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Significance of higher-order terms of the Williams expansion for plastic zone extent estimation demonstrated on a mixed-mode geometry

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Abstract

In this work, the multi-parameter description of the crack-tip fields is employed. The Williams expansion is introduced for approximation of the stress/displacement distribution. The over-deterministic method is used for calculation of the coefficients of the power series terms because it requires only a conventional FE analysis. Particularly, the plastic zone extent in a cracked specimen subjected to I+II mixed mode of loading is investigated using Rankine criterion. Results determined via finite element analysis are compared to those calculated by means of the stress distribution expressed via the Williams expansion under consideration of various numbers of initial terms of the series. The size of the plastic zone is calculated for various values of tensile strength of the material modeled in the numerical simulations. It is shown that the higher-order terms of the Williams expansion can be significant when the extent of the nonlinear zone is large enough in comparison to the typical structural dimensions.

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Keywords: Crack-tip fields approximation; Williams expansion; multi-parameter approach; over-deterministic method; plastic zone extent

1. Introduction

Classical linear elastic fracture mechanics is certainly the most common tool for assessment of fracture behavior of various structures and materials. It considers the stress intensity factor (SIF) as the single parameter expressing

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the amplitude of the near-crack tip stress field and controlling the (un)stable crack growth. This well-known theory is derived for brittle materials and is rather sufficiently described. Nevertheless, there exist materials where the failure occurs not only at the crack tip (and in its very vicinity) but it takes place in a large zone ahead of the crack tip. The material behavior in this region is mostly nonlinear and is not explained and described adequately. Moreover, if the zone with nonlinear behavior is of a large extent (comparable to the structural dimensions), the size/geometry/boundary effect cannot be omitted, see Ayatollahi and Akbardoost (2012), Duan et al. (2007) or Karihaloo et al. (2006).

It has been shown that the so-called multi-parameter approach for approximation of the crack-tip stress and displacement fields can help to express the stress distribution better than if the only one or two parameters are used (as it is usual), see e.g. Šestáková (2014a,b), Šestáková et al. (2013), Veselý et al. (2014) and Veselý et al. (2013) for more details. Further investigations in this area are introduced in this paper. The Williams expansion (WE), see Williams (1957), is used for the stress state description and subsequently also as an input for the fracture criterion enabling estimation the plastic zone size. It means that the criterion is used in a kind of their generalized form. Particularly, a cracked specimen subjected to I+II mixed mode of loading is investigated and it is discussed how the higher-order terms of the WE can be important. This study is only one part of the author's complex investigations devoted to influence of the higher-order terms of the WE on various fracture characteristics/phenomena (near-crack-tip stress field approximation, crack propagation angle, plastic zone extent, etc.), see e.g. the works cited above in this paragraph or (Šestáková) Malíková (2013), (Šestáková) Malíková and Veselý (2014) for more details.

2. Methodology

In this chapter, the basic terms and methodology are explained briefly.

2.1. Multi-parameter description of crack-tip stress fields

The basic idea of the multi-parameter description of crack-tip stress fields is using the Williams expansion, derived in Williams (1957), for approximation of the stress tensor and displacement vector components. The power series has been suggested for a linear-elastic material with a plane crack with traction-free faces subjected to arbitrary remote loading and it has the form (in case of the combination of loading modes I+II):

$$\sigma_{ij} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} A_n f_{ij}^{\sigma}(\theta,n) + \sum_{m=1}^{\infty} \frac{m}{2} r^{\frac{m}{2}-1} B_m g_{ij}^{\sigma}(\theta,m) \quad \text{for stress tensor components } (i,j \in \{x,y\}) \text{ and} \qquad (1)$$

$$u_i = \sum_{n=0}^{\infty} r^{n/2} A_n f_i^u(\theta,n,E,\nu) + \sum_{m=0}^{\infty} r^{m/2} B_m g_i^u(\theta,m,E,\nu) \quad \text{for displacement vector components } (i \in \{x,y\}). \qquad (2)$$

Note that the center of the polar coordinate system (r,θ) used in Eq. 1 and 2 is at the crack tip. Further symbols used in the relations above are: A_n and B_m correspond to the unknown coefficients of the WE terms of mode I and mode II of loading, respectively (they depend on the specimen geometry, relative crack length α and loading conditions) and they have to be determined numerically; E, ν represent material properties (Young's modulus and Poisson's ratio); f_{ij}^{σ}, g_{ij}^{σ} and f_i^u, g_i^u are known functions corresponding to the stress and displacement distribution (upper index σ or u) and to the loading mode I (f) and mode II (g), respectively. For the presented study the truncated form of the series was applied, i.e. N initial terms of the mode I expansion and M initial terms of the mode II expansion are considered.

2.2. Over-deterministic approach

When the WE shall be used, it is necessary to determine the coefficients of the WE terms. They can be calculated numerically and in this work, the over-deterministic method (ODM) is utilized. This procedure is advantageous because of its low demands on the finite element software. Unlike the other methods, ODM needs to know only the displacement field in a set of nodes (mostly at some radius) around the crack tip. These data together with Eq. (2) enable to calculate the coefficients of the WE terms. Of course, there exist several conditions that have to be satisfied and also some recommendations have been summarized for instance in Ayatollahi and Nejati (2011) or Šestáková (2013).

2.3. Plastic zone extent estimation - Rankine criterion

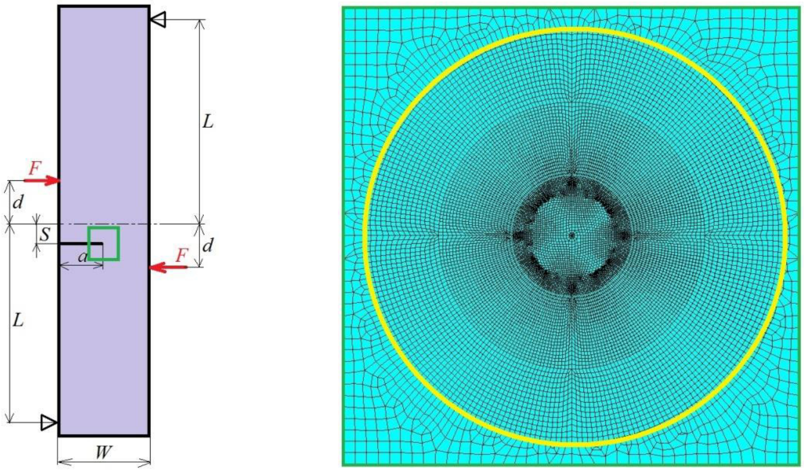
The Rankine criterion is also often referred to as maximum stress criterion. It is suitable especially for brittle/quasi-brittle materials and the plastic zone size is estimated from the relation between the principal stress, σ_1, and the critical stress value, σ_c, that is a material property (tensile strength, yield strength, etc.). Mathematically written:

$$\sigma_1 \geq \sigma_c \tag{3}$$

Into the criterion the stress tensor components approximated via the WE are substituted and the plastic zone size estimated for various numbers of the initial terms of the WE (N, M = 1, 2, 4, 7 and 10).

3. Cracked specimens under study

A mixed mode configuration was chosen for the presented study because of its generality in comparison to specimens loaded in pure mode I or pure mode II. Particularly, an eccentric antisymmetrical four-point bending specimen (EA4PB), see Fig. 1a, was modeled in Ansys with following dimensions: W = 40 mm and L = 100 mm.



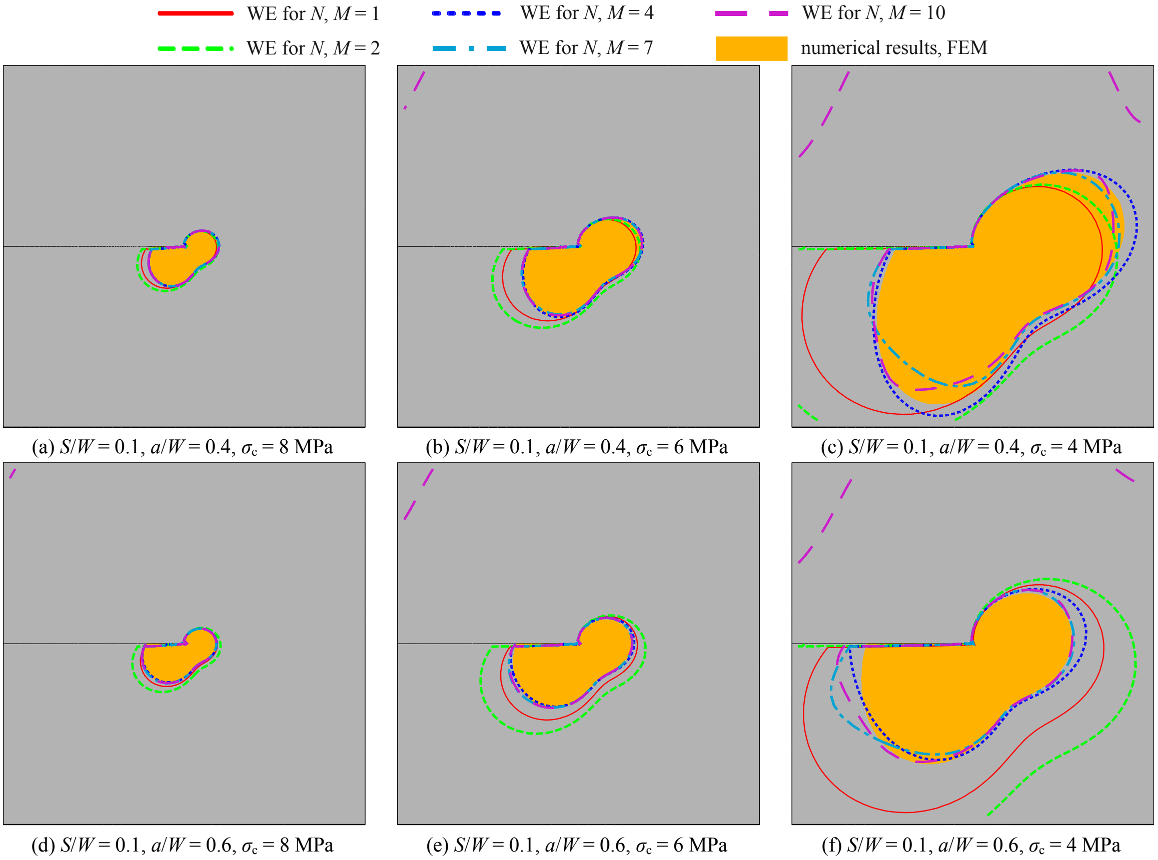
(a) (b)

Fig. 1. (a) schema of the EA4PB specimen with marked area of the investigated region near the crack tip;
(b) detail of the FE mesh around the crack tip with highlighted radius of nodes used for the ODM procedure.

Other dimensions (the crack length a, the distance of the forces from the axis of antisymmetry d and the crack eccentricity S) varied in some range in order to have enough data for the analysis. In the following, only two representative sets of results are presented: $d/W = 0.5$, $S/W = 0.1$, $a/W = 0.4$ and $d/W = 0.5$, $S/W = 0.1$, $a/W = 0.6$. They were chosen because of their similar ratio of stress intensity factors, particularly in both configurations mode II slightly prevails and the ratio between K_I and K_II is approximately 0.8. Moreover, the value of the loading force for the configuration with $a/W = 0.6$ was set to be $F = 50$ N/mm, which causes stress concentration at the crack tip with stress intensity factor values: $K_\mathrm{I} = 0.34\ \mathrm{MPa{\cdot}m^{1/2}}$ and $K_\mathrm{II} = 0.42\ \mathrm{MPa{\cdot}m^{1/2}}$. In order to obtain the same values of the stress intensity factors in the other configuration, the value of the loading force had to be $F = 84.8$ N/mm. Thus, two different configurations with the same values of K_I and K_{II} could be investigated. Material properties in the study were taken in order to correspond to concrete behavior: $E = 35$ GPa and $\nu = 0.2$. Three different values of the critical stress (which is necessary for application of the criteria for estimation of the plastic zone extent) were considered $\sigma_c = 4$, 6 and 8 MPa in order to show some trends in results. The plane strain conditions were applied in the finite element model. Based on the experience with ODM application, the distance of nodes (whose displacements were utilized for the ODM procedure) from the crack tip was chosen to be 7 mm, i.e. far enough from the crack tip, see Fig. 1b.

4. Results and discussion

In this paper, the plastic zone extent is investigated by means of the Rankine criterion for the selected EA4PB mixed-mode configuration. Particularly, the influence of the number of the WE terms considered for the stress components approximation on the plastic zone size estimation is studied. In Fig. 2 comparison of the plastic zone size calculated via FEM with results calculated from the same criterion by means of the WE with various numbers of the initial terms. For either configuration, three different cases are simulated: materials with the critical stress/tensile strength value $\sigma_c = 4$, 6 and 8 MPa are assumed. Note that the dimensions of the plotted area are 15.4 mm x 15.4 mm.



(a) $S/W = 0.1$, $a/W = 0.4$, $\sigma_c = 8$ MPa (b) $S/W = 0.1$, $a/W = 0.4$, $\sigma_c = 6$ MPa (c) $S/W = 0.1$, $a/W = 0.4$, $\sigma_c = 4$ MPa

(d) $S/W = 0.1$, $a/W = 0.6$, $\sigma_c = 8$ MPa (e) $S/W = 0.1$, $a/W = 0.6$, $\sigma_c = 6$ MPa (f) $S/W = 0.1$, $a/W = 0.6$, $\sigma_c = 4$ MPa

Fig. 2. Comparison of the plastic zone sizes calculated via FEM using Rankine criterion for the chosen EA4PB configurations with parameters $S/W = 0.1$, $a/W = 0.4$ (a-c) and $S/W = 0.1$, $a/W = 0.6$ (d-f) with results calculated from the same criterion when the stress components are approximated via the WE with various numbers of the initial terms and considering the critical stress value: (a,d) $\sigma_c = 8$ MPa; (b,e) $\sigma_c = 6$ MPa; (c,f) $\sigma_c = 4$ MPa.

First of all, the results presented in Fig. 2 show one important phenomenon (imperfection of the conventional one-parameter fracture mechanics concept), particularly that the plastic zone size is not dependent on the specimen size/boundary conditions etc. when only the stress intensity factor is used for its determination, see the red curves in the corresponding parts of Fig. 2: (a) vs. (d), (b) vs. (e) and (c) vs. (f).

Furthermore, it is well known that the plastic zone size is dependent on the value of the critical stress used as the limit in the Rankine criterion, i.e. in case of the concrete specimen investigated in this work, the tensile strength is important. It can be also seen in Fig. 2 that because of very low strength level used in these simulations the extent of the plastic zone can be really large. When the tensile strength is assumed to be 4 MPa (as it is common for concrete or other quasi-brittle materials), the plastic zone size in the specimen width direction is about 10 mm for the configuration with $d/W = 0.5$, $S/W = 0.1$, $a/W = 0.4$, which is 1/4 of the specimen width. Moreover, if a more advanced criterion taking into account also some stress redistribution ahead of the crack tip were applied, it can be expected that the plastic zone extent would be even larger. However, the results presented in Fig. 2 enable to show how important considering of more terms of the WE for the stress field approximation can be. When the plastic zone size is small in comparison to the specimen dimensions (in our case approximately 13 times smaller when $\sigma_c = 8$ MPa), the stress distribution necessary for application of the Rankine criterion needs to be known only very close to the crack tip, which can be described satisfactorily by means of the stress intensity factor, i.e. by means of only the first term of the WE, see Fig. 2a,d. On the other hand, when the PZ size starts to be comparable to the typical specimen dimensions, the classical fracture mechanics approach seems to be not sufficient. Fig. 2b,e and more obviously Fig. 2c,f show that the plastic zone size calculated by means of the one- or two-parameter approach overestimates the results calculated numerically via FE code. In order to describe the plastic zone better, some kind of a generalized Rankine criterion is to be recommended, i.e. several more initial terms of the WE should be taken into account for the crack-tip stress field reconstruction that is necessary for application of the selected criterion.

Note that the so-called fracture process zone (FPZ) is more typical than the plastic zone in quasi-brittle materials. Nevertheless, the qualitative nature of results is expected to be similar. Further investigations are going to be performed in this field as well as in the field of the stress redistribution ahead of the crack tip.

5. Conclusions

The plastic zone extent was investigated in an eccentric asymmetric four point bending specimen using the Rankine/maximum stress criterion. By means of varying the tensile strength of the material in the numerical model of the cracked specimen it is shown that when the nonlinear zone extent is small enough, the classical one- or two-parameter fracture mechanics concept can be applied. On the other hand, when the stress field needs to be investigated also in larger distances from the crack tip, i.e. the plastic zone extent is comparable to the typical specimen dimensions, using the WE with more higher-order terms for the stress field approximation can be very advantageous. Then, a generalized form of the criterion is spoken about and its utilization can contribute to better description of the occurring failure process.

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