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Well-posedness and inviscid limit behavior of solution for the generalized 1D Ginzburg–Landau equation

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Abstract

The Cauchy problem of the one-dimensional generalized Ginzburg–Landau (GGL) equation is considered. The local well-posedness is obtained for initial data in $H^s(\mathbb{R})$ with $s > 0$, and global result in $H^s(\mathbb{R})$ with $s > 0$ is also obtained under some conditions. Moreover, the relation between the solution for GGL equation and the solution for the derivative nonlinear Schrödinger (DNLS) equation is studied. It is proved that for some $T > 0$, the solution of Cauchy problem for the GGL equation converge to the solution of Cauchy problem for the DNLS in the natural space $C([0,T]; H^s)$ with $s > \frac{1}{2}$ if some coefficients tend to zero. Moreover, if initial data belong to H^2, the convergence holds in $C([0,T]; H^1)$ for any $T > 0$.

Résumé

Le problème de Cauchy pour l'équation de Ginzburg–Landau (GGL) unidimensionnel généralisé est considéré. Le problème bien posé local est obtenu pour les données initiales dans $H^s(\mathbb{R})$ avec $s > 0$ et le résultat global dans $H^s(\mathbb{R})$ avec $s > 0$ est aussi obtenu sous certaines conditions. De plus on étudié la relation entre la solution de GGL et la solution pour la dérivée de l'équation de Schrödinger (DNLS) non linéaire. On démontre que pour certaines valeurs de $T > 0$ la solution du problème de Cauchy de GGL converge vers la solution du problème de Cauchy de DNLS dans l'espace naturel $C([0,T]; H^s)$ avec $s > 1/2$ lorsque certains coefficients tendent vers zéro. De plus, si les données initiales sont dans H^2, la convergence est venue dans $C([0,T]; H^1)$ pour tout $T > 0$.

MSC: 35A05; 35K15; 35K55; 35Q55

Keywords: Generalized 1D Ginzburg–Landau equation; Derivative nonlinear Schrödinger equation; Well-posedness; Inviscid limit behavior

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1. Introduction

The Cauchy problem of the generalized one-dimensional Ginzburg–Landau equation is:

$$u_t - (\alpha + i)u_{xx} + \gamma_1|u|^2u_x + \gamma_2u^2\bar{u}_x + \gamma_3|u|^2u + \gamma_4|u|^4u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0(x) \in H^s, \tag{1.2}$$

where $\gamma_j = \alpha_j + i\beta_j$, α_j, β_j are real numbers, $j = 1,2,3,4$; $\alpha > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$. $\bar{u}(x,t)$ is the complex conjugate of $u(x,t)$. The aim of this work is to study its well-posedness and inviscid limit behavior of its solution.

The generalized 1D Ginzburg–Landau (GGL) equation arises as the envelope equation for a weakly subcritical bifurcation to counter-propagating waves. It is also of importance in the theory of interaction behavior, including complete interpenetration as well as partial annihilation for collision between localized solutions corresponding to a single particle and to a two particle state. For details of the physical backgrounds of the GGL equation, one refers to Brand and Deissler [1,4]. There are several papers [5,15] related to the well-posedness of Cauchy problem (1.1)–(1.2). Notice that these authors treated Eq. (1.1) as parabolic equations, used the time–space L^p–L^r estimate method or semigroup method to obtain the local results. Duan and Holmes [5] showed that the Cauchy problem (1.1)–(1.2) is globally well-posed in H^1 under the condition $4\alpha\alpha_4 > (\beta_1 - \beta_2)^2$. In fact, in this paper, we will prove that under this condition, the Cauchy problem (1.1)–(1.2) is globally well-posed in H^s with $s > 0$.

Taking $\gamma_1 = 0$, $\alpha = \beta_2 = \alpha_3 = \alpha_4 = 0$, Eq. (1.1) can be rewritten as

$$v_t - iv_{xx} + \alpha_2v^2\bar{v}_x + i\beta_3|v|^2v + i\beta_4|v|^4v = 0, \tag{1.3}$$

with initial data,

$$v(x,0) = v_0(x) \in H^s; \tag{1.4}$$

(1.3) is the well-known derivative nonlinear Schrödinger (DNLS) equation, which models Alfvén waves in plasma physics [7,10,13].

Recently, Takaoka [17] showed that the Cauchy problem (1.3)–(1.4) is locally well-posed in H^s with $s \geqslant \frac{1}{2}$. It is the best local well-posedness result of DNLS equation at present. In [14], the global well-posedness of DNLS equation is obtained in H^1 with assuming the smallness condition:

$$\|v_0\|_{L^2} \leqslant \eta, \quad \text{for some enough small number } \eta > 0. \tag{1.5}$$

The sharp global result was obtained in [3], where it was shown that under the condition (1.5), the Cauchy problem (1.3)–(1.4) with $\beta_3 = 0$ is globally well-posed in H^s with $s > \frac{1}{2}$, where a kind of almost conserved energy was introduced, so-called I-method.

It is natural to consider the question of inviscid limit. That is, if $u(t)$ and $v(t)$ are solutions of the Cauchy problems (1.1)–(1.2) and (1.3)–(1.4), respectively. What is the relation between the two solutions? Do the solution $u(t)$ of the GGL equation (1.1) converge (in an appropriate space norm) to the solution $v(t)$ of the DNLS equation (1.3) as u_0 tends to v_0 and the parameters $\alpha, \beta_2, \alpha_3, \alpha_4$ tend to zero?

For the generalized form of (1.3) (there exists a additional term $\alpha_1|u|^2u_x$ in Eq. (1.3)), B. Wang and Y. Wang [16] considered inviscid limit behavior between (1.1) and the generalized form of (1.3) with $u_0 \in \dot{H}^3 \cap \dot{H}^{-\frac{1}{2}}$ and $v_0 \in \dot{H}^2 \cap \dot{H}^{-\frac{1}{2}}$. In this paper, we will consider inviscid limit behavior when initial data u_0, v_0 belong to the natural space H^s.

Recently, Molinet and Ribaud [12] used the Bourgain's space with dissipation to consider the KdV–Burgers equation:

$$u_t + u_{xxx} - u_{xx} + uu_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{1.6}$$

They showed that it is globally well-posed in H^s with $s > -1$. Enlightened by some ideas in [12], we will use this method to study the Cauchy problem (1.1)–(1.2).

In this paper, we first consider the well-posedness of the Cauchy problem (1.1)–(1.2), and prove that it is locally well-posed in H^s with $s > 0$, and globally well-posed in H^s with $s > 0$ under some conditions. One notice that L^2 is critical space for Eq. (1.1). Therefore, our results about local and global well-posedness are sharp except the endpoint $s = 0$. Furthermore, we will study the inviscid limit of solution $u(t)$ for Eq. (1.1) with $\gamma_1 = 0$. The reason we consider the case $\gamma_1 = 0$ is that a derivative of the complex conjugate of solution $u(t)$ can be dealt with while a derivative of $u(t)$ cannot be done [14,17,18]. We will obtain the results as below: for $u_0, v_0 \in H^s$ with $s > \frac{1}{2}$ and some $T > 0$, $t \in (0, T)$, if $u(t)$ and $v(t)$ are solutions of the Cauchy problems (1.1)–(1.2) and (1.3)–(1.4), respectively, then $u(t)$ converges to $v(t)$ in the natural space $C([0, T]; H^s)$ under some conditions as $|\alpha|, |\beta_2|, |\alpha_3|, |\alpha_4|$ and $\|u_0 - v_0\|_{H^s}$ tend to zero. Moreover, if initial data u_0, v_0 belong to H^2, the convergence holds in $C([0, T]; H^1)$ for any $T > 0$.

1.1. Definitions and notations

The Cauchy problems (1.1)–(1.2) and (1.3)–(1.4) can be rewritten as the integral equivalent formulations:

$$u(x,t) = S_\alpha(t)u_0 - \int_0^t S_\alpha(t-t')\big(\gamma_1|u|^2u_x + \gamma_2 u^2\bar{u}_x + \gamma_3|u|^2u + \gamma_4|u|^4u\big)(t')\,dt', \tag{1.7}$$

$$v(x,t) = S_0(t)v_0 - \int_0^t S_0(t-t')\big(\alpha_2 v^2\bar{v}_x + i\beta_3|v|^2v + i\beta_4|v|^4v\big)(t')\,dt', \tag{1.8}$$

where $S_\alpha(t) = \mathcal{F}_x^{-1}e^{-it\xi^2}e^{-|t|\alpha\xi^2}\mathcal{F}_x$, $S_0(t) = \mathcal{F}_x^{-1}e^{-it\xi^2}\mathcal{F}_x$ are the semigroups associated to the linear GGL equation and Schrödinger equation, respectively.

For $s, b \in \mathbb{R}$, the standard spaces $X_{s,b}$ and $\overline{X}_{s,b}$ for the Schrödinger equation (1.3) are defined as the completion of the Schwartz function spaces on $\mathbb{R}^2$ with respect to the norms [2,8,9] respectively:

$$\|u\|_{X_{s,b}} = \big\|\langle\xi\rangle^s\langle\tau-\xi^2\rangle^b\hat{u}(\xi,\tau)\big\|_{L^2_\xi L^2_\tau}, \tag{1.9}$$

$$\|u\|_{\overline{X}_{s,b}} = \big\|\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b\hat{u}(\xi,\tau)\big\|_{L^2_\xi L^2_\tau}. \tag{1.10}$$

The Bourgain's spaces with dissipation for (1.1) are defined as follows [12],

$$\|u\|_{Y_{s,b}} = \big\|\langle\xi\rangle^s\langle i(\tau-\xi^2)+\alpha|\xi|^2\rangle^b\hat{u}(\xi,\tau)\big\|_{L^2_\xi L^2_\tau}, \tag{1.11}$$

$$\|u\|_{\overline{Y}_{s,b}} = \big\|\langle\xi\rangle^s\langle i(\tau+\xi^2)+\alpha|\xi|^2\rangle^b\hat{u}(\xi,\tau)\big\|_{L^2_\xi L^2_\tau}. \tag{1.12}$$

Notice that $\|\bar{u}\|_{\overline{X}_{s,b}} = \|u\|_{X_{s,b}}$, $\|\bar{u}\|_{\overline{Y}_{s,b}} = \|u\|_{Y_{s,b}}$. The spaces $Y_{s,b}$ and $\overline{Y}_{s,b}$ turn out to be very useful to consider the well-posedness of the dispersive equation with dissipative term, such as Eq. (1.1), (1.6), etc.

For $T \geqslant 0$, the corresponding localized spaces $X^T_{s,b}$ and $Y^T_{s,b}$ are endowed, respectively, with the norms:

$$\|u\|_{X^T_{s,b}} = \inf_{w\in X_{s,b}}\big\{\|w\|_{X_{s,b}}\colon w(t)=u(t) \text{ on } [0,T]\big\}, \tag{1.13}$$

$$\|u\|_{Y^T_{s,b}} = \inf_{w\in Y_{s,b}}\big\{\|w\|_{Y_{s,b}}\colon w(t)=u(t) \text{ on } [0,T]\big\}. \tag{1.14}$$

Define $A \sim B$ by using the statement: $A \leqslant C_1B$ and $B \leqslant C_1A$ for some constant $C_1 > 0$, and define $A \ll B$ through the statement: $A \leqslant \frac{1}{C_2}B$ for some large enough constant $C_2 > 0$.

Let $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp}\psi \subset [-1, 1]$, ψ is positive and even. Define $\psi_\delta(\cdot) = \psi(\delta^{-1}(\cdot))$ for some non-zero $\delta \in \mathbb{R}$.

Define the Fourier restriction operators:

$$P^N f = \int_{|\xi|\geqslant N} e^{ix\xi}\hat{f}(\xi)\,d\xi, \qquad P_N f = \int_{|\xi|\leqslant N} e^{ix\xi}\hat{f}(\xi)\,d\xi, \quad \forall N > 0. \tag{1.15}$$

1.2. Main methods and results

Actually, in order to obtain the local well-posedness of the Cauchy problem (1.1)–(1.2), we will apply a fixed point argument to the truncation version of (1.7) as below:

$$u(x,t) = \psi(t)S_\alpha(t)u_0 - \psi(t)\int_0^t S_\alpha(t-t')\big(\gamma_1|u|^2u_x + \gamma_2u^2\bar{u}_x + \gamma_3|u|^2u + \gamma_4|u|^4u\big)(t')\,dt', \tag{1.16}$$

for $u, \bar{u}$ with compact support in $[-T, T]$ in the integral of the right side of (1.16). Indeed, if $u(t)$ solves (1.16) then $u(t)$ is a solution of (1.7) on $[0, T]$ with $T < 1$. Therefore, following some ideas in [12], we mainly prove the trilinear and multilinear estimates as follows,

$$\big\||u|^2u_x\big\|_{Y_{s,-1/2+\delta}} \leqslant C_\delta\|u\|^3_{Y_{s,1/2}}, \tag{1.17}$$

$$\big\|u^2\bar{u}_x\big\|_{Y_{s,-1/2+\delta}} \leqslant C_\delta\|u\|^3_{Y_{s,1/2}}, \tag{1.18}$$

$$\big\||u|^2u\big\|_{Y_{s,-1/2+\delta}} \leqslant C_\delta\|u\|^3_{Y_{s,1/2}}, \tag{1.19}$$

$$\big\||u|^4u\big\|_{Y_{s,-1/2+\delta}} \leqslant C_\delta\|u\|^5_{Y_{s,1/2}}, \tag{1.20}$$

for any small $\delta > 0$. These estimates, which will be obtained in Section 3, together with the linear estimates obtained in Section 2, are used to obtain the local well-posedness. Then the global well-posedness will be obtained by some a priori estimates obtained in Section 4 and regularity of solution given in Lemma 2.6.

In order to consider the inviscid limit behavior, we first show that the solutions $u(x,t)$ of the Cauchy problem (1.1)–(1.2) for any $\alpha, \alpha_3, \alpha_4 \geqslant 0$, $\beta_2 \in \mathbb{R}$ and the solution $v(x,t)$ of the Cauchy problem (1.3)–(1.4) should exist in the same space $C([0,T]); H^s)$ with the same initial data. That is, the existence time T should be independent of $\alpha, \beta_2, \alpha_3, \alpha_4$. Next, we give some estimates of $u(x,t)$ uniformly for α by some a priori estimates to control $u(x,t)$. Moreover, we also need to consider the difference equation between (1.1) and (1.3), and treat the dissipative term as the perturbation to obtain the inviscid limit behavior.

Denote $Z_T = C([0,T]; H^s) \cap Y^T_{s,1/2}$, the main results of the paper are listed as below.

Theorem 1.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > 0$. Then there exists a constant $T > 0$, such that the Cauchy problem* (1.1)–(1.2) *admits a unique local solution $u(x,t) \in Z_T$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is smooth from H^s to Z_T and u belongs to $C((0,T); H^{+\infty})$.*

Theorem 1.2. *Let $u_0 \in H^s(\mathbb{R})$ with $s > 0$. Assume that $4\alpha\alpha_4 > (\beta_1 - \beta_2)^2$. Then for any $T > 0$, the Cauchy problem* (1.1)–(1.2) *admits a unique solution $u(x,t) \in Z_T$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is smooth from H^s to Z_T and u belongs to $C((0,+\infty); H^{+\infty})$.*

Theorem 1.3. *Let $u_0 \in H^s(\mathbb{R})$ with $s > 0$. Assume that $\|u_0\|_{L^2} \leqslant \eta$ for some enough small number $\eta > 0$. Moreover, we assume that*

$$|\beta_1|, |\beta_2| \leqslant 2\max\{\alpha, \alpha_4\}, \quad \alpha > 0, \alpha_3 > 0, \alpha_4 > 0, \quad \textit{and} \quad \max\big\{|\alpha|, |\beta_1|, |\beta_2|, |\alpha_3|, |\alpha_4|\big\} \leqslant C\alpha, \tag{1.21}$$

where the constant C depends on $\alpha_1, \alpha_2, \beta_3, \beta_4$.

Then for any $T > 0$, the Cauchy problem (1.1)–(1.2) *admits a unique solution $u(x,t) \in Z_T$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is smooth from H^s to Z_T and u belongs to $C((0,+\infty); H^{+\infty})$.*

Theorem 1.4. *Let $u_0, v_0 \in H^2(\mathbb{R})$. Assume that $\|u_0\|_{L^2}, \|v_0\|_{L^2} \leqslant \eta$ for some enough small number $\eta > 0$ (smallness assumption). Under the conditions of* (1.21)*, if $\gamma_1 = 0$, for any $T > 0$, $t \in (0,T)$, then the solution $u(x,t)$ of the Cauchy problem* (1.1)–(1.2) *converges to the solution $v(x,t)$ of the Cauchy problem* (1.3)–(1.4) *in $C([0,T]; H^1)$ as $\alpha, |\beta_2|, |\alpha_3|, |\alpha_4|$ and $\|u_0 - v_0\|_{H^1}$ tend to zero.*

We also prove that the Cauchy problem (1.1)–(1.2) is locally well-posed in H^s $(s > \frac{1}{2})$ uniformly for $\alpha, \beta_2, \alpha_3, \alpha_4$ in Section 5.1. Then for $u_0, v_0 \in H^s(\mathbb{R})$ $(s > \frac{1}{2})$, we have the following results on inviscid limit behavior.

Theorem 1.5. *Let $u_0, v_0 \in H^s(\mathbb{R})$ ($s > \frac{1}{2}$). Under the smallness assumption and* (1.21)*, if $\gamma_1 = 0$, for some $T > 0$, $t \in (0, T)$, then the solution $u(x,t)$ of the Cauchy problem* (1.1)–(1.2) *converges to the solution $v(x,t)$ of the Cauchy problem* (1.3)–(1.4) *in $C([0,T]; H^s)$ as α, $|\beta_2|$, $|\alpha_3|$, $|\alpha_4|$ and $\|u_0 - v_0\|_{H^s}$ tend to zero.*

2. Linear estimates

In this section, we give some linear estimates for Eqs. (1.1) and (1.3), similarly with the dissipative KdV equations [11,12]. In fact, Lemmas 2.1, 2.3 and 2.5 will be used to show local well-posedness of the Cauchy problem (1.1)–(1.2) with $\gamma_1 = 0$ uniformly for α; Lemmas 2.2, 2.4 and 2.6 will be used to study the global well-posedness of the Cauchy problem (1.1)–(1.2). The proofs of the following lemmas are similar with those of the corresponding lemmas in [11,12]. Here, we omit the details.

Lemma 2.1. *Let $s \in \mathbb{R}$ and $\alpha \geqslant 0$. Then*

$$\left\| \psi(t) S_\alpha(t) u_0 \right\|_{X_{s,1/2}} \leqslant C \|u_0\|_{H^s}, \tag{2.1}$$

where the positive constant C is independent of α.

Lemma 2.2. *Let $s \in \mathbb{R}$ and $\alpha \geqslant 0$. Then*

$$\left\| \psi(t) S_\alpha(t) u_0 \right\|_{Y_{s,1/2}} \leqslant C_\alpha \|u_0\|_{H^s}, \tag{2.2}$$

where constant $C_\alpha > 0$ depends on α.

Lemma 2.3. *Let $s \in \mathbb{R}$, $\alpha \geqslant 0$, $0 \leqslant b \leqslant 1$ and $0 \leqslant b' < 1/2$ with $b + b' \leqslant 1$. Then*

$$\left\| \psi(t) \int_0^t S_\alpha(t - t') f(t')\, dt' \right\|_{X_{s,b}} \leqslant C \|f\|_{X_{s,-b'}}, \tag{2.3}$$

where the positive constant C is independent of α.

Lemma 2.4. *Let $s \in \mathbb{R}$, $0 < \delta \ll \frac{1}{2}$ and $\alpha \geqslant 0$. Then*

$$\left\| \psi(t) \int_0^t S_\alpha(t - t') f(t')\, dt' \right\|_{Y_{s,1/2}} \leqslant C_{\delta,\alpha} \|f\|_{Y_{s,-1/2+\delta}}, \tag{2.4}$$

where the positive constant $C_{\delta,\alpha}$ depends on δ and α.

Lemma 2.5. *Let $s \in \mathbb{R}$, $\alpha \geqslant 0$ and $0 < \delta \ll \frac{1}{2}$. Then for $f \in X_{s,-1/2+\delta}$, we have*:

$$\int_0^t S_\alpha(t - t') f(t')\, dt' \in C\big(\mathbb{R}^+, H^s\big). \tag{2.5}$$

Moreover, if $\{f_n\}$ is a sequence in $X_{s,-1/2+\delta}$, satisfying that $f_n \to 0$ as $n \to \infty$. Then

$$\left\| \int_0^t S_\alpha(t - t') f_n(t')\, dt' \right\|_{L^\infty(\mathbb{R}^+, H^s)} \to 0, \quad \text{as } n \to \infty. \tag{2.6}$$

Lemma 2.6. *Let $s \in \mathbb{R}$, $\alpha > 0$, and $0 < \delta < \frac{1}{2}$. Then for $f \in Y_{s,-1/2+\delta}$, we have*:

$$\int_0^t S_\alpha(t - t') f(t')\, dt' \in C\big(\mathbb{R}^+, H^{s+2\delta}\big). \tag{2.7}$$

Moreover, if $\{f_n\}$ *is a sequence and* $f_n \to 0$ *in* $Y_{s,-1/2+\delta}$ *as* $n \to \infty$. *Then*

$$\left\| \int_0^t S_\alpha(t-t') f_n(t')\,dt' \right\|_{L^\infty(\mathbb{R}^+, H^{s+2\delta})} \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

3. Trilinear, multilinear estimates and local well-posedness

In this section, by the linear estimates in Section 2 and the trilinear and multilinear estimates below, we can obtain the local well-posedness for the Cauchy problem (1.1)–(1.2). This can complete the proof of Theorem 1.1. The trilinear and multilinear estimates will be obtained by using $[k; Z]$-multiplier method.

We firstly list some useful notations and properties for multi-linear expressions [19]. Let Z be any Abelian additive group with an invariant measure $d\xi$. For any integer $k \geqslant 2$, we denote $\Gamma_k(Z)$ by the "hyperplane":

$$\Gamma_k(Z) = \big\{(\xi_1, \ldots, \xi_k) \in Z^k \colon \xi_1 + \cdots + \xi_k = 0\big\},$$

which is endowed with the measure,

$$\int_{\Gamma_k(Z)} f = \int_{Z^{k-1}} f(\xi_1, \ldots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1})\,d\xi_1 \ldots d\xi_{k-1},$$

and define a $[k; Z]$-multiplier to be any function $m : \Gamma_k(Z) \to \mathbb{C}$.

If m is a $[k; Z]$-multiplier, we define $\|m\|_{[k;Z]}$ to be the best constant, such that the inequality,

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right| \leqslant \|m\|_{[k;Z]} \prod_{j=1}^k \|f_j\|_{L_2(Z)},$$

holds for all test functions f_j defined on Z. It is clear that $\|m\|_{[k;Z]}$ determines a norm on m, for test functions at least. We are interested in obtaining good bounds on this norm. We will also define $\|m\|_{[k;Z]}$ in situations where m is defined on all of Z^k by restricting to $\Gamma_k(Z)$.

We give some properties of $\|m\|_{[k;Z]}$, especially for the case $k = 3$. This corresponds to the bilinear $X_{s,b}$ estimates of Schrödinger equation ($Y_{s,b}$ estimates of GGL equation) since the multilinear estimates can be reduced to some bilinear estimates.

Let

$$\xi_1 + \xi_2 + \xi_3 = 0, \qquad \tau_1 + \tau_2 + \tau_3 = 0, \tag{3.1}$$

$$\tilde{\sigma}_j = \tau_j + h_j(\xi_j), \qquad h_j(\xi_j) = \pm\xi_j^2, \quad j = 1, 2, 3. \tag{3.2}$$

Then we will study the problem of obtaining:

$$\big\|m\big((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{3.3}$$

where $m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3))$ is some $[k; Z]$-multiplier in $\Gamma_3(\mathbb{R} \times \mathbb{R})$.

From (3.1) and (3.2), it follows that

$$\tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3 = h(\xi_1, \xi_2, \xi_3). \tag{3.4}$$

By symmetry, there are only two possibilities for the h_j: the $(+++)$ case,

$$h_1(\xi) = h_2(\xi) = h_3(\xi) = \xi^2, \tag{3.5}$$

and the $(++-)$ case,

$$h_1(\xi) = h_2(\xi) = \xi^2; \qquad h_3(\xi) = -\xi^2. \tag{3.6}$$

Among the two cases, the $(+++)$ case is substantially easier, because the resonance function,

$$h(\xi_1, \xi_2, \xi_3) := \xi_1^2 + \xi_2^2 + \xi_3^2, \tag{3.7}$$

does not vanish except at the origin. The $(++-)$ case is more delicate, because the resonance function,

$$h(\xi_1,\xi_2,\xi_3) := \xi_1^2+\xi_2^2-\xi_3^2, \tag{3.8}$$

vanishes when ξ_1 and ξ_2 are orthogonal.

By the dyadic decomposition of ξ_j, $\tilde{\sigma}_j$ and $h(\xi_1,\xi_2,\xi_3)$, we assume that $|\xi_j|\sim N_j$, $|\tilde{\sigma}_j|\sim L_j$ and $|h(\xi_1,\xi_2,\xi_3)|\sim H$. Where N_j, L_j and H are presumed to be dyadic, i.e. these variables range over numbers of form 2^k $(k\in\mathbb{Z})$.

It is convenient to define $N_{\max}\geqslant N_{\mathrm{med}}\geqslant N_{\min}$ to be the maximum, median, and minimum of N_1, N_2, N_3. Similarly define $L_{\max}\geqslant L_{\mathrm{med}}\geqslant L_{\min}$ whenever $L_1, L_2, L_3>0$. Without loss of generality, we can assume:

$$N_{\max}\gtrsim 1, \qquad L_{\min}\gtrsim 1. \tag{3.9}$$

We adopt following summation conventions.

Any summation of the form $L_{\max}\sim\ldots$ is sum over three dyadic variables $L_1, L_2, L_3\geqslant 1$. Therefore, denote for abbreviation, for instance,

$$\sum_{L_{\max}\sim H} := \sum_{L_1,L_2,L_3\geqslant 1:\ L_{\max}\sim H}.$$

Similarly, any summation of form $N_{\max}\sim\ldots$ sum over three dyadic variables $N_1, N_2, N_3>0$:

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N} := \sum_{N_1,N_2,N_3>0:\ N_{\max}\sim N_{\mathrm{med}}\sim N}.$$

By the dyadic decomposition of ξ_j, $\tilde{\sigma}_j$, as well as $h(\xi_1,\xi_2,\xi_3)$, we estimate the following expression to replace (3.3):

$$\Bigg\| \sum_{N_{\max}\geqslant 1}\sum_{H}\sum_{L_1,L_2,L_3\geqslant 1} m\big((N_1,L_1),(N_2,L_2),(N_3,L_3)\big) X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\Bigg\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1, \tag{3.10}$$

where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the multiplier,

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3}(\xi,\tau) := \chi_{|h(\xi)|\sim H}\prod_{j=1}^{3}\chi_{|\xi_j|\sim N_j}\chi_{|\tilde{\sigma}_j|\sim L_j}. \tag{3.11}$$

From the identities (3.1) and (3.4), $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless,

$$N_{\max}\sim N_{\mathrm{med}}, \tag{3.12}$$

and

$$L_{\max}\sim \max(H, L_{\mathrm{med}}). \tag{3.13}$$

By the comparison principle and Schur's test [19], it suffices to prove, for $N_{\max}\gtrsim 1$, that

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,L_3\gtrsim 1} m\big((N_1,L_1),(N_2,L_2),(N_3,L_3)\big)\|X_{N_1,N_2,N_3;L_{\max};L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1, \tag{3.14}$$

or

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}}\ \sum_{H\ll L_{\max}} m\big((N_1,L_1),(N_2,L_2),(N_3,L_3)\big)\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1. \tag{3.15}$$

Therefore, we only need to estimate:

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]}. \tag{3.16}$$

Then we have the following lemma about the boundedness of (3.16).

Lemma 3.1. *(See [19].) Let* $H, N_1, N_2, N_3, L_1, L_2, L_3>0$ *obey* (3.12), (3.13).

- *For the* $(+++)$ *case, let the dispersion relations be given by* (3.5)*, then* $H\sim N_{\max}^2$.

(1) *If $N_{\max} \sim N_{\min}$ and $L_{\max} \sim H$, then*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{\min}^{1/2} L_{\mathrm{med}}^{1/4}. \tag{3.17}$$

(2) *For other cases, we have*:

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} \min(N_{\max}N_{\min}, L_{\mathrm{med}})^{1/2}. \tag{3.18}$$

- *For the $(++-)$ case, let the dispersion relations be given by* (3.6), *then $H \sim N_1N_2$*:

(1) *If $N_{\max} \sim N_{\min}$ and $L_{\max} \sim H$, then*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{\min}^{1/2} L_{\mathrm{med}}^{1/4}. \tag{3.19}$$

(2) *If $N_1 \sim N_{\min}, L_1 \sim L_{\max} \sim H$ or $N_2 \sim N_{\min}, L_2 \sim L_{\max} \sim H$, then*

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{\min}^{1/2} N_{\min}^{-1/2} L_{\mathrm{med}}^{1/2}. \tag{3.20}$$

(3) *For other cases, we have*:

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} \min(N_{\max}N_{\min}, L_{\mathrm{med}})^{1/2}. \tag{3.21}$$

Lemma 3.2 *(Comparison principle). (See [19].) If m and M are $[k; Z]$ multipliers and satisfy $|m(\xi)| \leqslant |M(\xi)|$ for all $\xi \in \Gamma_k(Z)$. Then $\|m\|_{[k;Z]} \leqslant \|M\|_{[k;Z]}$. Also, if m is a $[k, Z]$ multiplier, and $a_1, \dots, a_k$ are functions from Z to $\mathbb{R}$, then*

$$\left\| m(\xi) \prod_{j=1}^{k} a_j(\xi_j) \right\|_{[k;Z]} \leqslant \|m\|_{[k;Z]} \prod_{j=1}^{k} \|a_j\|_{L_\infty}. \tag{3.22}$$

Lemma 3.3 *(Direct and semi-direct tensor products). (See [19].) Let Z_1, Z_2 be Abelian groups, with $Z_1 \times Z_2$ parameterized by (ξ^1, ξ^2) and m_1, m_2 be $[k; Z_1]$ and $[k; Z_2]$ multipliers respectively. Define the tensor product $m_1 \otimes m_2$ to be the $[k; Z_1 \times Z_2]$ multiplier*:

$$m_1 \otimes m_2\big(\big(\xi_1^1, \xi_1^2\big), \dots, \big(\xi_k^1, \xi_k^2\big)\big) = m_1\big(\xi_1^1, \dots, \xi_k^1\big) m_2\big(\xi_1^2, \dots, \xi_k^2\big).$$

Then we have:

$$\|m_1 \otimes m_2\|_{[k;Z_1\times Z_2]} = \|m_1\|_{[k;Z_1]} \|m_2\|_{[k;Z_2]}. \tag{3.23}$$

More generally, if m is a $[k; Z_1 \times Z_2]$ multiplier, define the $[k; Z_2]$ multiplier $m(\xi^1)$ for all $\xi^1 \in \Gamma_k(Z_1)$ by:

$$m\big(\xi^1\big)\big(\xi^2\big) := m\big(\big(\xi_1^1, \xi_1^2\big), \dots, \big(\xi_k^1, \xi_k^2\big)\big).$$

Then we have:

$$\|m\|_{[k;Z_1\times Z_2]} \leqslant \big\| \big\| m\big(\xi^1\big) \big\|_{[k;Z_2]} \big\|_{[k;Z_1]}. \tag{3.24}$$

Lemma 3.4 *(Tensored box lemma). (See [19].) Suppose $(R+\eta)_{\eta\in\Sigma}$ is a box covering of Z, and $m(\xi)$ is a function from Z to $\mathbb{R}$. Then for any $\eta \in \Sigma$, we have*:

$$\big\| m(\xi_1)\chi_{R+\eta}(\xi_2) \big\|_{[3;Z]} \sim \sup_{\eta'\in\Sigma} \|m\|_{L^2(R+\eta')}. \tag{3.25}$$

Also, we have:

$$\big\| m(\xi_1) \big\|_{[3;Z]} = \|m\|_{L^2}. \tag{3.26}$$

Lemma 3.5 *(Composition and TT^*). (See [19].) If $k_1, k_2 \geqslant 1$ and m_1, m_2 are functions on Z^{k_1} and Z^{k_2} respectively, then*

$$\begin{aligned} &\big\| m_1(\xi_1, \dots, \xi_{k_1}) m_2(\xi_{k_1+1}, \dots, \xi_{k_1+k_2}) \big\|_{[k_1+k_2;Z]} \\ &\quad \leqslant \big\| m_1(\xi_1, \dots, \xi_{k_1}) \big\|_{[k_1+1;Z]} \big\| m_2(\xi_1, \dots, \xi_{k_2}) \big\|_{[k_2+1;Z]}. \end{aligned} \tag{3.27}$$

As a special case, for all functions $m : Z^k \to \mathbb{R}$, we have the TT^* identity:

$$\left\| m(\xi_1, \ldots, \xi_k)\overline{m(-\xi_{k+1}, \ldots, -\xi_{2k})} \right\|_{[2k;Z]} = \left\| m(\xi_1, \ldots, \xi_k) \right\|^2_{[k+1;Z]}. \tag{3.28}$$

By using the lemmas above, we will give the main theorems in this section. We firstly give some notations about multilinear estimates. Define:

$$\begin{aligned} &\sigma_j = \tau_j - \xi_j^2, \qquad \bar{\sigma}_j = \tau_j + \xi_j^2, \quad j = 1, 2, \ldots, k, \\ &\xi_1 + \xi_2 + \cdots + \xi_k = 0, \qquad \tau_1 + \tau_2 + \cdots + \tau_k = 0. \end{aligned} \tag{3.29}$$

Denote $\tilde{\xi}_j$ and $\tilde{\tau}_j$ by variables different from $\xi_1, \xi_2, \ldots, \xi_k; \tau_1, \tau_2, \ldots, \tau_k$ respectively. Also define $\tilde{\sigma}_j = \tilde{\tau}_j - |\tilde{\xi}_j|^2$ or $\tilde{\tau}_j + |\tilde{\xi}_j|^2$.

$$|\sigma|_{\max} = \max\left\{|\sigma_{j_1}|, \ldots, |\sigma_{j_{k_1}}|; |\bar{\sigma}_{l_1}|, \ldots, |\bar{\sigma}_{l_{k_2}}|; |\tilde{\sigma}_{n_1}|, \ldots, |\tilde{\sigma}_{n_{k_3}}|\right\}, \tag{3.30}$$

$$|\sigma|_{\mathrm{med}} = \mathrm{med}\left\{|\sigma_{j_1}|, \ldots, |\sigma_{j_{k_1}}|; |\bar{\sigma}_{l_1}|, \ldots, |\bar{\sigma}_{l_{k_2}}|; |\tilde{\sigma}_{n_1}|, \ldots, |\tilde{\sigma}_{n_{k_3}}|\right\}, \tag{3.31}$$

$$|\xi|_{\max} = \max\left\{|\xi_{j_1}|, \ldots, |\xi_{j_{k_1}}|; |\tilde{\xi}_{l_1}|, \ldots, |\tilde{\xi}_{l_{k_2}}|\right\}, \tag{3.32}$$

$$|\xi|_{\mathrm{med}} = \mathrm{med}\left\{|\xi_{j_1}|, \ldots, |\xi_{j_{k_1}}|; |\tilde{\xi}_{l_1}|, \ldots, |\tilde{\xi}_{l_{k_2}}|\right\}. \tag{3.33}$$

For convenience, by the dyadic decomposition of ξ_j, σ_j, $\bar{\sigma}_j$; $\tilde{\xi}_j$, $\tilde{\sigma}_j$, we assume that $|\xi_j| \sim N_j$, $|\sigma_j| \sim L_j$, $|\bar{\sigma}_j| \sim L_j$; $|\tilde{\xi}_j| \sim \widetilde{N}_{\tilde{j}}$, $|\tilde{\sigma}_j| \sim \widetilde{L}_{\tilde{j}}$. Define $N_{\max} \geqslant N_{\mathrm{med}} \geqslant N_{\min}$ to be the maximum, median, and minimum of $\{N_{j_1}, N_{j_2}, \ldots, N_{j_{k_1}}; \widetilde{N}_{l_1}, \widetilde{N}_{l_2}, \ldots, \widetilde{N}_{l_{k_2}}\}$.

Similarly, define $L_{\max} \geqslant L_{\mathrm{med}} \geqslant L_{\min}$ to be the maximum, median, and minimum of $\{L_{j_1}, L_{j_2}, \ldots, L_{j_{k_1}}; \widetilde{L}_{l_1}, \widetilde{L}_{l_2}, \ldots, \widetilde{L}_{l_{k_2}}\}$. Notice that indices above $j_1, \ldots, j_{k_1}$; $l_1, \ldots, l_{k_2}$ and $n_1, \ldots, n_{k_3}$ are different in the following different cases.

Theorem 3.6 *(Trilinear estimates). Let $s > 0$ and $0 < \delta \ll \frac{1}{2}$, then*

$$\left\| \partial_x(u_1 u_2 \bar{u}_3) \right\|_{Y_{s,-1/2+\delta}} \leqslant C_{\delta,\alpha} \|u_1\|_{Y_{s,1/2}} \|u_2\|_{Y_{s,1/2}} \|u_3\|_{Y_{s,1/2}}, \tag{3.34}$$

$$\|u_1 u_2 \bar{u}_3\|_{Y_{s,-1/2+\delta}} \leqslant C_{\delta,\alpha} \|u_1\|_{Y_{s,1/2}} \|u_2\|_{Y_{s,1/2}} \|u_3\|_{Y_{s,1/2}}, \tag{3.35}$$

where the positive constant $C_{\delta,\alpha}$ depends on δ and α.

Proof. We only prove the estimates (3.34). The proof of (3.35) is easier than that of (3.34). In fact,we can prove that (3.35) holds for some negative s. However, we are only interested in $s > 0$ in this paper. In the following proof and that of Theorem 3.7, the proof of the claims (3.15) is similar with that of (3.14). For simplicity, we sometimes prove them without distinguishing and pointing out later. In other words, we define sometimes:

$$\sum_{L_1, L_2, L_3 \geqslant 1} := \sum_{L_1, L_2, L_3 \geqslant 1:\, L_{\max} \sim H} \quad \text{or} \quad \sum_{L_{\max} \sim L_{\mathrm{med}}} \sum_{H \ll L_{\max}}.$$

First, we prove (3.34). By duality and the Plancherel identity, it suffices to show

$$\begin{aligned} &\left\| m\big((\xi_1, \tau_1), \ldots, (\xi_4, \tau_4)\big) \right\|_{[4, \mathbb{R}\times\mathbb{R}]} \\ &:= \left\| \frac{K(\xi_1, \xi_2, \xi_3, \xi_4)}{\langle i\sigma_1 + \alpha|\xi_1|^2\rangle^{1/2} \langle i\sigma_2 + \alpha|\xi_2|^2\rangle^{1/2} \langle i\bar{\sigma}_3 + \alpha|\xi_3|^2\rangle^{1/2} \langle i\bar{\sigma}_4 + \alpha|\xi_4|^2\rangle^{1/2-\delta}} \right\|_{[4, \mathbb{R}\times\mathbb{R}]} \\ &\lesssim 1, \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} &K(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{|\xi_4| \langle \xi_4\rangle^s}{\langle \xi_1\rangle^s \langle \xi_2\rangle^s \langle \xi_3\rangle^s}, \\ &\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0, \qquad \tau_1 + \tau_2 + \tau_3 + \tau_4 = 0. \end{aligned} \tag{3.37}$$

By symmetry, we separately consider two cases:

$$\text{(A)}\quad |\xi_4| \lesssim |\xi_1| = \max\big\{|\xi_1|, |\xi_2|, |\xi_3|\big\}, \tag{3.38}$$

$$\text{(B)}\quad |\xi_4| \lesssim |\xi_3| = \max\big\{|\xi_1|, |\xi_2|, |\xi_3|\big\}. \tag{3.39}$$

First, we consider Case (A). Then it holds that

$$\begin{aligned} &m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big) \\ &\lesssim \frac{|\xi_4|}{\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2-\delta}\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}} \frac{\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_3+\alpha|\xi_3|^2\rangle^{1/2}} \\ &\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/4}} \frac{\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_3+\alpha|\xi_3|^2\rangle^{1/2}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/4-\delta}} \\ &:= m_{a-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)m_{a-2}\big((\xi_3,\tau_3),(\xi_4,\tau_4)\big). \end{aligned} \tag{3.40}$$

By Lemmas 3.2 and 3.5, it suffices to prove that

$$\begin{aligned} &\big\|m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]} \\ &\quad\lesssim \big\|m_{a-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\big\|m_{a-2}\big((\xi_3,\tau_3),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ &\quad\lesssim 1. \end{aligned} \tag{3.41}$$

We will prove the following two inequalities separately as below,

$$\big\|m_{a-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{3.42}$$

$$\big\|m_{a-2}\big((\xi_3,\tau_3),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{3.43}$$

Situation A-I. For $\|m_{a-1}((\xi_1,\tau_1),(\xi_2,\tau_2))\|_{[3,\mathbb{R}\times\mathbb{R}]}$, we choose two variables $\tilde{\xi}_3$ and $\tilde{\tau}_3$ such that $\xi_1+\xi_2+\tilde{\xi}_3=0$ and $\tau_1+\tau_2+\tilde{\tau}_3=0$. Let $\tilde{\sigma}_3=\tilde{\tau}_3-\tilde{\xi}_3^2$, from (3.29), one can conclude that it is the $(+++)$ case. Then $|\sigma_1+\sigma_2+\tilde{\sigma}_3| = |h(\xi_1,\xi_2,\tilde{\xi}_3)| \sim |\xi|_{\max}^2$, where $|\xi|_{\max} = \max\{|\xi_1|,|\xi_2|,|\tilde{\xi}_3|\}$.

We can separately consider four cases:

$$\begin{aligned} &\text{Case 1:}\quad |\xi_1|\sim|\xi_2|\sim|\tilde{\xi}_3|, &&\text{Case 2:}\quad |\xi_1|\sim|\xi_2|\gg|\tilde{\xi}_3|, \\ &\text{Case 3:}\quad |\xi_1|\sim|\tilde{\xi}_3|\gg|\xi_2|, &&\text{Case 4:}\quad |\xi_2|\sim|\tilde{\xi}_3|\gg|\xi_1|. \end{aligned}$$

Case A-I-1. Assume that $N_1\sim N_2\sim\widetilde{N}_3\sim N_{\max}\sim N_{\min}\sim N$.

If $L_{\max}\sim H\sim N_{\max}^2$, we apply (3.17) to obtain (3.42). Then, for $s\geqslant 5\varepsilon$ with any small enough $\varepsilon>0$, the left side of (3.42) is bounded by:

$$\begin{aligned} &\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{N^{-s}L_{\min}^{1/2}L_{\mathrm{med}}^{1/4}}{\langle L_1+\alpha N^2\rangle^{1/4}\langle L_2+\alpha N^2\rangle^{1/2}} \\ &\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{1}{N^s} \\ &\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{1}{L_{\min}^{\varepsilon}L_{\mathrm{med}}^{\varepsilon}N^{\varepsilon}} \\ &\quad\lesssim 1. \end{aligned} \tag{3.44}$$

If the case $L_{\max}\sim L_{\mathrm{med}}\gg N_{\max}^2$, then for $s\geqslant 5\varepsilon$, by (3.18), the left side of (3.42) is bounded by:

$$\begin{aligned} &\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H} \frac{N^{-s}L_{\min}^{1/2}N^{-1/2}\min(N^2,L_{\mathrm{med}})^{1/2}}{\langle L_1+\alpha N^2\rangle^{1/4}\langle L_2+\alpha N^2\rangle^{1/2}} \\ &\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H} \frac{N^{-s}L_{\min}^{1/2}N^{-1/2}\min(N^2,L_{\mathrm{med}})^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}\langle L_{\min}+\alpha N^2\rangle^{1/2}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H} \frac{N^{1/2-s}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}}\\
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H} \frac{N^{1/2-5\varepsilon}}{L_{\min}^{\varepsilon}L_{\mathrm{med}}^{\varepsilon}\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4-2\varepsilon}}\\
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H} \frac{1}{L_{\min}^{\varepsilon}L_{\mathrm{med}}^{\varepsilon}N^{\varepsilon}}\lesssim 1.
\end{aligned}\tag{3.45}$$

Case A-I-2. If $N\sim N_{\max}\sim N_1\sim N_2\gg \widetilde{N}_3\sim N_{\min}$, then for $s\geqslant 5\varepsilon$, similarly with the case above, by applying (3.18), the left side of (3.42) is bounded by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{N^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_1+\alpha N^2\rangle^{1/4}\langle L_2+\alpha N^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{N^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}\langle L_{\min}+\alpha N^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{N^{-s}N_{\min}^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}}\lesssim 1.
\end{aligned}\tag{3.46}$$

In fact, as we point out, the proof of the case $L_{\max}\sim L_{\mathrm{med}}\gg H\sim N_{\max}^2$ can be obtained similarly. We omit the details here. The following cases are same as what we point out.

Case A-I-3. If $N_{\max}\sim N_1\sim \widetilde{N}_3\gg N_2\sim N_{\min}$, then for $s\geqslant 5\varepsilon$, similarly with Case A-I-1, by applying (3.18), the left side of (3.42) is bounded by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_1+\alpha N^2\rangle^{1/4}\langle L_2+\alpha N_{\min}^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}\langle L_{\min}+\alpha N_{\min}^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{N_{\min}^{1/2}\langle N_{\min}\rangle^{-s}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,L_3\gtrsim 1} \frac{N_{\min}^{1/2-s}}{L_{\min}^{\varepsilon}L_{\mathrm{med}}^{\varepsilon}\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4-2\varepsilon}}\lesssim 1.
\end{aligned}\tag{3.47}$$

Case A-I-4. If $N_{\max}\sim N_2\sim \widetilde{N}_3\gg N_1\sim N_{\min}$, then for $s\geqslant 5\varepsilon$, similarly with the above case, by applying (3.18), the left side of (3.42) is bounded by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{\langle N\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_1+\alpha N_{\min}^2\rangle^{1/4}\langle L_2+\alpha N^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{N_{\min}^{1/2}\langle N\rangle^{-s}}{\langle L_{\mathrm{med}}+\alpha N_{\min}^2\rangle^{1/4}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_3\gtrsim 1} \frac{N_{\min}^{1/2}\langle N\rangle^{-s}}{\langle L_{\mathrm{med}}+\alpha N_{\min}^2\rangle^{1/4-2\varepsilon}L_{\min}^{\varepsilon}L_{\mathrm{med}}^{\varepsilon}}\lesssim 1.
\end{aligned}\tag{3.48}$$

Situation A-II. For $\|m_{a-2}((\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[3,\mathbb{R}\times\mathbb{R}]}$, we choose two variables $\tilde{\xi}_2$ and $\tilde{\tau}_2$ such that $\xi_3+\xi_4+\tilde{\xi}_2=0$ and $\tau_3+\tau_4+\tilde{\tau}_2=0$. If $\tilde{\sigma}_2=\tilde{\tau}_2+\tilde{\xi}_2^2$, then it follows that $|\tilde{\sigma}_2+\sigma_3+\sigma_4|=|h(\xi_3,\xi_4,\tilde{\xi}_2)|\sim|\xi|^2_{\max}$, where $|\xi|_{\max}=\max\{|\xi_3|,|\xi_4|,|\tilde{\xi}_2|\}$. It is the $(+++)$ case.

Similarly with Situation A-I, we can separately consider four cases:

$$\text{Case 1:}\quad |\xi_3|\sim|\xi_4|\sim|\tilde{\xi}_2|;\qquad \text{Case 2:}\quad |\xi_3|\sim|\tilde{\xi}_2|\gg|\xi_4|;$$
$$\text{Case 3:}\quad |\xi_3|\sim|\xi_4|\gg|\tilde{\xi}_2|;\qquad \text{Case 4:}\quad |\xi_4|\sim|\tilde{\xi}_2|\gg|\xi_3|.$$

Case A-II-1. Assume: $N_3\sim N_4\sim\widetilde{N}_2\sim N_{\max}\sim N_{\min}$.

If $L_{\max}\sim H\sim N^2_{\max}$, then similarly with Case A-I-1, for $s\geqslant 5\varepsilon+2\delta$, by applying (3.17), the left side of (3.43) is bounded by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_4,\widetilde{L}_2\gtrsim 1}\frac{N^{-s}L_{\min}^{1/2}L_{\mathrm{med}}^{1/4}}{\langle L_4+\alpha N^2\rangle^{1/4-\delta}\langle L_3+\alpha N^2\rangle^{1/2}}\\
&\lesssim\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_4,\widetilde{L}_2\gtrsim 1}\frac{N^{2\delta}}{N^s}\\
&\lesssim\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_4,\widetilde{L}_2\gtrsim 1}\frac{1}{L^{\varepsilon}_{\min}L^{\varepsilon}_{\mathrm{med}}N^{\varepsilon}}\lesssim 1.
\end{aligned}\tag{3.49}$$

If $L_{\max}\sim L_{\mathrm{med}}\gg H\sim N^2_{\max}$, then similarly with Case A-I-1, for $s\geqslant 5\varepsilon+2\delta$, we use (3.18) to bound the left side of (3.43) by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H}\frac{N^{-s}L_{\min}^{1/2}N^{-1/2}\min(N^2,L_{\mathrm{med}})^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4-\delta}\langle L_{\min}+\alpha N^2\rangle^{1/2}}\\
&\lesssim\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H}\frac{N^{1/2-2\delta-5\varepsilon}}{L^{\varepsilon}_{\min}L^{\varepsilon}_{\mathrm{med}}\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4-\delta-2\varepsilon}}\\
&\lesssim\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gg H}\frac{1}{L^{\varepsilon}_{\min}L^{\varepsilon}_{\mathrm{med}}N^{\varepsilon}}\lesssim 1.
\end{aligned}\tag{3.50}$$

Case A-II-2. If $N_{\max}\sim\widetilde{N}_2\sim N_3\gg N_4\sim N_{\min}$, then similarly with the above case, for $s\geqslant 5\varepsilon+2\delta$, we use (3.18) to bound the left side of (3.43) by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_4,\widetilde{L}_2\gtrsim 1}\frac{N^{-s}L_{\min}^{1/2}N^{-1/2}\min(N^2,L_{\mathrm{med}})^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2_{\min}\rangle^{1/4-\delta}\langle L_{\min}+aN^2\rangle^{1/2}}\\
&\lesssim\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_4,\widetilde{L}_2\gtrsim 1}\frac{N^{-2\delta-5\varepsilon}N_{\min}^{1/2}}{L^{\varepsilon}_{\min}L^{\varepsilon}_{\mathrm{med}}\langle L_{\mathrm{med}}+\alpha N^2_{\min}\rangle^{1/4-\delta-2\varepsilon}}\\
&\lesssim\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_4,\widetilde{L}_2\gtrsim 1}\frac{1}{L^{\varepsilon}_{\min}L^{\varepsilon}_{\mathrm{med}}N^{\varepsilon}}\lesssim 1.
\end{aligned}\tag{3.51}$$

For other cases:

$$N_{\max}\sim N_3\sim N_4\gg\widetilde{N}_2\sim N_{\min}\quad\text{and}\quad N_{\max}\sim\widetilde{N}_2\sim N_4\gg N_3\sim N_{\min},$$

similarly with Case A-I-2 and Case A-I-3, respectively, we can obtain the results for $s\geqslant 5\varepsilon+2\delta$.

Next, we consider Case (B): $|\xi_4| \lesssim |\xi_3| = \max\{|\xi_1|, |\xi_2|, |\xi_3|\}$. By Lemmas 3.2 and 3.5, we obtain that

$$\begin{aligned}
&\big\| m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]} \\
&\lesssim \left\| \frac{|\xi_4|}{\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{1/2-\delta}\langle i\bar\sigma_3+\alpha|\xi_3|^2\rangle^{1/2}} \frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \\
&\lesssim \left\| \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar\sigma_3+\alpha|\xi_3|^2\rangle^{1/4}} \cdot \frac{\langle\xi_1\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{1/4-\delta}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \\
&\lesssim \big\| m_{b-1}\big((\xi_2,\tau_2),(\xi_3,\tau_3)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \cdot \big\| m_{b-2}\big((\xi_1,\tau_1),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}.
\end{aligned} \tag{3.52}$$

Situation B-I. In this situation, we will prove:

$$\big\| m_{b-1}\big((\xi_2,\tau_2),(\xi_3,\tau_3)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{3.53}$$

We choose two variables $\tilde\xi_1$ and $\tilde\tau_1$ such that $\tilde\xi_1+\xi_2+\xi_3=0$ and $\tilde\tau_1+\tau_2+\tau_3=0$. Let $\tilde\sigma_1=\tilde\tau_1-\tilde\xi_1^2$ or $\tilde\tau_1+\tilde\xi_1^2$ in the different cases. It follows that $|\tilde\sigma_1+\sigma_2+\bar\sigma_3| = |h(\tilde\xi_1,\xi_2,\xi_3)| \lesssim |\xi|_{\max}$, where $|\xi|_{\max}=\max\{|\tilde\xi_1|,|\xi_2|,|\xi_3|\}$. It is the $(++-)$ case. Similarly with Situation A, we can separately consider four cases:

$$\text{Case 1:}\quad |\xi_2|\sim|\xi_3|\sim|\tilde\xi_1|; \qquad \text{Case 2:}\quad |\xi_2|\sim|\xi_3|\gg|\tilde\xi_1|;$$
$$\text{Case 3:}\quad |\tilde\xi_1|\sim|\xi_3|\gg|\xi_2|; \qquad \text{Case 4:}\quad |\tilde\xi_1|\sim|\xi_2|\gg|\xi_3|.$$

Case B-I-1. If $\widetilde N_1\sim N_2\sim N_3\sim N_{\max}\sim N_{\min}$, then for $s\geqslant 5\varepsilon$, we can obtain (3.53) similarly with Case A-I-1.

Case B-I-2. If $N_{\max}\sim N_2\sim N_3\gg \widetilde N_1\sim N_{\min}$, then $H\sim NN_{\min}$.

Subcase B-I-2-1. If $\widetilde L_1\sim L_{\max}\sim NN_{\min}$, then for $s\geqslant 5\varepsilon$, we use (3.20) to bound the left side of (3.53) by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_2,\widetilde L_1\gtrsim 1} \frac{\langle N\rangle^{-s}L_{\min}^{1/2}N_{\min}^{-1/2}L_{\mathrm{med}}^{1/2}}{\langle L_3+\alpha N^2\rangle^{1/4}\langle L_2+\alpha N^2\rangle^{1/2}} \\
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_2,\widetilde L_1\gtrsim 1} \frac{N^{1/2-s}}{\langle L_3+\alpha N^2\rangle^{1/4}} \\
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_2,\widetilde L_1\gtrsim 1} \frac{1}{N^s} \\
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_2,\widetilde L_1\gtrsim 1} \frac{1}{L_{\min}^{\varepsilon}L_{\mathrm{med}}^{\varepsilon}N^{\varepsilon}} \lesssim 1.
\end{aligned} \tag{3.54}$$

Subcase B-I-2-2. For other cases, for $s\geqslant 5\varepsilon$, we use (3.21) to bound the left side of (3.53) by:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_2,\widetilde L_1\gtrsim 1} \frac{\langle N\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}\langle L_{\min}+\alpha N^2\rangle^{1/2}} \\
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_2,\widetilde L_1\gtrsim 1} \frac{\langle N\rangle^{-s}N_{\min}^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}} \\
&\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_3,L_2,\widetilde L_1\gtrsim 1} \frac{N^{-s}N_{\min}^{1/2}}{L_{\min}^{\varepsilon}L_{\mathrm{med}}^{\varepsilon}\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4-2\varepsilon}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{L_3,L_2,\widetilde{L}_1\gtrsim 1} \frac{N_{\min}^{1/2}}{L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}N^{1/2-4\varepsilon}N^{s}}\\
&\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{L_3,L_2,\widetilde{L}_1\gtrsim 1} \frac{1}{L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}N^{\varepsilon}} \lesssim 1.
\end{aligned}
\tag{3.55}
$$

Case B-I-3. If $N_{\max}\sim N_3\sim \widetilde{N}_1 \gg N_2\sim N_{\min}$, then we choose $\tilde{\sigma}_1=\tilde{\tau}_1+\tilde{\xi}_1^2$, it follows that $|\tilde{\sigma}_1+\sigma_2+\bar{\sigma}_3| = |h(\tilde{\xi}_1,\xi_2,\xi_3)|\sim|\xi|_{\max}^2$. Then, for $s\geqslant 5\varepsilon$, we use (3.21) to bound the left side of (3.53) by:

$$
\begin{aligned}
&\sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{L_2,L_3,\widetilde{L}_1\gtrsim 1} \frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\text{med}})^{1/2}}{\langle L_3+\alpha N^2\rangle^{1/4}\langle L_2+\alpha N_{\min}^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{L_2,L_3,\widetilde{L}_1\gtrsim 1} \frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\text{med}})^{1/2}}{\langle L_{\text{med}}+\alpha N^2\rangle^{1/4}\langle L_{\min}+\alpha N_{\min}^2\rangle^{1/2}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{L_2,L_3,\widetilde{L}_1\gtrsim 1} \frac{\langle N_{\min}\rangle^{-s}N_{\min}^{1/2}}{\langle L_{\text{med}}+\alpha N^2\rangle^{1/4-2\varepsilon}L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}}\\
&\quad\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{L_2,L_3,\widetilde{L}_1\gtrsim 1} \frac{1}{L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}N^{\varepsilon}} \lesssim 1.
\end{aligned}
\tag{3.56}
$$

Case B-I-4. If $N_{\max}\sim N_2\sim\widetilde{N}_1\gg N_3\sim N_{\min}$, then we choose $\tilde{\sigma}_1=\tilde{\tau}_1-\tilde{\xi}_1^2$ such that $|\tilde{\sigma}_1+\sigma_2+\bar{\sigma}_3|\sim|\xi_2|^2\sim|\xi|_{\max}^2$. Then similarly with the above, for $s\geqslant 5\varepsilon$, we use (3.21) to bound the left side of (3.53) by:

$$
\sum_{N_{\max}\sim N_{\text{med}}\sim N} \sum_{L_2,L_3,\widetilde{L}_1\gtrsim 1} \frac{\langle N\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\text{med}})^{1/2}}{\langle L_3+\alpha N_{\min}^2\rangle^{1/4}\langle L_2+\alpha N^2\rangle^{1/2}} \lesssim 1.
\tag{3.57}
$$

Situation B-II. In this situation, we will prove:

$$
\big\|m_{b-2}\big((\xi_1,\tau_1),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1.
\tag{3.58}
$$

In fact, we choose two variables $\tilde{\xi}_0$ and $\tilde{\tau}_0$ such that $\tilde{\xi}_0+\xi_1+\xi_4=0$ and $\tilde{\tau}_0+\tau_1+\tau_4=0$. Let $\tilde{\sigma}_0=\tilde{\tau}_0-\tilde{\xi}_0^2$ or $\tilde{\sigma}_0=\tilde{\tau}_0+\tilde{\xi}_0^2$. Then we can obtain (3.58) similarly with the proof of (3.53) for $s\geqslant 5\varepsilon+2\delta$.

This completes the proof of Theorem 3.6. □

Theorem 3.7 *(Multilinear estimate). Let* $s>0$ *and* $0<\delta\ll\frac{1}{2}$.

$$
\|u_1u_2u_3\bar{u}_4\bar{u}_5\|_{Y_{s,-1/2+\delta}}\leqslant C_{\delta,\alpha}\|u_1\|_{Y_{s,1/2}}\|u_2\|_{Y_{s,1/2}}\|u_3\|_{Y_{s,1/2}}\|u_4\|_{Y_{s,1/2}}\|u_5\|_{Y_{s,1/2}},
\tag{3.59}
$$

where the positive constant $C_{\delta,\alpha}$ *depends on* δ *and* α.

Proof. Similarly with the proof of Theorem 3.6, by duality and the Plancherel identity, it suffices to show

$$
\begin{aligned}
&\big\|m\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big)\big\|_{[6,\mathbb{R}\times\mathbb{R}]}\\
&\quad=\left\|\frac{\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{-1/2}\langle i\bar{\sigma}_5+\alpha|\xi_5|^2\rangle^{-1/2}\langle i\bar{\sigma}_6+\alpha|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\sigma_3+\alpha|\xi_3|^2\rangle^{1/2}}K(\xi_1,\dots,\xi_6)\right\|_{[6,\mathbb{R}\times\mathbb{R}]}\\
&\quad\lesssim 1,
\end{aligned}
\tag{3.60}
$$

where

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)=\frac{\langle\xi_6\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\xi_5\rangle^s},$$
$$\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0,\qquad \tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0. \tag{3.61}$$

By symmetry, we separately consider two cases

$$\text{(C)}\quad |\xi_6|\lesssim|\xi_1|=\max\bigl\{|\xi_1|,|\xi_2|,|\xi_3|,|\xi_4|,|\xi_5|\bigr\};$$
$$\text{(D)}\quad |\xi_6|\lesssim|\xi_4|=\max\bigl\{|\xi_1|,|\xi_2|,|\xi_3|,|\xi_4|,|\xi_5|\bigr\}.$$

In fact, the proofs of the cases $|\xi_6|\lesssim|\xi_2|$ and $|\xi_6|\lesssim|\xi_3|$ are similar with that of Case (C). The proof of the case $|\xi_6|\lesssim|\xi_5|$ is similar with that of Case (D).

First, we consider Case (C). It follows that

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\leqslant\frac{1}{\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\xi_5\rangle^s} \tag{3.62}$$

and

$$\begin{aligned}
&m\bigl((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\bigr)\\
&\quad\leqslant\frac{\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{-1/2}\langle i\bar\sigma_5+\alpha|\xi_5|^2\rangle^{-1/2}\langle i\bar\sigma_6+\alpha|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\sigma_3+\alpha|\xi_3|^2\rangle^{1/2}}\frac{1}{\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\xi_5\rangle^s}\\
&\quad\leqslant\frac{\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{-1/2}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}}\frac{\langle i\bar\sigma_5+\alpha|\xi_5|^2\rangle^{-1/2}\langle i\bar\sigma_6+\alpha|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_3+\alpha|\xi_3|^2\rangle^{1/2}\langle\xi_3\rangle^s\langle\xi_5\rangle^s}\\
&\quad:=m_{c-1}\bigl((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\bigr)m_{c-2}\bigl((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\bigr).
\end{aligned} \tag{3.63}$$

By Lemmas 3.2 and 3.5, it suffices to prove:

$$\begin{aligned}
&\bigl\|m\bigl((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\bigr)\bigr\|_{[6,\mathbb{R}\times\mathbb{R}]}\\
&\quad\lesssim\bigl\|m_{c-1}\bigl((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\bigr)\bigr\|_{[4,\mathbb{R}\times\mathbb{R}]}\bigl\|m_{c-2}\bigl((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\bigr)\bigr\|_{[4,\mathbb{R}\times\mathbb{R}]}\\
&\quad\lesssim 1.
\end{aligned} \tag{3.64}$$

Situation C-I. We first prove

$$\bigl\|m_{c-1}\bigl((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\bigr)\bigr\|_{[4,\mathbb{R}\times\mathbb{R}]}\lesssim 1. \tag{3.65}$$

We choose two variables $\tilde\xi_3$ and $\tilde\tau_3$ such that $\xi_1+\xi_2+\tilde\xi_3+\xi_4=0$ and $\tau_1+\tau_2+\tilde\tau_3+\tau_4=0$. Let $\tilde\sigma_3=\tilde\tau_3+\tilde\xi_3^2$, it follows that $|\sigma_1+\sigma_2+\tilde\sigma_3+\sigma_4|=|h(\xi_1,\xi_2,\tilde\xi_3,\xi_4)|\lesssim|\xi|^2_{\max}$, where $|\xi|_{\max}=\max\{|\xi_1|,|\xi_2|,|\tilde\xi_3|,|\xi_4|\}$. Moreover, we have,

$$|\sigma|_{\max}\sim|\sigma|_{\mathrm{med}}\gg\bigl|h(\xi_1,\xi_2,\tilde\xi_3,\xi_4)\bigr|, \tag{3.66}$$

or

$$|\sigma|_{\max}\sim\bigl|h(\xi_1,\xi_2,\tilde\xi_3,\xi_4)\bigr|, \tag{3.67}$$

where $|\sigma|_{\max}=\max\{|\sigma_1|,|\sigma_2|,|\tilde\sigma_3|,|\sigma_4|\}$. Without loss of generality, we can assume that

$$|\sigma|_{\max}\sim|\sigma|_{\mathrm{med}}\gtrsim|\xi|^2_{\max}\sim|\xi|^2_{\mathrm{med}}, \tag{3.68}$$

or

$$|\sigma|_{\max}\lesssim|\xi|^2_{\max}\sim|\xi|^2_{\mathrm{med}}. \tag{3.69}$$

First, we consider the case: $|\sigma|_{\max}\sim|\sigma|_{\mathrm{med}}\gtrsim|\xi|^2_{\max}$.

Case C-I-1. If $|\sigma_1| = |\sigma|_{\max}$ or $|\sigma|_{\text{med}}$, then it follows that

$$\begin{aligned} &m_{c-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big) \\ &\quad\lesssim \frac{\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}} \\ &\quad\lesssim \frac{\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/4}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}}. \end{aligned} \tag{3.70}$$

Then we can obtain (3.65) similarly with Situation A in proof of Theorem 3.6.

Case C-I-2. If $|\sigma_2| = |\sigma|_{\max}$ or $|\sigma|_{\text{med}}$, then it follows that

$$\begin{aligned} &m_{c-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big) \\ &\quad\lesssim \frac{\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}} \\ &\quad\lesssim \frac{\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/4}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}} \\ &\quad\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/4}}\cdot\frac{\langle\xi_4\rangle^{-s}}{\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2}\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}} \\ &\quad:= m_{c-11}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)\cdot m_{c-12}\big((\tilde{\xi}_3,\tilde{\tau}_3),(\xi_4,\tau_4)\big). \end{aligned} \tag{3.71}$$

In order to prove (3.65), by Lemmas 3.2 and 3.5, it suffices to prove:

$$\begin{aligned} &\big\|m_{c-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]} \\ &\quad\lesssim \big\|m_{c-11}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\cdot\big\|m_{c-12}\big((\tilde{\xi}_3,\tilde{\tau}_3),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ &\quad\lesssim 1. \end{aligned} \tag{3.72}$$

Similarly with Situation A-I in proof of Theorem 3.6, we can obtain that

$$\big\|m_{c-12}\big((\tilde{\xi}_3,\tilde{\tau}_3),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1. \tag{3.73}$$

Then we only need to prove:

$$\big\|m_{c-11}\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1. \tag{3.74}$$

We can choose the two variables $\tilde{\xi}_0$ and $\tilde{\tau}_0$ such that $\tilde{\xi}_0+\xi_1+\xi_2=0$ and $\tilde{\tau}_0+\tau_1+\tau_2=0$. Let $\tilde{\sigma}_0=\tilde{\tau}_0-\tilde{\xi}_0^2$, then $|\tilde{\sigma}_0+\sigma_1+\sigma_2|=|h(\tilde{\xi}_0,\xi_1,\xi_2)|\sim|\xi|_{\max}^2$, where $|\xi|_{\max}=\max\{|\tilde{\xi}_0|,|\xi_1|,|\xi_2|\}$. It is the $(+++)$ case.

Similarly with Case A-I-1 in proof of Theorem 3.6, we separately consider four cases:

Case 1: $|\xi_1|\sim|\xi_2|\sim|\tilde{\xi}_0|$; Case 2: $|\xi_1|\sim|\xi_2|\gg|\tilde{\xi}_0|$;
Case 3: $|\xi_1|\sim|\tilde{\xi}_0|\gg|\xi_2|$; Case 4: $|\xi_2|\sim|\tilde{\xi}_0|\gg|\xi_1|$.

Subcase C-I-2-1. If $N_{\max}\sim N_{\min}$, then for $s\geqslant 5\varepsilon$ with any small enough $\varepsilon>0$, we can obtain (3.74) similarly with Case A-I-1 in the proof of Theorem 3.6.

Subcase C-I-2-2. If $N\sim N_{\max}\sim N_1\sim N_2\gg\widetilde{N}_0\sim N_{\min}$, then for $s\geqslant 5\varepsilon$, we can obtain (3.74) similarly with Case A-I-2 in the proof of Theorem 3.6.

Subcase C-I-2-3. If $N\sim N_{\max}\sim N_1\sim\widetilde{N}_0\gg N_2\sim N_{\min}$, then we use (3.18) to bound the left side of (3.74) by:

$$\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(N^2,L_{\text{med}})^{1/2}}{\langle L_1+\alpha N^2\rangle^{1/2}\langle L_2+\alpha N_{\min}^2\rangle^{1/4}}\lesssim 1. \tag{3.75}$$

If $L_{\max} \sim H \sim N^2$, then for $s \geqslant 5\varepsilon$, the left side of (3.74) is bounded by:

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_1+\alpha N^2\rangle^{1/2}\langle L_2+\alpha N_{\min}^2\rangle^{1/4}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/4}N_{\min}^{1/2}}{\langle L_1+\alpha N^2\rangle^{1/2}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{\langle N_{\min}\rangle^{-s}N_{\min}^{1/2}}{\langle L_1+\alpha N^2\rangle^{1/4}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{1}{N^s}\lesssim 1. \tag{3.76}$$

If $L_{\max} \sim L_{\mathrm{med}} \gg H \sim N^2$, then for $s \geqslant 5\varepsilon$, the left side of (3.74) is bounded by:

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_{\max}\sim L_{\mathrm{med}}\gg H\sim N^2}\frac{\langle N_{\min}\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_{\min}+\alpha N^2\rangle^{1/2}\langle L_{\mathrm{med}}+\alpha N_{\min}^2\rangle^{1/4}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_{\max}\sim L_{\mathrm{med}}\gg H\sim N^2}\frac{\langle N_{\min}\rangle^{-s}N_{\min}^{1/2}}{\langle L_{\mathrm{med}}+\alpha N_{\min}^2\rangle^{1/4}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_{\max}\sim L_{\mathrm{med}}\gg H\sim N^2}\frac{\langle N_{\min}\rangle^{-s}N_{\min}^{1/2}}{\langle L_{\mathrm{med}}+\alpha N_{\min}^2\rangle^{1/4-2\varepsilon}L_{\mathrm{med}}^{\varepsilon}L_{\min}^{\varepsilon}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_{\max}\sim L_{\mathrm{med}}\gg H\sim N^2}\frac{N_{\min}^{1/2-5\varepsilon}}{N^{1/2-4\varepsilon}L_{\mathrm{med}}^{\varepsilon}L_{\min}^{\varepsilon}}\lesssim 1. \tag{3.77}$$

Subcase C-I-2-4. If $N \sim N_{\max} \sim N_2 \sim \widetilde{N}_0 \gg N_1 \sim N_{\min}$, then for $s \geqslant 5\varepsilon$, we use (3.18) to bound the left side of (3.74) by:

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{\langle N\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_1+\alpha N_{\min}^2\rangle^{1/2}\langle L_2+\alpha N^2\rangle^{1/4}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{\langle N\rangle^{-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\mathrm{med}})^{1/2}}{\langle L_{\min}+\alpha N_{\min}^2\rangle^{1/2}\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{\langle N\rangle^{-s}N_{\min}^{1/2}}{\langle L_{\mathrm{med}}+\alpha N^2\rangle^{1/4}}$$
$$\lesssim \sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\sum_{L_1,L_2,\widetilde{L}_0\gtrsim 1}\frac{1}{N^s}\lesssim 1. \tag{3.78}$$

Case C-I-3. If $|\sigma_4| = |\sigma|_{\max}$ or $|\sigma|_{\mathrm{med}}$, then

$$m_{c-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)$$
$$\lesssim \frac{\langle i\tilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}\langle\xi_2\rangle^{-s}\langle\xi_4\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\overline{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2}\langle i\widetilde{\sigma}_3+\alpha|\tilde{\xi}_3|^2\rangle^{1/4}}$$

$$\begin{aligned}
&\lesssim \frac{\langle \xi_2\rangle^{-s}\langle \xi_4\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{1/4}\langle i\tilde\sigma_3+\alpha|\tilde\xi_3|^2\rangle^{1/4}}\\
&\lesssim \frac{\langle \xi_2\rangle^{-s}}{\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\tilde\sigma_3+\alpha|\tilde\xi_3|^2\rangle^{1/4}}\cdot\frac{\langle \xi_4\rangle^{-s}}{\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{1/4}\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}}\\
&:= m_{c-11}\big((\xi_2,\tau_2),(\tilde\xi_3,\tilde\tau_3)\big)\cdot m_{c-12}\big((\xi_1,\tau_1),(\xi_4,\tau_4)\big).
\end{aligned}\tag{3.79}$$

In order to prove (3.65), by Lemmas 3.2 and 3.5, it suffices to prove:

$$\begin{aligned}
&\big\|m_{c-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]}\\
&\quad\lesssim \big\|m_{c-11}\big((\xi_2,\tau_2),(\tilde\xi_3,\tilde\tau_3)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\cdot\big\|m_{c-12}\big((\xi_1,\tau_1),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\\
&\quad\lesssim 1.
\end{aligned}\tag{3.80}$$

Similarly with Situation B-I in the proof of Theorem 3.6, for $s \geqslant 5\varepsilon$, we have:

$$\big\|m_{c-11}\big((\xi_2,\tau_2),(\tilde\xi_3,\tilde\tau_3)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1.\tag{3.81}$$

Enlightened by some ideas in Situation B-I, similarly with Case C-I-2, we can obtain, for $s \geqslant 5\varepsilon$, that

$$\big\|m_{c-12}\big((\xi_1,\tau_1),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1.\tag{3.82}$$

Next, we consider the case: $|\sigma|_{\max}\lesssim|\xi|^2_{\max}\sim|\xi|^2_{\mathrm{med}}$. In fact, we can obtain (3.65) by considering the following cases:

$$\begin{aligned}
&|\xi_1|\sim|\xi|_{\max}\sim|\xi|_{\mathrm{med}}\quad \text{corresponding to Case C-I-1;}\\
&|\xi_2|\sim|\xi|_{\max}\sim|\xi|_{\mathrm{med}}\quad \text{corresponding to Case C-I-2;}\\
&|\xi_4|\sim|\xi|_{\max}\sim|\xi|_{\mathrm{med}}\quad \text{corresponding to Case C-I-3.}
\end{aligned}$$

Situation C-II. In this situation, we will prove:

$$\big\|m_{c-2}\big((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]}\lesssim 1.\tag{3.83}$$

We choose two variables $\tilde\xi_4$ and $\tilde\tau_4$ such that $\xi_3+\tilde\xi_4+\xi_5+\xi_6=0$ and $\tau_3+\tilde\tau_4+\tau_5+\tau_6=0$. Let $\tilde\sigma_4=\tilde\tau_4+\tilde\tau_4^2$. Similarly with Situation C-I, we can obtain (3.83) for $s>2\delta+5\varepsilon$. Gathering (3.65) and (3.83), we obtain (3.64).

Next, we consider Case (D). It follows that

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\leqslant\frac{1}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_5\rangle^s},\tag{3.84}$$

and

$$\begin{aligned}
&m_d\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big)\\
&\quad\leqslant\frac{\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{-1/2}\langle i\bar\sigma_5+\alpha|\xi_5|^2\rangle^{-1/2}\langle i\bar\sigma_6+\alpha|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\sigma_3+\alpha|\xi_3|^2\rangle^{1/2}}\frac{1}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_5\rangle^s}\\
&\quad\leqslant\frac{\langle i\bar\sigma_4+\alpha|\xi_4|^2\rangle^{-1/2}\langle\xi_2\rangle^{-s}\langle\xi_1\rangle^{-s}}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}}\cdot\frac{\langle i\bar\sigma_5+\alpha|\xi_5|^2\rangle^{-1/2}\langle i\bar\sigma_6+\alpha|\xi_6|^2\rangle^{-1/2+\delta}}{\langle i\sigma_3+\alpha|\xi_3|^2\rangle^{1/2}\langle\xi_3\rangle^s\langle\xi_5\rangle^s}\\
&\quad:= m_{d-1}\big((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_4,\tau_4)\big)m_{d-2}\big((\xi_3,\tau_3),(\xi_5,\tau_5),(\xi_6,\tau_6)\big).
\end{aligned}\tag{3.85}$$

In fact, by symmetry about σ_j and $\bar\sigma_j$, similarly with Case (C), we can obtain:

$$\big\|m_d\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big)\big\|_{[6,\mathbb{R}\times\mathbb{R}]}\lesssim 1.\tag{3.86}$$

Gathering (3.64) and (3.86), we have (3.60). This completes the proof of Theorem 3.7. □

Corollary 3.8. *Let $s > 0$ and $0 < \delta \ll \frac{1}{2}$. Then*

$$\|u_1 u_2 \partial_x \bar{u}_3\|_{Y_{s,-1/2+\delta}} \leqslant C_{\delta,\alpha} \|u_1\|_{Y_{s,1/2}} \|u_2\|_{Y_{s,1/2}} \|u_3\|_{Y_{s,1/2}}, \tag{3.87}$$

$$\|\partial_x u_1 u_2 \bar{u}_3\|_{Y_{s,-1/2+\delta}} \leqslant C_{\delta,\alpha} \|u_1\|_{Y_{s,1/2}} \|u_2\|_{Y_{s,1/2}} \|u_3\|_{Y_{s,1/2}}, \tag{3.88}$$

where the positive constant $C_{\delta,\alpha}$ depends on δ and α.

Proof. First, we prove (3.87). By duality and the Plancherel identity, it suffices to show

$$\begin{aligned}
&\big\| m\big((\xi_1,\tau_1),\ldots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]} \\
&:= \left\| \frac{K(\xi_1,\xi_2,\xi_3,\xi_4)}{\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_3+\alpha|\xi_3|^2\rangle^{1/2}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2-\delta}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \\
&\lesssim 1,
\end{aligned} \tag{3.89}$$

where

$$\begin{aligned}
K(\xi_1,\xi_2,\xi_3,\xi_4) &= \frac{|\xi_3|\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}, \\
\xi_1+\xi_2+\xi_3+\xi_4 = 0, &\qquad \tau_1+\tau_2+\tau_3+\tau_4 = 0.
\end{aligned} \tag{3.90}$$

Without loss of generality, we can assume $|\xi_3| \sim |\xi|_{\max} \sim |\xi|_{\mathrm{med}}$, where $|\xi|_{\max} = \max\{|\xi_1|, |\xi_2|, |\xi_3|, |\xi_4|\}$.

Case 1. If $|\xi_3| \sim |\xi_4|$, then we can obtain (3.89) similarly with the proof of Theorem 3.6.

Case 2. If $|\xi_3| \sim |\xi_1| \sim |\xi|_{\max}$, then

$$\begin{aligned}
&m\big((\xi_1,\tau_1),\ldots,(\xi_4,\tau_4)\big) \\
&\quad\lesssim \frac{|\xi_3|}{\langle i\bar{\sigma}_3+\alpha|\xi_3|^2\rangle^{1/2}\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/2}} \frac{\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s}}{\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/2-\delta}} \\
&\quad\lesssim \frac{\langle\xi_2\rangle^{-s}}{\langle i\sigma_2+\alpha|\xi_2|^2\rangle^{1/2}\langle i\sigma_1+\alpha|\xi_1|^2\rangle^{1/4}} \frac{\langle\xi_3\rangle^{-s}}{\langle i\bar{\sigma}_3+\alpha|\xi_3|^2\rangle^{1/4}\langle i\bar{\sigma}_4+\alpha|\xi_4|^2\rangle^{1/4-\delta}} \\
&\quad:= m_1\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big) m_2\big((\xi_3,\tau_3),(\xi_4,\tau_4)\big).
\end{aligned} \tag{3.91}$$

Similarly with Situation A-I in the proof of Theorem 3.6, we have:

$$\big\| m_1\big((\xi_1,\tau_1),(\xi_2,\tau_2)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{3.92}$$

Similarly with Case C-I-2 in the proof of Theorem 3.7, we have:

$$\big\| m_2\big((\xi_3,\tau_3),(\xi_4,\tau_4)\big)\big\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{3.93}$$

Case 3. If $|\xi_3| \sim |\xi_2| \sim |\xi|_{\max}$, then similarly with above, we can obtain (3.89).

Similarly with the proof of (3.87), we can obtain (3.88). This completes the proof of Corollary 3.8. □

4. Some a priori estimates and global well-posedness

In this section, we first give some a priori estimates for Eq. (1.1). Furthermore, we prove that the local solution obtained in Section 3 can be extended to the global one by using Lemma 2.6 and the a priori estimates.

We often use the following inequalities in this section.

The Gagliardo–Nirenberg's inequalities:

$$\|u\|_{L^\infty} \leqslant \|u\|_{L^2}^{1/2} \|u_x\|_{L^2}^{1/2}, \tag{4.1}$$

$$\big\|D^s u\big\|_{L^2} \leqslant \big\|D^{s_0} u\big\|_{L^2}^{\theta} \big\|D^{s_1} u\big\|_{L^2}^{1-\theta}, \qquad s = \theta s_0 + (1-\theta)s_1, \quad s_0, s_1 \in \mathbb{R},\ 0 < \theta < 1. \tag{4.2}$$

The Young's inequality:

$$ab \leqslant \frac{\varepsilon^p}{p}a^p + \frac{1}{\varepsilon^q q}b^q \quad \text{for any } \varepsilon > 0,\ a, b, p, q > 0, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.3}$$

Taking $\alpha = \beta_1 = \beta_2 = \alpha_3 = \alpha_4 = 0$, Eq. (1.1) can be rewritten as

$$v_t - i v_{xx} + \alpha_1 |v|^2 v_x + \alpha_2 v^2 \bar{v}_x + i\beta_3 |v|^2 v + i\beta_4 |v|^4 v = 0. \tag{4.4}$$

Lemma 4.1. *(See [6,14].) Let $v(t)$ be a smooth solution of the Cauchy problem* (4.4)–(1.4)*. Then*

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2}, \tag{4.5}$$

$$E_1(v(t)) = E_1(v_0), \tag{4.6}$$

$$E_2(v(t)) = E_2(v_0) + \int_0^t G(t')\,dt', \tag{4.7}$$

for some function $G(t)$ satisfying $|G(t)| \leqslant C(\|v_0\|_{H^1})\|v_{xx}\|_{L^2}^2$, where

$$E_1(v(t)) = \|v_x(t)\|_{L^2}^2 - \frac{\alpha_1 + \alpha_2}{2}\,\mathrm{Im}\int_{\mathbb{R}} |v|^2 v \bar{v}_x\,dx + \left(\frac{(\alpha_1 + \alpha_2)\alpha_2}{6} + \frac{\beta_4}{3}\right)\|v(t)\|_{L^6}^6 + \frac{\beta_3}{2}\|v(t)\|_{L^4}^4, \tag{4.8}$$

$$E_2(v(t)) = \|v_{xx}(t)\|_{L^2}^2 + \frac{\alpha_2 + 2\alpha_1}{2}\,\mathrm{Im}\int_{\mathbb{R}} \left(|v|^2\bar{v}\right)_x v_{xx}\,dx - \frac{4\alpha_1 - 3\alpha_2}{2}\,\mathrm{Im}\int_{\mathbb{R}} \bar{v}^2 v_x v_{xx}\,dx. \tag{4.9}$$

Lemma 4.2. *Let $u(t)$ be a smooth solution of the Cauchy problem* (1.1)–(1.2)*, and assume that $|\beta_1|, |\beta_2| \leqslant 2\max\{\alpha, \alpha_4\}$ and $\alpha > 0, \alpha_3 > 0, \alpha_4 > 0$. Then*

$$\|u(t)\|_{L^2} \lesssim \|u_0\|_{L^2}. \tag{4.10}$$

Proof. Rewrite Eq. (1.1) by:

$$\begin{aligned} &u_t - i u_{xx} + \alpha_1|u|^2 u_x + \alpha_2 u^2 \bar{u}_x + i\beta_3|u|^2 u + i\beta_4|u|^4 u \\ &\quad = \alpha u_{xx} - i\beta_1|u|^2 u_x - i\beta_2 u^2\bar{u}_x - \alpha_3|u|^2 u - \alpha_4|u|^4 u. \end{aligned} \tag{4.11}$$

For the sake of convenience, let

$$-F(x,t) = -i u_{xx} + \alpha_1|u|^2 u_x + \alpha_2 u^2\bar{u}_x + i\beta_3|u|^2 u + i\beta_4|u|^4 u. \tag{4.12}$$

Then

$$u_t = F(x,t) + \alpha u_{xx} - i\beta_1|u|^2 u_x - i\beta_2 u^2\bar{u}_x - \alpha_3|u|^2 u - \alpha_4|u|^4 u, \tag{4.13}$$

$$\bar{u}_t = \overline{F(x,t)} + \alpha\bar{u}_{xx} + i\beta_1|u|^2\bar{u}_x + i\beta_2\bar{u}^2 u_x - \alpha_3|u|^2\bar{u} - \alpha_4|u|^4\bar{u}. \tag{4.14}$$

From (4.5) (the sum of terms with $F(x,t)$ and $\overline{F(x,t)}$ is zero), it follows that

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{L^2}^2 &= \int_{\mathbb{R}} (u\bar{u}_t + \bar{u}u_t)\,dx \\ &= \int_{\mathbb{R}} u\left(\overline{F(x,t)} + \alpha\bar{u}_{xx} + i\beta_1|u|^2\bar{u}_x + i\beta_2\bar{u}^2 u_x - \alpha_3|u|^2\bar{u} - \alpha_4|u|^4\bar{u}\right)dx \\ &\quad + \int_{\mathbb{R}} \bar{u}\left(F(x,t) + \alpha u_{xx} - i\beta_1|u|^2 u_x - i\beta_2 u^2\bar{u}_x - \alpha_3|u|^2 u - \alpha_4|u|^4 u\right)dx \end{aligned}$$

$$= -2\alpha \int_{\mathbb{R}} |u_x|^2\,dx + 2\beta_1 \operatorname{Im}\int_{\mathbb{R}} \bar u|u|^2 u_x\,dx + 2\beta_2 \operatorname{Im}\int_{\mathbb{R}} \bar u u^2 \bar u_x\,dx$$
$$- 2\alpha_3 \int_{\mathbb{R}} |u|^4\,dx - 2\alpha_4 \int_{\mathbb{R}} |u|^6\,dx. \tag{4.15}$$

By Hölder's inequality and Young's inequality, we have:

$$\left|2\beta_1 \operatorname{Im}\int_{\mathbb{R}} \bar u|u|^2 u_x\,dx\right| \leqslant |2\beta_1| \|u_x\|_{L^2}\|u\|_{L^6}^3 \leqslant |\beta_1|\big(\|u_x\|_{L^2}^2 + \|u_x\|_{L^6}^6\big), \tag{4.16}$$

$$\left|2\beta_2 \operatorname{Im}\int_{\mathbb{R}} \bar u u^2 \bar u_x\,dx\right| \leqslant |\beta_2|\big(\|u_x\|_{L^2}^2 + \|u_x\|_{L^6}^6\big). \tag{4.17}$$

Using the fact $|\beta_1| + |\beta_2| \leqslant 2\max\{\alpha, \alpha_4\}$, $\alpha > 0$, $\alpha_3 > 0$ and $\alpha_4 > 0$, we have

$$\frac{d}{dt}\big\|u(t)\big\|_{L^2}^2 \leqslant 0, \tag{4.18}$$

which yields (4.10). This completes the proof of Lemma 4.2. □

Lemma 4.3. *Assume that $\|u_0\|_{L^2} \leqslant \eta$ for some small enough $\eta > 0$. Let $u(t)$ be a smooth solution of the Cauchy problem* (1.1)–(1.2). *Moreover, we assume that*

$$|\beta_1|, |\beta_2| \leqslant 2\max\{\alpha, \alpha_4\}, \quad \alpha > 0,\ \alpha_3 > 0,\ \alpha_4 > 0 \quad \text{and} \quad \max\big\{|\alpha|, |\beta_1|, |\beta_2|, |\alpha_3|, |\alpha_4|\big\} \leqslant C\alpha,$$

where the constant C depends on $\alpha_1, \alpha_2, \beta_3, \beta_4$. Then it holds that

$$\big\|u_x(t)\big\|_{L^2} + \alpha\big\|u_{xx}(t)\big\|_{L^2_{t\in[0,T]}L^2_x} \leqslant C\big(T, \|u_0\|_{L^2}, \|u_{0x}\|_{L^2}\big), \tag{4.19}$$
$$\big\|u_{xx}(t)\big\|_{L^\infty_{t\in[0,T]}L^2_x} + \alpha\big\|u_{xxx}(t)\big\|_{L^2_{t\in[0,T]}L^2_x} \leqslant C\big(T, \|u_0\|_{L^2}, \|u_{0x}\|_{L^2}, \|u_{0xx}\|_{L^2}\big). \tag{4.20}$$

Proof. First, we prove (4.19). From the definition of $E_1(u(t))$ in (4.8), it follows that

$$\frac{d}{dt}E_1\big(u(t)\big) = -\int_{\mathbb{R}} (u_{xx}\bar u_t + \bar u_{xx}u_t)\,dx - \frac{\alpha_1+\alpha_2}{2}\operatorname{Im}\int_{\mathbb{R}} \big\{2|u|^2\bar u_x u_t + u^2\bar u_x\bar u_t - \big(u^2\bar u\big)_x\bar u_t\big\}\,dx$$
$$+ \left(\frac{(\alpha_1+\alpha_2)\alpha_2}{6} + \frac{\beta_4}{3}\right)\int_{\mathbb{R}} \big\{3|u|^4\bar u u_t + 3|u|^4 u\bar u_t\big\}\,dx + \frac{\beta_3}{2}\int_{\mathbb{R}} \big\{2|u|^2\bar u u_t + 2|u|^2 u\bar u_t\big\}\,dx$$
$$:= -\mathcal{I}_1 - \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4. \tag{4.21}$$

$$\mathcal{I}_1 = \int_{\mathbb{R}} (u_{xx}\bar u_t + \bar u_{xx}u_t)\,dx$$
$$= \int_{\mathbb{R}} u_{xx}\big\{\overline{F(x,t)} + \alpha\bar u_{xx} + i\beta_1|u|^2\bar u_x + i\beta_2\bar u^2 u_x - \alpha_3|u|^2\bar u - \alpha_4|u|^4\bar u\big\}\,dx$$
$$+ \int_{\mathbb{R}} \bar u_{xx}\big\{F(x,t) + \alpha u_{xx} - i\beta_1|u|^2 u_x - i\beta_2 u^2\bar u_x - \alpha_3|u|^2 u - \alpha_4|u|^4 u\big\}\,dx$$
$$= \int_{\mathbb{R}} \big\{2\operatorname{Re}\bar u_{xx}F(x,t) + 2\alpha|u_{xx}|^2 + 2\beta_1\operatorname{Im}\bar u_{xx}|u|^2 u_x$$
$$+ 2\beta_2\operatorname{Im}\bar u_{xx}u^2\bar u_x - 2\alpha_3\operatorname{Re}\bar u_{xx}|u|^2 u - 2\alpha_4\operatorname{Re}\bar u_{xx}|u|^4 u\big\}\,dx, \tag{4.22}$$

$$\begin{aligned}\mathcal{I}_2 = {} & \frac{\alpha_1+\alpha_2}{2}\operatorname{Im}\int_{\mathbb{R}} 2|u|^2\bar{u}_x\big\{F(x,t)+\alpha u_{xx}-i\beta_1|u|^2u_x-i\beta_2u^2\bar{u}_x-\alpha_3|u|^2u-\alpha_4|u|^4u\big\}\,dx\\ & +\frac{\alpha_1+\alpha_2}{2}\operatorname{Im}\int_{\mathbb{R}} u^2\bar{u}_x\big\{\overline{F(x,t)}+\alpha\bar{u}_{xx}+i\beta_1|u|^2\bar{u}_x+i\beta_2\bar{u}^2u_x-\alpha_3|u|^2\bar{u}-\alpha_4|u|^4\bar{u}\big\}\,dx\\ & -\frac{\alpha_1+\alpha_2}{2}\operatorname{Im}\int_{\mathbb{R}} \big(u^2\bar{u}\big)_x\big\{\overline{F(x,t)}+\alpha\bar{u}_{xx}+i\beta_1|u|^2\bar{u}_x+i\beta_2\bar{u}^2u_x-\alpha_3|u|^2\bar{u}-\alpha_4|u|^4\bar{u}\big\}\,dx, \end{aligned}\tag{4.23}$$

$$\begin{aligned}\mathcal{I}_3 = {} & \left(\frac{(\alpha_1+\alpha_2)\alpha_2}{6}+\frac{\beta_4}{3}\right)\int_{\mathbb{R}} 3|u|^4\bar{u}\big\{F(x,t)+\alpha u_{xx}-i\beta_1|u|^2u_x-i\beta_2u^2\bar{u}_x-\alpha_3|u|^2u-\alpha_4|u|^4u\big\}\,dx\\ & +\left(\frac{(\alpha_1+\alpha_2)\alpha_2}{6}+\frac{\beta_4}{3}\right)\int_{\mathbb{R}} 3|u|^4u\big\{\overline{F(x,t)}+\alpha\bar{u}_{xx}+i\beta_1|u|^2\bar{u}_x+i\beta_2\bar{u}^2u_x-\alpha_3|u|^2\bar{u}-\alpha_4|u|^4\bar{u}\big\}\,dx, \end{aligned}\tag{4.24}$$

$$\begin{aligned}\mathcal{I}_4 = {} & \frac{\beta_3}{2}\int_{\mathbb{R}} 2|u|^2\bar{u}\big\{F(x,t)+\alpha u_{xx}-i\beta_1|u|^2u_x-i\beta_2u^2\bar{u}_x-\alpha_3|u|^2u-\alpha_4|u|^4u\big\}\,dx\\ & +\frac{\beta_3}{2}\int_{\mathbb{R}} 2|u|^2u\big\{\overline{F(x,t)}+\alpha\bar{u}_{xx}+i\beta_1|u|^2\bar{u}_x+i\beta_2\bar{u}^2u_x-\alpha_3|u|^2\bar{u}-\alpha_4|u|^4\bar{u}\big\}\,dx. \end{aligned}\tag{4.25}$$

For simplicity, let $\tilde{u}_j = u$ or $\bar{u}$, $j = 1,2,\dots,6$. By Hölder's inequality, Gagliardo–Nirenberg's inequality and Lemma 4.2, we have:

$$\begin{aligned}\left|\int_{\mathbb{R}}(\partial_{xx}\tilde{u}_1)\tilde{u}_2\tilde{u}_3(\partial_x\tilde{u}_4)\,dx\right| & \leqslant \|\partial_{xx}\tilde{u}_1\|_{L^2}\|\partial_x\tilde{u}_4\|_{L^2}\|\tilde{u}_2\|_{L^\infty}\|\tilde{u}_3\|_{L^\infty} \leqslant \|u_{xx}\|_{L^2}\|u_x\|_{L^2}^2\|u\|_{L^2}\\ & \leqslant \|u_{xx}\|_{L^2}^2\|u_0\|_{L^2}^2, \end{aligned}\tag{4.26}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\partial_{xx}\tilde{u}_1|\tilde{u}_2|^2\tilde{u}_3\,dx\right| & \leqslant \|\partial_{xx}\tilde{u}_1\|_{L^2}\|\tilde{u}_3\|_{L^2}\|\tilde{u}_2\|_{L^\infty}^2 \leqslant \|u_{xx}\|_{L^2}\|u_x\|_{L^2}\|u\|_{L^2}^2\\ & \leqslant \|u_{xx}\|_{L^2}^{3/2}\|u_0\|_{L^2}^{5/2}, \end{aligned}\tag{4.27}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\partial_{xx}\tilde{u}_1|\tilde{u}_2|^4\tilde{u}_3\,dx\right| & \leqslant \|\partial_{xx}\tilde{u}_1\|_{L^2}\|\tilde{u}_3\|_{L^2}\|\tilde{u}_2\|_{L^\infty}^4 \leqslant \|u_{xx}\|_{L^2}\|u_x\|_{L^2}^2\|u\|_{L^2}^3\\ & \leqslant \|u_{xx}\|_{L^2}^2\|u_0\|_{L^2}^4, \end{aligned}\tag{4.28}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\tilde{u}_1\tilde{u}_2(\partial_x\tilde{u}_3)\tilde{u}_4\tilde{u}_5(\partial_x\tilde{u}_6)\,dx\right| & \leqslant \|\partial_x u\|_{L^2}^2\|u\|_{L^\infty}^4 \leqslant \|\partial_x u\|_{L^2}^4\|u\|_{L^2}^2\\ & \leqslant \|u_{xx}\|_{L^2}^2\|u_0\|_{L^2}^4, \end{aligned}\tag{4.29}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\tilde{u}_1\tilde{u}_2(\partial_x\tilde{u}_3)|\tilde{u}_4|^2\tilde{u}_5\,dx\right| & \leqslant \|u_x\|_{L^2}\|u\|_{L^2}\|u\|_{L^\infty}^4 \leqslant \|u_x\|_{L^2}^3\|u\|_{L^2}^3\\ & \leqslant \|u_{xx}\|_{L^2}^{3/2}\|u\|_{L^2}^{9/2}, \end{aligned}\tag{4.30}$$

$$\left|\int_{\mathbb{R}} \tilde{u}_1\tilde{u}_2(\partial_x\tilde{u}_3)|\tilde{u}_4|^4\tilde{u}_5\,dx\right| \leqslant \|u_x\|_{L^2}\|u\|_{L^2}\|u\|_{L^\infty}^6 \leqslant \|u_x\|_{L^2}^4\|u\|_{L^2}^4$$
$$\leqslant \|u_{xx}\|_{L^2}^2\|u_0\|_{L^2}^6, \tag{4.31}$$

$$\left|\int_{\mathbb{R}} |\tilde{u}_1|^4\tilde{u}_2|\tilde{u}_3|^2\tilde{u}_4\,dx\right| \leqslant \|u\|_{L^2}^2\|u\|_{L^\infty}^6 \leqslant \|u_x\|_{L^2}^3\|u\|_{L^2}^5$$
$$\leqslant \|u_{xx}\|_{L^2}^{3/2}\|u\|_{L^2}^{13/2}, \tag{4.32}$$

$$\left|\int_{\mathbb{R}} |\tilde{u}_1|^4\tilde{u}_2|\tilde{u}_3|^4\tilde{u}_4\,dx\right| \leqslant \|u\|_{L^2}^2\|u\|_{L^\infty}^8 \leqslant \|u_x\|_{L^2}^4\|u\|_{L^2}^6$$
$$\leqslant \|u_{xx}\|_{L^2}^2\|u_0\|_{L^2}^8. \tag{4.33}$$

Using (4.26)–(4.33) and the fact that $\|u_0\|_{L^2} \leqslant \eta$ for some enough small number $\eta > 0$ (without loss of generality, we can assume $\eta \leqslant 1$), by (4.6) (the sum of terms with $F(x,t)$ and $\overline{F(x,t)}$ is zero), we have:

$$-\mathcal{I}_1 - \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 \leqslant -2\alpha\|u_{xx}\|_{L^2}^2 + C(\alpha_1,\alpha_2,\beta_3,\beta_4)\max\bigl\{|\alpha|,|\beta_1|,|\beta_2|,|\alpha_3|,|\alpha_4|\bigr\}\eta\|u_{xx}\|_{L^2}^2. \tag{4.34}$$

We take $|\alpha|, |\beta_1|, |\beta_2|, |\alpha_3|$ and $|\alpha_4|$ small enough such that

$$C(\alpha_1,\alpha_2,\beta_3,\beta_4)\max\bigl\{|\alpha|,|\beta_1|,|\beta_2|,|\alpha_3|,|\alpha_4|\bigr\}\eta \leqslant \alpha. \tag{4.35}$$

Then

$$\frac{d}{dt}E_1\bigl(u(t)\bigr) \leqslant -\alpha\|u_{xx}\|_{L^2}^2, \tag{4.36}$$

which yields (4.19).

Next, we prove (4.20). From the definition of $E_2(u(t))$ in (4.9), it follows that

$$\begin{aligned}\frac{d}{dt}E_2\bigl(u(t)\bigr) ={}& \int_{\mathbb{R}} \bigl\{\bar{u}_{xxxx}u_t + u_{xxxx}\bar{u}_t\bigr\}\,dx\\
&+ \frac{\alpha_2+2\alpha_1}{2}\operatorname{Im}\int_{\mathbb{R}} \bigl\{2uu_{xxx}\bar{u}\bar{u}_t + \bar{u}^2u_{xxx}u_t - \bigl(|u|^2\bar{u}\bigr)_{xxx}u_t\bigr\}\,dx\\
&- \frac{4\alpha_1-3\alpha_2}{2}\operatorname{Im}\int \bigl\{2u_xu_{xx}\bar{u}\bar{u}_t - \bigl(\bar{u}^2u_{xx}\bigr)_x u_t + \bigl(\bar{u}^2u_x\bigr)_{xx}u_t)\bigr\}\,dx\\
:={}& \mathcal{II}_1 + \mathcal{II}_2 - \mathcal{II}_3.\end{aligned} \tag{4.37}$$

$$\begin{aligned}\mathcal{II}_1 ={}& \int_{\mathbb{R}} \{\bar{u}_{xxxx}u_t + u_{xxxx}\bar{u}_t\}\,dx\\
={}& \int_{\mathbb{R}} \bar{u}_{xxxx}\bigl\{F(x,t) + \alpha u_{xx} - i\beta_1|u|^2u_x - i\beta_2u^2\bar{u}_x - \alpha_3|u|^2u - \alpha_4|u|^4u\bigr\}\,dx\\
&+ \int_{\mathbb{R}} u_{xxxx}\bigl\{\overline{F(x,t)} + \alpha\bar{u}_{xx} + i\beta_1|u|^2\bar{u}_x + i\beta_2\bar{u}^2u_x - \alpha_3|u|^2\bar{u} - \alpha_4|u|^4\bar{u}\bigr\}\,dx,\end{aligned} \tag{4.38}$$

$$\begin{aligned}\mathcal{II}_2 ={}& \frac{\alpha_2+2\alpha_1}{2}\operatorname{Im}\int_{\mathbb{R}} \bigl\{2uu_{xxx}\bar{u}\bar{u}_t + \bar{u}^2u_{xxx}u_t - \bigl(|u|^2\bar{u}\bigr)_{xxx}u_t\bigr\}\,dx\\
={}& \frac{\alpha_2+2\alpha_1}{2}\operatorname{Im}\int_{\mathbb{R}} 2uu_{xxx}\bar{u}\bigl\{\overline{F(x,t)} + \alpha\bar{u}_{xx} + i\beta_1|u|^2\bar{u}_x + i\beta_2\bar{u}^2u_x - \alpha_3|u|^2\bar{u} - \alpha_4|u|^4\bar{u}\bigr\}\,dx\end{aligned}$$

$$+\frac{\alpha_2+2\alpha_1}{2}\operatorname{Im}\int_{\mathbb{R}}\bar{u}^2u_{xxx}\big\{F(x,t)+\alpha u_{xx}-i\beta_1|u|^2u_x-i\beta_2u^2\bar{u}_x-\alpha_3|u|^2u-\alpha_4|u|^4u\big\}\,dx$$
$$-\frac{\alpha_2+2\alpha_1}{2}\operatorname{Im}\int_{\mathbb{R}}\big(|u|^2\bar{u}\big)_{xxx}\big\{F(x,t)+\alpha u_{xx}-i\beta_1|u|^2u_x-i\beta_2u^2\bar{u}_x-\alpha_3|u|^2u-\alpha_4|u|^4u\big\}\,dx, \tag{4.39}$$

$$\begin{aligned}\mathcal{I}\mathcal{I}_3&=\frac{4\alpha_1-3\alpha_2}{2}\operatorname{Im}\int\big\{2u_xu_{xx}\bar{u}\bar{u}_t-\big(\bar{u}^2u_{xx}\big)_xu_t+\big(\bar{u}^2u_x\big)_{xx}u_t)\big\}\,dx\\&=\frac{4\alpha_1-3\alpha_2}{2}\operatorname{Im}\int_{\mathbb{R}}2u_xu_{xx}\bar{u}\big\{\overline{F(x,t)}+\alpha\bar{u}_{xx}+i\beta_1|u|^2\bar{u}_x+i\beta_2\bar{u}^2u_x-\alpha_3|u|^2\bar{u}-\alpha_4|u|^4\bar{u}\big\}\,dx\\&\quad-\frac{4\alpha_1-3\alpha_2}{2}\operatorname{Im}\int_{\mathbb{R}}\big(\bar{u}^2u_{xx}\big)_x\big\{F(x,t)+\alpha u_{xx}-i\beta_1|u|^2u_x-i\beta_2u^2\bar{u}_x-\alpha_3|u|^2u-\alpha_4|u|^4u\big\}\,dx\\&\quad+\frac{4\alpha_1-3\alpha_2}{2}\operatorname{Im}\int_{\mathbb{R}}\big(\bar{u}^2u_x\big)_{xx}\big\{F(x,t)+\alpha u_{xx}-i\beta_1|u|^2u_x-i\beta_2u^2\bar{u}_x-\alpha_3|u|^2u-\alpha_4|u|^4u\big\}\,dx.\end{aligned} \tag{4.40}$$

For simplicity, let $\tilde{u}_j=u$ or $\bar{u}$, $j=1,2,\dots,6$. By Hölder's inequality, Gagliardo–Nirenberg's inequality and Lemma 4.2, we have:

$$\begin{aligned}\left|\int_{\mathbb{R}}\big(\partial_x^4\tilde{u}_1\big)\tilde{u}_2\tilde{u}_3(\partial_x\tilde{u}_4)\,dx\right|&\leqslant\|u_{xxx}\|_{L^2}\|u_{xx}\|_{L^2}\|u\|_{L^\infty}^2+\|u_{xxx}\|_{L^2}\|u_x\|_{L^2}\|u_x\|_{L^\infty}\|u\|_{L^\infty}\\&\leqslant 2\|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^2,\end{aligned} \tag{4.41}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\big(\partial_x^4\tilde{u}_1\big)|\tilde{u}_2|^2\tilde{u}_3\,dx\right|&\leqslant\|u_{xxx}\|_{L^2}\|u_x\|_{L^2}\|u\|_{L^\infty}^2\\&\leqslant\|u_{xxx}\|_{L^2}^{5/3}\|u_0\|_{L^2}^{7/3},\end{aligned} \tag{4.42}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\big(\partial_x^4\tilde{u}_1\big)|\tilde{u}_2|^4\tilde{u}_3\,dx\right|&\leqslant\|u_{xxx}\|_{L^2}\|u_x\|_{L^2}\|u\|_{L^\infty}^4\\&\leqslant\|u_{xxx}\|_{L^2}^{2}\|u_0\|_{L^2}^{4},\end{aligned} \tag{4.43}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\big(\partial_x^3\tilde{u}_1\big)\tilde{u}_2\tilde{u}_3\big(\partial_x^2\tilde{u}_4\big)\,dx\right|&\leqslant\|u_{xxx}\|_{L^2}\|u_{xx}\|_{L^2}\|u\|_{L^\infty}^2\\&\leqslant\|u_{xxx}\|_{L^2}^{2}\|u_0\|_{L^2}^{2},\end{aligned} \tag{4.44}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\big(\partial_x^3\tilde{u}_1\big)\tilde{u}_2\tilde{u}_3\tilde{u}_4\tilde{u}_5(\partial_x\tilde{u}_6)\,dx\right|&\leqslant\|u_{xxx}\|_{L^2}\|u_x\|_{L^2}\|u\|_{L^\infty}^4\\&\leqslant\|u_{xxx}\|_{L^2}^{2}\|u_0\|_{L^2}^{4},\end{aligned} \tag{4.45}$$

$$\begin{aligned}\left|\int_{\mathbb{R}}\big(\partial_x^3\tilde{u}_1\big)\tilde{u}_2\tilde{u}_3|\tilde{u}_4|^2\tilde{u}_5\,dx\right|&\leqslant\|u_{xxx}\|_{L^2}\|u\|_{L^2}\|u\|_{L^\infty}^4\\&\leqslant\|u_{xxx}\|_{L^2}^{5/3}\|u_0\|_{L^2}^{13/3},\end{aligned} \tag{4.46}$$

$$\left|\int_{\mathbb{R}} \big(\partial_x^3\tilde{u}_1\big)\tilde{u}_2\tilde{u}_3|\tilde{u}_4|^4\tilde{u}_5\,dx\right| \leqslant \|u_{xxx}\|_{L^2}\|u\|_{L^2}\|u\|_{L^\infty}^6$$

$$\leqslant \|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^6, \tag{4.47}$$

$$\left|\int_{\mathbb{R}} (\partial_{xx}\tilde{u}_1)(\partial_x\tilde{u}_2)\tilde{u}_3(\partial_{xx}\tilde{u}_4)\,dx\right| \leqslant \|u_{xx}\|_{L^2}^2\|u_x\|_{L^\infty}\|u\|_{L^\infty}$$

$$\leqslant \|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^2, \tag{4.48}$$

$$\left|\int_{\mathbb{R}} (\partial_{xx}\tilde{u}_1)(\partial_x\tilde{u}_2)\tilde{u}_3\tilde{u}_4\tilde{u}_5(\partial_x\tilde{u}_6)\,dx\right| \leqslant \|u_{xx}\|_{L^2}\|u_x\|_{L^2}\|u_x\|_{L^\infty}\|u\|_{L^\infty}^3$$

$$\leqslant \|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^4, \tag{4.49}$$

$$\left|\int_{\mathbb{R}} (\partial_{xx}\tilde{u}_1)(\partial_x\tilde{u}_2)\tilde{u}_3|\tilde{u}_4|^2\tilde{u}_5\,dx\right| \leqslant C\|u_{xx}\|_{L^2}\|u_x\|_{L^2}\|u\|_{L^\infty}^4$$

$$\leqslant C\|u_{xxx}\|_{L^2}^{5/3}\|u_0\|_{L^2}^{13/3}, \tag{4.50}$$

$$\left|\int_{\mathbb{R}} (\partial_{xx}\tilde{u}_1)(\partial_x\tilde{u}_2)\tilde{u}_3|\tilde{u}_4|^4\tilde{u}_5\,dx\right| \leqslant C\|u_{xx}\|_{L^2}\|u_x\|_{L^2}\|u\|_{L^\infty}^6$$

$$\leqslant C\|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^6, \tag{4.51}$$

$$\left|\int_{\mathbb{R}} (\partial_x\tilde{u}_1)(\partial_x\tilde{u}_2)(\partial_x\tilde{u}_3)(\partial_{xx}\tilde{u}_4)\,dx\right| \leqslant \|u_{xx}\|_{L^2}\|u_x\|_{L^2}\|u_x\|_{L^\infty}^2$$

$$\leqslant \|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^2, \tag{4.52}$$

$$\left|\int_{\mathbb{R}} (\partial_x\tilde{u}_1)(\partial_x\tilde{u}_2)(\partial_x\tilde{u}_3)\tilde{u}_4\tilde{u}_5(\partial_x\tilde{u}_6)\,dx\right| \leqslant \|u_x\|_{L^2}^2\|u_x\|_{L^\infty}^2\|u\|_{L^\infty}^2$$

$$\leqslant \|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^4, \tag{4.53}$$

$$\left|\int_{\mathbb{R}} (\partial_x\tilde{u}_1)(\partial_x\tilde{u}_2)(\partial_x\tilde{u}_3)|\tilde{u}_4|^2\tilde{u}_5\,dx\right| \leqslant \|u_x\|_{L^2}^2\|u_x\|_{L^\infty}\|u\|_{L^\infty}^3$$

$$\leqslant \|u_{xxx}\|_{L^2}^{5/3}\|u_0\|_{L^2}^{13/3}, \tag{4.54}$$

$$\left|\int_{\mathbb{R}} (\partial_x\tilde{u}_1)(\partial_x\tilde{u}_2)(\partial_x\tilde{u}_3)|\tilde{u}_4|^4\tilde{u}_5\,dx\right| \leqslant \|u_x\|_{L^2}^2\|u_x\|_{L^\infty}\|u\|_{L^\infty}^5$$

$$\leqslant \|u_{xxx}\|_{L^2}^2\|u_0\|_{L^2}^6. \tag{4.55}$$

Using (4.41)–(4.55) and the fact that $\|u_0\|_{L^2} \leqslant \eta$ for some enough small number $\eta > 0$, by (4.7), we have:

$$\mathcal{II}_1 + \mathcal{II}_2 - \mathcal{II}_3 = -2\alpha\|u_{xxx}\|_{L^2}^2 + \int_0^t \big|G(t')\big|\,dt'$$

$$+ C(\alpha_1, \alpha_2)\max\big\{|\alpha|, |\beta_1|, |\beta_2|, |\alpha_3|, |\alpha_4|\big\}\eta\|u_{xxx}\|_{L^2}^2, \tag{4.56}$$

where $|G(t')| \leqslant C(\alpha_1, \alpha_2, \beta_3, \beta_4, \|u_0\|_{H^1})\|u_{xx}\|_{L^2}^2$. We take $|\alpha|, |\beta_1|, |\beta_2|, |\alpha_3|$ and $|\alpha_4|$ small enough such that

$$C(\alpha_1, \alpha_2, \beta_3, \beta_4) \max\{|\alpha|, |\beta_1|, |\beta_2|, |\alpha_3|, |\alpha_4|\}\eta \leqslant \alpha. \tag{4.57}$$

Then, we have:

$$\frac{d}{dt} E_2(u(t)) \leqslant -\alpha\|u_{xxx}\|_{L^2}^2 + C(\alpha_1, \alpha_2, \beta_3, \beta_4, \|u_0\|_{H^1}) \int_0^t \|u_{xx}\|_{L^2}^2, \tag{4.58}$$

which yields (4.20) by using the Gronwall's inequality similarly with the above.

This completes the proof of Lemma 4.3. □

Lemma 4.4. *(See [5].) Assume that* $4\alpha\alpha_4 > (\beta_1 - \beta_2)^2$. *Let* $u(t)$ *be a smooth solution of the Cauchy problem* (1.1)–(1.2). *Then*

$$\|u_x(t)\|_{L^2} \leqslant C(T, \|u_0\|_{L^2}, \|u_{0x}\|_{L^2}). \tag{4.59}$$

Therefore, by Lemma 2.6, we can extend the local solution obtained in Section 3 to the global one. In fact, by Lemmas 4.2, 4.4, we can obtain Theorem 1.2; by Lemmas 4.2, 4.3, we have Theorem 1.3.

5. Inviscid limit behavior for Eq. (1.1) with $\gamma_1 = 0$

In this section, we will consider the inviscid limit behavior of the solution $u(x,t)$ for the Cauchy problem (1.1)–(1.2), and prove that for some $T > 0$, $t \in (0,T)$, solution $u(x,t)$ converges to the solution $v(x,t)$ for Cauchy problem (1.3)–(1.4) in the space $C([0,T]; H^s)$ if $|\alpha|, |\beta_2|, |\alpha_3|, |\alpha_4| \to 0$ and $\|u_0 - v_0\|_{H^s} \to 0$ with $s > \frac{1}{2}$. Moreover, if initial data $u_0, v_0 \in H^2$, the convergence holds in $C([0,T]; H^1)$ for any $T > 0$. For achieving the results, we first choose the same working space X_T, where the solutions $u(x,t)$ and $v(x,t)$ should exist for the same $T > 0$ and the same initial data. From Lemmas 2.1 and 2.3, we can choose the standard space $X_{s,1/2}^T$ as the working space, then we can first obtain the local well-posedness in $X_{s,1/2}^T$ for Eq. (1.1) uniformly for $\alpha, \beta_2, \alpha_3, \alpha_4$. Next, we control solution $u(t)$ by the uniform estimate of $u(t)$, which will be obtained by Lemmas 4.2 and 4.3. Then, we also need to consider difference equations between (1.1) and (1.3), where we can treat the dissipative terms as perturbations and then use the uniform estimates of solutions to get the inviscid limit behavior in $X_{s,1/2}^T$. Finally, by Lemma 2.5, we can obtain the inviscid limit behavior in $C([0,T]; H^s)$.

5.1. Local well-posedness for Eq. (1.1) uniformly for any $\alpha \geqslant 0$

By the trilinear, multilinear estimates as below, Lemmas 2.1 and 2.3, we can obtain the uniform local well-posedness.

Lemma 5.1. *(See [17].) Let* $\tilde{u}_1 = u_1$ *or* $\bar{u}_1$, $\tilde{u}_2 = u_2$ *or* $\bar{u}_2$. *Then for* $b_1, b_2 > \frac{3}{8}$, *we have*:

$$\|\tilde{u}_1\tilde{u}_2\|_{L^2} \leqslant \|u_1\|_{X_{0,b_1}}\|u_2\|_{X_{0,b_2}}. \tag{5.1}$$

Remark. By multi-linear expressions in Section 3, it means that

$$\left\| \frac{1}{\langle\tilde{\sigma}_1\rangle^{b_1}\langle\tilde{\sigma}_2\rangle^{b_2}} \right\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{5.2}$$

where $\tilde{\sigma}_j = \tau_j - \xi_j^2$ or $\tau_j + \xi_j^2$.

Lemma 5.2. *Let $s > \frac{1}{2}$ and $0 < \delta \ll \frac{1}{2}$. Then there exist $C_\delta > 0$ such that*

$$\big\|u_1u_2(\partial_x\bar{u}_3)\big\|_{X_{s,-1/2+\delta}} \leqslant C_\delta\|u_1\|_{X_{s,1/2}}\|u_2\|_{X_{s,1/2}}\|u_3\|_{X_{s,1/2}}, \tag{5.3}$$

$$\|u_1u_2\bar{u}_3\|_{X_{s,-1/2+\delta}} \leqslant C_\delta\|u_1\|_{X_{s,1/2}}\|u_2\|_{X_{s,1/2}}\|u_3\|_{X_{s,1/2}}, \tag{5.4}$$

$$\|u_1u_2u_3\bar{u}_4\bar{u}_5\|_{X_{s,-1/2+\delta}} \leqslant C_\delta\|u_1\|_{X_{s,1/2}}\|u_2\|_{X_{s,1/2}}\|u_3\|_{X_{s,1/2}}\|u_4\|_{X_{s,1/2}}\|u_5\|_{X_{s,1/2}}, \tag{5.5}$$

where the positive constant C_δ depends on δ.

Proof. We first prove (5.3). The proof of (5.4) is easier than that of (5.3). By duality and the Plancherel equality, it suffices to show that

$$\big\|m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]} := \left\|\frac{K(\xi_1,\xi_2,\xi_3,\xi_4)}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\bar{\sigma}_3\rangle^{1/2}\langle\bar{\sigma}_4\rangle^{1/2-\delta}}\right\|_{[4,\mathbb{R}\times\mathbb{R}]} \lesssim 1, \tag{5.6}$$

where

$$K(\xi_1,\xi_2,\xi_3,\xi_4) = \frac{|\xi_3|\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s},$$

$$\xi_1+\xi_2+\xi_3+\xi_4 = 0, \qquad \tau_1+\tau_2+\tau_3+\tau_4 = 0, \tag{5.7}$$

$$|\sigma_1+\sigma_2+\bar{\sigma}_3+\bar{\sigma}_4| = 2|\xi_1+\xi_4||\xi_2+\xi_4|. \tag{5.8}$$

Case 1. If $|\xi_4+\xi_1| \leqslant 1$ or $|\xi_4+\xi_2| \leqslant 1$, by symmetry, we can assume that $|\xi_4+\xi_1| \leqslant 1$, then $|\xi_4| \sim |\xi_1|$ and $|\xi_2| \sim |\xi_3|$. It follows that for $s \geqslant \frac{1}{2}$,

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \leqslant C. \tag{5.9}$$

By Lemmas 3.2, 3.5 and (5.2), we have for $\delta < \frac{1}{8}$,

$$\begin{aligned}\big\|m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]} &\lesssim \left\|\frac{1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\bar{\sigma}_3\rangle^{1/2}\langle\bar{\sigma}_4\rangle^{1/2-\delta}}\right\|_{[4,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim \left\|\frac{1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\left\|\frac{1}{\langle\bar{\sigma}_3\rangle^{1/2}\langle\bar{\sigma}_4\rangle^{1/2-\delta}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim 1.\end{aligned} \tag{5.10}$$

Case 2. Assume: $|\xi_4+\xi_1| \geqslant 1$ and $|\xi_4+\xi_2| \geqslant 1$.

Subcase 2-1. If $|\xi_4| \ll |\xi_3|$, then from (5.7) it follows that

$$|\xi_3| \sim |\xi_3+\xi_4| \leqslant |\xi_1+\xi_2| \leqslant \max\big\{|\xi_1|,|\xi_2|\big\}. \tag{5.11}$$

By symmetry, we can assume $\max\{|\xi_1|,|\xi_2\} = |\xi_2|$.

If $|\xi_4| \lesssim |\xi_1|$, then we have for $s \geqslant \frac{1}{2}$,

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \leqslant C. \tag{5.12}$$

We can obtain the result similarly with Case 1.

If $|\xi_4| \gg |\xi_1|$, then for $s > \frac{1}{2}$,

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \lesssim \frac{1}{\langle\xi_1\rangle^s\langle\xi_4\rangle^{s-1}}. \tag{5.13}$$

By Lemmas 3.2 and 3.5, it suffices to show that

$$\begin{aligned}
&\big\|m\big((\xi_1,\tau_1),\ldots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]}\\
&\lesssim \left\|\frac{1}{\langle\xi_1\rangle^s\langle\xi_4\rangle^{s-1}}\frac{1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\bar\sigma_3\rangle^{1/2}\langle\bar\sigma_4\rangle^{1/2-\delta}}\right\|_{[4,\mathbb{R}\times\mathbb{R}]}\\
&\lesssim \left\|\frac{1}{\langle\bar\sigma_3\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\left\|\frac{1}{\langle\sigma_1\rangle^{1/2}\langle\bar\sigma_4\rangle^{1/2-\delta}}\frac{1}{\langle\xi_1\rangle^s\langle\xi_4\rangle^{s-1}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\\
&\lesssim 1. \qquad (5.14)
\end{aligned}$$

Using (5.2), we have:

$$\left\|\frac{1}{\langle\bar\sigma_3\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1. \qquad (5.15)$$

Next, we will prove:

$$\left\|\frac{1}{\langle\sigma_1\rangle^{1/2}\langle\bar\sigma_4\rangle^{1/2-\delta}}\frac{1}{\langle\xi_1\rangle^s\langle\xi_4\rangle^{s-1}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\lesssim 1. \qquad (5.16)$$

We choose two variables $\tilde\xi_0$ and $\tilde\tau_0$ such that $\tilde\xi_0+\xi_1+\xi_4=0$ and $\tilde\tau_0+\tau_1+\tau_4=0$. Since $|\xi_4|\gg|\xi_1|$, we let $\tilde\sigma_0=\tilde\tau_0+\tilde\xi_0^2$ such that $|\tilde\sigma_0+\sigma_1+\sigma_4|=|h(\tilde\xi_0,\xi_1,\xi_4)|\sim|\xi|_{\max}^2\sim|\xi_4|^2\sim|\tilde\xi_0|^2$, where $|\xi|_{\max}=\max\{|\tilde\xi_0|,|\xi_1|,|\xi_4|\}$. It is $(++-)$ case. By dyadic decomposition, we assume that $N\sim N_{\max}\sim N_4\sim\widetilde N_0\gg N_1\sim N_{\min}$ and $|h(\tilde\xi_0,\xi_1,\xi_4)|\sim H\sim N_{\max}^2$.

(1) If $L_{\max}\sim H\sim N_{\max}^2$, then for $s\geqslant\frac12+\delta+3\varepsilon$, we obtain the boundedness of the left side of (5.16), by applying (3.21), as follows

$$\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_4,\widetilde L_0\gtrsim 1}\frac{\langle N\rangle^{1-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\text{med}})^{1/2}}{\langle N_{\min}\rangle^s\langle L_1\rangle^{1/2}\langle L_4\rangle^{1/2-\delta}}. \qquad (5.17)$$

If $NN_{\min}\leqslant L_{\text{med}}$, then for $s\geqslant\frac12+\delta+3\varepsilon$ with any small enough $\varepsilon>0$, it holds that

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_4,\widetilde L_0\gtrsim 1}\frac{\langle N\rangle^{1-s}L_{\min}^{1/2}N^{-1/2}\min(NN_{\min},L_{\text{med}})^{1/2}}{\langle N_{\min}\rangle^s\langle L_{\min}\rangle^{1/2}\langle L_{\text{med}}\rangle^{1/2-\delta}}\\
&\lesssim\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_4,\widetilde L_0\gtrsim 1}\frac{\langle N\rangle^{1-s}N_{\min}^{1/2}}{\langle N_{\min}\rangle^s(NN_{\min})^{1/2-\delta-2\varepsilon}L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}}\\
&\lesssim\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_4,\widetilde L_0\gtrsim 1}\frac{N_{\min}^{\delta+2\varepsilon}}{\langle N_{\min}\rangle^s N^{s-1/2-\delta-2\varepsilon}L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}}\\
&\lesssim\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_4,\widetilde L_0\gtrsim 1}\frac{N_{\min}^{\delta+2\varepsilon}}{\langle N_{\min}\rangle^s N^{\varepsilon}L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}}\\
&\lesssim 1. \qquad (5.18)
\end{aligned}$$

If $NN_{\min}>L_{\text{med}}$, then for $s\geqslant\frac12+\delta+3\varepsilon$, we have:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_4,\widetilde L_0\gtrsim 1}\frac{L_{\text{med}}^{\delta}}{N^{s-1/2}\langle N_{\min}\rangle^s}\\
&\lesssim\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_1,L_4,\widetilde L_0\gtrsim 1}\frac{1}{\langle N_{\min}\rangle^{s-2\varepsilon-\delta}N^{\varepsilon}L_{\min}^{\varepsilon}L_{\text{med}}^{\varepsilon}}\\
&\lesssim 1. \qquad (5.19)
\end{aligned}$$

(2) If $L_{\max} \sim L_{\text{med}} \gg H \sim N_{\max}^2$, then for $s \geqslant \max\{\frac{1}{2}+\varepsilon, 2\delta+5\varepsilon\}$, we have:

$$\begin{aligned}
&\sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_{\max}\sim L_{\text{med}}\gg H} \frac{\langle N\rangle^{1-s} L_{\min}^{1/2} N^{-1/2} \min(N N_{\min}, L_{\text{med}})^{1/2}}{\langle N_{\min}\rangle^s \langle L_{\min}\rangle^{1/2} \langle L_{\text{med}}\rangle^{1/2-\delta}} \\
&\quad\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_{\max}\sim L_{\text{med}}\gg H} \frac{\langle N\rangle^{1-s} N_{\min}^{1/2}}{\langle N_{\min}\rangle^s \langle L_{\text{med}}\rangle^{1/2-\delta}} \\
&\quad\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_{\max}\sim L_{\text{med}}\gg H} \frac{\langle N\rangle^{1-s} N_{\min}^{1/2}}{\langle N_{\min}\rangle^s N^{1-2\delta-4\varepsilon} L_{\text{med}}^{\varepsilon} L_{\min}^{\varepsilon}} \\
&\quad\lesssim \sum_{N_{\max}\sim N_{\text{med}}\sim N}\ \sum_{L_{\max}\sim L_{\text{med}}\gg H} \frac{N^{2\delta+4\varepsilon} N_{\min}^{1/2}}{\langle N_{\min}\rangle^s N^s L_{\text{med}}^{\varepsilon} L_{\min}^{\varepsilon}} \\
&\quad\lesssim 1.
\end{aligned} \tag{5.20}$$

Subcase 2-2. If $|\xi_4| \gg |\xi_3|$, then from (5.7) it follows that

$$|\xi_4| \sim |\xi_3+\xi_4| \leqslant |\xi_1+\xi_2| \leqslant \max\bigl\{|\xi_1|, |\xi_2\bigr\}. \tag{5.21}$$

By symmetry, we can assume $\max\{|\xi_1|, |\xi_2\} = |\xi_2|$.

If $|\xi_3| \lesssim |\xi_1|$, then we have for $s \geqslant \frac{1}{2}$,

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \leqslant C. \tag{5.22}$$

We can obtain the result similarly with Case 1.

If $|\xi_3| \gg |\xi_1|$, then for $s > \frac{1}{2}$,

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \lesssim \frac{1}{\langle\xi_1\rangle^s \langle\xi_3\rangle^{s-1}}. \tag{5.23}$$

We can obtain the result similarly with Subcase 2-1.

Subcase 2-3. If $|\xi_4| \sim |\xi_3|$, then by (5.7) we have the estimate either

$$\min\bigl\{|\xi_1|, |\xi_2|\bigr\} \gtrsim |\xi_3| \sim |\xi_4|, \tag{5.24}$$

or

$$\min\bigl\{|\xi_4+\xi_1|, |\xi_4+\xi_2|\bigr\} \gtrsim |\xi_3| \sim |\xi_4| \gg \min\bigl\{|\xi_1|, |\xi_2|\bigr\}. \tag{5.25}$$

In the first case, for $s \geqslant \frac{1}{2}$, we have:

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \leqslant C. \tag{5.26}$$

We can obtain the result similarly with Case 1.

In the second case, if $|\xi_3| \lesssim \max\{|\xi_1|, |\xi_2|\}$, then we have:

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \lesssim \frac{|\xi_3|^{1-s}}{\min\{\langle\xi_1\rangle, \langle\xi_2\rangle\}^s}. \tag{5.27}$$

Similarly with Subcase 2-1, we can obtain the result.

If $|\xi_4| \sim |\xi_3| \gg \max\{|\xi_1|, |\xi_2|\}$, then from (5.25), it follows that

$$\max\bigl\{|\sigma_1|, |\sigma_2|, |\bar\sigma_3|, |\bar\sigma_4|\bigr\} \gtrsim |\sigma_1+\sigma_2+\bar\sigma_3+\bar\sigma_4| = 2|\xi_1+\xi_4||\xi_2+\xi_4| \gtrsim |\xi_4|^2 \sim |\xi_3|^2. \tag{5.28}$$

By symmetry, we can assume:

$$|\sigma_1| = \max\bigl\{|\sigma_1|, \sigma_2|, |\bar\sigma_3|, |\bar\sigma_4|\bigr\} \gtrsim |\xi_4|^2 \sim |\xi_3|^2. \tag{5.29}$$

For $0 < \varsigma \ll \frac{1}{2}$ (ς depends δ and ε), we have:

$$K(\xi_1,\xi_2,\xi_3,\xi_4) \lesssim \frac{|\xi_3|^{1/2+\varsigma}|\xi_4|^{1/2-\varsigma}}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s}. \tag{5.30}$$

By Lemmas 3.2 and 3.5, it suffices to show that

$$\begin{aligned}\big\|m\big((\xi_1,\tau_1),\dots,(\xi_4,\tau_4)\big)\big\|_{[4,\mathbb{R}\times\mathbb{R}]} &\lesssim \bigg\|\frac{|\xi_3|^{1/2+\varsigma}|\xi_4|^{1/2-\varsigma}}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s}\frac{1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\bar\sigma_3\rangle^{1/2}\langle\bar\sigma_4\rangle^{1/2-\delta}}\bigg\|_{[4,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim \bigg\|\frac{|\xi_3|^{1/2+\varsigma}}{\langle\xi_1\rangle^s\langle\sigma_1\rangle^{1/2}\langle\bar\sigma_3\rangle^{1/2}}\bigg\|_{[3,\mathbb{R}\times\mathbb{R}]}\bigg\|\frac{|\xi_4|^{1/2-\varsigma}}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{1/2}\langle\bar\sigma_4\rangle^{1/2-\delta}}\bigg\|_{[3,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim 1.\end{aligned} \tag{5.31}$$

Similarly with Subcase 2-1, for $\varsigma \geqslant \delta + 3\varepsilon$, we obtain:

$$\bigg\|\frac{|\xi_4|^{1/2-\varsigma}}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{1/2}\langle\bar\sigma_4\rangle^{1/2-\delta}}\bigg\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{5.32}$$

Similarly with Subcase 2-1, for $\varsigma + 5\varepsilon \leqslant 1/2$, we have:

$$\bigg\|\frac{|\xi_3|^{1/2+\varsigma}}{\langle\xi_1\rangle^s\langle\sigma_1\rangle^{1/2}\langle\bar\sigma_3\rangle^{1/2}}\bigg\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{5.33}$$

In fact, if $|\sigma_4| = \max\{|\sigma_1|, \sigma_2|, |\bar\sigma_3|, |\bar\sigma_4|\}$, we need to take $\varsigma + \delta + 5\varepsilon \leqslant 1/2$.

Next, we prove (5.5). By duality and the Plancherel equality, it suffices to show that

$$\begin{aligned}&\big\|m\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big)\big\|_{[6,\mathbb{R}\times\mathbb{R}]}\\ &:= \bigg\|\frac{\langle\bar\sigma_4\rangle^{-1/2}\langle\bar\sigma_5\rangle^{-1/2}\langle\bar\sigma_6\rangle^{-1/2+\delta}}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\sigma_3\rangle^{1/2}}K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\bigg\|_{[6,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim 1,\end{aligned} \tag{5.34}$$

where

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = \frac{\langle\xi_6\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\xi_5\rangle^s}, \tag{5.35}$$

$$\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6 = 0, \qquad \tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6 = 0. \tag{5.36}$$

By symmetry, we can assume:

$$|\xi_6| \lesssim |\xi_5| = \max\big\{|\xi_1|, |\xi_2|, |\xi_3|, |\xi_4|, |\xi_5|\big\}.$$

Then

$$K(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) \lesssim \frac{1}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s}. \tag{5.37}$$

By Lemmas 3.2 and 3.5, it suffices to show that

$$\begin{aligned}&\big\|m\big((\xi_1,\tau_1),\dots,(\xi_6,\tau_6)\big)\big\|_{[6,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim \bigg\|\frac{\langle\bar\sigma_4\rangle^{-1/2}\langle\bar\sigma_5\rangle^{-1/2}\langle\bar\sigma_6\rangle^{-1/2+\delta}}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}\langle\sigma_3\rangle^{1/2}}\frac{1}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s\langle\xi_4\rangle^s}\bigg\|_{[6,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim \bigg\|\frac{\langle\bar\sigma_5\rangle^{-1/2}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\bigg\|_{[4,\mathbb{R}\times\mathbb{R}]}\cdot\bigg\|\frac{\langle\bar\sigma_4\rangle^{-1/2}\langle\bar\sigma_6\rangle^{-1/2+\delta}}{\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\sigma_3\rangle^{1/2}}\bigg\|_{[4,\mathbb{R}\times\mathbb{R}]}\\ &\lesssim 1.\end{aligned} \tag{5.38}$$

We first prove,

$$\left\| \frac{\langle\bar{\sigma}_4\rangle^{-1/2}\langle\bar{\sigma}_6\rangle^{-1/2+\delta}}{\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\sigma_3\rangle^{1/2}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{5.39}$$

We take two variables $\tilde{\xi}_5$ and $\tilde{\tau}_5$ such that $\xi_3+\xi_4+\tilde{\xi}_5+\xi_6=0$ and $\tau_3+\tau_4+\tilde{\tau}_5+\tau_6=0$. Let $\tilde{\sigma}_5=\tilde{\tau}_5-\tilde{\xi}_5^2$. By symmetry, we assume $|\sigma_4|=\max\{|\sigma_3|,|\sigma_4|\}$.

By Lemmas 3.2 and 3.5, we take small enough $\varepsilon>0$ such that $1/2-\varepsilon>\frac{3}{8}$. Then by using (5.2), Lemmas 3.3 and 3.4, for $s>\frac{1}{2}$, we have:

$$\begin{aligned} &\left\| \frac{\langle\bar{\sigma}_4\rangle^{-1/2+\varepsilon}\langle\bar{\sigma}_6\rangle^{-1/2+\delta}}{\langle\xi_3\rangle^s\langle\xi_4\rangle^s\langle\sigma_3\rangle^{1/2+\varepsilon}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \\ &\lesssim \left\| \frac{1}{\langle\bar{\sigma}_4\rangle^{1/2-\varepsilon}\langle\bar{\sigma}_6\rangle^{1/2-\delta}\langle\xi_4\rangle^s} \right\|_{[3,\mathbb{R}\times\mathbb{R}]} \cdot \left\| \frac{1}{\langle\sigma_3\rangle^{1/2+\varepsilon}\langle\xi_3\rangle^s} \right\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ &\lesssim 1. \end{aligned} \tag{5.40}$$

Similarly with above, for $s>\frac{1}{2}$, we also have:

$$\left\| \frac{\langle\bar{\sigma}_5\rangle^{-1/2}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{5.41}$$

This completes the proof of Lemma 5.2. □

Corollary 5.3. *Let $0<\delta\ll\frac{1}{2}$. Then there exist $\mu, C_\delta>0$ such that for $u_1,u_2,u_3\in X_{s,1/2}$, $\bar{u}_3,\bar{u}_4,\bar{u}_5\in\overline{X}_{s,1/2}$ with compact support in $[-T,T]$,*

$$\|u_1u_2(\partial_x\bar{u}_3)\|_{X_{s,-1/2+\delta}} \leqslant C_\delta T^\mu\|u_1\|_{X_{s,1/2}}\|u_2\|_{X_{s,1/2}}\|u_3\|_{X_{s,1/2}}, \tag{5.42}$$

$$\|u_1u_2\bar{u}_3\|_{X_{s,-1/2+\delta}} \leqslant C_\delta T^\mu\|u_1\|_{X_{s,1/2}}\|u_2\|_{X_{s,1/2}}\|u_3\|_{X_{s,1/2}}, \tag{5.43}$$

$$\|u_1u_2u_3\bar{u}_4\bar{u}_5\|_{X_{s,-1/2+\delta}} \leqslant C_\delta T^\mu\|u_1\|_{X_{s,1/2}}\|u_2\|_{X_{s,1/2}}\|u_3\|_{X_{s,1/2}}\|u_4\|_{X_{s,1/2}}\|u_5\|_{X_{s,1/2}}, \tag{5.44}$$

where the positive constant C_δ depends on δ.

Remark. In fact, from Lemma 5.2, we can complete the proof by the following inequality for $f(t)$ with compact support in $[-T,T]$,

$$\left\| \mathcal{F}^{-1}\frac{\hat{f}(\tau,\xi)}{\langle\tau-\xi^2\rangle^\delta} \right\|_{L^2} \leqslant C_\delta T^\mu\|f\|_{L^2}, \quad \text{for any } \delta>0. \tag{5.45}$$

5.2. The proofs of Theorems 1.4 and 1.5

For $u_0,v_0\in H^s$ $(s>\frac{1}{2})$; $u(t), v(t), \bar{u}(t)$ and $\bar{v}(t)$ with compact support in $[-T,T]$, we define the operators and the sets:

$$\begin{aligned} \Phi(u) &= \psi_T(t)S_\alpha(t)u_0-\psi_T(t)\int_0^t S_\alpha(t-t') \\ &\quad\times\big((\alpha_2+i\beta_2)u^2\bar{u}_x+(\alpha_3+i\beta_3)|u|^2u+(\alpha_4+i\beta_4)|u|^4u\big)(t')\,dt', \end{aligned} \tag{5.46}$$

$$\Psi(v)=\psi_T(t)S_0(t)v_0-\psi_T(t)\int_0^t S_0(t-t')\big(\alpha_2v^2\bar{v}_x+i\beta_3|v|^2v+i\beta_4|v|^4v\big)(t')\,dt', \tag{5.47}$$

$$\mathcal{B}=\big\{u\in X_{s,1/2}\colon\ \|u\|_{X_{s,1/2}}\leqslant 2C\|u_0\|_{H^s}\big\}, \tag{5.48}$$

$$\mathcal{C}=\big\{v\in X_{s,1/2}\colon\ \|v\|_{X_{s,1/2}}\leqslant 2C\|v_0\|_{H^s}\big\}. \tag{5.49}$$

We consider the inviscid limit behavior of solution when $\alpha \to 0$, $|\beta_2| \to 0$, $|\alpha_3| \to 0$ and $|\alpha_4| \to 0$. Without loss of generality, we can assume that $|\beta_2| \leqslant |\alpha_2|$, $|\alpha_3| \leqslant |\beta_3|$ and $|\alpha_4| \leqslant |\beta_4|$. By Lemmas 2.1, 2.3 and Corollary 5.3, we have:

$$\big\|\Phi(u)\big\|_{X_{s,1/2}} \leqslant C\|u_0\|_{H^s} + C\max\big\{|\alpha_2|, |\beta_3|\big\}T^\mu\|u\|^3_{X_{s,1/2}} + C|\beta_4|T^\mu\|u\|^5_{X_{s,1/2}}, \tag{5.50}$$

$$\big\|\Psi(v)\big\|_{X_{s,1/2}} \leqslant C\|v_0\|_{H^s} + C\max\big\{|\alpha_2|, |\beta_3|\big\}T^\mu\|u\|^3_{X_{s,1/2}} + C|\beta_4|T^\mu\|u\|^5_{X_{s,1/2}}. \tag{5.51}$$

Therefore, if we fix T such that

$$\begin{cases} C\max\big\{|\alpha_2|, |\beta_3|\big\}T^\mu\|u\|^2_{X_{s,1/2}} \leqslant 2C\max\big\{|\alpha_2|, |\beta_3|\big\}T^\mu\|u_0\|^2_{H^s} \leqslant \dfrac{1}{4}, \\ C|\beta_4|T^\mu\|u\|^4_{X_{s,1/2}} \leqslant 2C|\beta_4|T^\mu\|u_0\|^4_{H^s} \leqslant \dfrac{1}{4}. \end{cases} \tag{5.52}$$

$$\begin{cases} C\max\big\{|\alpha_2|, |\beta_3|\big\}T^\mu\|v\|^2_{X_{s,1/2}} \leqslant 2C\max\big\{|\alpha_2|, |\beta_3|\big\}T^\mu\|v_0\|^2_{H^s} \leqslant \dfrac{1}{4}, \\ C|\beta_4|T^\mu\|v\|^4_{X_{s,1/2}} \leqslant 2C|\beta_4|T^\mu\|v_0\|^4_{H^s} \leqslant \dfrac{1}{4}. \end{cases} \tag{5.53}$$

Then Φ and Ψ are contraction mapping on $\mathcal{B}$ and $\mathcal{C}$, respectively. This means that the existence time T of the local solutions $u(t)$ and $v(t)$ is independent of $\alpha, \beta_2, \alpha_3, \alpha_4$. The constants C appearing in the following part depend on $\alpha_2, \beta_3, \beta_4$, for simplicity, denote $C(\alpha_2, \beta_3, \beta_4) = C$.

Now, we turn to the proof of Theorem 1.4. Assume that $u(t)$ and $v(t)$ are solutions for (1.1) and (1.3) with initial data $u_0 \in H^2$ and $v_0 \in H^2$, respectively. Let $w = u - v$ and $w_0 = u_0 - v_0$, then we obtain the difference equation as follows:

$$\begin{aligned} &w_t - iw_{xx} - \alpha u_{xx} + i\beta_2 u^2\bar{u}_x + \alpha_3|u|^2u + \alpha_4|u|^4u + \alpha_2\big(u^2\bar{w}_x + (u+v)w\bar{v}_x\big) \\ &\quad + i\beta_3\big(u^2\bar{w} + (u+v)w\bar{v}\big) + i\beta_4\big(u^3(\bar{u}+\bar{v})\bar{w} + \bar{v}^2w\big(u^2+uv+v^2\big)\big) = 0, \end{aligned} \tag{5.54}$$

$$w_0 = u_0 - v_0. \tag{5.55}$$

We treat αu_{xx} as a perturbation term for the equation above. Then we consider the equivalent integral formulation of the problem above,

$$\begin{aligned} w(x,t) = S_0(t)w_0 - \int_0^t S_0(t-t')\big\{&-\alpha u_{xx} + i\beta_2u^2\bar{u}_x + \alpha_3|u|^2u + \alpha_4|u|^4u \\ &+ \alpha_2\big(u^2\bar{w}_x + (u+v)w\bar{v}_x\big) + i\beta_3\big(u^2\bar{w} + (u+v)w\bar{v}\big) \\ &+ i\beta_4\big(u^3(\bar{u}+\bar{v})\bar{w} + \bar{v}^2w\big(u^2+uv+v^2\big)\big)\big\}(t')\,dt'. \end{aligned} \tag{5.56}$$

By Lemmas 2.1, 2.3, Corollary 5.3, (5.52) and (5.53), we have:

$$\begin{aligned} \big\|w(x,t)\big\|_{X^T_{1,1/2}} \lesssim{}& \|w_0\|_{H^1} + \alpha\|u_{xx}\|_{X^T_{1,-1/2+\delta}} + |\beta_2|T^\mu\|u\|^3_{X^T_{1,1/2}} + |\alpha_3|T^\mu\|u\|^3_{X^T_{1,1/2}} + |\alpha_4|T^\mu\|u\|^5_{X^T_{1,1/2}} \\ &+ |\alpha_2|T^\mu\big(\|u\|^2_{X^T_{1,1/2}}\|w\|_{X^T_{1,1/2}} + \big(\|u\|_{X^T_{1,1/2}} + \|v\|_{X^T_{1,1/2}}\big)\|w\|_{X^T_{1,1/2}}\|v\|_{X^T_{1,1/2}}\big) \\ &+ |\beta_3|T^\mu\big(\|u\|^2_{X^T_{1,1/2}}\|w\|_{X^T_{1,1/2}} + \big(\|u\|_{X^T_{1,1/2}} + \|v\|_{X^T_{1,1/2}}\big)\|w\|_{X^T_{1,1/2}}\|v\|_{X^T_{1,1/2}}\big) \\ &+ |\beta_4|T^\mu\big(\|u\|^3_{X^T_{1,1/2}}\big(\|u\|_{X^T_{1,1/2}} + \|v\|_{X^T_{1,1/2}}\big)\|w\|_{X^T_{1,1/2}} \\ &+ \|v\|^2_{X^T_{1,1/2}}\|w\|_{X^T_{1,1/2}}\big(\|u\|^2_{X^T_{1,1/2}} + \|u\|_{X^T_{1,1/2}}\|v\|_{X^T_{1,1/2}} + \|v\|^2_{X^T_{1,1/2}}\big)\big) \\ \lesssim{}& \|w_0\|_{H^1} + |\alpha|\|u_{xx}\|_{L^2_tH^1_x} + |\beta_2|\|u_0\|_{H^1} + |\alpha_3|\|u_0\|_{H^1} + |\alpha_4|\|u_0\|_{H^1} \\ &+ \frac{1}{4}\|w\|_{X^T_{1,1/2}} + \frac{1}{4}\|w\|_{X^T_{1,1/2}}. \end{aligned} \tag{5.57}$$

From Lemma 4.3 and (5.57), it follows that

$$\|w(x,t)\|_{X^T_{1,1/2}} \lesssim \|w_0\|_{H^1} + \alpha^{1/2} C\big(\|u_0\|_{L^2}, \|u_0\|_{H^1}, \|u_0\|_{H^2}, T\big) + \max\big\{|\beta_2|, |\alpha_3|, |\alpha_4|\big\}\|u_0\|_{H^1}. \tag{5.58}$$

From (5.58), we obtain that

$$\|w(x,t)\|_{X^T_{1,1/2}} \to 0, \quad \text{for } T \leqslant 1, \quad \text{if } \alpha, |\beta_2|, |\alpha_3|, |\alpha_4| \to 0 \quad \text{and} \quad \|w_0\|_{H^1} \to 0. \tag{5.59}$$

Moreover, for the solution w above on $[0,T]$, we have:

$$\begin{aligned}\|w(x,t)\|_{C([0,T];H^1)} \lesssim{}& \|w_0\|_{H^1} + \alpha T^{1/2}\|u_{xx}\|_{L^2_t([0,T])H^1_x} \\ &+ \bigg\| \int_0^t S_0(t-t')\big\{i\beta_2 u^2\bar{u}_x + \alpha_3|u|^2u + \alpha_4|u|^4u + \alpha_2\big(u^2\bar{w}_x + (u+v)w\bar{v}_x\big) \\ &+ i\beta_3\big(u^2\bar{w} + (u+v)w\bar{v}\big) \\ &+ i\beta_4\big(u^3(\bar{u}+\bar{v})\bar{w} + \bar{v}^2 w\big(u^2+uv+v^2\big)\big)\big\}(t')\,dt'\bigg\|_{L^\infty([0,T];H^1)}.\end{aligned} \tag{5.60}$$

Define:

$$\begin{aligned}F(\beta_2,\alpha_3,\alpha_4)(t) ={}& i\beta_2 u^2\bar{u}_x + \alpha_3|u|^2u + \alpha_4|u|^4u + \alpha_2\big(u^2\bar{w}_x + (u+v)w\bar{v}_x\big) \\ &+ i\beta_3\big(u^2\bar{w} + (u+v)w\bar{v}\big) + i\beta_4\big(u^3(\bar{u}+\bar{v})\bar{w} + \bar{v}^2 w\big(u^2+uv+v^2\big)\big).\end{aligned} \tag{5.61}$$

On the other hand, by Lemma 2.3, Corollary 5.3, (5.52) and (5.53), similarly with (5.57), we have:

$$\begin{aligned}&\big\|F(\beta_2,\alpha_3,\alpha_4)(t)\big\|_{X^T_{1,-1/2+\delta}} \lesssim |\beta_2|\|u_0\|_{H^1} + |\alpha_3|\|u_0\|_{H^1} + |\alpha_4|\|u_0\|_{H^1} + \frac{1}{4}\|w\|_{X^T_{1,1/2}} + \frac{1}{4}\|w\|_{X^T_{1,1/2}} \to 0, \\ &\text{if } |\beta_2|, |\alpha_3|, |\alpha_4| \to 0 \quad \text{and} \quad \|w(x,t)\|_{X^T_{1,1/2}} \to 0.\end{aligned} \tag{5.62}$$

Then by using Lemma 2.5, we have:

$$\begin{aligned}&\bigg\|\int_0^t S_0(t-t')F(\beta_2,\alpha_3,\alpha_4)(t')\,dt'\bigg\|_{L^\infty([0,T];H^1)} \to 0, \\ &\text{if } |\beta_2|, |\alpha_3|, |\alpha_4| \to 0 \quad \text{and} \quad \|w(x,t)\|_{X^T_{1,1/2}} \to 0.\end{aligned} \tag{5.63}$$

From (5.60) and (5.63), it follows that

$$\|w(x,t)\|_{C([0,T];H^1)} \to 0, \quad \text{for } T \leqslant 1, \quad \text{if } \alpha, |\beta_2|, |\alpha_3|, |\alpha_4| \to 0 \quad \text{and} \quad \|w_0\|_{H^1} \to 0. \tag{5.64}$$

From Theorem 1.3, it follows that (5.64) holds for any $T > 0$.

This completes the proof of Theorem 1.4.

Next, we give the proof of Theorem 1.5. Notice that the inviscid limit behavior of the solution above is considered for $u_0, v_0 \in H^2$, while the solutions $u(t), v(t)$ exist in space $C([0,T]; H^s)$ with $u_0, v_0 \in H^s$ $(s > \frac{1}{2})$. Next, we consider the inviscid limit behavior in space $C([0,T]; H^s)$ with $u_0, v_0 \in H^s$ $(s > \frac{1}{2})$.

For convenience, we define solution operators of the Cauchy problem (1.1)–(1.2) as well as the Cauchy problem (1.3)–(1.4) as below: $\mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)u_0 = u(t)$ and $\mathcal{A}_{(0,0,0,0)}(t)v_0 = v(t)$. Notice that $P_N u_0, P_N v_0 \in H^2$ for fixed N if $u_0, v_0 \in H^s$ $(s > \frac{1}{2})$. Then

$$\begin{aligned}&\big\|\mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)u_0 - \mathcal{A}_{(0,0,0,0)}(t)v_0\big\|_{C(0,T;H^s)} \\ &\quad\leqslant \big\|\mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)(P_N u_0) - \mathcal{A}_{(0,0,0,0)}(t)(P_N v_0)\big\|_{C(0,T;H^s)} && := \mathcal{J}_1 \\ &\qquad+ \big\|\mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)u_0 - \mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)(P_N u_0)\big\|_{C(0,T;H^s)} && := \mathcal{J}_2 \\ &\qquad+ \big\|\mathcal{A}_{(0,0,0,0)}(t)v_0 - \mathcal{A}_{(0,0,0,0)}(t)(P_N v_0)\big\|_{C(0,T;H^s)}. && := \mathcal{J}_3\end{aligned} \tag{5.65}$$

For the first term $\mathcal{J}_1$, by (5.64), we have:

$$\mathcal{J}_1 \to 0 \quad \text{if } \alpha, |\beta_2|, |\alpha_3|, |\alpha_4| \to 0 \quad \text{and} \quad \|P_N(u_0 - v_0)\|_{H^1} \leqslant N^{1-s}\|P_N(u_0 - v_0)\|_{H^s} \to 0. \tag{5.66}$$

For $\mathcal{J}_2$ and $\mathcal{J}_3$, by using the fact that solution operators $\mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)$ and $\mathcal{A}_{(0,0,0,0)}(t)$ are continuous with respect to initial data u_0 and v_0 [11,17], respectively; that is, for $\forall\varepsilon > 0$, $\exists\zeta > 0$, such that if

$$\|u_0 - (P_N u_0)\|_{H^s} \leqslant \zeta, \tag{5.67}$$

$$\|v_0 - (P_N v_0)\|_{H^s} \leqslant \zeta, \tag{5.68}$$

then

$$\mathcal{J}_2 = \|\mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)u_0 - \mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)(P_N u_0)\|_{C(0,T;H^s)} \leqslant \varepsilon, \tag{5.69}$$

$$\mathcal{J}_3 = \|\mathcal{A}_{(0,0,0,0)}(t)v_0 - \mathcal{A}_{(0,0,0,0)}(t)P_N v_0\|_{C(0,T;H^s)} \leqslant \varepsilon. \tag{5.70}$$

In fact, (5.67) and (5.68) hold for any small $\zeta > 0$ by taking enough large N.

Therefore, we conclude that if $\|u_0 - v_0\|_{H^s} \to 0$, $s > \frac{1}{2}$ and $\alpha, |\beta_2|, |\alpha_3|, |\alpha_4| \to 0$, then

$$\|u - v\|_{C(0,T;H^s)} = \|\mathcal{A}_{(\alpha,\beta_2,\alpha_3,\alpha_4)}(t)u_0 - \mathcal{A}_{(0,0,0,0)}(t)v_0\|_{C(0,T;H^s)} \to 0. \tag{5.71}$$

This completes the proof of Theorem 1.5.

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