( m \geq 1 \). Choose an arbitrary vertex \( x \) of \( G \) as the root and orient the edges of \( G \) so that it is considered as the naturally oriented covering graph of a poset with minimum element \( x \). By adding a chain of \( m-1 \) new elements below \( x \), the tree poset \( P \) that we obtain satisfies \( G_m(P) = G \).

Now let \( C_n \) be the cycle with \( n \) vertices. \( C_3 \) is not an \( m \)-chain graph for any \( m \geq 1 \) since \( G_m(P) \) must be triangle-free. However, \( C_4 \) is an \( m \)-chain graph for all \( m \geq 1 \). The posets \( P \) for which \( G_m(P) = C_4 \) are given in Fig. 1.

![Fig. 1](image_url)
A tedious argument will show that $C_5$ is not an $m$-chain graph for all $m \geq 2$ but observe that $C_n$, $n \geq 2m + 2$ is the $m$-chain graph of the poset in Fig. 2.

3. Two-chain graphs of tree posets

In this section we initiate the study of 2-chain graphs by restricting our attention to tree posets. A poset is a tree poset if its covering graph is a tree. Note that any orientation of the edges of a tree $T$ gives rise to a tree poset by viewing the Hasse diagram simply as an oriented covering graph. Also, it is clear that all tree posets having covering graph $T$ can be formed this way.

We will call a tree an oriented tree if each of its edges is oriented. From the above, we thus have that oriented trees are precisely tree posets and this leads to the fact that the underlying graph of the line digraph of an oriented tree is just the 2-chain graph of the associated tree poset.

The following theorem of Chartrand will be required for our first characterization.

**Theorem 1 ([1, p. 78]).** A graph is the line graph of a tree if and only if it is connected, each block is a complete subgraph, and each cut vertex lies in exactly two blocks.

Let $\tilde{T}$ denote an oriented tree with underlying tree $T$. We will show that the line digraph of $\tilde{T}$ can be obtained from the line graph of $T$ by a simple operation of splitting its blocks as follows. Let $B = \{e_1, \ldots, e_k\}$ be a block of the line graph of $T$. By Theorem 1, $B$ is a complete subgraph so it follows that the edges $e_1, \ldots, e_k$ are all incident to some vertex $v \in T$. In $\tilde{T}$, the vertex $v$ is either the head or tail of the corresponding arcs $\tilde{e}_1, \ldots, \tilde{e}_k$ so that this defines a partition of $\{\tilde{e}_1, \ldots, \tilde{e}_k\}$ into one or two subsets. Clearly these subsets are independent sets in the line digraph and all arcs exist between them.

A graph $H$ is derived by block splitting from a graph $G$ if $H$ arises by replacing
Fig. 3.

each block of $G$ by a complete bipartite graph or independent set on the same vertex set. The above discussion establishes our next theorem.

**Theorem 2.** Let $G$ be a graph. The following are equivalent:

(i) $G$ is the $2$-chain graph of some tree poset

(ii) $G$ is the underlying graph of the line digraph of some oriented tree

(iii) $G$ is derived by block splitting from a graph whose blocks are complete subgraphs and each cut vertex lies in exactly two blocks

Our main result is a characterization of $2$-chain graphs of tree posets in terms of forbidden induced subgraphs.

**Theorem 3.** A graph $G$ is the $2$-chain graph of some tree poset if and only if $G$ does not contain any of the graphs $C_n$ $(n \neq 4)$, $H_1$, $H_2$, $H_8$, or $H_{2k+8}$ $(k \geq 1)$ shown in Fig. 3 as induced subgraphs.

**Proof.** Let $G$ be a $2$-chain graph of a tree poset. By property (iii) of Theorem 2, $G$ is a bipartite graph such that its blocks are complete bipartite graphs. Thus the two-connected graphs $C_n$ $(n \neq 4)$, $H_1$, and $H_2$ are not induced subgraphs of $G$.

Suppose that $G$ contains $H_{2k+8}$ (or $H_8$) as an induced subgraph and that $G$ is derived by block splitting from some graph $G^*$. It is easy to check that one of the vertices $x_0, x_1, \ldots, x_k$ ($x_0$ if $G$ contains $H_8$) is a cut vertex of $G^*$ contained in more than two blocks of $G^*$, which contradicts property (iii) of Theorem 2.

Conversely, assume that $G$ is a graph that does not contain the graphs stated in the theorem as induced subgraphs.

**Claim 1.** $G$ is bipartite. Indeed, the smallest cycle of odd length in $G$ is an induced cycle $C_n$ $(n \neq 4)$, which is excluded from $G$.

**Claim 2.** The blocks of $G$ are complete bipartite graphs. Let $C$ be a cycle of $G$
(not necessarily induced.) From Claim 1, \( G \) is bipartite, so the length of \( C \) is \( 2p \), \( p \geq 2 \). We show by induction on the length of \( C \) that \( C \) induces a complete bipartite subgraph, \( K_{p,p} \) of \( G \). This is clearly true for \( p = 2 \). Hence assume \( p > 2 \) and that the cycles \( C \) of length less than \( 2p \) induce complete bipartite subgraphs of \( G \). Obviously, \( C \) contains a chord \( xy \) which splits \( C \) into two smaller cycles \( C_1 \) and \( C_2 \) sharing the common edge \( xy \). By induction, \( C_1 \) and \( C_2 \) are complete bipartite graphs \( K_{m,m} \) and \( K_{n,n} \), respectively, with \( m, n < p \). Let \( a_1b_1 \neq xy \) and \( a_2b_2 \neq xy \) be edges of \( C_1 \) and \( C_2 \), respectively. Since the subgraph of \( G \) induced by \( \{a_1, b_1, x, y, a_2, b_2\} \) contains a 6-cycle and it cannot be \( H_1 \) or \( H_2 \) by assumption, \( \{a_1, b_1, a_2, b_2\} \) must induce a 4-cycle. This is true for any \( a_1b_1 \neq xy \in E(C_1) \) and \( a_2b_2 \neq xy \in E(C_2) \). Consequently, \( C = K_{p,p} \) as stated.

To finish the proof of the theorem, we define a graph \( G^* \) on the same vertex set as \( G \). First let \( G' \) be the graph obtained from \( G \) by adding new edges within each block such that all blocks of \( G' \) are complete subgraphs. Then \( G' \) is a tree-like structure. The argument is by induction on \( n \), the number of nontrivial blocks of \( G' \). Suppose \( G' \) has at most one nontrivial block. That is, \( G' \) is a tree or the union of pairwise disjoint rooted trees plus a complete graph formed by the roots of these trees and possibly some other points not contained in any of the rooted trees. See Fig. 4 above.

These trees can be partitioned into edge disjoint stars such that each vertex of \( G' \) belongs to at most two stars. Add new edges to each star to get a complete graph. The resulting graph \( G^* \) obviously satisfies property (iii) of Theorem 2.

Assume \( G' \) contains \( n \) nontrivial blocks, \( n \geq 2 \) and that from a graph \( G \) we can construct \( G^* \) with property (iii) whenever \( G' \) contains less than \( n \) non-trivial blocks. Let \( B_0 \) and \( B_k \) be two nontrivial blocks of \( G' \) which are the closest possible, that is, the unique path \( x_0, x_1, \ldots, x_k \) connecting them (where \( x_0 \in B_0 \) and \( x_k \in B_k \)) is such that the vertices \( x_1, \ldots, x_{k-1} \) are only contained in trivial blocks of \( G' \). In particular, \( G \) will contain the graph in Fig. 5 as an induced subgraph.

![Fig. 4.](image_url)

![Fig. 5.](image_url)
Clearly, one of $x_0, x_1, \ldots, x_k$ must be covered by just two blocks in $G$, otherwise $G$ would have $H_{2k+8}$ ($k \geq 1$) as an induced subgraph. Suppose that $x_i$ (0 $\leq i \leq k$) is a cut vertex of $G'$ dividing $G'$ into parts $G'_1$ and $G'_2$ and sharing common vertex $x_i$. Since $G'_1$ and $G'_2$ contain less than $n$ nontrivial blocks we can construct graphs $G'_1^*$ and $G'_2^*$ satisfying property (iii) from each respectively, and hence $G^* = G'_1^* \cup G'_2^*$ satisfies property (iii) of Theorem 2. The result now follows by Theorem 2. $\square$

References