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A perturbed differential resultant based implicitization algorithm for linear DPPEs

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ABSTRACT

Let $\mathbb{K}$ be an ordinary differential field with derivation ∂. Let $\mathcal{P}$ be a system of n linear differential polynomial parametric equations in $n-1$ differential parameters, with implicit ideal ID. Given a nonzero linear differential polynomial A in ID, we give necessary and sufficient conditions on A for $\mathcal{P}$ to be $n-1$ dimensional. We prove the existence of a linear perturbation $\mathcal{P}_\phi$ of $\mathcal{P}$, so that the linear complete differential resultant ∂CRes_ϕ associated to $\mathcal{P}_\phi$ is nonzero. A nonzero linear differential polynomial in ID is obtained, from the lowest degree term of ∂CRes_ϕ, and used to provide an implicitization for $\mathcal{P}$.

1. Introduction

The use of algebraic elimination techniques, such as Gröbner bases and multivariate resultants, to obtain the implicit equation of a unirational algebraic variety is well known (see for instance (Cox et al., 1997, 1998)). The development of similar techniques in the differential case is an active field of research. In Gao (2003), characteristic set methods were used to solve the differential implicitization problem for differential rational parametric equations and, alternative methods are emerging to treat special cases. In Rueda and Sendra (2010), linear complete differential resultants were used to compute the implicit equation of a set of linear differential polynomial parametric equations (linear DPPEs). As in the algebraic case, differential resultants often vanish under specialization and we are left with no candidate for the implicit equation. This reason prevented us from giving an algorithm for differential implicitization in Rueda and Sendra (2010). Motivated by Canny's method (Canny, 1990) and its generalizations in D'Andrea and Emiris (2001) and Rojas (1999), in the present work, we consider a linear perturbation of a given system of linear DPPEs, and use linear complete differential resultants to give a candidate for the implicit equation of the system.

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Given a system $\mathcal{P}(X, U)$ of n linear ordinary differential polynomial parametric equations $x_1 = P_1(U), \ldots, x_n = P_n(U)$, in $n-1$ differential parameters $u_1, \ldots, u_{n-1}$ (we give a precise statement of the problem in Section 2), we give an algorithm to decide if the dimension of the implicit ideal ID of $\mathcal{P}$ is $n-1$ and, in the affirmative case, provide the implicit equation of $\mathcal{P}$.

The linear complete differential resultant $\partial\text{CRes}(x_1 - P_1(U), \ldots, x_n - P_n(U))$ is the algebraic resultant of Macaulay, of a set of differential polynomials with L elements. It was defined in Rueda and Sendra (2010), as a generalization of Carra'-Ferro's differential resultant (Carra'Ferro, 1997) (in the linear case), in order to adjust the number L, of differential polynomials, to the order of the derivatives of the variables $u_1, \ldots, u_{n-1}$ in $F_i = x_i - P_i(U)$.

In this paper, we provide a perturbation $\mathcal{P}_\phi(X, U)$ of $\mathcal{P}(X, U)$, so that the linear differential polynomials $F_1 - p\phi_1(U), \ldots, F_n - p\phi_n(U)$ have nonzero linear complete differential resultant $\partial\text{CRes}_\phi(p)$, which is a polynomial depending on p. It will be shown that the coefficient of the lowest degree term of $\partial\text{CRes}_\phi(p)$ is a nonzero linear differential polynomial, which belongs to the implicit ideal ID of $\mathcal{P}(X, U)$. In fact, if $\partial\text{CRes}_\phi(p)$ has a nonzero constant term, with respect to p, it equals $\partial\text{CRes}(F_1, \ldots, F_n)$ and, as proved in Rueda and Sendra (2010), it gives the implicit equation of $\mathcal{P}(X, U)$.

The main result of this paper generalizes the result previously mentioned from Rueda and Sendra (2010). Given a nonzero linear differential polynomial A in ID, necessary and sufficient conditions on A are provided so that $A(X) = 0$ is the implicit equation of $\mathcal{P}(X, U)$. The higher order terms in the equations of $\mathcal{P}(X, U)$ and the rank of the coefficient matrix, of the set of L polynomials used to construct the differential resultant $\partial\text{CRes}(F_1, \ldots, F_n)$, play a significant role in this theory. The fact that we are dealing with linear differential polynomials will be also relevant, allowing us to treat them by means of differential operators.

The paper is organized as follows. In Section 2, we introduce the main notions and notation. Next we review the definition of the linear complete differential (homogeneous) resultant in Section 3. Definitions regarding linear differential polynomials in ID are given in Section 4. The next section contains the main result of the paper, namely a characterization of the implicit equation of ID, in the $n-1$ dimensional case, is provided in Section 5. In Section 6, we give a perturbation $\mathcal{P}_\phi(X, U)$ of $\mathcal{P}(X, U)$, with nonzero differential resultant, and use it to obtain a nonzero linear differential polynomial in ID, candidate to provide the implicit equation. Finally, in Section 7, we give an implicitization algorithm and examples.

2. Basic notions and notation

This section is devoted to the introduction of the terminology, the notation and the basic notions (as in Rueda and Sendra (2010)) that will be used throughout the paper. We refer to Kolchin (1973) and Ritt (1950) for further concepts and results on differential algebra.

Let $\mathbb{K}$ be an ordinary differential field with derivation ∂, (e.g. $\mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$). By $\mathbb{N}_0$ we mean the natural numbers including 0. Given a set Y of differential indeterminates over $\mathbb{K}$, we denote by $\{Y\}$ the set of derivatives of the elements of Y, $\{Y\} = \{\partial^k y \mid y \in Y, \ k \in \mathbb{N}_0\}$, and by $\mathbb{K}\{Y\}$ the ring of differential polynomials in the differential indeterminates Y, which is a differential ring with derivation ∂. Let $X = \{x_1, \ldots, x_n\}$ and $U = \{u_1, \ldots, u_{n-1}\}$ be sets of differential indeterminates over $\mathbb{K}$. For $k \in \mathbb{N}_0$, we denote by $x_{i,k}$ the kth derivative of x_i and for $x_{i,0}$ we simply write x_i. Observe that

$$\mathbb{K}\{X\} = \mathbb{K}[x_{i,k} \mid i = 1, \ldots, n, k \in \mathbb{N}_0]$$

is a differential domain with derivation ∂. The differential rings $\mathbb{K}\{U\}$ and $\mathbb{K}\{X \cup U\}$, which will be also used throughout the paper, can be defined analogously.

As defined in Rueda and Sendra (2010), we consider the system of linear DPPEs

$$\mathcal{P}(X, U) = \begin{cases} x_1 &= P_1(U), \\ &\vdots \\ x_n &= P_n(U), \end{cases} \tag{1}$$

where $P_1, \ldots, P_n \in \mathbb{K}\{U\}$, with degree at most 1 and not all $P_i \in \mathbb{K}$, $i = 1, \ldots, n$. There exists $a_i \in \mathbb{K}$ and a homogeneous differential polynomial $H_i \in \mathbb{K}\{U\}$ such that

$$F_i(X, U) = x_i - P_i(U) = x_i - a_i + H_i(U).$$

Given $P \in \mathbb{K}\{X \cup U\}$ and $y \in X \cup U$, we denote by $\mathrm{ord}(P, y)$ the order of P in the variable y. If P does not have a term in y then we define $\mathrm{ord}(P, y) = -1$. To ensure that the number of parameters is $n-1$, we assume that for each $j \in \{1, \ldots, n-1\}$ there exists $i \in \{1, \ldots, n\}$ such that $\mathrm{ord}(F_i, u_j) \geq 0$.

The implicit ideal of the system (1) is the differential prime ideal

$$\mathrm{ID} = \{f \in \mathbb{K}\{X\} \mid f(P_1(U), \ldots, P_n(U)) = 0\}.$$

Given a characteristic set $\mathcal{C}$ of ID, then $n-|\mathcal{C}|$ is the (differential) dimension of ID. By abuse of notation, we will also speak about the dimension of a DPPE system, meaning the dimension of its implicit ideal.

If $\dim(\mathrm{ID}) = n-1$, then $\mathcal{C} = \{A(X)\}$ for some irreducible differential polynomial $A \in \mathbb{K}\{X\}$. The polynomial A is called a characteristic polynomial of ID. An implicit equation of a $(n-1)$-dimensional system of DPPEs, in n differential indeterminates $X = \{x_1, \ldots, x_n\}$, is defined as the equation $A(X) = 0$, where A is any characteristic polynomial of the implicit ideal ID of the system.

Let $\mathbb{K}[\partial]$ be the ring of differential operators with coefficients in $\mathbb{K}$. If $\mathbb{K}$ is not a field of constants with respect to ∂, then $\mathbb{K}[\partial]$ is not commutative but $\partial k - k\partial = \partial(k)$, for all $k \in \mathbb{K}$. The ring $\mathbb{K}[\partial]$ of differential operators with coefficients in $\mathbb{K}$ is left Euclidean (and also right Euclidean). Given $\mathcal{L}, \mathcal{L}' \in \mathbb{K}[\partial]$, by applying the left division algorithm we obtain $q, r \in \mathbb{K}[\partial]$, the left quotient and the left reminder of $\mathcal{L}$ and $\mathcal{L}'$ respectively, such that $\mathcal{L} = \mathcal{L}'q + r$ where $\deg(r) < \deg(\mathcal{L}')$.

3. Linear complete differential resultants

We review next the results on linear complete differential resultants from Rueda and Sendra (2010), which will be used in this paper.

Let $\mathbb{D}$ be a differential integral domain. Let $f_i \in \mathbb{D}\{U\}$ be a linear ordinary differential polynomial of order o_i, $i = 1, \ldots, n$. We assume that the polynomials $f_1, \ldots, f_n$ are distinct. For each $j \in \{1, \ldots, n-1\}$, let $\mathcal{O}(f_i, u_j) = \mathrm{ord}(f_i, u_j)$, if $\mathrm{ord}(f_i, u_j) \geq 0$ and $\mathcal{O}(f_i, u_j) = 0$, if $\mathrm{ord}(f_i, u_j) = -1$. We define the positive integers

$$\gamma_j(f_1, \ldots, f_n) := \min\{o_i - \mathcal{O}(f_i, u_j) \mid i \in \{1, \ldots, n\}\},$$
$$\gamma(f_1, \ldots, f_n) := \sum_{j=1}^{n-1} \gamma_j(f_1, \ldots, f_n).$$

Let $N = \sum_{i=1}^{n} o_i$, the completeness index $\gamma(f_1, \ldots, f_n)$ verifies $\gamma(f_1, \ldots, f_n) \leq N - o_i$, for $i = 1, \ldots, n$.

We defined the linear complete differential resultant $\partial\mathrm{CRes}(f_1, \ldots, f_n)$ in Rueda and Sendra (2010), as the Macaulay's algebraic resultant of the differential polynomial set

$$\mathrm{PS}(f_1, \ldots, f_n) := \{\partial^{N-o_i-\gamma} f_i, \ldots, \partial f_i, f_i \mid i = 1, \ldots, n, \gamma = \gamma(f_1, \ldots, f_n)\}.$$

Since the differential polynomials $f_1, \ldots, f_n$ are distinct, the set $\mathrm{PS}(f_1, \ldots, f_n)$ contains $L = \sum_{i=1}^{n}(N - o_i - \gamma + 1)$ polynomials in the following set $\mathcal{V}$ of $L-1$ differential variables

$$\mathcal{V} = \{u_j, u_{j,1}, \ldots, u_{j,N-\gamma_j-\gamma} \mid \gamma_j = \gamma_j(f_1, \ldots, f_n), j = 1, \ldots, n-1\}.$$

Let $h_i \in \mathbb{D}\{U\}$ be a linear ordinary differential homogeneous polynomial of order o_i, $i = 1, \ldots, n$, with $N = \sum_{i=1}^{n} o_i \geq 1$. We assume that the polynomials $h_1, \ldots, h_n$ are distinct. We define the differential polynomial set

$$\mathrm{PS}^h(h_1, \ldots, h_n) := \{\partial^{N-o_i-\gamma-1} h_i, \ldots, \partial h_i, h_i \mid i \in \{1, \ldots, n\}, N - o_i - \gamma - 1 \geq 0, \gamma = \gamma(h_1, \ldots, h_n)\}.$$

Observe that $N \geq 1$ implies $\mathrm{PS}^h(h_1, \ldots, h_n) \neq \emptyset$. The linear complete differential homogenous resultant $\partial\mathrm{CRes}^h(h_1, \ldots, h_n)$ is the Macaulay's algebraic resultant of the set $\mathrm{PS}^h(h_1, \ldots, h_n)$. Since the differential polynomials are distinct, the set $\mathrm{PS}^h(h_1, \ldots, h_n)$ contains $L^h = \sum_{i=1}^{n}(N - o_i - \gamma)$ polynomials in the set $\mathcal{V}^h$ of L^h differential variables

$$\mathcal{V}^h = \{u_j, u_{j,1}, \ldots, u_{j,N-\gamma_j-\gamma-1} \mid \gamma_j = \gamma_j(h_1, \ldots, h_n), j = 1, \ldots, n-1\}.$$

We review next the matrices that will allow the use of determinants to compute $\partial\text{CRes}(f_1,\dots,f_n)$ and $\partial\text{CRes}^h(h_1,\dots,h_n)$. The order $u_1 < \cdots < u_{n-1}$ induces an orderly ranking on U (i.e. an order on $\{U\}$) as follows (see Kolchin, 1973, page 75): $u_{i,j} < u_{k,l} \Leftrightarrow (j,i) <_{\text{lex}} (l,k)$. We set $1 < u_1$.

For $i = 1,\dots,n$, $\gamma = \gamma(f_1,\dots,f_n)$ and $k = 0,\dots,N - o_i - \gamma$, we define the positive integer $l(i,k) = (i-1)(N-\gamma) - \sum_{h=1}^{i-1} o_h + i + k$ in $\{1,\dots,L\}$. The complete differential resultant matrix $M(L)$ is the $L \times L$ matrix containing the coefficients of $\partial^{N-o_i-\gamma-k} f_i$, as a polynomial in $\mathbb{D}[\mathcal{V}]$, in the $l(i,k)$th row, where the coefficients are written in decreasing order with respect to the orderly ranking on U. In this situation:

$$\partial\text{CRes}(f_1,\dots,f_n) = \det(M(L)).$$

If $N \geq 1$, for $\gamma = \gamma(h_1,\dots,h_n)$, $i \in \{1,\dots,n\}$, $N - o_i - \gamma - 1 \geq 0$ and $k = 0,\dots,N - o_i - \gamma - 1$, define the positive integer $l^h(i,k) = (i-1)(N-\gamma-1) - \sum_{h=1}^{i-1} o_h + i + k$ in $\{1,\dots,L^h\}$. The complete differential homogeneous resultant matrix $M(L^h)$ is the $L^h \times L^h$ matrix containing the coefficients of $\partial^{N-o_i-\gamma-k-1} h_i$, as a polynomial in $\mathbb{D}[\mathcal{V}^h]$, in the $l^h(i,k)$th row, where the coefficients are written in decreasing order with respect to the orderly ranking on U. In this situation:

$$\partial\text{CRes}^h(h_1,\dots,h_n) = \det(M(L^h)).$$

Throughout the remaining parts of the paper, we will say differential (homogeneous) resultant always meaning linear complete differential (homogeneous) resultant.

3.1. Linear complete differential resultants from linear DPPEs

We highlight, in this section, some facts on differential resultants of the differential polynomials F_i and H_i, obtained from a system of linear DPPEs as in Section 2.

Let $\gamma = \gamma(F_1,\dots,F_n) = \gamma(H_1,\dots,H_n)$ and $\mathbb{D} = \mathbb{K}\{X\}$. The differential resultants $\partial\text{CRes}(F_1,\dots,F_n)$ and $\partial\text{CRes}^h(H_1,\dots,H_n)$ are closely related, as shown in Rueda and Sendra (2010), Section 5. Since $F_i(X,U) = x_i - a_i + H_i(U)$, if $N \geq 1$ the matrix $M(L^h)$ is a submatrix of $M(L)$, obtained by removing n specific rows and columns. This fact together with the identities below allowed us to prove that (when $N \geq 1$)

$$\partial\text{CRes}(F_1,\dots,F_n) = 0 \Leftrightarrow \partial\text{CRes}^h(H_1,\dots,H_n) = 0.$$

The next matrices will play an important role in the remaining parts of the paper.

- Let S be the $n \times (n-1)$ matrix whose entry (i,j) is the coefficient of $u_{n-j,o_i-\gamma_{n-j}}$ in F_i, $i \in \{1,\dots,n\}$, $j \in \{1,\dots,n-1\}$. We call S the leading matrix of $\mathcal{P}(X,U)$. For $i \in \{1,\dots,n\}$, let S_i be the $(n-1)\times(n-1)$ matrix obtained by removing the ith row of S.
- Let M_{L-1} be the $L \times (L-1)$ principal submatrix of $M(L)$. We call M_{L-1} the principal matrix of $\mathcal{P}(X,U)$.

Let $\mathcal{X} = \{x_i, x_{i,1},\dots,x_{i,N-o_i-\gamma} \mid i = 1,\dots,n\}$. Given $x \in \mathcal{X}$, say $x = x_{i,k}$ with $k \in \{0,1,\dots,N - o_i - \gamma\}$, let M_x be referred to as the submatrix of M_{L-1} obtained by removing the row corresponding to the coefficients of $\partial^k F_i = x_{i,k} + \partial^k(H_i(U) - a_i)$. Then, developing the determinant of $M(L)$ by the last column, we obtain

$$\partial\text{CRes}(F_1,\dots,F_n) = \sum_{i=1}^{n} \sum_{k=0}^{N-o_i-\gamma} b_{ik} \det(M_{x_{i,k}})(x_{i,k} - \partial^k a_i), \tag{2}$$

with $b_{ik} = \pm 1$, according to the row index of $x_{i,k} - \partial^k a_i$ in the matrix $M(L)$.

4. The implicit ideal ID

Let $\mathcal{P}(X,U)$, F_i, H_i be as in Section 2. Let PS $=$ PS$(F_1,\dots,F_n)$ and let ID be the implicit ideal of $\mathcal{P}(X,U)$. In this section, we review the computation of ID in terms of characteristic sets (see Gao, 2003; Rueda and Sendra, 2010) and give some definitions, related with linear differential polynomials in ID, that will be important in the remaining parts of the paper.

Let $\mathcal{X}$ and $\mathcal{V}$ be as in Section 3.1 and observe that PS $\subset \mathbb{K}[\mathcal{X}][\mathcal{V}]$. Let (PS) be the ideal generated by PS in $\mathbb{K}[\mathcal{X}][\mathcal{V}]$ and, let [PS] be the differential ideal generated by PS in $\mathbb{K}\{X\}\{U\}$.

Let $(i,j), (k,l) \in \mathbb{N}_0^2$ be distinct. We write $(i,j) \prec (k,l)$ if $i > k$, or $i = k$ and $j < l$. The order $x_n < \cdots < x_1$, induces a ranking on X (see Kolchin, 1973, page 75), using the monomial order $\prec$: $x_{i,j} < x_{k,l} \Leftrightarrow (i,j) \prec (k,l)$.

- We call $\mathcal{R}$ the ranking on $X \cup U$ that eliminates X with respect to U, that is $\partial^k x > \partial^{k^\star} u$, for all $x \in X$, $u \in U$ and $k, k^\star \in \mathbb{N}_0$.
- We call $\mathcal{R}^\star$ the ranking on $X \cup U$ that eliminates U with respect to X, that is $\partial^k x < \partial^{k^\star} u$, for all $x \in X$, $u \in U$ and $k, k^\star \in \mathbb{N}_0$.

Note that, because of the particular structure of F_i, with respect to $\mathcal{R}$ then PS is a chain (see Ritt, 1950, page 3) of L differential polynomials, with L as in Section 3. Let $\mathcal{A}$ be a characteristic set of [PS] with respect to $\mathcal{R}^\star$ and $\mathcal{A}_0 = \mathcal{A} \cap \mathbb{K}\{X\}$. By Gao (2003), Lemma 3.2 and Theorem 3.1 then

$$\mathrm{ID} = [\mathrm{PS}] \cap \mathbb{K}\{X\} = [\mathcal{A}_0].$$

To compute a characteristic set of [PS] w.r.t. $\mathcal{R}^\star$, we will use the reduced Gröbner basis of (PS) with respect to lex monomial order induced by the ranking $\mathcal{R}^\star$. We call $\mathcal{G}$ the Gröbner basis associated to the system $\mathcal{P}(X, U)$.

We are dealing with a linear system of polynomials and computing a Gröbner basis is equivalent to performing Gaussian elimination. Some details on this computation were given in Rueda and Sendra (2010), and we include them below to be used further in this paper. Let M_{2L} be the $L \times (2L)$ matrix whose row $l(i,k)$, with $i = 1, \ldots, n$ and $k = 0, \ldots, N - o_i - \gamma$, contains the coefficients of $\partial^{N-o_i-\gamma-k} F_i$, as a polynomial in $\mathbb{K}[\mathcal{X}][\mathcal{V}]$, and where the coefficients are written in decreasing order w.r.t. $\mathcal{R}^\star$,

$$M_{2L} = \begin{bmatrix} M_{L-1} & \begin{matrix} 1 & & & & & & & -\partial^{N-o_1-\gamma} a_1 \\ & \ddots & & & & & & \vdots \\ & & 1 & & & & & -a_1 \\ & & & \ddots & & & & \vdots \\ & & & & 1 & & & -\partial^{N-o_n-\gamma} a_n \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 & -a_n \end{matrix} \end{bmatrix}.$$

The polynomials corresponding to the rows of the reduced echelon form E_{2L} of M_{2L} are the elements of the Gröbner basis associated to $\mathcal{P}(X, U)$. Let $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. Given $P \in \mathbb{K}\{X \cup U\}$, the lead of P is the highest derivative present in P w.r.t. $\mathcal{R}^\star$, we denote it by lead(P). From Rueda and Sendra (2010), Lemma 20(1):

Lemma 4.1. *The Gröbner basis associated to the system $\mathcal{P}(X, U)$ is a set of linear differential polynomials $\mathcal{G} = \{B_0, B_1, \ldots, B_{L-1}\}$, where $B_0 < B_1 < \cdots < B_{L-1}$ with respect to the ranking $\mathcal{R}^\star$ and $B_0 \in \mathcal{G}_0$. Hence,*

1. lead$(B_0) <$ lead$(B_1) < \cdots <$ lead(B_{L-1}) *w.r.t. the ranking $\mathcal{R}^\star$ and,*
2. (PS) $\cap \mathbb{K}\{X\} = (\mathcal{G}_0)$ *is nonzero.*

Given $P, Q \in \mathbb{K}\{X \cup U\}$, we denote by prem$(P, Q)$ the pseudo-remainder of P with respect to Q, (Ritt, 1950, page 7). Given a chain $\mathcal{A} = \{A_1, \ldots, A_t\}$ of elements of $\mathbb{K}\{X \cup U\}$ then prem$(P, \mathcal{A}) =$ prem(prem$(P, A_t), \{A_1, \ldots, A_{t-1}\})$ and prem$(P, \emptyset) = P$.

Algorithm 4.2. Given the set of differential polynomials PS, the next algorithm returns a subset $\mathcal{A}$ of [PS].

1. Compute the reduced Gröbner basis $\mathcal{G}$ of (PS) with respect to lex monomial order induced by $\mathcal{R}^\star$.
2. Assume that the elements of $\mathcal{G}$ are arranged in increasing order $B_0 < B_1 < \cdots < B_{L-1}$ w.r.t. $\mathcal{R}^\star$. Let $\mathcal{A}^0 := \{B_0\}$. For i from 1 to $L - 1$ do $R_i :=$ prem$(B_i, \mathcal{A}^{i-1})$, if $R_i \neq 0$ then $\mathcal{A}^i := \mathcal{A}^{i-1} \cup \{R_i\}$ else $\mathcal{A}^i := \mathcal{A}^{i-1}$. $\mathcal{A} := \mathcal{A}^{L-1}$.

Lemma 4.3. *With the notation used in Algorithm 4.2, let us assume that, if $R_i \neq 0$ then $\text{lead}(R_i) = \text{lead}(B_i)$. Then Algorithm 4.2 returns a characteristic set $\mathcal{A}$ of* [PS] *with respect to $\mathcal{R}^{\star}$.*

Proof. We prove first that $\mathcal{A}$ is an autoreduced set w.r.t. $\mathcal{R}^{\star}$. Given $A \in \mathcal{A}$, there exists $i \in \{1, \ldots, L-1\}$ such that $\mathcal{A}^i = \mathcal{A}^{i-1} \cup \{A\}$ and $A = \text{prem}(B_i, \mathcal{A}^{i-1})$. Therefore, A is reduced w.r.t. every polynomial in $\mathcal{A}^{i-1}$. Let us assume that $\mathcal{A}^i \neq \mathcal{A}$. Given $B \in \mathcal{A}\backslash\mathcal{A}^i$, $B = \text{prem}(B_j, \mathcal{A}^{j-1})$ for some $j \in \{i+1, \ldots, L-1\}$. By assumption and Lemma 4.1(1), $\text{lead}(B) = \text{lead}(B_j) > \text{lead}(B_i) = \text{lead}(A)$, which shows that A is reduced w.r.t. B. We have proved that A is reduced w.r.t. every polynomial in $\mathcal{A}$ and therefore $\mathcal{A}$ is autoreduced.

Since $\mathcal{A}$ is autoreduced and linear, $[\mathcal{A}]$ is a prime differential ideal, with $\mathcal{A}$ as its characteristic set. From $[\text{PS}] = [\mathcal{A}]$, it follows that $\mathcal{A}$ is a characteristic set of [PS]. □

4.1. On linear differential polynomials in ID

In this section, we give some definitions that will play an important role throughout the paper. Let us consider the linear span over $\mathbb{K}$ of the polynomials in $\mathcal{PS} = \{\partial^k F_i \mid k \in \mathbb{N}_0, i = 1, \ldots, n\}$, that is

$$\text{span}_{\mathbb{K}}\mathcal{PS} = \left\{\sum_{i=1}^{n} \mathcal{F}_i(F_i(X, U)) \mid \mathcal{F}_i \in \mathbb{K}[\partial], i = 1, \ldots, n\right\}.$$

Observe that $\text{span}_{\mathbb{K}}\mathcal{PS}$ is a subset of the set of linear polynomials in [PS].

Lemma 4.4. 1. *Given a nonzero B in $\text{span}_{\mathbb{K}}\mathcal{PS}$, there exist unique differential operators $\mathcal{F}_1, \ldots, \mathcal{F}_n$ in $\mathbb{K}[\partial]$ such that*

$$B(X, U) = \sum_{i=1}^{n} \mathcal{F}_i(F_i(X, U)).$$

2. *Given a nonzero linear differential polynomial B in* ID *then B belongs to $\text{span}_{\mathbb{K}}\mathcal{PS}$. Furthermore, there exist unique differential operators $\mathcal{F}_i \in \mathbb{K}[\partial]$, $i = 1, \ldots, n$ such that*

$$B(X) = \sum_{i=1}^{n} \mathcal{F}_i(x_i - a_i) \quad \textit{and} \quad \sum_{i=1}^{n} \mathcal{F}_i(H_i(U)) = 0.$$

Proof. 1. Let us suppose that there exist $\mathcal{E}_i \in \mathbb{K}[\partial]$, $i = 1, \ldots, n$ such that $B(X, U) = \sum_{i=1}^{n} \mathcal{E}_i (F_i(X, U))$. Then $\sum_{i=1}^{n} \mathcal{E}_i(x_i) = \sum_{i=1}^{n} \mathcal{F}_i(x_i)$. Thus for $i = 1, \ldots, n$, the linear polynomials $\mathcal{E}_i(x_i) = \mathcal{F}_i(x_i)$, which implies $\mathcal{E}_i = \mathcal{F}_i$.
2. Given a linear B in $\text{ID} = [\text{PS}] \cap \mathbb{K}\{X\}$, $B \in \mathbb{K}\{X\}$ implies that there exist $\mathcal{F}_i \in \mathbb{K}[\partial]$, $i = 1, \ldots, n$ and $a \in \mathbb{K}$ such that $B(X) = a + \sum_{i=1}^{n} \mathcal{F}_i(x_i)$. By definition of ID, $B(P_1(U), \ldots, P_n(U)) = 0$ but

$$B(P_1(U), \ldots, P_n(U)) = a + \sum_{i=1}^{n} \mathcal{F}_i(a_i) + \sum_{i=1}^{n} \mathcal{F}_i(-H_i(U)) = 0.$$

Thus $a = \sum_{i=1}^{n} \mathcal{F}_i(-a_i)$ and $\sum_{i=1}^{n} \mathcal{F}_i(H_i(U)) = 0$. Since $F_i(X, U) = x_i - a_i + H_i(U)$, $i = 1, \ldots, n$, this proves the result. □

Remark 4.5. Let a_i, $i = 1, \ldots, n$, be as in Section 2. If $a_i = 0$, $i = 1, \ldots, n$, then, for all linear B in [PS], $B \in \text{span}_{\mathbb{K}}\mathcal{PS}$.

If B belongs to (PS) then $\text{ord}(B, x_i) \leq N - o_i - \gamma$, $i = 1, \ldots, n$. Let $\text{span}_{\mathbb{K}}\text{PS}$ be the linear span over $\mathbb{K}$ of the polynomials in PS, that is

$$\text{span}_{\mathbb{K}}\text{PS} = \left\{\sum_{i=1}^{n} \mathcal{F}_i(F_i(X, U)) \mid \mathcal{F}_i \in \mathbb{K}[\partial], \deg(\mathcal{F}_i) \leq N - o_i - \gamma, i = 1, \ldots, n\right\}.$$

Observe that $\text{span}_{\mathbb{K}}\text{PS}$ is a subset of the set of linear differential polynomials in (PS).

Remark 4.6. The Gröbner basis $\mathcal{G}$ associated to $\mathcal{P}(X, U)$ is obtained from M_{2L} by Gaussian elimination, thus $\mathcal{G} \subset \text{span}_{\mathbb{K}}\text{PS}$. Also, by Lemma 4.4(2), the linear differential polynomials in $(\mathcal{G}_0) = (\text{PS}) \cap \mathbb{K}\{X\}$ belong to $\text{span}_{\mathbb{K}}\text{PS}$.

Definition 4.7. Given a nonzero differential polynomial B in $\text{span}_{\mathbb{K}}\text{PS}$, with $B(X,U) = \sum_{i=1}^{n} \mathcal{F}_i (F_i(X,U))$, $\mathcal{F}_i \in \mathbb{K}[\partial]$.

1. We define the co-order of B in (PS) as the highest positive integer $\text{c}(B)$ such that $\partial^{\text{c}(B)}B \in (\text{PS})$. Observe that
$$\text{c}(B) = \min\{N - o_i - \gamma - \deg(\mathcal{F}_i) \mid i \in \{1,\ldots,n\}, \mathcal{F}_i \neq 0\}.$$
2. For $i \in \{1,\ldots,n\}$, let α_i be the coefficient of $\partial^{N-o_i-\gamma-\text{c}(B)}$ in $\mathcal{F}_i$, if $\mathcal{F}_i \neq 0$ and $\alpha_i = 0$, if $\mathcal{F}_i = 0$. We call $(\alpha_1,\ldots,\alpha_n)$ the leading coefficients vector of B in (PS) and we denote it by $\text{l}(B)$.

Let S be the leading matrix of the system $\mathcal{P}(X,U)$. Denote by S^T the transpose matrix of S.

Remark 4.8. Given a nonzero $B \in \text{span}_{\mathbb{K}}\text{PS}$.

1. The ith row of S consists of the coefficients of $u_{n-j,N-\gamma_{n-j}-\gamma-\text{c}(B)}$, $j \in \{1,\ldots,n-1\}$ in $\partial^{N-o_i-\gamma-\text{c}(B)} F_i(X,U)$ (alternative description to the one given in Section 3.1).
2. By 1, if $\text{ord}(B,u_j) < N - \gamma_j - \gamma - \text{c}(B)$, for $j = 1,\ldots,n-1$, then $\text{l}(B)S = 0$, that is $\text{l}(B)^T \in \text{Ker}(S^T)$.

Definition 4.9. Given a nonzero linear differential polynomial B in ID, with $B = \sum_{i=1}^{n} \mathcal{F}_i(x_i - a_i)$, $\mathcal{F}_i \in \mathbb{K}[\partial]$.

1. We define the ID-content of B as a greatest common left divisor of $\mathcal{F}_1,\ldots,\mathcal{F}_n$ (we write $\text{gcld}(\mathcal{F}_1,\ldots,\mathcal{F}_n)$). We denote it by $\text{IDcont}(B)$.
2. There exist $\mathcal{L}_i \in \mathbb{K}[\partial]$ such that $\mathcal{F}_i = \text{IDcont}(B)\mathcal{L}_i$, $i = 1,\ldots,n$, and $\mathcal{L}_1,\ldots,\mathcal{L}_n$ are coprime (we write $(\mathcal{L}_1,\ldots,\mathcal{L}_n) = 1$). We define an ID-primitive part of B as
$$\text{IDprim}(B)(X,U) = \sum_{i=1}^{n} \mathcal{L}_i(x_i - a_i).$$
3. If $\text{IDcont}(B) \in \mathbb{K}$ then we say that B is ID-primitive .

Given $B \in (\text{PS}) \cap \mathbb{K}\{X\}$, by Remark 4.6, $B \in \text{span}_{\mathbb{K}}\text{PS}$. If $A = \text{IDprim}(B)$ then $\text{c}(A) \geq \deg(\text{IDcont}(B))$ and $\deg(\mathcal{L}_i) \leq N - o_i - \gamma - \text{c}(A)$, $i = 1,\ldots,n$.

Lemma 4.10. *Given a nonzero linear differential polynomial* $B \in (\text{PS}) \cap \mathbb{K}\{X\}$, *it holds that* $\text{IDprim}(B) \in (\text{PS}) \cap \mathbb{K}\{X\}$.

Proof. For $i = 1,\ldots,n$ and $j = 1,\ldots,n-1$, there exist differential operators $\mathcal{L}_{ij} \in \mathbb{K}[\partial]$ such that $H_i(U) = \sum_{j=1}^{n-1} \mathcal{L}_{ij}(u_j)$. If $B(X,U) = \sum_{i=1}^{n} \mathcal{F}_i(x_i - a_i)$ then $\sum_{i=1}^{n} \mathcal{F}_i(H_i(U)) = 0$. As a consequence, $\sum_{i=1}^{n} \mathcal{F}_i(\mathcal{L}_{ij}(u_j)) = 0$ for $j \in \{1,\ldots,n-1\}$. Let $\mathcal{L} = \text{IDcont}(B)$ then $\mathcal{F}_i = \mathcal{L}\mathcal{L}_i$, with $\mathcal{L}_i \in \mathbb{K}[\partial]$ and $\text{IDprim}(B) = \sum_{i=1}^{n} \mathcal{L}_i(x_i - a_i)$. Thus $\mathcal{L}\sum_{i=1}^{n} \mathcal{L}_i\mathcal{L}_{ij} = 0$ and $\mathcal{L} \neq 0$ so the differential operator $\sum_{i=1}^{n} \mathcal{L}_i\mathcal{L}_{ij} = 0$. We conclude that $\sum_{i=1}^{n} \mathcal{L}_i(H_i(U)) = 0$. Therefore $\text{IDprim}(B) = \sum_{i=1}^{n} \mathcal{L}_i(F_i(X,U)) \in (\text{PS})$, which proves the lemma. □

5. Conditions for dim(ID) = $n - 1$

Let $\mathcal{P}(X,U)$, F_i, H_i be as in Section 2. Let $\text{PS} = \text{PS}(F_1,\ldots,F_n)$ and let ID be the implicit ideal of $\mathcal{P}(X,U)$. Let S and M_{L-1} be the leading and principal matrices of $\mathcal{P}(X,U)$ respectively, as defined in Section 3.1. Let $\mathcal{G}$ be the Gröbner basis associated to the system $\mathcal{P}(X,U)$, $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$ and denote by $|\mathcal{G}_0|$ the number of elements of $\mathcal{G}_0$.

By Lemma 4.1, the ideal $(\mathcal{G}_0)$ is nonzero. Given a nonzero linear differential polynomial A in $(\mathcal{G}_0)$, by Remark 4.6 we can talk about its co-order $\text{c}(A)$. In this section, if A is ID-primitive, we provide necessary and sufficient conditions on S, M_{L-1} and $\text{c}(A)$ for $A(X) = 0$ to be the implicit equation of $\mathcal{P}(X,U)$.

5.1. *Necessary conditions for* dim(ID) = $n - 1$

If the dimension of ID is $n-1$ then $\text{ID} = [A]$ and A is a characteristic polynomial of ID. By Lemma 4.1, A is linear and $A \in (\mathcal{G}_0)$. We give some more requirements for A in the next theorem.

Lemma 5.1. 1. $|\mathcal{G}_0| = L - \text{rank}(M_{L-1})$.
2. *For every nonzero linear* $B \in (\mathcal{G}_0)$, $|\mathcal{G}_0| \geq \text{c}(B) + 1$.

Proof. 1. Let M_{2L} be the $L \times 2L$ matrix defined in Section 4 and E_{2L} its reduced echelon form. The number of elements of $\mathcal{G}_0$ is the number of rows in E_{2L} with zeros in the first $L-1$ columns. Thus $|\mathcal{G}_0| = L - \text{rank}(M_{L-1})$.
2. Given a nonzero linear $B \in (\mathcal{G}_0) = (\text{PS}) \cap \mathbb{K}\{X\}$, by definition of $\text{c}(B)$ then $\partial B, \ldots, \partial^{\text{c}(B)} B \in (\text{PS}) \cap \mathbb{K}\{X\}$. Also, there exists $k \in \{1, \ldots, n\}$ such that $\text{ord}(B, x_k) = N - o_k - \gamma - \text{c}(B)$. We can assume that the coefficient of $x_{k,N-o_k-\gamma-\text{c}(B)}$ in B is 1. Thus M_{2L} is row equivalent to an $L \times 2L$ matrix with $\partial^{\text{c}(B)} B, \ldots, \partial B, B$ in the last $\text{c}(B) + 1$ rows. Namely, replace the row of M_{2L} corresponding to the coefficients of $\partial^{N-o_k-\gamma-t} F_k$ by $\partial^{\text{c}(B)-t} B$, $t = 0, \ldots, \text{c}(B)$, and reorder the rows of the obtained matrix. Therefore $|\mathcal{G}_0| \geq \text{c}(B) + 1$. □

Theorem 5.2. *Let* $\mathcal{G}$ *be the Gröbner basis associated to the system* $\mathcal{P}(X, U)$ *with implicit ideal* ID, $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. *If* $\dim \text{ID} = n - 1$ *then* $\text{ID} = [A]$, *where* A *is a nonzero linear differential polynomial verifying:*

1. *A is an* ID*-primitive differential polynomial in* $(\mathcal{G}_0)$.
2. $|\mathcal{G}_0| = \text{c}(A) + 1$.

Proof. By Lemma 4.1, $\mathcal{G}_0 = \{B_0, B_1, \ldots, B_m\}$, with $m \in \{0, \ldots, L-2\}$. Since $B_0 \in \text{ID}$, $B_0 = \mathcal{D}_0(A)$ for a nonzero $\mathcal{D}_0 \in \mathbb{K}[\partial]$, which implies that A is a linear polynomial in ID.

1. Let $\mathcal{L} = \text{IDcont}(A)$ and $A' = \text{IDprim}(A)$. If A is not ID-primitive then $\deg(\mathcal{L}) \geq 1$ and $A = \mathcal{L}(A')$, contradicting that $\{A\}$ is a characteristic set of ID. By Lemma 4.4(2), $A \in \text{span}_{\mathbb{K}} \mathcal{P}\mathcal{S}$. There exist unique differential operators $\mathcal{F}_i \in \mathbb{K}[\partial]$, $i = 1, \ldots, n$, such that $A(X) = \sum_{i=1}^{n} \mathcal{F}_i(x_i - a_i)$. Since $B_0(X) = \sum_{i=1}^{n} \mathcal{D}_0(\mathcal{F}_i(x_i - a_i)) \in \text{span}_{\mathbb{K}}\text{PS}$, $\deg(\mathcal{F}_i) \leq N - o_i - \gamma$ and $A \in \text{span}_{\mathbb{K}}\text{PS}$. In particular, $A \in (\text{PS}) \cap \mathbb{K}\{X\} = (\mathcal{G}_0)$.
2. Recall that $B_0 < B_1 < \cdots < B_m$, therefore
$$\mathcal{G}_0 = \{\mathcal{D}_0(A), \mathcal{D}_1(A), \ldots, \mathcal{D}_m(A)\},$$
with $\mathcal{D}_i \in \mathbb{K}[\partial]$, $\deg(\mathcal{D}_i) > \deg(\mathcal{D}_{i-1})$, $i = 1, \ldots, m$. Now, $A \in (\mathcal{G}_0)$ implies $A = \gamma_0 \mathcal{D}_0(A) + \gamma_1 \mathcal{D}_1(A) + \cdots + \gamma_m \mathcal{D}_m(A)$. Therefore, $\gamma_0 \mathcal{D}_0 + \gamma_1 \mathcal{D}_1 + \cdots + \gamma_m \mathcal{D}_m = 1$, which implies $\gamma_1 = \cdots = \gamma_m = 0$ and $\mathcal{D}_0 \in \mathbb{K}$. Thus $m \leq \text{c}(A)$ and, by Lemma 5.1, $|\mathcal{G}_0| = \text{c}(A) + 1$. □

Observe that, if $\dim \text{ID} = n - 1$, given A and B nonzero linear ID-primitive differential polynomials in $(\mathcal{G}_0)$, then $\text{ID} = [A] = [B]$ and $\text{c}(A) = \text{c}(B)$.

Corollary 5.3. *Let* $\mathcal{G}$ *be the Gröbner basis associated to the system* $\mathcal{P}(X, U)$ *with implicit ideal* ID, $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. *If* $\dim \text{ID} = n - 1$, *for every nonzero linear* ID*-primitive differential polynomial* A *in* $(\mathcal{G}_0)$, *then* $\text{ID} = [A]$ *and* $|\mathcal{G}_0| = \text{c}(A) + 1$.

If $N = \sum_{i=1}^{n} o_i = 0$ then $\mathcal{P}(X, U)$ is a system of n linear equations in $n - 1$ indeterminates.

Lemma 5.4. *If* $N = 0$, *then* $\dim \text{ID} = n - 1$ *if and only if* $\text{rank}(S) = n - 1$.

Proof. The matrix $M(L)$ is the $n \times n$ matrix whose $n \times (n-1)$ principal submatrix is S and, whose last column contains $x_i - a_i$ in the ith row, $i = 1, \ldots, n$. For linear $U_i \in \mathbb{K}\{X\}$, the following statement holds,
$$\text{rank}(S) = n - 1 \Leftrightarrow \mathcal{G} = \{B_0, u_1 - U_1(X), \ldots, u_{n-1} - U_{n-1}(X)\}.$$
Equivalently, $\{B_0\}$ is a characteristic set of ID. □

The next example shows that, if $N > 0$ and $n > 2$ then $\text{rank}(S) = n - 1$ is not a necessary condition for $\dim \text{ID} = n - 1$.

Example 5.5. Let $\mathbb{K} = \mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$ and consider the system $\mathcal{P}(X, U)$, of linear DPPEs, providing the set of differential polynomials

$$F_1(X, U) = x_1 - u_1 - u_{1,1} - u_{2,1},$$
$$F_2(X, U) = x_2 + 2u_2 - 2u_{1,1} - 2u_{2,1},$$
$$F_3(X, U) = x_3 - 2u_2 + u_{1,1} + u_{2,1},$$

in $\mathbb{K}\{x_1, x_2, x_3\}\{u_1, u_2\}$. The set $\mathrm{PS}(F_1, F_2, F_3)$ contains $L = 9$ differential polynomials and $\gamma = 0$. The leading matrix S of $\mathcal{P}(X, U)$ has rank $1 < n - 1 = 2$ and equals

$$S = \begin{bmatrix} -1 & -1 \\ -2 & -2 \\ 1 & 1 \end{bmatrix}.$$

The Gröbner basis associated to the system $\mathcal{P}(X, U)$ is $\mathcal{G} = \{B_0, B_1, \ldots, B_8\}$,

$$\mathcal{G} = \Big\{x_{1,1} - \frac{1}{2}x_{2,1} - x_2 - x_3, x_{1,2} - \frac{1}{2}x_{2,2} - x_{2,1} - x_{3,1}, u_1 - x_1 + x_2 + x_3,$$
$$u_2 - \frac{1}{2}x_2 - x_3, u_{1,1} + \frac{1}{2}x_{2,1} - x_2 + x_{3,1} - x_3, u_{2,1} - \frac{1}{2}x_{2,1} - x_{3,1},$$
$$u_{1,2} + \frac{1}{2}x_{2,2} - x_{2,1} + x_{3,2} - x_{3,1}, u_{2,2} - \frac{1}{2}x_{2,2} - x_{3,2}, u_{2,3} + u_{1,3} - x_{2,2} - x_{3,2}\Big\}.$$

Using Algorithm 4.2, by Lemma 4.3, a characteristic set of [PS] equals $\mathcal{A} = \{B_0, B_2, B_3\}$. Thus ID has dimension $n - 1$ and $B_0(X) = 0$ is an implicit equation of ID.

5.2. *Sufficient conditions for* $\dim(\mathrm{ID}) = n - 1$

In this section, we will assume $\mathrm{rank}(S) = n - 1$, to prove that the necessary conditions given in Theorem 5.2 are also sufficient conditions, for a nonzero linear differential polynomial A in $(\mathcal{G}_0)$ to be a characteristic polynomial of ID.

Recall that, by Remark 4.6, the Gröbner basis associated to $\mathcal{P}(X, U)$ is a subset of $\mathrm{span}_{\mathbb{K}}\mathrm{PS}$, and thus Definition 4.7 applies.

Lemma 5.6. *Let $\mathcal{G}$ be the Gröbner basis associated to the system $\mathcal{P}(X, U)$ with implicit ideal* ID, *$\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. Let S be the leading matrix of $\mathcal{P}(X, U)$. Given a nonzero linear* ID-*primitive differential polynomial A in $(\mathcal{G}_0)$, with $|\mathcal{G}_0| = \mathrm{c}(A) + 1$, the following statements hold.*

1. *For $j = 0, 1, \ldots, \mathrm{c}(A)$, there exist $\mathcal{D}_j \in \mathbb{K}[\partial]$, with $\deg(\mathcal{D}_j) = j$, such that $\mathcal{G}_0 = \{B_0 = \mathcal{D}_0(A), B_1 = \mathcal{D}_1(A), \ldots, B_{\mathrm{c}(A)} = \mathcal{D}_{\mathrm{c}(A)}(A)\}$.*
2. *If $\mathrm{rank}(S) = n - 1$. Given $B \in \mathcal{G}\backslash\mathcal{G}_0$, let us suppose there exists a positive integer e_B, such that $1 \leq e_B \leq \mathrm{c}(A) + 1$ and $\mathrm{ord}(B, u_j) \leq N - \gamma_j - \gamma - e_B$, $j = 1, \ldots, n - 1$. Then there exists a linear differential polynomial $\overline{B} \in (\mathcal{G}_0)$, such that $\mathrm{c}(B - \overline{B}) \geq e_B$.*

Proof. 1. Since $|\mathcal{G}_0| = L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A) + 1$, there exists an echelon form E of M_{2L} whose last $\mathrm{c}(A)+1$ rows contain the coefficients of $\partial^{\mathrm{c}(A)}A, \ldots, \partial A, A$. Then the last $\mathrm{c}(A)+1$ rows of the reduced echelon form E_{2L} of M_{2L} contain the coefficients of $B_{\mathrm{c}(A)} = \mathcal{D}_{\mathrm{c}(A)}(A), \ldots, B_1 = \mathcal{D}_1(A), B_0 = \mathcal{D}_0(A)$, for some $\mathcal{D}_j \in \mathbb{K}[\partial]$, $\deg(\mathcal{D}_j) = j$, $j = 0, 1, \ldots, \mathrm{c}(A)$. Therefore $\mathcal{G}_0 = \{B_0, B_1, \ldots, B_{\mathrm{c}(A)}\}$.

2. Let $s \in \{0, \ldots, e_B - 1\}$. By 1, the co-order of $B_{\mathrm{c}(A)-s}$ equals $\mathrm{c}(B_{\mathrm{c}(A)-s}) = s$. Since $B_{\mathrm{c}(A)-s} \in \mathbb{K}\{X\}$, $\mathrm{ord}(B_{\mathrm{c}(A)-s}, u_j) < N - \gamma_j - \gamma - s$, hence by Remark 4.8, $\mathrm{l}(B_{\mathrm{c}(A)-s})^T \in \mathrm{Ker}(S^T)$. Given $B \in \mathcal{G}\backslash\mathcal{G}_0$, we will prove by induction on s that, for $s = 0, \ldots, e_B - 1$, there exists a linear $C_s \in (\mathcal{G}_0)$ such that $\mathrm{c}(B - C_s) \geq s + 1$. The linear differential polynomial in $(\mathcal{G}_0)$ we were looking for is $\overline{B} = C_{e_B-1}$.

   By Remark 4.6, there exist $\mathcal{F}_i \in \mathbb{K}[\partial]$, with $\deg(\mathcal{F}_i) \leq N - o_i - \gamma$, $i = 1, \ldots, n$, such that $B = \sum_{i=1}^{n} \mathcal{F}_i(F_i(X, U))$. Let β_i be the coefficient of $\partial^{N-o_i-\gamma}$ in $\mathcal{F}_i$. By assumption, $\mathrm{ord}(B, u_j) \leq N - \gamma_j - \gamma - e_B < N - \gamma_j - \gamma$, so $\beta^T = (\beta_1, \ldots, \beta_n)^T \in \mathrm{Ker}(S^T)$. Now $\mathrm{rank}(S) = n - 1$, which means that $\dim \mathrm{Ker}(S^T) = 1$, so there exists $\mu \in \mathbb{K}$ such that $\beta = \mu l(B_{\mathrm{c}(A)})$. Let $C_0 = \mu B_{\mathrm{c}(A)}$, then $\mathrm{c}(B - C_0) \geq 1$. This proves the claim for $s = 0$.

Assuming the claim is true for $s-1$, $s \geq 1$, there exists a linear $C_{s-1} \in (\mathcal{G}_0)$ such that $\mathrm{c}(B-C_{s-1}) \geq s$. Then there exists $\mathcal{F}_i^s \in \mathbb{K}[\partial]$, with $\deg(\mathcal{F}_i^s) \leq N - o_i - \gamma - s$, $i = 1, \dots, n$, such that $B - C_{s-1} = \sum_{i=1}^{n} \mathcal{F}_i^s(F_i(X, U))$. Let β_i^s be the coefficient of $\partial^{N-o_i-\gamma-s}$ in $\mathcal{F}_i^s$. By assumption, $\mathrm{ord}(B-C_{s-1}, u_j) \leq N-\gamma_j-\gamma-e_B < N-\gamma_j-\gamma-s$, so $(\beta^s)^T = (\beta_1^s, \dots, \beta_n^s)^T \in \mathrm{Ker}(S^T)$. Now there exists $\mu_s \in \mathbb{K}$ such that $\beta^s = \mu_s l(B_{\mathrm{c}(A)-s})$. Let $C_s = C_{s-1} + \mu_s B_{\mathrm{c}(A)-s}$, then $\mathrm{c}(B - C_s) \geq s + 1$. □

Theorem 5.7. *Let $\mathcal{G}$ be the Gröbner basis associated to the system $\mathcal{P}(X, U)$ with implicit ideal* ID, $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. *Let S be the leading matrix of $\mathcal{P}(X, U)$. If* $\mathrm{rank}(S) = n - 1$, *and if there exists a nonzero linear* ID*-primitive differential polynomial A in $(\mathcal{G}_0)$, with $|\mathcal{G}_0| = \mathrm{c}(A) + 1$, then A is a characteristic polynomial of* ID.

Proof. If $\mathrm{c}(A) = 0$ then $\mathrm{rank}(M_{L-1}) = L-1$. By Theorem 18(2) and Lemma 20(4) in Rueda and Sendra (2010), and Lemma 5.6(1), A is a characteristic polynomial of ID.

Let us suppose that $\mathrm{c}(A) > 0$. We use Algorithm 4.2 to prove the result. By Lemma 5.6(1), $\mathcal{G}_0 = \{B_0 = \mathcal{D}_0(A), B_1 = \mathcal{D}_1(A), \dots, B_{\mathrm{c}(A)} = \mathcal{D}_{\mathrm{c}(A)}(A)\}$. For $j \in \{1, \dots, n-1\}$, let

$$\Gamma_j = \{B \in \mathbb{K}\{X, U\} \mid \mathrm{lead}(B) = u_{j,k}, k \in \mathbb{N}_0\}.$$

Then $\mathcal{G} \backslash \mathcal{G}_0 = \{B_{\mathrm{c}(A)+1}, \dots, B_{L-1}\} = \cup_{j=1}^{n-1}(\mathcal{G} \cap \Gamma_j)$. Given $i \in \{\mathrm{c}(A) + 1, \dots, L-1\}$, $B_i \in \Gamma_{j_i}$, for some $j_i \in \{1, \dots, n-1\}$. The proof is based on the following claims:

1. For $i \in \{1, \dots, \mathrm{c}(A)\}$, by Lemma 5.6(1), $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) = 0$. Thus $\{B_0\} = \mathcal{A}^0 = \mathcal{A}^1 = \cdots = \mathcal{A}^{\mathrm{c}(A)}$.
2. Let $R_i = \mathrm{prem}(B_i, \mathcal{A}^{i-1})$, $i = \mathrm{c}(A) + 1, \dots, L-1$. We will prove below, by induction on i, that either $R_i = 0$ or $\mathrm{lead}(R_i) = \mathrm{lead}(B_i)$.

From the previous statements and Lemma 4.3, it follows that $\mathcal{A}$ is a characteristic set of [PS] w.r.t. $\mathcal{R}^\star$. Furthermore, in the ith iteration of Algorithm 4.2(2) either an element not in $\mathbb{K}\{X\}$ is added to $\mathcal{A}^{i-1}$ or no element is added to $\mathcal{A}^{i-1}$. This proves that

$$\mathcal{A}_0 = \mathcal{A} \cap \mathbb{K}\{X\} = \mathcal{A}^0 = \{\mathcal{D}_0(A)\}$$

and ultimately that A is a characteristic polynomial of ID.

Proof of 2. Observe that $\Gamma_{j_{\mathrm{c}(A)+1}} \cap \mathcal{A}^{\mathrm{c}(A)} = \emptyset$, which implies $\mathrm{lead}(R_{\mathrm{c}(A)+1}) = \mathrm{lead}(B_{\mathrm{c}(A)+1})$. Given $i \in \{\mathrm{c}(A)+2, \dots, L-1\}$, if $\Gamma_{j_i} \cap \mathcal{A}^{i-1} = \emptyset$ then $\mathrm{lead}(R_i) = \mathrm{lead}(B_i)$. Let us assume that $\Gamma_{j_i} \cap \mathcal{A}^{i-1} \neq \emptyset$. We will prove next that $R_i = 0$.

By induction hypothesis $\mathcal{B} = \{B_{\mathrm{c}(A)+1}, \dots, B_{i-1}\} \cap \Gamma_{j_i} \neq \emptyset$. We prove first that there exists $B \in \mathcal{B}$ such that

$$1 \leq e_B = \mathrm{ord}(B_i, u_{j_i}) - \mathrm{ord}(B, u_{j_i}) \leq \mathrm{c}(A) + 1. \tag{3}$$

Let us suppose that $e_B > \mathrm{c}(A) + 1$, for all $B \in \mathcal{B}$. Then $e = \min\{e_B \mid B \in \mathcal{B}\} > \mathrm{c}(A) + 1$. Let $o = \mathrm{ord}(B_i, u_{j_i})$ and let E_{2L} be the $L \times 2L$ matrix in echelon form (as in Section 4), whose rows are the coefficients of the polynomials in $\mathcal{G}$. By definition of e, no row of E_{2L} has a pivot position in the columns indexed by $u_{j_i,o-1}, \dots, u_{j_i,o-e+1}$. Thus $\mathrm{rank}(M_{L-1}) \leq L-1-(e-1) < L-1-\mathrm{c}(A)$, contradicting that $L - \mathrm{rank}(M_{L-1}) = |\mathcal{G}_0| = \mathrm{c}(A) + 1$.

By (3), $\mathrm{ord}(B, u_{j_i}) \leq N - \gamma_{j_i} - \gamma - e_B$ and, $B \in \Gamma_{j_i}$ implies $\mathrm{ord}(B, u_j) \leq \mathrm{ord}(B, u_{j_i})$, for $j = 1, \dots, n-1$. By Lemma 5.6(2), there exists a linear polynomial $\overline{B} \in (\mathcal{G}_0)$ such that $\mathrm{c}(B - \overline{B}) \geq e_B$. Let $C_i = \mathrm{prem}(B_i, B - \overline{B})$, which is a linear polynomial. Recall that $B \in \mathcal{B}$ and $\overline{B} \in \mathbb{K}\{X\}$. This implies that $B \in \Gamma_{j_i}$ and $\mathrm{lead}(B - \overline{B}) = \mathrm{lead}(B) < \mathrm{lead}(B_i)$. Thus, by definition of C_i and the fact that both $B_i, B \in \Gamma_{j_i}$, the inequality $\mathrm{lead}(C_i) < \mathrm{lead}(B_i)$ holds. By definition of e_B, there exists a differential operator $\mathcal{F} \in \mathbb{K}[\partial]$, with $\deg(\mathcal{F}) \leq e_B$, such that $C_i = B_i - \mathcal{F}(B - \overline{B})$. Now, $\mathrm{c}(B - \overline{B}) \geq e_B$ guarantees that $\mathcal{F}(B - \overline{B}) \in (\mathrm{PS})$, which implies $C_i \in (\mathrm{PS})$.

To finish, we use $B_i = C_i + \mathcal{F}(B - \overline{B})$ to prove that $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) = 0$. We proved in the previous paragraph that, the linear polynomial $C_i \in (\mathrm{PS})$ and $\mathrm{lead}(C_i) < \mathrm{lead}(B_i)$. Also, $\mathcal{G} = \{B_0, B_1, \dots, B_{L-1}\}$ is the reduced Gröbner basis of (PS), with $\mathrm{lead}(B_0) < \mathrm{lead}(B_1) < \cdots < \mathrm{lead}(B_{L-1})$. Then $C_i = \gamma_0 B_0 + \cdots + \gamma_{i-1} B_{i-1}$, with $\gamma_0, \gamma_1, \dots, \gamma_{i-1} \in \mathbb{K}$. It holds that, $B_0, \dots, B_{\mathrm{c}(A)} \in [\mathcal{A}^0] \subset [\mathcal{A}^{i-1}]$. For every $l \in \{\mathrm{c}(A) + 1, \dots, i-1\}$, the linear polynomial $B_l = \mathrm{prem}(B_l, \mathcal{A}^{l-1}) + P$, for some linear polynomial $P \in [\mathcal{A}^{l-1}]$, hence $B_l \in [\mathcal{A}^l] \subseteq [\mathcal{A}^{i-1}]$. Thus $C_i \in [\mathcal{A}^{i-1}]$. Recall that $B \in \mathcal{B}$, so $B = B_{l_0}$, with $\mathrm{c}(A)+1 \leq l_0 \leq i-1$. Observe that, the linear polynomial $\overline{B} \in (\mathcal{G}_0) \subset [\mathcal{A}^0] \subset [\mathcal{A}^{i-1}]$ and $B_{l_0} \in [\mathcal{A}^{i-1}]$, then $\mathcal{F}(B-\overline{B}) \in [\mathcal{A}^{i-1}]$. At this point, we have proved that B_i is a polynomial in $[\mathcal{A}^{i-1}]$, which implies that $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) = 0$. □

Corollary 5.8. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs, with implicit ideal* ID*. Let S and M_{L-1} be the leading and principal matrices of $\mathcal{P}(X,U)$ respectively. If* $\mathrm{rank}(S) = n-1$ *then the following statements are equivalent.*

1. *The dimension of* ID *is $n-1$.*
2. *There exists a nonzero linear* ID*-primitive differential polynomial A such that $L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A)+1$.*

In such situation, $A(X) = 0$ is the implicit equation of $\mathcal{P}(X,U)$.

Proof. By Theorem 5.2, (1)⇒(2). By Theorem 5.7, (2)⇒(1). □

Given $\mathcal{P}(X,U)$ with implicit ideal ID and leading matrix S. If $\mathrm{rank}(S) < n-1$, it is natural to wonder if there exists a linear system of DPPEs $\mathcal{P}'(X,U)$, with implicit ideal ID$'$ and leading matrix S', such that ID $=$ ID$'$ and $\mathrm{rank}(S') = n-1$. We will not deal with this question in this paper but, we show next how in Example 5.5 this question is easily solved.

Example 5.9. We continue with Example 5.5. Let $\eta(U) = u_1 - u_2$ and let us replace u_1 by $\eta(U)$ in $\mathcal{P}(X,U)$ to obtain the system

$$\mathcal{P}'(X,U) = \mathcal{P}(x_1, x_2, x_3, \eta(U), u_2).$$

From $\mathcal{P}'(X,U)$ we obtain the polynomials

$$\begin{aligned} F_1'(X,U) &= x_1 - u_1 + u_2 - u_{1,1}, \\ F_2'(X,U) &= x_2 + 2u_2 - 2u_{1,1}, \\ F_3'(X,U) &= x_3 - 2u_2 + u_{1,1}. \end{aligned}$$

In this case, the completeness index $\gamma' = \gamma_2' = 1$ and, the leading matrix S' of $\mathcal{P}'(X,U)$ has rank $n-1=2$ and equals

$$S' = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ -2 & 1 \end{bmatrix}.$$

In fact, $A = \partial\mathrm{CRes}(F_1', F_2', F_3') = 4x_3 + 2x_{2,1} - 4x_{1,1} + 4x_2$ and by Rueda and Sendra (2010), Theorem 21, $A(X) = 0$ is the implicit equation of $\mathcal{P}'(X,U)$. Therefore, ID $= [B_0] = [A] =$ ID$'$, with B_0 as in Example 5.5.

6. Linear perturbations of $\mathcal{P}(X,U)$

Let $\mathcal{P}(X,U)$, F_i, H_i be as in Section 2. Let p be an algebraic indeterminate over $\mathbb{K}$, thus $\partial(p) = 0$. Denote $\mathbb{K}_p = \mathbb{K}\langle p\rangle$ the differential field extension of $\mathbb{K}$ by p. A linear perturbation of the system $\mathcal{P}(X,U)$ is a new system

$$\mathcal{P}_\phi(X,U) = \begin{cases} x_1 &= P_1(U) + p\,\phi_1(U), \\ &\vdots \\ x_n &= P_n(U) + p\,\phi_n(U), \end{cases}$$

where the linear perturbation $\phi = (\phi_1(U), \ldots, \phi_n(U))$ is a family of linear differential polynomials in $\mathbb{K}\{U\}$. For $i = 1, \ldots, n$, let

$$F_i^\phi(X,U) = F_i(X,U) - p\,\phi_i(U) \quad \text{and} \quad H_i^\phi(U) = H_i(U) - p\,\phi_i(U).$$

The set $\mathrm{PS}_\phi = \mathrm{PS}(F_1^\phi, \ldots, F_n^\phi)$ is a set of linear differential polynomials in $\mathbb{K}_p[\mathcal{X}][\mathcal{V}] \subset \mathbb{K}_p\{X \cup U\}$. Let (PS_ϕ) be the ideal generated by PS_ϕ in $\mathbb{K}_p[\mathcal{X}][\mathcal{V}]$. We prove next the existence of a linear perturbation ϕ such that $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) \neq 0$.

Let us suppose that $o_n \geq o_{n-1} \geq \cdots \geq o_1$ to define the perturbation $\phi = (\phi_1(U), \ldots, \phi_n(U))$ by

$$\phi_i(U) = \begin{cases} u_{n-1,o_1-\gamma_{n-1}}, & i = 1, \\ u_{n-i,o_i-\gamma_{n-i}} + u_{n-i+1}, & i = 2, \ldots, n-1, \\ u_1, & i = n. \end{cases} \tag{4}$$

We use this perturbation to prove that $\partial\text{CRes}(F_1^\phi, \ldots, F_n^\phi) \neq 0$ but, as expected, the linear perturbation that makes $\partial\text{CRes}(F_1^\phi, \ldots, F_n^\phi) \neq 0$ is not unique (see Sections 7.1 and 7.3).

Let us suppose that $N \geq 1$. We denote by $M_\phi(L^h)$ the complete differential homogeneous resultant matrix, for the set of linear differential polynomials $H_1^\phi, \ldots, H_n^\phi$. The matrix $M_\phi(L^h)$ is $L^h \times L^h$, with elements in $\mathbb{K}[p]$, and there exists an $L^h \times L^h$ matrix M_ϕ, with elements in $\mathbb{K}$, such that $M_\phi(L^h) = M(L^h) - p\,M_\phi$. Then

$$\partial\text{CRes}^h(H_1^\phi, \ldots, H_n^\phi) = \det(M_\phi(L^h)) = \det(M(L^h) - p\,M_\phi). \tag{5}$$

Let S^ϕ be the leading matrix of $\mathcal{P}_\phi(X, U)$. For $i \in \{1, \ldots, n\}$, let S_i^ϕ be the $(n-1) \times (n-1)$ matrix obtained by removing the ith row of S^ϕ.

Proposition 6.1. *Given a system $\mathcal{P}(X, U)$ of linear DPPEs and the perturbation ϕ defined by (4), the following statements hold.*

1. *The determinant of S_n^ϕ is nonzero and it has degree $n-1$ in p.*
2. *If $N \geq 1$ then $\partial\text{CRes}^h(H_1^\phi, \ldots, H_n^\phi)$ is a polynomial in $\mathbb{K}[p]$, of degree L^h and not identically zero.*

Proof. 1. Observe that S_n^ϕ has p's in the main diagonal.

2. By (5), we can write $\det(M_\phi(L^h)) = p^{L^h} \det((1/p)M(L^h) - M_\phi)$. If we set $y = 1/p$ then the matrix obtained from $yM(L^h) - M_\phi$ at $y = 0$ is $-M_\phi$. Thus $\det(-M_\phi)$ is the coefficient of p^{L^h} in $\det(M_\phi(L^h))$. The remaining part of the proof is devoted to show that $\det(M_\phi) \neq 0$, which ensures that the degree of $\det(M_\phi(L^h))$ in p is L^h.

   The matrix M_ϕ contains in its $l^h(i, k)$th row the coefficients, as a polynomial in $\mathbb{D}[\mathcal{V}^h]$, of $\partial^{N-o_i-\gamma-k-1}\phi_i(U)$, $i = 1, \ldots, n$, $N - o_i - \gamma - 1 \geq 0$, $k = 0, \ldots, N - o_i - \gamma - 1$. We will prove that $\det(M_\phi) \neq 0$ in two steps:

   2.1. We reorganize the rows of M_ϕ to get a matrix M, which has ones in the main diagonal and, in every row at most one nonzero entry not in the main diagonal, equal to 1.

   2.2. We perform row operations on M to get an upper triangular matrix M', with 1's in the main diagonal.

   2.1. We define $O_j := o_j - \gamma_{n-j}$, $j = 1, \ldots, n-1$ and $O_n := o_n$. The matrix M_ϕ has $L^h = (n-1)(N - o_n - \gamma) + (n-1)o_n - \gamma$ rows where

$$L^h = (n-1)\sum_{j=1}^{n-1} O_j + \sum_{j=1}^{n-1}(o_n - \gamma_{n-j}). \tag{6}$$

   If $N - o_n - \gamma = 0$ then $O_j = 0$ and $N - o_j - \gamma = o_n - o_j = o_n - \gamma_{n-j}$, for $j = 1, \ldots, n-1$. If $N - o_n - \gamma > 0$, the assumption $o_n \geq \cdots \geq o_1$ to define (4) implies $0 \leq N - o_n - \gamma - 1 \leq N - o_j - \gamma - 1, j = 1, \ldots, n-1$.

   Let $\mathcal{J} := \{j \in \{1, \ldots, n-1\} \mid o_n - \gamma_{n-j} > 0\}$. Let Γ be the submatrix of M_ϕ, whose rows contain the coefficients of the $o_n(n-1) - \gamma = \sum_{j=1}^{n-1}(o_n - \gamma_{n-j})$ differential polynomials in the set

$$\{\partial^{N-o_j-\gamma-1}\phi_j, \ldots, \partial^{N-O_n-\gamma-O_j}\phi_j \mid j \in \mathcal{J}\},$$

   where $N - O_n - \gamma - O_j = N - o_j - \gamma - (o_n - \gamma_{n-j})$. Each one of the previous rows has a 1, respectively, in the column indexed by the monomials in the set

$$\Omega_n := \{u_{n-j,N-\gamma_{n-j}-\gamma-k} \mid j \in \mathcal{J}, k = 1, \ldots, o_n - \gamma_{n-j}\}.$$

Now we reorganize the rows of Γ. For $j \in \mathcal{J}$ and $k = 1, \ldots, o_n - \gamma_{n-j}$, let us suppose that $u_{n-j,N-\gamma_{n-j}-\gamma-k}$ is in column $c_{jk} \in \{1, \ldots, (n-1)o_n - \gamma\}$ of Γ and, let Δ_n be the $((n-1)o_n - \gamma) \times L^h$ matrix whose row c_{jk} contains the coefficients of $\partial^{N-o_j-\gamma-k}\phi_j$. Thus Δ_n has ones in the main diagonal of its $(n-1)o_n - \gamma$ principal submatrix and, in each row at most one nonzero entry not in the mentioned diagonal, equal to 1. If $N - o_n - \gamma = 0$, by (6), then $\Gamma = M_\phi$ and the matrix $M := \Delta_n$.

We assume that $N - o_n - \gamma > 0$. Observe that, if $o_1 = 0$ then $\gamma_j = 0, j = 1, \ldots, n-1$, since $\gamma_j \leq o_1$. Let $\mathcal{I} := \{i \in \{1, \ldots, n-1\} \mid O_i > 0\}$. Given $i \in \mathcal{I}$, for $i \neq 1$ we define the polynomials

$$\psi^i_{j,k} := \begin{cases} \partial^{N-O_j-\sum_{h=i+1}^n O_h-\gamma-k}\phi_j, & j = 1, \ldots, i-1, \\ \partial^{N-\sum_{h=i+1}^n O_h-\gamma-k}\phi_j, & j = i+1, \ldots, n, \end{cases}$$

$k = 1, \ldots, O_i$, and for $i = 1$ we define the polynomials

$$\psi^1_{j,k} := \partial^{O_1-k}\phi_j, \quad j = 2, \ldots, n,\ k = 1, \ldots, O_1.$$

Let us denote by $r(\psi^i_{j,k})$ the row vector of M_ϕ containing the coefficients of $\psi^i_{j,k}$. Let $\Delta_{i,k}$, $k = 1, \ldots, O_i$, be the submatrix of M_ϕ obtained by stacking $n-1$ of these row vectors as follows

$$\Delta_{i,k} := \begin{cases} \text{stack}(r(\psi^i_{1,k}), \ldots, r(\psi^i_{i-1,k}), r(\psi^i_{i+1,k}), \ldots, r(\psi^i_{n,k})), & i \neq 1, \\ \text{stack}(r(\psi^i_{2,k}), \ldots, r(\psi^i_{n,k})), & i = 1. \end{cases}$$

The matrix $\Delta_i := \text{stack}(\Delta_{i,1}, \ldots, \Delta_{i,O_i})$ has $O_i(n-1)$ rows. The lth row of Δ_i has a 1 in the column indexed by the lth monomial in the set

$$\Omega_i := \begin{cases} \{u_{j,N-\sum_{h=i+1}^n O_h-\gamma-k} \mid j = n-1, \ldots, 1,\ k = 1, \ldots, O_i\}, & i \neq 1, \\ \{u_{n-1,O_1-k}, \ldots, u_{1,O_1-k} \mid k = 1, \ldots, O_1\}, & i = 1. \end{cases}$$

The monomials in Ω_i are arranged in decreasing order w.r.t. the orderly ranking on U, as in Section 3. This is the order of the monomials in the set $\mathcal{V}^h$ indexing the columns of M_ϕ.

Finally, the union of Ω_n with the sets Ω_i, $i \in \mathcal{I}$, equals the set $\mathcal{V}^h$. Let us suppose that $\mathcal{I}$ has $I \geq n-1$ elements, $\mathcal{I} = \{i_1, \ldots, i_I\}$, with $i_I > \cdots > i_1$. Stacking the matrix Δ_n with the matrices in the set $\{\Delta_i \mid i \in \mathcal{I}\}$, we obtain

$$M := \text{stack}(\Delta_n, \Delta_{i_I}, \ldots, \Delta_{i_1}),$$

with 1's in the main diagonal and, in every row only one nonzero entry not in the main diagonal, equal to 1.

2.2. The matrix M has three kinds of rows, which we will name as follows: right-row (left-row), with two nonzero entries, both equal to 1, one in the main diagonal of M and the other to the right (left) of the main diagonal of M; diag-row, with only one nonzero entry, equal to 1, which is in the main diagonal of M.

If $N - o_n - \gamma = 0$, by (4), all the rows of $M = \Delta_n$ are right-rows, thus $M' := M$. We assume in the remaining parts of the proof that $N - o_n - \gamma > 0$. Given $u \in \mathcal{V}^h$, let $r(u)$ denote the coefficient vector of u as a polynomial in $\mathbb{K}[\mathcal{V}^h]$, whose L^h entries are all zero except for a 1 in the column indexed by u. Given a matrix T, let $r(T)$ denote the set of row vectors of T.

Observe that, blocks Δ_n and Δ_{n-1} (if $O_{n-1} > 0$) of M have only right-rows and diag-rows. Blocks Δ_i, $i \in \{n-2, \ldots, 1\} \cap \mathcal{I}$ have also left-rows and our goal is to replace them by diag-rows using row operations. Given $i \in \mathcal{I}$, let $\Delta^n_i := \Delta_i$, which contains the diag-rows

$$r(\psi^i_{n,k}) = r(u_{1,N-\sum_{h=i+1}^n O_h-\gamma-k}), \quad k = 1, \ldots, O_i. \tag{7}$$

If $O_{n-1} > 0$, set $\Delta'_{n-1} := \Delta_{n-1}$. For $i \in \{n-2, \ldots, 1\} \cap \mathcal{I}$, we replace the left-rows of Δ_i by diag-rows, to obtain a matrix Δ'_i. It holds

$$\{n-2, \ldots, 1\} \cap \mathcal{I} = \{i_1, \ldots, i_H\},$$

with $H = I$ if $n-1 \notin \mathcal{I}$ and $H = I-1$ if $n-1 \in \mathcal{I}$. For $t = H, \ldots, 1$, set $i := i_t$. For $j = n-1, \ldots, i+1$ replace $r(\psi^i_{j,k})$, $k = 1, \ldots, O_i$, by

$$r(u_{n-j+1,N-\sum_{h=i+1}^n O_h-\gamma-k}) = r(\psi^i_{j,k}) - r(u_{n-j,O_j+N-\sum_{h=i+1}^n O_h-\gamma-k}), \tag{8}$$

to obtain a matrix Δ_i^j. Set $\Delta_i' := \Delta_i^{i+1}$. To finish

$$M' := \text{stack}(\Delta_n, \Delta_{i_I}', \ldots, \Delta_{i_1}').$$

It remains to prove that the r.h.s. of (8) is an operation with rows in $\cup_{l\in\{j,\ldots,i\}\cap\mathcal{I}} r(\Delta_l^{j+1})$. For $i \in \{n-2,\ldots,1\}\cap\mathcal{I}, j \in \{n-1,\ldots,i+1\}$, the set $\{r(u_{n-j,O_j+N-\sum_{h=i+1}^n O_h-\gamma-k}) \mid k = 1,\ldots,O_i\}$ is included in

$$\mathcal{S}_i^j := \left\{ r(u_{n-j,N-\sum_{h=j+1}^n O_h-\gamma-k}) \mid k = 1,\ldots,\sum_{h=i}^j O_h \right\}.$$

For $t = H,\ldots,1$, set $i := i_t$. We prove next, by induction on t, that for $j = n-1,\ldots,i+1$,

$$\mathcal{S}_i^j \subset \cup_{l\in\{j,\ldots,i\}\cap\mathcal{I}} r(\Delta_l^{j+1}). \tag{9}$$

Observe that $\{n-1,\ldots,i_H\}\cap\mathcal{I} = \{i_I, i_H\}$. By equation (7), $\mathcal{S}_{i_H}^{n-1} \subset \cup_{l\in\{i_I,i_H\}} r(\Delta_l^n)$. If $i_H < n-2$, for $j = n-2,\ldots,i_H+1$ then $\{j,\ldots,i_H\}\cap\mathcal{I} = \{i_H\}$ and by the l.h.s. of (8), $\mathcal{S}_{i_H}^j \subset r(\Delta_{i_H}^{j+1})$.

Let $t \in \{H-1,\ldots,1\}$, set $i := i_t$ and let us assume that (9) is true for $t+1$

$$\mathcal{S}_{i_{t+1}}^j \subset \cup_{l\in\{j,\ldots,i_{t+1}\}\cap\mathcal{I}} r(\Delta_l^{j+1}), j = n-1,\ldots,i_{t+1}+1. \tag{10}$$

By Eq. (7), $\mathcal{S}_i^{n-1} \subset \cup_{l\in\{n-1,\ldots,i\}\cap\mathcal{I}} r(\Delta_l^n)$. Given $j \in \{n-2,\ldots,i+2\}$, by the l.h.s. of (8), $r(\Delta_i^{j+1})$ contains rows

$$r(u_{n-j,N-\sum_{h=i+1}^n O_h-\gamma-k}), \quad k = 1,\ldots,O_i. \tag{11}$$

Observe that, if $i_{t+1} > i+1$ then $O_{i_{t+1}-1} = \cdots = O_{i+1} = 0$. Thus (10) together with (11) proves (9) for $j = n-2,\ldots,i+2$. Now take $j = i+1$. By the l.h.s. of (8), $r(\Delta_i^{i+2})$ contains

$$\{r(u_{n-(i+1),N-\sum_{h=i+1}^n O_h-\gamma-k}) \mid k = 1,\ldots,O_i\}. \tag{12}$$

If $i+1 \notin \mathcal{I}$, then $O_{i+1} = 0$ and S_i^{i+1} equals (12). If $i+1 \in \mathcal{I}$ then, by the l.h.s. of (8), $r(\Delta_{i+1}^{i+2})$ contains

$$\{r(u_{n-(i+1),N-\sum_{h=i+2}^n O_h-\gamma-k}) \mid k = 1,\ldots,O_{i+1}\}. \tag{13}$$

Thus S_i^{i+1} is the union of (12) and (13), which proves $S_i^{i+1} \subset r(\Delta_{i+1}^{i+2}) \cup r(\Delta_i^{i+2})$. This proves (9) for the chosen i and $j = n-1,\ldots,i+1$.

Theorem 6.2. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs, there exists a linear perturbation ϕ such that the differential resultant $\partial\text{CRes}(F_1^\phi,\ldots,F_n^\phi)$ is a nonzero polynomial in $\mathbb{K}[p]\{X\}$ and $\det(S_n^\phi) \neq 0$.*

Proof. Let ϕ be the perturbation defined by (4). By Proposition 6.1, $\det(S_n^\phi) \neq 0$. If $N = 0$, the result follows from

$$\partial\text{CRes}(F_1^\phi,\ldots,F_n^\phi) = \sum_{i=1}^n (-1)^{i+n} \det(S_i^\phi)(x_i - a_i).$$

If $N \geq 1$ then $\partial\text{CRes}^h(H_1^\phi,\ldots,H_n^\phi) \neq 0$, by Proposition 6.1. This is equivalent, by Rueda and Sendra (2010), Theorem 18(2), to $\partial\text{CRes}(F_1^\phi,\ldots,F_n^\phi) \neq 0$. □

If nonzero, $\partial\text{CRes}(F_1^\phi,\ldots,F_n^\phi)$ is a polynomial in p, whose coefficients are linear differential polynomials in $\mathbb{K}\{X\}$. We focus our attention next on the coefficient of the lowest degree term, in p, of $\partial\text{CRes}(F_1^\phi,\ldots,F_n^\phi)$.

Theorem 6.3. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs, let ϕ be a linear perturbation such that $\partial\text{CRes}(F_1^\phi,\ldots,F_n^\phi) \neq 0$ and $\det(S_n^\phi) \neq 0$. The following statements hold.*

1. *There exists a linear differential polynomial P in $(\text{PS}_\phi) \cap \mathbb{K}_p\{X\}$, with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$, such that $\alpha_n \neq 0$, where $l(P) = (\alpha_1,\ldots,\alpha_n)$.*

2. *There exists $a \in \mathbb{N}$ such that*

$$\alpha_n \partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi}) = (-1)^a \det(S_n^{\phi}) \partial\mathrm{CRes}^h(H_1^{\phi}, \ldots, H_n^{\phi}) P(X). \quad (14)$$

Furthermore, $\frac{\det(S_n^{\phi})\partial\mathrm{CRes}^h(H_1^{\phi},\ldots,H_n^{\phi})}{\alpha_n} \in \mathbb{K}[p]$.

Proof. 1. By Lemma 4.1, there exists a nonzero linear differential polynomial $B \in (\mathrm{PS}_{\phi}) \cap \mathbb{K}_p\{X\}$. There exists $c \in \mathbb{K}_p$ such that $P(X) = c\partial^{\mathrm{c}(B)} B(X)$ has coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$. Observe that $\mathrm{c}(P) = 0$. Since $P \in \mathbb{K}_p\{X\}$ then $\mathrm{ord}(P, u_j) < N - \gamma_j - \gamma$ and, by Remark 4.8, it holds $l(P)^T \in \mathrm{Ker}((S^{\phi})^T)$. If $\alpha_n = 0$ then $\det(S_n^{\phi}) \neq 0$ implies $\alpha_i = 0, i = 1, \ldots, n$. This contradicts that $\mathrm{c}(P) = 0$ and therefore $\alpha_n \neq 0$.

2. Eq. (14) follows from Rueda and Sendra (2010), Theorem 18(1). Since the content of $P(X)$ belongs to $\mathbb{K}$, α_n divides $\det(S_n^{\phi})\partial\mathrm{CRes}^h(H_1^{\phi}, \ldots, H_n^{\phi})$ in $\mathbb{K}[p]$. □

Let D_{ϕ} be the lowest degree of p in $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi})$ and let $A_{D_{\phi}}$ be the coefficient of $p^{D_{\phi}}$ in $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi})$. We call D_{ϕ} the degree of the perturbed system $\mathcal{P}_{\phi}(X, U)$. Observe that

$$D_{\phi} = 0 \Leftrightarrow \partial\mathrm{CRes}(F_1, \ldots, F_n) \neq 0.$$

We write $D_{\phi} = -1$ if $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi}) = 0$ and so

$$D_{\phi} \geq 0 \Leftrightarrow \partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi}) \neq 0.$$

We assume that $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi}) \neq 0$, in the remaining parts of this section. Let $\mathrm{PS} = \mathrm{PS}(F_1, \ldots, F_n)$ and let ID be the implicit ideal of $\mathcal{P}(X, U)$. We will use $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi})$ to provide a nonzero ID-primitive differential polynomial A_{ϕ} in $(\mathrm{PS}) \cap \mathbb{K}\{X\}$.

Lemma 6.4. *The linear differential polynomial $A_{D_{\phi}}$ belongs to* $(\mathrm{PS}) \cap \mathbb{K}\{X\}$.

Proof. By Rueda and Sendra (2010), Proposition 16, $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi}) \in (\mathrm{PS}_{\phi}) \cap \mathbb{K}_p\{X\}$. Furthermore, by definition of F_i^{ϕ}, $i = 1, \ldots, n$, $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi})$ is the determinant of a matrix with entries in $\mathbb{K}[p]\{X\}$ and it equals $p^{D_{\phi}}(A_{D_{\phi}} + pA')$, for some $A' \in \mathbb{K}[p]\{X\}$. Therefore, the linear polynomial $A_{D_{\phi}} + pA' \in (\mathrm{PS}_{\phi}) \cap \mathbb{K}[p]\{X\}$ and there exist $\mathcal{F}_i \in \mathbb{K}[p][\partial]$, with $\deg(\mathcal{F}_i) \leq N - o_i - \gamma$, such that $A_{D_{\phi}}(X) + pA'(X) = \sum_{i=1}^{n} \mathcal{F}_i(F_i^{\phi}(X, U))$. Then $A_{D_{\phi}}(X) + pA'(X) = \sum_{i=1}^{n} \mathcal{F}_i(x_i - a_i)$ and $\sum_{i=1}^{n} \mathcal{F}_i(H_i^{\phi}(U)) = 0$.

For each $i \in \{1, \ldots, n\}$, there exist unique $\mathcal{L}_i \in \mathbb{K}[\partial]$ and $\mathcal{F}_i' \in \mathbb{K}[p][\partial]$ such that $\mathcal{F}_i = \mathcal{L}_i + p\mathcal{F}_i'$. Then $A_{D_{\phi}}(X) + pA'(X) = \sum_{i=1}^{n} \mathcal{L}_i(x_i - a_i) + p\sum_{i=1}^{n} \mathcal{F}_i'(x_i - a_i)$ and $A_{D_{\phi}}(X) = \sum_{i=1}^{n} \mathcal{L}_i(x_i - a_i)$. On the other hand,

$$0 = \sum_{i=1}^{n} \mathcal{F}_i(H_i^{\phi}(U)) = \sum_{i=1}^{n} \mathcal{L}_i(H_i(U)) + p\sum_{i=1}^{n} \mathcal{L}_i(\phi_i(U)) + p\sum_{i=1}^{n} \mathcal{F}_i'(H_i^{\phi}(U)),$$

which implies $\sum_{i=1}^{n} \mathcal{L}_i(H_i(U)) = 0$. Thus, $A_{D_{\phi}}(X) = \sum_{i=1}^{n} \mathcal{L}_i(F_i(X, U))$, with $\deg(\mathcal{L}_i) \leq N - o_i - \gamma$, showing that $A_{D_{\phi}} \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$. □

Let A_{ϕ} be the ID-primitive part of $A_{D_{\phi}}$. We call A_{ϕ} the differential polynomial associated to $\mathcal{P}_{\phi}(X, U)$. We relate D_{ϕ} with $\mathrm{c}(A_{\phi})$ and give conditions, on D_{ϕ}, for $A_{\phi}(X) = 0$ to be the implicit equation of $\mathcal{P}(X, U)$.

Remark 6.5. Let $\mathcal{P}_{\phi}(X, U)$ and $\mathcal{P}_{\psi}(X, U)$ be two different linear perturbations of $\mathcal{P}(X, U)$, with degrees $D_{\phi} \geq 0$ and $D_{\psi} \geq 0$. Let A_{ϕ} and A_{ψ} be the associated differential polynomials.

1. As illustrated in Example 1 of Section 7, the degrees D_{ϕ} and D_{ψ} may be different.
2. If $\dim \mathrm{ID} = n - 1$ then $A_{\phi} = \gamma A_{\psi}$, for some $\gamma \in \mathbb{K}$.

Theorem 6.6. *Let $\mathcal{P}_{\phi}(X, U)$ be a perturbed system of $\mathcal{P}(X, U)$, with degree $D_{\phi} \geq 0$. Let $\mathcal{G}$ be the Gröbner basis associated to $\mathcal{P}(X, U)$ and $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. Then $|\mathcal{G}_0| - 1 \leq D_{\phi}$.*

Proof. Let $M_\phi(L)$ be the differential resultant matrix of $F_1^\phi, \dots, F_n^\phi$ and M_{L-1}^ϕ the $L \times (L-1)$ principal submatrix of $M_\phi(L)$. By equation (2),

$$\partial\text{CRes}(F_1^\phi, \dots, F_n^\phi) = \sum_{i=1}^{n} \sum_{k=0}^{N-o_i-\gamma} b_{ik} \det(M_{x_{i,k}}^\phi)(x_{i,k} - \partial^k a_i),$$

with $M_{x_{i,k}}^\phi$ an $(L-1)\times(L-1)$ submatrix of M_{L-1}^ϕ and $b_{ik} = \pm 1$, according to the row index of $x_{i,k} - \partial^k a_i$ in the matrix $M_\phi(L)$. Let $M_{x_{i,k}}$ be the $(L-1)\times(L-1)$ submatrix of the principal matrix M_{L-1} of $\mathcal{P}(X, U)$. Let $r_{ik} = \text{rank}(M_{x_{i,k}})$. There exists an invertible matrix E_{ik}, of order $L-1$ and entries in $\mathbb{K}$, such that the last $L-1-r_{ik}$ rows of $E_{ik}M_{x_{i,k}}$ are zero. If we divide each one of the last $L-1-r_{ik}$ rows of $E_{ik}M_{x_{i,k}}^\phi$ by p, we obtain a matrix N_{ik}^ϕ and $c_{ik} \in \mathbb{K}$ such that

$$\det(M_\phi(L)) = \sum_{i=1}^{n} \sum_{k=0}^{N-o_i-\gamma} c_{ik} p^{L-1-r_{ik}} \det(N_{ik}^\phi)(x_{i,k} - \partial^k a_i).$$

Let $r_{i_0k_0} = \max\{r_{i,k} \mid i \in \{1, \dots, n\}, k \in \{0, 1, \dots, N - o_i - \gamma\}, \det(N_{ik}^\phi) \neq 0\}$. This proves that $L - 1 - r_{i_0k_0} \leq D_\phi$. Now, $\text{rank}(M_{L-1}) \geq r_{i_0k_0}$ so

$$|\mathcal{G}_0| - 1 = L - \text{rank}(M_{L-1}) - 1 \leq L - 1 - r_{i_0k_0} \leq D_\phi. \quad \square$$

We showed that $D_\phi \geq |\mathcal{G}_0| - 1 = L - \text{rank}(M_{L-1}) - 1 \geq \text{c}(A_\phi)$ and, in general, the equality does not hold (see examples in Section 7).

Corollary 6.7. *Let $\mathcal{P}(X, U)$ be a system of linear DPPEs with implicit ideal* ID *and leading matrix S. Let $\mathcal{P}_\phi(X, U)$ be a perturbed system of $\mathcal{P}(X, U)$, with degree $D_\phi \geq 0$. Let A_ϕ be the differential polynomial associated to $\mathcal{P}_\phi(X, U)$. If* $\text{rank}(S) = n - 1$ *and* $D_\phi = \text{c}(A_\phi)$ *then* ID *has dimension $n-1$ and $A_\phi(X) = 0$ is the implicit equation of $\mathcal{P}(X, U)$.*

Proof. If $D_\phi = \text{c}(A_\phi)$ then, by Theorem 6.6, we have $|\mathcal{G}_0| \leq \text{c}(A_\phi) + 1$. By Lemma 5.1, $|\mathcal{G}_0| = \text{c}(A_\phi) + 1$ and, by Corollary 5.8, the result follows. $\square$

7. Implicitization algorithm for linear DPPEs and examples

Let $\mathcal{P}(X, U)$ be a system of linear DPPEs with implicit ideal ID. Let S and M_{L-1} be the leading and principal matrices of $\mathcal{P}(X, U)$ respectively. Assuming that $\text{rank}(S) = n - 1$, in this section, we give an algorithm that decides whether the dimension of ID is $n - 1$ and, in the affirmative case, returns the implicit equation of $\mathcal{P}(X, U)$. As defined in Section 6, given a perturbed system $\mathcal{P}_\phi(X, U)$ of $\mathcal{P}(X, U)$, of degree $D_\phi \geq 0$, let A_{D_ϕ} be the coefficient of p^{D_ϕ} in $\partial\text{CRes}(F_1^\phi, \dots, F_n^\phi)$ and A_ϕ the differential polynomial associated to $\mathcal{P}_\phi(X, U)$.

Algorithm 7.1.
- Given a system $\mathcal{P}(X, U)$ of linear DPPEs whose leading matrix verifies $\text{rank}(S) = n - 1$.
- Decide whether the dimension is $n - 1$ and, in the affirmative case,
- Return a characteristic polynomial of ID.

1. Compute $\mathcal{P}_\phi(X, U)$ with perturbation ϕ given by (4).
2. Compute $\partial\text{CRes}(F_1^\phi, \dots, F_n^\phi)$, D_ϕ and A_{D_ϕ}.
3. If $D_\phi = 0$ RETURN A_{D_ϕ}.
4. Compute A_ϕ and $\text{c}(A_\phi)$.
5. If $D_\phi = \text{c}(A_\phi)$ RETURN A_ϕ.
6. Compute $\text{rank}(M_{L-1})$.
7. If $L - \text{rank}(M_{L-1}) > \text{c}(A_\phi) + 1$ RETURN "dimension less than $n - 1$".
8. If $L - \text{rank}(M_{L-1}) = \text{c}(A_\phi) + 1$ RETURN A_ϕ.

The previous algorithm follows from the results proved throughout this paper. Namely, step 3 follows from Rueda and Sendra (2010), Theorem 21, step 5 from Corollary 6.7, step 7 from

Corollary 5.3 and step 8 from Corollary 5.8. Also, in step 1, we could use any perturbation ϕ such that $\partial\mathrm{CRes}(F_1^{\phi}, \ldots, F_n^{\phi}) \neq 0$. We refer to ϕ given by (4) because it is the perturbation we used to prove Theorem 6.2.

The next examples were computed with Maple. The computation of differential resultants was carried out with our Maple implementation of the linear complete differential resultant, available at Rueda (2008).

7.1. Example 1

Let $\mathbb{K} = \mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$ and consider the system $\mathcal{P}(X, U)$, of linear DPPEs, providing the set of differential polynomials in $\mathbb{K}\{x_1, x_2, x_3\}\{u_1, u_2\}$,

$$F_1(X, U) = x_1 + u_1 - u_2 + u_{1,1} - u_{1,2} - 4u_{2,1} - 3u_{2,2},$$
$$F_2(X, U) = x_2 + u_2 + u_{1,1} - u_{2,2},$$
$$F_3(X, U) = x_3 + u_2 + u_{1,1} + u_{2,1}.$$

The set $\mathrm{PS}(F_1, F_2, F_3)$ contains $L = 13$ differential polynomials and $\gamma = 0$. The leading matrix S of $\mathcal{P}(X, U)$ has rank 2 and equals

$$S = \begin{bmatrix} -3 & -1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We consider the perturbation $\phi = (\phi_1(U), \phi_2(U), \phi_3(U))$ with

$$\phi_i(U) = \begin{cases} u_{1,2} + u_2, & i = 1, \\ u_1, & i = 2, \\ u_{2,1}, & i = 3. \end{cases}$$

There exists a differential polynomial $P \in (\mathrm{PS}_{\phi}) \cap \mathbb{K}_p\{X\}$, with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$, such that the determinant of the 13×13 matrix $M_{\phi}(13)$ equals

$$\begin{aligned} \partial\mathrm{CRes}(F_1^{\phi}, F_2^{\phi}, F_3^{\phi}) &= \partial\mathrm{CRes}^h(H_1^{\phi}, H_2^{\phi}, H_3^{\phi})P(X) \\ &= p\,(1 + 4p + 4p^4 - p^5 + 2p^7 + 11p^3 + p^9 - 12p^2 - 4p^6 + p^8)P(X). \end{aligned}$$

Then $D_{\phi} = 1$ and the coefficient of p in $\partial\mathrm{CRes}(F_1^{\phi}, F_2^{\phi}, F_3^{\phi})$ is

$$A_{D_{\phi}} = x_{1,2} + x_{1,3} - x_2 - 3x_{2,1} - 4x_{2,2} - 2x_{2,3} + x_3 + 2x_{3,1} + 2x_{3,2} + 2x_{3,3} + x_{3,4}.$$

We have $A_{D_{\phi}} = \mathcal{L}_1(x_1) + \mathcal{L}_2(x_2) + \mathcal{L}_3(x_3)$, with

$$\mathcal{L}_1 = \partial^2 + \partial^3 = \partial^2(1 + \partial),$$
$$\mathcal{L}_2 = -1 - 3\partial - 4\partial^2 - 2\partial^3 = -(\partial + 1)(2\partial^2 + 2\partial + 1),$$
$$\mathcal{L}_3 = 1 + 2\partial + 2\partial^2 + 2\partial^3 + \partial^4 = (\partial^2 + 1)(\partial + 1)^2.$$

Therefore $\mathcal{L} = \mathrm{gcld}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = 1 + \partial$ and $A_{\phi} = x_{1,2} - x_2 - 2x_{2,1} - 2x_{2,2} + x_{3,3} + x_{3,2} + x_{3,1} + x_3$, with $\mathrm{c}(A_{\phi}) = 1$. Then $D_{\phi} = \mathrm{c}(A_{\phi})$. We conclude that the dimension of ID is $n - 1 = 2$ and its implicit equation $A_{\phi}(X) = 0$.

If we consider the perturbation $\psi = (\psi_1(U), \psi_2(U), \psi_3(U))$, given by (4), with

$$\psi_i(U) = \begin{cases} u_{2,2}, & i = 1, \\ u_{1,2} + u_2, & i = 2, \\ u_1, & i = 3. \end{cases}$$

There exists a differential polynomial $P \in (\mathrm{PS}_{\phi}) \cap \mathbb{K}_p\{X\}$, with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$, such that the determinant of the 13×13 matrix $M_{\psi}(13)$ equals

$$\begin{aligned} \partial\mathrm{CRes}(F_1^{\psi}, F_2^{\psi}, F_3^{\psi}) &= \partial\mathrm{CRes}^h(H_1^{\psi}, H_2^{\psi}, H_3^{\psi})P(X) \\ &= p^2(-1 + p)(p^7 + 8p^6 + 15p^5 + 15p^4 - 50p^3 - 5p^2 + 13p - 3)P(X). \end{aligned}$$

Then $D_{\psi} = 2$ but the coefficient of p^2 is $A_{D_{\psi}} = 3A_{D_{\phi}}$. Therefore $A_{\psi} = A_{\phi}$ as expected.

7.2. Example 2

Let $\mathbb{K} = \mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$ and consider the system $\mathcal{P}(X, U)$, of linear DPPEs, providing the set of differential polynomials in $\mathbb{K}\{x_1, x_2, x_3, x_4\}\{u_1, u_2, u_3\}$,

$$\begin{aligned} F_1(X, U) &= x_1 + 2u_1 - u_3 - u_{1,1} + 3u_{2,2} - u_{3,2}, \\ F_2(X, U) &= x_2 + 2u_1 - u_3 + 2u_{2,1} + u_{3,2}, \\ F_3(X, U) &= x_3 + 2u_1 - u_3 + 3u_{2,1} - 2u_{3,1}, \\ F_4(X, U) &= x_4 - 2u_1 + u_3 - 3u_{2,1} + u_{3,1}. \end{aligned}$$

The set $\mathrm{PS}(F_1, F_2, F_3, F_4)$ contains $L = 18$ differential polynomials and $\gamma = \gamma_1 = 1$. The leading matrix S of $\mathcal{P}(X, U)$ has rank 3 and equals

$$S = \begin{bmatrix} -1 & 3 & -1 \\ 1 & 0 & 0 \\ -2 & 3 & 2 \\ 1 & -3 & -2 \end{bmatrix}.$$

We consider the perturbation $\phi = (\phi_1(U), \phi_2(U), \phi_3(U))$, given by (4), with

$$\phi_i(U) = \begin{cases} u_{3,2}, & i = 1, \\ u_{2,2} + u_3, & i = 2, \\ u_1 + u_2, & i = 3, \\ u_1, & i = 4. \end{cases}$$

There exists a differential polynomial $P \in (\mathrm{PS}_\phi) \cap \mathbb{K}_p\{X\}$, with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$, such that the determinant of the 18×18 matrix $M_\phi(18)$ equals

$$\begin{aligned} \partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi, F_4^\phi) &= \partial\mathrm{CRes}^h(H_1^\phi, H_2^\phi, H_3^\phi, H_4^\phi)P(X) \\ &= p^3(-9009p^3 - 233p^8 + p^{11} + 1917 + 7p^{10} - 553p^7 - 20p^9 \\ &\quad - 2070p + 9828p^4 + 12033p^2 - 4198p^5 - 1680p^6)P(X). \end{aligned}$$

Then $D_\phi = 3$ and the coefficient of p^3 in $\partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi, F_4^\phi)$ is

$$A_{D_\phi} = -1917(2x_{1,2} - 6x_{2,2} + 9x_{2,3} - 5x_{3,2} + 9x_{3,4} - 9x_{4,2} + 8x_{4,3} + 9x_{4,4}).$$

We have $A_{D_\phi} = \mathcal{L}_1(x_1) + \mathcal{L}_2(x_2) + \mathcal{L}_3(x_3) + \mathcal{L}_4(x_4)$, with

$$\begin{aligned} \mathcal{L}_1 &= -1917\partial^2 2, \\ \mathcal{L}_2 &= -1917\partial^2(9\partial - 6), \\ \mathcal{L}_3 &= -1917\partial^2(9\partial^2 - 5), \\ \mathcal{L}_4 &= -1917\partial^2(9\partial^2 + 8\partial - 9). \end{aligned}$$

Therefore $\mathcal{L} = -1917\partial^2$ and $A_\phi = 2x_1 + 9x_{2,1} - 6x_2 + 9x_{3,2} - 5x_3 + 9_{4,2} + 8x_{4,1} - 9x_4$, with $\mathrm{c}(A_\phi) = 2$. Then $D_\phi > \mathrm{c}(A_\phi)$.

Replace p in $M_\phi(18)$ by zero to obtain $M(18)$, whose principal 18×17 submatrix is M_{L-1}. Compute $L - \mathrm{rank}(M_{L-1}) = 3$. Then $\mathrm{c}(A_\phi) + 1 = L - \mathrm{rank}(M_{L-1})$. We conclude that the dimension of ID is $n - 1 = 3$ and its implicit equation $A_\phi(X) = 0$.

7.3. Example 3

Let $\mathbb{K} = \mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$ and consider the system $\mathcal{P}(X, U)$, of linear DPPEs, providing the set of differential polynomials in $\mathbb{K}\{x_1, x_2, x_3\}\{u_1, u_2\}$,

$$\begin{aligned} F_1(X, U) &= x_1 - 3 + u_{1,1} + u_{1,2} - u_2 - 4u_{2,1} - 3u_{2,2}, \\ F_2(X, U) &= x_2 + u_{1,1} + u_2 - u_{2,2}, \\ F_3(X, U) &= x_3 + 2 + u_{1,1} + tu_2 + u_{2,1}. \end{aligned}$$

Then the set $\mathrm{PS}(F_1, F_2, F_3)$ contains $L = 13$ differential polynomials and $\gamma = 0$. The leading matrix S of $\mathcal{P}(X, U)$ has rank 2 and equals

$$S = \begin{bmatrix} -3 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We consider the perturbation $\phi = (\phi_1(U), \phi_2(U), \phi_3(U))$, given by (4), with

$$\phi_i(U) = \begin{cases} u_{2,2}, & i = 1, \\ u_{1,2} + u_2, & i = 2, \\ u_1, & i = 3. \end{cases}$$

There exists a differential polynomial $P \in (\mathrm{PS}_\phi) \cap \mathbb{K}_p\{X\}$, with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$, such that the determinant of the 13×13 matrix $M_\phi(13)$ equals $\partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi) = -p(p+1)\,P(X)$. In this case, $\alpha_3\, p(p+1) = -\det(S_3^\phi)\partial\mathrm{CRes}^h(H_1^\phi, H_2^\phi, H_3^\phi)$ where $l(P) = (\alpha_1, \alpha_2, \alpha_3)$ is the leading vector of P. Then $D_\phi = 1$ and the coefficient A_{D_ϕ} of p in $\partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi)$ equals

$$A_{D_\phi} = \mathcal{L}_1(x_1 - 3) + \mathcal{L}_2(x_2) + \mathcal{L}_3(x_3 + 2)$$

with

$$\begin{aligned}
\mathcal{L}_1 &= -664 - 2t^3 + 228t - 16t^2 + (312 - 2t^4 + 187t^2 - 19t^3 - 510t)\partial \\
&\quad + (-2t^3 - 464 + 198t - 15t^2)\partial^2 + (-21t^2 + 156t - 2t^3 - 308)\partial^3, \\
\mathcal{L}_2 &= 16t^2 + 2t^3 - 228t + 664 + (-139t^2 + 25t^3 + 1680 + 2t^4 - 174t)\partial \\
&\quad + (-75t^2 + 2t^4 - 640t + 2164 + 31t^3)\partial^2 + (-624t + 84t^2 + 8t^3 + 1232)\partial^3, \\
\mathcal{L}_3 &= (-32t^2 + 456t - 4t^3 - 1328)\partial + (952t - 97t^2 - 2012 - 10t^3)\partial^2 \\
&\quad + (-4t^3 - 48t^2 - 460 + 270t)\partial^3 + (21t^2 - 156t + 2t^3 + 308)\partial^4.
\end{aligned}$$

Using the Maple package OreTools, we check that $\mathcal{L} = \mathrm{gcld}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ equals

$$\mathcal{L} = \frac{-(6t^5 + 125t^4 + 302t^3 - 2988t^2 + 1424t + 464)}{2t^6 + 47t^5 + 37t^4 - 2484t^3 + 11042t^2 - 18248t + 11704} + \partial.$$

Then $A_{D_\phi} = \mathcal{L}(A_\phi)$, with $\mathrm{c}(A_\phi) = 1$ and $D_\phi = \mathrm{c}(A_\phi)$. We conclude that the dimension of ID is $n-1 = 2$ and its implicit equation $A_\phi(X) = 0$.

If we take a different perturbation $\psi = (\psi_1(U), \psi_2(U), \psi_3(U))$ with

$$\psi_i(U) = \begin{cases} u_{2,2} + u_1, & i = 1, \\ u_2, & i = 2, \\ u_{1,1}, & i = 3. \end{cases}$$

We obtain $D_\psi = 1$ and the coefficient of p in $\partial\mathrm{CRes}(F_1^\psi, F_2^\psi, F_3^\psi)$ is

$$A_{D_\psi} = \mathcal{K}_1(x_1 - 3) + \mathcal{K}_2(x_2) + \mathcal{K}_3(x_3 + 2)$$

with

$$\begin{aligned}
\mathcal{K}_1 &= (t^5 - 156 + 11t^4 + 12t + 154t^2 - 76t^3)\partial \\
&\quad +(t^4 - 100 - 94t^2 + 8t^3 + 204t)\partial^2 + (-12 + 88t - 58t^2 + 12t^3 + t^4)\partial^3, \\
\mathcal{K}_2 &= 664 + 2t^3 - 228t + 16t^2 + (-13t^4 + 820 + 90t^2 - 904t + 62t^3 - t^5)\partial \\
&\quad + (412 - 17t^4 + 14t^3 + 346t^2 - t^5 - 892t)\partial^2 \\
&\quad + (48 - 352t + 232t^2 - 48t^3 - 4t^4)\partial^3, \\
\mathcal{K}_3 &= -664 - 16t^2 - 2t^3 + 228t + (-664 + 892t - 244t^2 + 2t^4 + 14t^3)\partial \\
&\quad + (676t - 406t^2 + 5t^4 + 54t^3 - 156)\partial^2 + (-80t^2 + 2t^4 + 60t + 28t^3 + 64)\partial^3 \\
&\quad + (12 - 88t + 58t^2 - 12t^3 - t^4)\partial^4.
\end{aligned}$$

Using the Maple package OreTools we check that $\mathcal{K} = \text{gcld}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ equals

$$\mathcal{K} = \frac{-(5t^6 + 114t^5 + 286t^4 - 3692t^3 + 9732t^2 - 12024t + 6208)}{t^7 + 25t^6 + 58t^5 - 1108t^4 + 3908t^3 - 5880t^2 + 3824t - 456} + \partial.$$

Thus $A_{D_\psi} = \mathcal{K}(A_\psi)$ and $c(A_\psi) = 1$. Therefore, as it should be, we obtain the same conclusion, with

$$A_\psi = \frac{12 - 88t + 58t^2 - 12t^3 - t^4}{308 - 156t + 21t^2 + 2t^3} A_\phi.$$

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