Comparison of finite element models for piezoelectric materials

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Abstract

The presented work is focused on simulation of piezoelectric effect in thin ceramic layers using finite element method. The layers are the primary components of piezoelectric patches – flexural transducers used e.g. in vibration control or structural health monitoring. At first the model of piezoelectric bimorph beam was investigated. The beam is composed of two actuators loaded by opposite electric potentials which cause the beam to bend. Further, the influence of approximation of electric potential through the thickness was tested on a simple piezoelectric cantilever beam loaded by external tip force. The numerical results obtained using several finite element types are compared mutually and with simplified analytical solution.

Keywords: Piezoelectricity; Bimorph beam; Finite element method; Ansys

1. Introduction

Piezoelectric effect, firstly demonstrated and explained in 1880 by the Curie brothers, is the ability of crystal to convert mechanical energy of deformation to electric charge. The crystals with piezoelectricity have also the capability of converse effect – to change their shape regarding to the applied electric field. This is why these materials are used as the primary components of transducers in wide area of application like aviation, automotive, defense industry etc. Although some of the natural crystals are piezoelectric, i.e. tourmaline or Rochelle salt; artificially prepared ceramics and polymers with higher piezoelectric capabilities are used for purposes of sensing and actuating. For instance commonly used “patch” transducers are composed of a silver-plated layer of piezoelectric ceramic (PZT) of thickness in a micrometer range, embedded in a polymeric protective foil.

Although the transducers are basically a useful experimental tool, when involved in a complex “smart structure” (e.g. structural health monitoring, active damping etc.), a reliable computational model for the prediction of system behavior is needed. The papers dealing with the numerical simulation of piezoelectric phenomena by the finite element analysis have been published since early ’70. Finite elements with an additional degree of freedom for electric potential were introduced in various types like beams, plates, solid or shells. In [2] a comprehensive survey of development of finite elements for piezoelectricity can be found. Some of the finite elements have been already implemented into commercial FEM packages such as Ansys or MSC.Marc [1].
This paper attempts to present and compare the results of simple problems of piezoelectricity obtained by various finite elements available in Ansys 13. Although the finite elements of the coupled field family can be also found in other FEM packages, Ansys offers wide selection of dimensions, shapes and order of the element shape functions [1].

2. Modeling piezoelectric materials

2.1. Constitutive equations of piezoelectricity

Piezoelectric behavior is described by the system of constitutive equations:

\[
\sigma = C \varepsilon - e^T E, \quad (1)
\]

\[
D = e \varepsilon + \mu E, \quad (2)
\]

where \( \sigma \) is stress matrix, \( D \) is vector of electric displacements, \( \varepsilon \) is the strain vector, \( C \) is matrix of elastic coefficients, \( e \) is the stress piezoelectric matrix, \( \mu \) is the dielectric matrix with the coefficients of electric permittivity on its diagonal. Components of electric field intensity \( E \) is linked with the electric potential \( \phi \) by relation

\[
E = -\nabla \phi. \quad (3)
\]

The system of equations in (1) and (2) can be written also as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\tau_{23} \\
\tau_{13} \\
\tau_{12} \\
D_1 \\
D_2 \\
D_3
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{12} \\
C_{21} & C_{22} & C_{23} \\
C_{21} & C_{32} & C_{22} \\
C_{44} \\
C_{55} \\
C_{45} \\
e_{15} & \mu_{11} \\
e_{24} & \mu_{22} \\
e_{31} & e_{32} & e_{33}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
e_{24} \\
\gamma_{13} \\
\gamma_{12} \\
e_{15} \\
\mu_{33} \end{bmatrix} + \begin{bmatrix}
-e_{31} \\
e_{32} \end{bmatrix}. \quad (4)
\]

2.2. Finite element model

The analytical model can be transformed to the finite element model. Using the variation principle we obtain

\[
\begin{bmatrix}
M & 0 & \ddot{u} \\
0 & 0 & \dot{\phi}
\end{bmatrix} +
\begin{bmatrix}
K_{uu} & K_{up} \\
K_{up}^T & K_{pp}
\end{bmatrix}
\begin{bmatrix}
u \\
\phi
\end{bmatrix} =
\begin{bmatrix}
F \\
Q
\end{bmatrix} \quad (5)
\]

The submatrices in Equation (5), mass matrix \( M \), mechanical stiffness matrix \( K_{uu} \), coupling stiffness matrix \( K_{up} \) and dielectric stiffness matrix \( K_{pp} \) are defined as

\[
M = \rho \int_N \mathbf{N}^T \mathbf{N} \text{d}V \quad (6)
\]
\[
K_{uu} = \int_{V} B^T C B dV \quad (7)
\]
\[
K_{up} = -\int_{V} B^T e \Phi dV \quad (8)
\]
\[
K_{\phi p} = -\int_{V} \Phi^T e \Phi dV \quad (9)
\]

where \( \rho \) is the density, \( \mathbf{N} \) is the matrix of element shape functions, \( \mathbf{B} \) denotes the displacement–strain matrix, \( \mathbf{\Phi} \) is the electric potential–electric field intensity matrix. On the right side of Equation (5) \( \mathbf{F} \) is the vector of forces and \( \mathbf{Q} \) is the electric charge vector.

3. Numerical results

3.1. Bimorph beam deflection

Bimorph beam is a body composed of two thin layers of PZT clamped on one side. The layers have either opposite polarization (P) while being subjected to the same electric field intensity, or, vice versa, the polarization is the same and electric field intensity is opposite. The latter approach is used herein. Upper and lower faces of the beam and the layer interface are considered as electrodes. When the voltage is applied to electrodes, one of layer contracts and the other extends, which results in bending of the beam. The deflection in bending is much higher than the longitudinal deformation of each layer, for this reason mechanism of bimorph beam is used as a bending actuator (see Figure 1).

![Fig. 1. Scheme of the bimorph beam problem](image)

The formula for bimorph beam deflection \( u \) in a place \( x \) found assuming Euler-Bernoulli beam theory and linear variation of electric potential through the thickness of the beam, is

\[
u = 3 \frac{e_{31} U}{E h^2} x^2 \quad (10)
\]

where \( e_{31} \) is the component of piezoelectric matrix, \( U \) is applied voltage, \( h \) is the thickness of the bimorph beam and \( E \) is Young’s modulus.

The numerical model of the bimorph beam (Fig. 2) with dimensions 0.1 × 0.01 × 0.001 m was created in Ansys software using various finite elements of the coupled field family. With respect to the regular geometry, only brick and quad elements were analyzed. The shell elements, which are the object of current research, have not yet been implemented to the package.

In contrast to solely static mechanical or electrostatic analysis, the coupled field problem requires the knowledge the coupling properties, here represented by the piezoelectric stress matrix \( \mathbf{e} \). This is generally matrix 3×6 for three-dimensional problems, (or 2×3 for plane stress problem) whereas its rows correspond to the electric field directions and columns to
components of mechanical deformations. In the problems analyzed in this paper the piezoelectric coefficients corresponding to the shear deformation, i.e. $e_{15}$ and $e_{14}$, were neglected and set to zero.

![Computational model of the bimorph beam in Ansys](image1)

![Figure 2](image2)

**Table 1. Material properties of PZT**

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus [GPa]</td>
<td>61.8</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>0.3</td>
</tr>
<tr>
<td>Electric permittivity [F/m]</td>
<td>1.062x10^-10</td>
</tr>
<tr>
<td>Piezoelectric constant $e_{31}$ [C/m²]</td>
<td>5.6</td>
</tr>
<tr>
<td>Piezoelectric constant $e_{33}$ [C/m²]</td>
<td>-12.8</td>
</tr>
</tbody>
</table>

According to Equation (10) the deflection of beam in distance $x$ from the clamping is directly proportional to the applied voltage, elastic properties and the thickness of beam. Theoretically, only the component $e_{31}$ of the piezoelectric stress matrix is sufficient for the calculation of the deflection and all other components can be set to zero. The matrix containing only one non-zero coefficient ($e_{31}$) is designated as simplified in the text. On the contrary, the full piezoelectric stress matrix of PZT contains also $e_{32}$ ($e_{32} = e_{31}$) and $e_{33}$ components. For the purpose of testing the coefficient $e_{31}$ was varied in range 0.046 ≤ $e_{31}$ ≤ 5.6 and $e_{33}$ was calculated to maintain a mutual ratio $e_{33} = e_{31} \times (5.6 / 12.8)$ using the real properties of PZT (see Table 1).

![Deflection of the beam with respect to piezoelectric constant $e_{31}$](image3)

**Fig. 3. The deflection of the beam with respect to piezoelectric constant $e_{31}$.**
The calculation with simplified matrix is appropriate only when the distribution of the electric potential through the thickness of each layer is strictly linear. As can be seen from Figure 3 the resulting deflection of linear brick and quad elements is lower than that obtained from formula (10). This is caused by shear-locking effect, described e.g. in [4], which means an unreal stiffness increase caused by insufficient approximation of displacement fields. Generally, in bending problems more finite elements per thickness or an option “extra shape functions” (ESF) have to be used. Then the results correspond to those obtained by the quadratic elements.

Assuming the quadratic distribution of the electric potential over thickness of layer, full matrix must be set to provide meaningful results. The simplified matrix leads to incorrect non-linear behavior or an incorrect deformation in general (see also Figure 3). The linear bricks with extra shape functions and quadratic bricks with the full piezoelectric matrix provide results close to the simplified solution (10).

3.2. Bending of beam with short-circuited electrodes

The approximation of electric potential through the thickness was further investigated in a problem of short-circuited piezoelectric (not bimorph) cantilever beam of dimensions 1 × 0.1 × 0.1 m with tip force loading. The electric potentials on its upper and lower face are set to zero. Due to the mechanical deformation, electric potential inside the beam is generated (Figures 4-5). Obviously this effect cannot be observed using one layer of the linear finite elements [3].

![Fig. 4. The scheme of the short-circuited piezoelectric beam loaded by a force](image)

Results of analysis using various solid finite elements and their number through the thickness are presented in Table 2. The values of electric potential are compared along the center line in selected cross-section at \( x = 0.01 \) m. The corresponding distributions through the thickness are plotted in Figure 6. Assuming linear distribution through the thickness of both the displacements and electric potential (linear brick), the beam is too stiff due to shear-locking. This effect further causes lower maximum of electric field intensity, but the nodal value of the electric potential is relatively close to the solution obtained from analysis with fine quadratic mesh. Model composed of one layer of quadratic bricks seems to be also too stiff and leads to higher values of electric potential. Increasing number of element per thickness or using extra shapes function is expected to converge to accurate results, however, the analysis becomes very demanding in terms of CPU time.

<table>
<thead>
<tr>
<th>Finite element</th>
<th>Number of FE through thickness</th>
<th>Deflection ( u ) [mm]</th>
<th>Electric potential ( \varphi ) [V]</th>
<th>Maximum of electric field intensity ( E_z ) [V/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear brick</td>
<td>2</td>
<td>( 3.13 \times 10^{-2} )</td>
<td>18.6</td>
<td>445</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>( 3.33 \times 10^{-2} )</td>
<td>19.0</td>
<td>634</td>
</tr>
<tr>
<td>Linear brick + ESF</td>
<td>2</td>
<td>( 3.40 \times 10^{-2} )</td>
<td>18.2</td>
<td>814</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>( 3.40 \times 10^{-2} )</td>
<td>18.9</td>
<td>827</td>
</tr>
<tr>
<td>Quadratic brick</td>
<td>1</td>
<td>( 3.40 \times 10^{-2} )</td>
<td>19.4</td>
<td>851</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( 3.40 \times 10^{-2} )</td>
<td>18.9</td>
<td>824</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>( 3.41 \times 10^{-2} )</td>
<td>18.7</td>
<td>812</td>
</tr>
</tbody>
</table>
Fig. 5. Resulting electric potential calculated in Ansys using quadratic bricks

Fig. 6. Comparison of induced electric potential through the thickness of the piezoelectric beam.

4. Conclusion

Two problems of piezoelectric analysis were solved using various finite elements and configuration in Ansys. The bimorph beam bending requires correct setting of piezoelectric stress matrix, so that the results were comparable to the simplified linear theory. The distribution of the electric potential in piezoelectric body was further investigated in a problem of short-circuited PZT cantilever beam. Model composed of two layers of quadratic elements or the linear elements with increased order of shape functions can be considered as sufficient for this problem.

Acknowledgements

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References