Liouville's theorem in conformal geometry

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Abstract

Liouville's theorem states that all conformal transformations of E^n and S^n ($n \geqslant 3$) are restrictions of Möbius transformations. As a generalization, we determine all conformal mappings of semi-Riemannian manifolds preserving pointwise the Ricci tensor. It turns out that, up to isometries, they are essentially of the same type as in the classical case but they can exist for metrics different from the Euclidean metric and spherical metric.
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Résumé

Le théorème de Liouville assure qu'une transformation conforme d'un ouvert de l'espace euclidien E^n ou de la sphère S^n ($n \geqslant 3$) est la restriction d'une transformation de Möbius. Nous généralisons ce résultat en déterminant les transformations conformes d'une variété semi-riemannienne qui préservent le tenseur de Ricci. Ce sont les mêmes que dans le cas classique, mais il en existe pour d'autres métriques que les métriques euclidienne et sphérique.
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In two dimensions conformal mappings are nothing but holomorphic functions between two open parts of the complex numbers. In higher dimensions, the situation is much more rigid. The classical theorem of Liouville [29] states that a conformal mapping between two open parts of Euclidean 3-space is the composition of a similarity and an inversion. This work was motivated by conformal maps in cartography. Lie [28] generalized this to the case of Euclidean n-space, for expositions in textbooks see [35, p. 173] and [2, Thm. A.3.7], other references are [31,17] and, under the assumption of minimal possible regularity, [33,4]. In a slightly modified form it holds also in the case of pseudo-Euclidean n-space of arbitrary signature, see [15], [36, p. 209] and, recently, [13]. It is also well known that the conformal mappings of (pseudo-)Euclidean space are precisely those preserving the sets of hyperspheres and hyperplanes. This aspect of Liouville's theorem is emphasized in [25,26,3].

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In this note we point out a more general version of Liouville's theorem in the context of semi-Riemannian geometry, namely, for conformal mappings preserving pointwise the Ricci tensor. Particular cases are the flat case (pseudo-Euclidean space) and the Ricci-flat case (special Einstein spaces). In any case we have the same type of phenomenon, sketched in [11]: Up to isometries, the mapping is necessarily of one of the following three types: A *similarity* (or dilatation) which fixes one point or limit point, an *inversion* sending one point or limit point to infinity, or a special type of a transformation with a parallel isotropic vector field as the gradient of the conformal factor. The analysis and the classification of these cases is based on an ODE reduction of the differential equation $Ric_{\bar{g}} = Ric_g$ for the two Ricci tensors of two conformally equivalent metrics $\bar{g}$ and g. Global results about complete spaces admitting such conformal mappings were given in [10] and in our previous paper [20]. Here the results are semi-local in some sense because only the homotheties can be defined on a complete space.

Notations and definitions. Let (M, g) be an n-dimensional semi-Riemannian manifold with a metric tensor g of arbitrary signature $(k, n - k)$. A *conformal diffeomorphism* $f : (M, g) \to (\overline{M}, \overline{g})$ is a smooth mapping such that the induced metric $f^*\overline{g}$ is everywhere a positive multiple of g. It is convenient to introduce this positive factor as φ^{-2} with a function $\varphi : M \to \mathbb{R}_+$. So on M we can consider two conformally equivalent metrics g and $\overline{g} = \varphi^{-2}g$. Such a transformation f is called an *isometry* if $\varphi = 1$, it is called a *homothety* or *similarity* if φ is constant. The classical case of a dilatation in Euclidean space is the mapping $x \mapsto cx$ with a real constant c, the standard inversion is the mapping $x \mapsto x/||x||^2$, other inversions are the mappings $x \mapsto cx/||x||^2$ for a real constant c. Ric or Ric_g denotes the Ricci tensor of the metric g. By definition $Ric(X, Y)$ is nothing but the trace of the transformation $R(-, X)Y$ where R denotes the curvature tensor of type $(1, 3)$. Note that the Ricci tensor is invariant under scaling, i.e., $Ric_{cg} = Ric_g$ for any positive constant c. Let ∇ denote the Levi-Civita connection induced by g. For any given smooth function φ on (M, g) let $\nabla \varphi$ denote the *gradient* of φ, $\nabla^2 \varphi$ denotes the *Hessian* $(0, 2)$-tensor, $\Delta \varphi = trace_g \nabla^2 \varphi$ is the *Laplacian* of φ. The expression $(\nabla^2 \varphi)^0 = \nabla^2 \varphi - \frac{\Delta \varphi}{n} \cdot g$ denotes the traceless part of the Hessian. A *conformal vector field* V is generated by a local 1-parameter group of conformal mappings. V is conformal if and only if $\mathcal{L}_V g = 2\psi \cdot g$ for a real function ψ. By the equation $\mathcal{L}_{\nabla \varphi} g = 2\nabla^2 \varphi$ a gradient field $\nabla \varphi$ is conformal if and only if $(\nabla^2 \varphi)^0 = 0$. This equation has been studied in many papers. One crucial case is the equation $\nabla^2 \varphi = \pm \varphi \cdot g$, compare [37,18]. For general facts about conformal transformations of semi-Riemannian manifolds we refer to [8] and [16].

We call a non-isometric conformal diffeomorphism $f : (M, g) \to (\overline{M}, \overline{g})$ a *Liouville mapping* if the equation $Ric_{f^*\overline{g}} = Ric_g$ holds as a pointwise identity on M. If $M = \overline{M}$ then we call a conformal transformation $g \mapsto \overline{g}$ a *Liouville transformation* of the metric if the equation $Ric_{\bar{g}} = Ric_g$ holds everywhere. A conformal vector field inducing a 1-parameter group of Liouville mappings is called a *conformal Liouville vector field* (an example of type 3 will given at the end of the paper). A Liouville vector field V is a particular case of what is called a *Ricci collineation* $\mathcal{L}_V Ric = 0$, compare [23]. If in a spacetime the stress-energy momentum tensor is preserved by a certain Liouville mapping then the Einstein field equations are also preserved since $Ric_{\bar{g}} = Ric_g$ implies $\overline{S}\,\overline{g} = Sg$. This gives a certain conformal interpretation of these equations even if they are not conformally invariant in general. In the vacuum case this reduces to the statement that any conformal mapping between vacuum spacetimes is a Liouville mapping. This case was studied in [5,9,30].

Obviously, the identity map between (M, g) and (M, cg) is a Liouville transformation if c is constant. Moreover, any homothetic mapping is a Liouville mapping. Therefore, throughout this paper we focus on the case of non-homothetic conformal mappings. The homothetic case was studied in [1] and [7]. In [11] a Liouville mapping is called a *quasi-similarity*, a Liouville transformation is called a quasi-homothety. We prefer our terminology because it is the aim of the present paper to show that all these considerations are in fact generalizations of Liouville's theorem in conformal geometry. In the sequel we throughout consider *semi-Riemannian manifolds M of dimension n* which are sufficiently smooth in order to admit all the derivatives we are using.

Lemma 1. *For a conformal transformation $\bar{g} = \varphi^{-2}g$ of class C^2 the following conditions are equivalent*:

(1) $Ric_{\bar{g}} - Ric_g$ *is a scalar multiple of g*;
(2) $\nabla^2 \varphi$ *is a scalar multiple of g*;
(3) $(\nabla^2 \varphi)^0 = 0$;
(4) φ *is a concircular scalar field in the sense of* [37,11].

Proof. The proof follows directly from the following standard formula for the relation between the two Ricci tensors of $\bar{g}$ and g (for a proof see [19, p. 107]):

$$Ric_{\bar{g}} - Ric_g = \varphi^{-2}\big((n-2) \cdot \varphi \cdot \nabla^2\varphi + \big[\varphi \cdot \Delta\varphi - (n-1) \cdot g(\nabla\varphi, \nabla\varphi)\big] \cdot g\big).$$

The notion of a *concircular scalar field* is motivated by the so-called *concircular transformations* preserving circles. Here a *circle* is defined to be a curve with constant geodesic curvature and vanishing geodesic torsion.   □

Lemma 2. *Under the same assumptions the following conditions are equivalent*:

(1) $Ric_{\bar{g}} - Ric_g = 0$;
(2) $\nabla^2\varphi = \frac{\Delta\varphi}{n} \cdot g$ *and* $2\varphi \cdot \Delta\varphi = n \cdot g(\nabla\varphi, \nabla\varphi)$;
(3) *the conformal transformation* $g \mapsto \bar{g}$ *is a Liouville transformation* (*or quasi-homothetic in the sense of* [11]).

Lemma 2 follows from Lemma 1 by taking the trace of the equation $Ric_{\bar{g}} - Ric_g = 0$.

By Lemma 2 a Liouville transformation is characterized by the equations $(\nabla^2\varphi)^0 = 0$ and $2\varphi \cdot \Delta\varphi = n \cdot g(\nabla\varphi, \nabla\varphi)$. From the second equation it becomes immediately clear that we have to distinguish between two cases (unless φ is constant):

$$\text{\textit{Case} 1: } g(\nabla\varphi, \nabla\varphi) \neq 0 \text{ on an open set,}$$

$$\text{\textit{Case} 2: } g(\nabla\varphi, \nabla\varphi) = 0 \text{ and } \nabla\varphi \neq 0 \text{ on an open set.}$$

In the second case the gradient of φ is isotropic and necessarily parallel because $Xg(\nabla\varphi, \nabla\varphi) = 2g(\nabla_X\nabla\varphi, \nabla\varphi) = 0$ and $g(\nabla_X\nabla\varphi, Y) = 0$ for X orthogonal to $\nabla\varphi$. In the context of conformal mappings between Einstein spaces Case 1 was called *proper* in [5], Case 2 was called *improper*. The results for these two types are quite different.

Lemma 3. *In a certain neighborhood of any point with* $g(\nabla\varphi, \nabla\varphi) \neq 0$ *the following conditions are equivalent*:

(1) $(\nabla^2\varphi)^0 = 0$;
(2) $\nabla\varphi$ *is a conformal vector field*;
(3) g *is a warped product metric* $g = \eta\, dt^2 + (\frac{d\varphi}{dt})^2 g_*$ *where* $\eta = \pm 1$ *denotes the sign of* $g(\nabla\varphi, \nabla\varphi)$ *and where the metric* g_* *is independent of* t.

The condition (3) *implies that* φ *is a function only of the real parameter* t *and that*

$$\nabla\varphi = \frac{d\varphi}{dt} \cdot \frac{\partial}{\partial t} \quad \text{and} \quad \mathcal{L}_{\nabla\varphi}g = 2\nabla^2\varphi = 2\frac{d^2\varphi}{dt^2} \cdot g.$$

Proof. Actually Lemma 3 is a standard result, originally due to Brinkmann [5] and Fialkow [12], compare [37,11,21]. (1) ⇔ (2) is trivial, (3) ⇒ (1) is easily obtained by the calculation of $\nabla_{\frac{\partial}{\partial t}}$ in terms of Christoffel symbols. For the proof of (1) ⇒ (3) one introduces φ as one coordinate function. Then for one particular φ-level M_* one takes geodesic parallel coordinates. It is easily seen that the parallel levels of the same distance coincide with the φ-levels. If we choose t as the arclength on the trajectories of $\nabla\varphi$, then we obtain a form $g = \eta\, dt^2 + g_{ij}(t)$ for the metric, $i, j = 1, \ldots, n-1$. The proof is completed by calculating $\frac{\partial}{\partial t}((\frac{d\varphi}{dt})^{-2}g_{ij}(t)) = 0$ from the equation $\nabla^2\varphi = \frac{d^2\varphi}{dt^2} \cdot g$. This implies that $g_{*ij} = (\frac{d\varphi}{dt})^{-2}g_{ij}$ is independent of t.   □

Corollary 1. *In a certain neighborhood of any point with* $g(\nabla\varphi, \nabla\varphi) \neq 0$ *the following conditions are equivalent*:

(1) $g \mapsto \varphi^{-2}g$ *is a Liouville transformation*;
(2) g *is a warped product metric* $g = \eta\, dt^2 + (at+b)^2 g_*$ *where* $a \neq 0$ *and* b *are real constants*.

In particular, in this case $\nabla\varphi$ *is a homothetic vector field satisfying* $\mathcal{L}_{\nabla\varphi}g = 2a \cdot g$.

Proof. By combining Lemma 3 with Lemma 2, we find that the differential equation for the function $\varphi(t)$ can be reduced to the ODE $2\varphi\varphi'' = (\varphi')^2$. The solutions are precisely the squares of linear functions: $\varphi(t) = (At + B)^2$ with $A \neq 0$ since φ is not constant by assumption. Consequently, $\varphi'(t) = 2A(At + B)$ is linear, say $\varphi'(t) = at + b$, and there is a common zero of φ and φ'. The equation $\nabla^2\varphi = a \cdot g$ is easily verified. $\quad\square$

In some sense this phenomenon of a common zero of φ and φ' is the crucial point in Liouville's theorem. It leads to the following generalization:

Theorem 1 (*Generalized Liouville theorem*). *Let (M, g) be a connected semi-Riemannian manifold and let $f : (M_1, g) \to (M, g)$ be a Liouville mapping of class C^3 for some open $M_1 \subset M$. Assume that the induced conformal factor φ with $f^*g = \varphi^{-2}g$ satisfies $g(\nabla\varphi, \nabla\varphi) \neq 0$ everywhere. Then (M_1, g) is isometric to an open subset of a warped product $M_2 = (0, \infty) \times_t M_*$ with a cone-like metric $g = \eta\, dt^2 + t^2 g_*$ with $\eta = \pm 1$ and where (M_*, g_*) is independent of t. Furthermore, up to an isometry, in these coordinates the mapping f appears as $f(t, x) = (\frac{2}{t}, x)$. Consequently, the mapping f is the composition of an isometry and an inversion.*

If in addition M is assumed to be Einstein then it is Ricci flat and, consequently, any conformal mapping is a Liouville mapping. In this case it follows that (M_*, g_*) is an Einstein metric with the same Ricci curvature as a space of constant sectional curvature η. If in addition the dimension of M is not greater than 4 then g is flat.

Corollary 2. *Let (M, g) be a complete Riemannian manifold and assume that there is a Liouville mapping $f : M \setminus \{p\} \to M \setminus \{p\}$ for one point $p \in M$. Then (M, g) is isometric with the Euclidean space.*

Proof of Theorem 1. From Corollary 1 we obtain a local representation of the metric as $g = \eta\, dt^2 + (at + b)^2 g_*$ with $a \neq 0$. By a shift of the parameter we may assume that $b = 0$ and that the factor a is incorporated into the metric g_*. This leads to a local expression $g = \eta\, dt^2 + t^2 g_*$ in the neighborhood of every point which satisfies the assumptions. Since g_* is independent of t, it cannot change on a connected open subset. Therefore there is one maximal "level space" (M_*, g_*). The maximum possible interval for t is either $(0, \infty)$ or $(-\infty, 0)$. Hence M_1 can be regarded as an open subset of a warped product $M_2 := (0, \infty) \times_t M_*$ with a cone-like metric. Then the function φ is a function of t only, more precisely we have $\varphi(t) = t^2/2$. The conformal factor of the mapping is $\varphi^{-2} = 4t^{-4}$, hence the apex $t = 0$ of the cone cannot belong to the domain of φ in any case. Note however, that this apex is a removable singularity only if (M_*, g_*) is isometric with the standard sphere in Euclidean space if $\eta = 1$ or with a standard quadric in pseudo-Euclidean space if $\eta = -1$. In any case by removing the singularity we obtain the (pseudo-)Euclidean metric in geodesic polar coordinates around the origin, see [21]. If M is assumed to be Ricci flat then we obtain information about the metric g_* from the equation $Ric = t^{-2}(Ric_* - (n - 2)g_*)$. In any case the vector field $\nabla\varphi = t \cdot \frac{\partial}{\partial t}$ is a homothetic vector field satisfying the equation $\mathcal{L}_{t\frac{\partial}{\partial t}} g = 2\nabla^2\varphi = 2g$.

Now let us consider the conformal point transformation $f : M_1 \to M$ with the conformal factor φ^{-2}. The inversion $f_0 : M_2 \to M_2$ defined by $f_0(t, x) = (\frac{2}{t}, x)$ is also conformal with the same conformal factor $\varphi^{-2} = 4t^{-4}$. This implies that $f_0 \circ f^{-1} : f(M) \to M_2$ and $f \circ f_0^{-1} : f_0(M_1) \to M$ are isometric mappings.

Now assume that in addition g is Einstein. It is well known which Einstein spaces admit a representation as a warped product. In any case the level M_* has to be an Einstein space, too. In our case the normalized scalar curvatures S and S_* of g and g_* satisfy the equation $1 + \eta St^2 = \eta S_*$, see [22], Lemma 3.3. This is possible only for $S = 0$ and $S_* = \eta$. A peculiar example for such an Einstein space is given by the non-standard Einstein metrics on spheres (appropriately scaled), see [23]. In this case the cone has no topological singularity at the apex, just a metrical singularity. If the dimension of M is not greater than 4 then the level metric g_* is of constant curvature and, therefore, g is of constant curvature. $\quad\square$

Proof of Corollary 2. In the Riemannian case we have $g(\nabla\varphi, \nabla\varphi) \neq 0$ on an open subset M' unless φ is constant. If φ is globally constant then f is a homothety, and the assertion is well known. Otherwise, by Theorem 1 $M' \setminus \{p\}$ is an open part of a warped product $(0, \infty) \times_t M_*$ with $g = dt^2 + t^2 g_*$. Here (M_*, g_*) must be complete because otherwise g would not be complete. Again by the assumption of completeness the apex $t = 0$ of the cone must be an ordinary (non-singular) point of M, hence M_* is a standard unit sphere and M is the Euclidean space in standard polar coordinates. $\quad\square$

Remarks. (1) There are, however, examples in higher dimensions which are Ricci flat but not flat. Take any 4-dimensional level metric g_* which is Einstein with scalar curvature η and then take $M^5 = (0, \infty) \times M_*$ with the metric $g = \eta \, dt^2 + t^2 g_*$.

(2) If the isometry mentioned in Theorem 1 preserves the t-levels then it is easily seen to be induced by an isometry of M_*. Otherwise the metric is of a special type and admits a parallel vector field, see Corollary 3 below. If the metric is Riemannian and if M_* is complete then this cannot happen unless the metric is flat, see [14]. The case of a conformal mapping moving the apex p and not preserving the t-levels can really occur: Consider a Euclidean translation composed by an ordinary inversion at p. This defines a conformal mapping of $E^n \setminus \{p\}$ into E^n which does not preserve the levels of the associated conformal factor. These levels are concentric spheres.

(3) In addition to the obvious self-similarity of cones there are other types of self-similarities which do not satisfy the assumptions of Theorem 1, see [1,7].

(4) The classical Liouville theorem, as stated in [36, p. 209], [2, Thm. A.3.7], [15,34], follows from Theorem 1 by restriction to the (pseudo-)Euclidean metric:

Any conformal mapping f of class C^3 from a connected and open part of Euclidean space into Euclidean space can be written (up to motions in source and target) as a dilatation or as an inversion. Similarly, any conformal mapping of class C^3 from a connected and open part of pseudo-Euclidean space into pseudo-Euclidean space can be written (up to motions) as either a dilatation or an inversion or the composition of two distinct inversions.

Proposition 1. *If $\varphi : M \to \mathbb{R}$ is a function satisfying $\nabla^2 \varphi = c \cdot g$ with a constant $c \neq 0$ (i.e., $\nabla \varphi$ is homothetic) and if $F : M \to M$ is an isometry which does not preserve φ (i.e., $\varphi \circ F \neq \varphi$) then M carries a parallel vector field.*

Proof. Let $\psi = \varphi \circ F$. Then we have $\nabla \psi|_p = DF^{-1}|_p (\nabla \varphi|_{F(p)})$ and $\Delta \psi|_p = \Delta \varphi|_{F(p)}$. It follows that $\nabla^2 \psi = c \cdot g$ and, consequently, $\nabla^2(\psi - \varphi) = 0$. Hence $\nabla(\psi - \varphi)$ is parallel vector field. It remains to show that it does not vanish. If $\nabla \psi = \nabla \varphi$ then we have:

$$2\varphi\big(F(p)\big) \cdot c = \frac{2}{n} \varphi\big(F(p)\big) \Delta \varphi\big(F(p)\big) = g(\nabla \varphi|_{F(p)}, \nabla \varphi|_{F(p)}) = g(\nabla \psi|_p, \nabla \psi|_p) = g(\nabla \varphi|_p, \nabla \varphi|_p)$$

$$= \frac{2}{n} \varphi(p) \Delta \varphi(p) = 2\varphi(p) \cdot c$$

since F is an isometry. Consequently $\nabla \psi = \nabla \varphi$ implies that $\psi = \varphi$ which was excluded by our assumption. $\quad\square$

Key example. In Euclidean space we find translations and inversions simultaneously. Normally we describe inversions by polar coordinates. So it is instructive to look at translations in polar coordinates. In the Euclidean plane with $x = r \cos \varphi$, $y = r \sin \varphi$ a fixed vector $V = x_0 \partial_x + y_0 \partial_y$ can be expressed as $V = (x_0 \cos \varphi + y_0 \sin \varphi) \partial_r + \frac{1}{r}(y_0 \cos \varphi + x_0 \sin \varphi) \partial_\varphi$. In particular the coefficient in r-direction is independent of r, the one in φ-direction is r^{-1} times some expression which is independent of r. This does not only generalize to higher dimensional Euclidean spaces but also to any situation where we have a Liouville transformation and in addition a parallel vector field, see Proposition 2 and Corollary 3 below. The metric does not have to be flat in this case, compare the example after Corollary 5.

Proposition 2. *Assume that a cone metric $g = \eta \, dt^2 + t^2 g_*$ is given. Then locally the following conditions are equivalent:*

(1) *There exists a parallel vector field V;*
(2) *On the level M_* there exists a non-constant function α with $g_*(\nabla_* \alpha, \nabla_* \alpha) \neq 0$ which satisfies the equation $\nabla_*^2 \alpha = -\eta \alpha g_*$.*

Proof. (1) $\Rightarrow$ (2) Let V be a parallel vector field. We can decompose it into $V(t, x) = \alpha(t, x) \partial_t + W(t, x)$ where W is orthogonal to ∂_t and where α is not the zero function because V cannot be orthogonal on every t-line simultaneously. Since the t-lines are geodesics, V has a constant angle with ∂_t along them. Consequently, α does not depend on t. So we can regard α as a function on any of the levels. From $\nabla_{\partial_t} V = 0$ we obtain $\nabla_{\partial_t} W = 0$. Therefore $W(t, x)$ is parallel along the t-lines, and we can set $W(t, x) = \frac{1}{t} W(x)$ where $W(x)$ is the lift of a vector field on M_*. Altogether we obtain $V(t, x) = \alpha(x) \partial_t + \frac{1}{t} W(x)$.

Next, if X denotes the lift of a vector field on M_*, we have for any fixed t:

$$\begin{aligned}
\nabla_X^* W &= \nabla_X W + g(X, W) t^{-1} \eta \partial_t \\
&= \nabla_X \big(t V(t, x) - t\alpha(x)\partial_t \big) + g_*(X, W) t\eta \partial_t \\
&= -t X(\alpha)\partial_t - t\alpha(x)\nabla_X \partial_t + g_*(X, W) t\eta \partial_t \\
&= t \big(g_*(X, W)\eta - g_*(\nabla^* \alpha, X) \big)\partial_t - t\alpha(x) t^{-1} X.
\end{aligned}$$

Since the ∂_t-component of $\nabla_X^* W$ must vanish for any X we obtain first the equation,

$$\nabla^* \alpha = \eta W,$$

and consequently,

$$\nabla_X^* \nabla^* \alpha = \eta \nabla_X^* W = -\eta \alpha(x) \cdot X,$$

which is the assertion. In particular, α is not constant. Moreover, since

$$g(V, V) = \eta \alpha^2 + g_*(W, W) = \eta \alpha^2 + g_*(\nabla_* \alpha, \nabla_* \alpha)$$

is constant, it follows that $g_*(\nabla_* \alpha, \nabla_* \alpha)$ cannot vanish on any open set. Consequently, $\|W\|^2$ cannot vanish either.

For the converse direction $(2) \Rightarrow (1)$ we assume that α satisfies this differential equation and define V by the equations above. It then follows that V is parallel. □

Corollary 3. *If a cone metric $g = \eta \, dt^2 + t^2 g_*$ carries a parallel vector field then g_* can be written as a warped product $g_* = \epsilon \, du^2 + \alpha'^2(u) g_{**}$ where $\epsilon = \pm 1$ and the function $\alpha(u)$ satisfies the equation $\alpha'' + \epsilon \eta \alpha = 0$. Conversely, if a metric of the form $g = \eta \, dt^2 + t^2(\epsilon \, du^2 + \alpha'^2(u) g_{**})$ with such an α is given then g carries a parallel vector field.*

This follows from Lemma 3 in connection with Proposition 2. Along the trajectories of the gradient of α is not a null vector. With $\nabla_* \alpha = \epsilon \alpha'(u)\partial_u$ the function satisfies the ODE $\alpha'' + \epsilon \eta \alpha = 0$. The solutions are linear combinations of sin, cos if $\epsilon \eta = 1$ and of sinh, cosh is $\epsilon \eta = -1$.

Corollary 4. *(See [14].) If $g = dt^2 + t^2 g_*$ is positive definite and if g_* is complete then g is either irreducible or flat.*

Proof. This follows from the well known theorem attributed to Obata [32,27] that on a complete Riemannian manifold (M_*, g_*) the equation $\nabla_*^2 \alpha = -\alpha g$ admits a non-constant solution only if the manifold is a round sphere of radius 1. Note, however, that the same theorem was proved by Tashiro in [37] (which was submitted before [32] but appeared later). Compare also [19, Thm. 25]. So if g carries a parallel vector field then this equation on (M_*, g_*) is satisfied by Proposition 2. It follows that the level M_* is the unit sphere and, consequently, g is flat. □

Corollary 5. *Assume that the cone metric with $g = \eta \, dt^2 + t^2 g_*$ is an Einstein n-manifold (necessarily Ricci flat) and carries a parallel vector field. Then for $n \leqslant 5$ g is necessarily flat. In contrast, there is an example of a 6-dimensional Ricci flat cone which is not flat.*

Example. Let h be a complete Ricci flat space with a positive definite metric which is not flat (one example is the Calabi–Yau metric on the $K3$ surface). Then the Lorentzian warped product metric,

$$g = -dt^2 + t^2(du^2 + e^{2u} h),$$

is Ricci flat but not flat. It carries the homothetic vector field $t\partial_t$ and, in addition, the parallel null vector field $V = e^u(\partial_t + \frac{1}{t}\partial_u)$. The level $g_* = du^2 + e^{2u} h$ is complete if h is.

Lemma 4. *Assume that an n-dimensional manifold (M, g) admits a Liouville mapping of class C^3 such that the conformal factor φ has a gradient $\nabla\varphi \neq 0$ everywhere. Then the following conditions are equivalent:*

(1) $g(\nabla\varphi, \nabla\varphi) = 0$ *everywhere*;

(2) $\nabla^2 \varphi = 0$ *everywhere*;

(3) $\nabla \varphi$ *is a parallel vector field*;

(4) *in a suitable coordinate system the metric g can be written as $g = 2\,d\varphi\,d\vartheta + g^{\#}(\varphi)$ where $g^{\#}(\varphi)$ is an $(n-2)$-dimensional metric which is independent of ϑ.*

Proof. $(4) \Rightarrow (1)$ holds because in these coordinates we have $\frac{\partial}{\partial\vartheta} = \nabla\varphi$.

$(1) \Leftrightarrow (2)$ holds by Lemma 2.

$(2) \Leftrightarrow (3)$ is trivial.

The main part is the proof of $(1) \Rightarrow (4)$ which is originally due to Brinkmann in [5, pp. 132–133], for a modern treatment see also [6]. Because φ is nowhere constant we can introduce φ as one of the coordinates. Then $\nabla\varphi$ is a parallel vector field by (3), therefore we can introduce a second coordinate function ϑ by the condition $\frac{\partial}{\partial\vartheta} = \nabla\varphi$. It follows that $g(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}) = 1$ and, in particular, that $\frac{\partial}{\partial\varphi}$ and $\frac{\partial}{\partial\vartheta}$ are linearly independent everywhere. Now let us fix one particular φ-ϑ-level with a non-degenerate induced metric $g_{\#}(\varphi, \vartheta)$. Then introduce geodesic normal coordinates $\varphi, \vartheta, x_1, \ldots, x_{n-2}$ around this level. It follows that the metric $g^{\#}$ is independent of ϑ because $\frac{\partial}{\partial\vartheta}$ is parallel. By a transformation of the metric one can assume that $g(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi})$ is also zero. This leads to the expression $g = 2\,d\varphi\,d\vartheta + g^{\sharp}(\varphi)$ for the metric. $\quad\square$

Corollary 6. *If the conditions in Lemma 4 are satisfied then the ϑ-lines (or the trajectories generated by the parallel vector field $\nabla\varphi$) are null geodesics.*

Proof. We have to show that the parallel isotropic vector $\nabla\varphi$ is mapped by the Liouville mapping onto a parallel isotropic vector. This follows directly from the formula,

$$\overline{\nabla}_X V - \nabla_X V = -X(\log\varphi)V - V(\log\varphi)X + g(X, V)\nabla(\log\varphi),$$

for the conformal change of the metric $\bar{g} = \varphi^{-2}g$ and any X, V. In our case we have $V = \nabla\varphi$ which implies $V(\log\varphi) = 0$ and $X(\log\varphi)\nabla\varphi = g(X, \nabla\varphi)\nabla(\log\varphi)$. $\quad\square$

Definition. A nowhere vanishing and lightlike vector field V defines a family of integral curves, which is also called a null congruence, cf. [8, p. 56]. If these curves are geodesics they form a geodesic null congruence. The Proof of Corollary 6 shows that a geodesic null congruence generated by a parallel vector field is mapped under a Liouville transformation onto a geodesic null congruence generated by a parallel vector field.

Lemma 5. *Let (M, g) be an n-dimensional manifold carrying two geodesic null congruences generated by two linearly independent and parallel vector fields. Then the manifold is (locally) reducible to a product $(\mathbb{R}^2 \times M_*, 2\,du\,dv + g_*)$. Moreover, there is an isometry mapping one null congruence to the other.*

This follows because the two linearly independent isotropic vector fields generate a 2-dimensional distribution which is integrable and parallel. Therefore, a 2-dimensional factor splits off which is a flat Lorentzian plane.

In view of Lemma 5 and up to an isometry, one has to consider only the case that the null congruence is preserved.

Theorem 2 *(The isotropic case in Liouville's theorem). Assume that an n-dimensional manifold (M, g) admits a Liouville mapping $F: M \to M$ of class C^3 such that the conformal factor φ has an isotropic gradient $\nabla\varphi \neq 0$ everywhere. Assume further that F preserves the geodesic null congruence given by the parallel null vector $\partial_v = \nabla\varphi$. Then in certain coordinates u, v, x_k $(k = 1, \ldots, n-2)$ the metric has the form,*

$$g = -2\,du\,dv + \sum_{i,j} g^{\#}_{ij}(u, x_k)\,dx^i\,dx^j,$$

and, up to an isometry, F has the form,

$$F(u, v, x_k) = \left(-\frac{1}{cu}, cv + \zeta(u, x_k), \xi_1(u, x_k), \ldots, \xi_{n-2}(u, x_k)\right),$$

with a constant c and with a certain function ζ, where for any fixed u, v the transformation,

$$(x_1, \ldots, x_{n-2}) \mapsto (\xi_1, \ldots, \xi_{n-2}),$$

is a homothety with respect to the metric $g^\#$. The conformal factor of F is the function $\varphi(u, v, x_k) = u$, i.e., $F^* g = u^{-2} g$.

Conversely, Let h be any metric on an $(n-2)$-dimensional space M_* admitting a 1-parameter group Φ_u of similarities (homothetic transformations) with $\Phi_u^* h = u^{-2} h$. Then on $M = \mathbb{R}_+ \times \mathbb{R} \times M_*$ the metric $g = -2 \, du \, dv + \Phi_u^* h$ admits a conformal mapping F such that the conformal factor u has an isotropic gradient $\nabla u = \partial_v$. In this case F acts on M_* by the similarities Φ_u.

Proof. The particular form of the metric is obtained by Lemma 4 which also determines the conformal factor as $\varphi = u$. We compute the differential of F with respect to the basis $\partial_u, \partial_v, \partial_1, \ldots, \partial_{n-2}$ as follows. By assumption F preserves the null congruence given by the v-lines. This implies that at every point the differential of F transforms the vector ∂_v into some vector linearly dependent on ∂_v. This implies that the component functions $\kappa, \xi_1, \ldots, \xi_{n-2}$ in

$$F(u, v, x_1, \ldots, x_{n-2}) = (\kappa, \lambda, \xi_1, \ldots, \xi_{n-2})$$

are independent of the variable v. Consequently, we have:

$$\begin{aligned}
F_u &= \big(\kappa_u, \lambda_u, (\xi_1)_u, \ldots, (\xi_{n-2})_u\big), \\
F_v &= (0, \lambda_v, 0, \ldots, 0), \\
F_k &= \big(\kappa_k, \lambda_k, (\xi_1)_k, \ldots, (\xi_{n-2})_k\big) \quad \text{for } k = 1, \ldots, n-2,
\end{aligned}$$

where the indices indicate the partial derivatives with respect to the corresponding coordinates.

From the equation $g(F_u, F_v) = u^{-2} g(\partial_u, \partial_v) = -u^{-2}$ we obtain $\kappa_u \lambda_v = u^{-2}$. Differentiating once more leads to the equation $\kappa_u \lambda_{vv} = 0$.

Similarly, from the equation $g(F_k, F_v) = u^{-2} g(\partial_k, \partial_v) = 0$ we obtain $\kappa_k \lambda_v = 0$. Since λ_v cannot vanish identically, we get $\kappa_k = 0$ for all k. As a consequence, κ_u does not vanish identically. It follows that λ_{vv} must vanish identically. Hence $\lambda_v = c$ is constant and $\kappa_u = \frac{1}{cu^2}$ and $\kappa = -\frac{1}{cu}$. It also follows that $\lambda = cv + \eta(u, x_k)$ with some function η.

Now let u_0 and v_0 be fixed, and consider the mapping Φ defined by:

$$(x_1, \ldots, x_{n-2}) \mapsto \big(\xi_1(u_0, x_1, \ldots, x_{n-2}), \ldots, \xi_{n-2}(u_0, x_1, \ldots, x_{n-2})\big).$$

From the equation,

$$\Phi^* g^\#(\partial_i, \partial_j) = g^\#(F_i, F_j) = g(F_i, F_j) = u_0^{-2} g(\partial_i, \partial_j) = u_0^{-2} g^\#(\partial_i, \partial_j),$$

we see that Φ is homothetic.

Conversely, if (M_*, g_*) is given with a 1-parameter group Φ_u of similarities then we can define F by:

$$F(u, v, \mathbf{x}) = \left(-\frac{1}{cu}, cv + \zeta(u, \mathbf{x}), \Phi_u(\mathbf{x}) \right).$$

The function ζ is determined by the condition that F_u must be orthogonal to the derivative of F by any coordinate in the $(n-2)$-dimensional space M_* and, furthermore, that $g(F_u, F_u) = 0$. $\quad\square$

Remark. In the case of a vacuum spacetime our extra assumption on preserving the null congruence is always satisfied unless the manifold is flat.

Examples. (1) Let $g = -2 \, du \, dv + \sum_i \epsilon_i x_i^2$ be the metric of the semi-euclidean space. Then the following mapping is a conformal involution:

$$F(u, v, x_1, \ldots, x_{n-2}) = \left(-u^{-1}, v + (2u)^{-1} \sum_i \epsilon_i x_i^2, u^{-1} x_1, \ldots, u^{-1} x_{n-2} \right).$$

A local 1-parameter group of conformal mappings is:

$$\Phi_t(u, v, x_1, \ldots, x_{n-2}) = \frac{1}{1 - tu} \left(u, v(1 - tu) + \frac{t}{2} \sum_i \epsilon_i x_i^2, x_1, \ldots, x_{n-2} \right).$$

The expression in the first coordinate differs from the form given in Theorem 2 but that is just up to a conjugation with a shift $u \mapsto u + u_0$. It generates the following conformal vector field,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t = \left(u^2, \frac{1}{2}\sum_i \epsilon_i x_i^2, ux_1, \ldots, ux_{n-2}\right),$$

which is a standard special conformal vector field also in the general context of *pp*-waves, see [24]. Here it can be considered as a Liouville vector field where every member of the 1-parameter family is of type 3.

(2) Let $g = -2\,\mathrm{d}u\,\mathrm{d}v + u^2\sum_i \epsilon_i x_i^2;$. Then the following involution is conformal:

$$F(u, v, x_1, \ldots, x_{n-2}) = \left(-u^{-1}, v + \frac{u}{2}\sum_i \epsilon_i x_i^2, ux_1, \ldots, ux_{n-2}\right).$$

(3) More generally, let h be any metric and let $g = -2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}t^2 + t^2 h$. Then the following involution is conformal:

$$F(u, v, t, x_k) = \left(-u^{-1}, v - \frac{1}{2}u^{-1}t^2, u^{-1}t, x_k\right).$$

Similarly, if $g = -2\,\mathrm{d}u\,\mathrm{d}v + u^2(\mathrm{d}t^2 + t^2 h)$. Then the following involution is conformal:

$$F(u, v, t, x_k) = \left(-u^{-1}, v + \frac{u}{2}t^2, tu, x_k\right).$$

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References

[1] D.V. Alekseevskii, Selfsimilar Lorentzian manifolds, Ann. Glob. Anal. Geom. 3 (1985) 59–84.

[2] R. Benedetti, V. Petronio, Lectures on Hyperbolic Geometry, Springer, Berlin, 1992.

[3] W. Benz, Der Satz von Liouville über räumliche orthogonaltreue (winkeltreue) Abbildungen für beliebige Signatur, Aequationes Math. 21 (1980) 257–282.

[4] B. Bojarski, T.M. Iwaniec, Another approach to Liouville theorem, Math. Nachr. 107 (1982) 253–262.

[5] H.W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math. Ann. 94 (1925) 119–145.

[6] D.A. Catalano, Closed conformal vector fields on pseudo-Riemannian manifolds, Internat. J. Math. Math. Sci. 2006 (2006), Article ID 36545, 8 pages.

[7] A.A. Coley, Self-similarity and cosmology, Fields Inst. Commun. 15 (1997) 19–42.

[8] K.L. Duggal, R. Sharma, Symmetries of Spacetimes and Riemannian Manifolds, Kluwer, Dordrecht, 1999.

[9] D. Eardley, J. Isenberg, J. Marsden, V. Moncrief, Homothetic and conformal symmetries of solutions to Einstein's equations, Commun. Math. Phys. 106 (1986) 137–158.

[10] J. Ferrand, Sur une classe de morphismes conformes, C. R. Acad. Sci. Paris 296 (1983) 681–684.

[11] J. Ferrand, Concircular transformations of Riemannian manifolds, Ann. Acad. Sci. Fenn., Ser. A. I. Math. 10 (1985) 163–171.

[12] A. Fialkow, Conformal geodesics, Trans. Amer. Math. Soc. 45 (1939) 443–473.

[13] C. Frances, Une preuve du théorème de Liouville en géométrie dans le cas analytique, L'Enseign. Math. (2) 49 (2003) 95–100.

[14] S. Gallot, Équations différentielles caractéristiques de la sphère, Ann. Scient. Ec. Norm. Sup. (4) 12 (1979) 235–267.

[15] J. Haantjes, Conformal representations of an *n*-dimensional Euclidean space with a non-definite fundamental form on itself, Proc. Kon. Ned. Akad. Amsterdam 40 (1937) 700–705.

[16] G.S. Hall, Symmetries and Curvature Structure in General Relativity, World Scientific, Singapore, 2004.

[17] H. Jacoobowitz, Two notes on conformal geometry, Hokkaido Math. J. 20 (1991) 313–329.

[18] Y. Kerbrat, Transformations conformes des variétés pseudo-Riemanniennes, J. Differential Geom. 11 (1976) 547–571.

[19] W. Kühnel, Conformal transformations between Einstein spaces, in: R.S. Kulkarni, U. Pinkall (Eds.), Conformal Geometry, in: Aspects Math., vol. E12, Vieweg, 1988, pp. 105–146.

[20] W. Kühnel, H.-B. Rademacher, Conformal diffeomorphisms preserving the Ricci tensor, Proc. Amer. Math. Soc. 123 (1995) 2841–2848.

[21] W. Kühnel, H.-B. Rademacher, Essential conformal fields in pseudo-Riemannian geometry, J. Math. Pures Appl. (9) 74 (1995) 453–481;
W. Kühnel, H.-B. Rademacher, Essential conformal fields in pseudo-Riemannian geometry, Part II, J. Math. Sci. Univ. Tokyo 4 (1997) 649–662.

[22] W. Kühnel, H.-B. Rademacher, Conformal vector fields on pseudo-Riemannian spaces, Differential Geom. Appl. 7 (1997) 237–250.

[23] W. Kühnel, H.-B. Rademacher, Conformal Ricci collineations of space–times, Gen. Rel. Grav. 33 (2001) 1905–1914.

[24] W. Kühnel, H.-B. Rademacher, Conformal geometry of gravitational plane waves, Geom. Dedicata 109 (2004) 175–188.

[25] N.H. Kuiper, On conformally-flat spaces in the large, Ann. of Math. (2) 50 (1949) 916–924.

[26] R.S. Kulkarni, Conformal structures and Möbius structures, in: R.S. Kulkarni, U. Pinkall (Eds.), Conformal Geometry, in: Aspects Math., vol. E12, Vieweg, 1988, pp. 1–39.

[27] J. Lafontaine, Sur la géométrie d'une généralisation de l'équation différentielle d'Obata, J. Math. Pures Appl. (9) 62 (1983) 63–72.

[28] S. Lie, Ueber Complexe, insbesondere Linien- und Kugel-Complexe mit Anwendung auf die Theorie partieller Differentialgleichungen, Math. Ann. 5 (1872) 145–246.

[29] J. Liouville, Extension au cas des trois dimensions de la question du tracé géographique, Note VI, in: G. Monge (Ed.), Applications de l'analyse à la géométrie, Bachelier, Paris, 1850, pp. 609–617.

[30] R. Maartens, S.D. Maharaj, Conformal symmetries of *pp*-waves, Class. Quantum Grav. 8 (1991) 503–514.

[31] W. Moore, J.D. Zund, A note on Liouville's theorem in conformal geometry, Tensor (N.S.) 44 (1987) 204–207.

[32] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan 14 (1962) 333–340.

[33] Yu.G. Reshetnyak, Liouville's theorem on conformal mappings for minimal regularity assumptions, Sib. Math. J. 8 (1967) 631–634, transl. from Sib. Mat. Zh. 8 (1967) 835–840.

[34] M. Schottenloher, A Mathematical Introduction to Conformal Field Theory, Springer, Berlin, 1997.

[35] J.A. Schouten, Der Ricci-Kalkül, Springer, Berlin, 1924.

[36] J.A. Schouten, D.J. Struik, Einführung in die neueren Methoden der Differentialgeometrie, Band 2, Noordhoff, Groningen, 1938.

[37] Y. Tashiro, Complete Riemannian manifolds and some vector fields, Trans. Amer. Math. Soc. 117 (1965) 251–275.