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Note

Subsets without q-Separation and Binomial Products of Fibonacci Numbers

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The total number of subsets of $\{1, 2, 3, ..., n\}$ without q-separation is expressed in terms of binomial products of Fibonacci numbers. Several new combinatorial identities are derived. © 1991 Academic Press, Inc.

1. INTRODUCTION

Kaplansky [2] proved that the number of k-subsets of $\{1, 2, 3, ..., n\}$ not containing a pair of consecutive integers (i.e., i and $i+1$) is

$$\binom{n+1-k}{k}.$$

Konvalina [3] proved that the number of k-subsets of $\{1, 2, 3, ..., n\}$ not containing a pair of uniseparate integers (i.e., i and $i+2$) is

$$\begin{cases} \displaystyle\sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} & \text{if} \quad n \geqslant 2(k-1) \\ 0 & \text{if} \quad n < 2(k-1). \end{cases}$$

Prodinger [5] and Hwang, Korner, and Wei [1] generalized Konvalina's result and determined explicit formulae for the number of k-subsets of $\{1, 2, 3, ..., n\}$ without q-separation. Two integers are called *q-separate* $(q \geqslant 1)$ if their difference is q. In this paper we consider the total number of subsets of $\{1, 2, 3, ..., n\}$ without q-separation and prove the unexpected result that this number can be concisely expressed in terms of the product of powers of Fibonacci numbers arising from the binomial theorem.

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The Fibonacci numbers have a well-known combinatorial interpretation in terms of subsets of $\{1, 2, 3, ..., n\}$ not containing a pair of consecutive integers. Specifically, let F_n denote the nth Fibonacci number determined by the recurrence relation

$$F_1 = 1$$

$$F_2 = 1$$

$$F_{n+2} = F_{n+1} + F_n \qquad (n \geqslant 1).$$

The total number of subsets of $\{1, 2, 3, ..., n\}$ not containing a pair of consecutive integers is F_{n+2}.

Recently, Konvalina and Liu [4] showed that the total number of subsets $\{1, 2, 3, ..., n\}$ without unit separation ($q = 2$) can be expressed in terms of the Fibonacci numbers. Specifically, let T_n denote the total number of subsets of $\{1, 2, 3, ..., n\}$ without unit separation; then the following identities hold:

$$T_{2n} = F_{n+2}^2$$

$$T_{2n+1} = F_{n+2} F_{n+3}.$$

In this paper we will generalize the result to q-separation. Let $T(n, q)$ denote the total number of subsets of $\{1, 2, 3, ..., n\}$ without q-separation. We will show that if n is written in the form $n = mq + r$, where $0 \leqslant r < q$, then $T(n, q)$ is a binomial product (arising from the binomial theorem) of Fibonacci numbers:

$$T(n, q) = F_{m+2}^{q-r} F_{m+3}^{r}.$$

The result by Konvalina and Liu is the special case $q = 2$. The classical result is the special case $q = 1$. Several corollaries, including some interesting combinatorial identities, are derived.

2. The Main Result

THEOREM. *Let $T(n, q)$ denote the number of subsets of $\{1, 2, 3, ..., n\}$ without q-separation. If $n = mq + r$, where $0 \leqslant r < q$, then*

$$T(n, q) = F_{m+2}^{q-r} F_{m+3}^{r}.$$

Proof. Partition the set $\{1, 2, 3, ..., n\}$ into the q disjoint subsets $S_1, S_2, ..., S_q$ defined as follows:

$$S_i = \begin{cases} \{i, i+q, i+2q, ..., i+mq\} & \text{if } 1 \leqslant i \leqslant r \\ \{i, i+q, i+2q, ..., i+(m-1)q\} & \text{if } r < i \leqslant q \end{cases}$$

Let $|S_i|$ denote the cardinality of S_i. Then we have

$$|S_i| = \begin{cases} m+1 & \text{if } \ 1 \leqslant i \leqslant r \\ m & r < i \leqslant q. \end{cases}$$

The total number of subsets of $\{1, 2, 3, ..., n\}$ without q-separation is equal to the product of the number of subsets of each S_i not containing a pair of consecutive elements. Thus we have reduced the problem to the classical case $(q=1)$.

If $1 \leqslant i \leqslant r$, then the number of subsets of S_i not containing a pair of consecutive elements is F_{m+3}, since $|S_i| = m+1$. If $i > r$, then this number is F_{m+2}, since $|S_i| = m$.

Finally, forming the product with r subsets of cardinality $m+1$ and $q-r$ subsets of cardinality m, we obtain the result:

$$T(n, q) = F^{q-r}_{m+2} F^{r}_{m+3}.$$

COROLLARY 1.

$$T(n, 1) = F_{n+2}$$

$$\sum_{k \geqslant 0} \binom{n+1-k}{k} = F_{n+2}. \tag{1}$$

Proof. If $q=1$, the theorem reduces to the classical result. If we sum over all k-subsets and apply Kaplansky's result we obtain the well-known combinatorial identity (1).

COROLLARY 2.

$$T(n, 2) = F^{2-r}_{m+2} F^{r}_{m+3}$$

$$\sum_{k \geqslant 0} \sum_{i=0}^{[k/2]} \binom{n+1-k-2i}{k-2i} = F^{2-r}_{m+2} F^{r}_{m+3}. \tag{2}$$

Proof. If $q=2$, we obtain the result by Konvalina and Liu [4], and summing over all k-subsets without unit separation we obtain the identity (2).

COROLLARY 3. *Let $f_q(n, k)$ denote the number of k-subsets of $\{1, 2, 3, ..., n\}$ without q-separation. Then*

$$\sum_{k \geqslant 0} f_q(n, k) = F^{q-r}_{m+2} F^{r}_{m+3}$$

or

$$\sum_{k\geqslant 0}\sum(-1)^{\lambda_1+m\lambda_3+(m+1)\lambda_4}\binom{q-2+\lambda_1}{\lambda_1}\binom{n+q-2k+\lambda_2}{\lambda_2}\binom{r}{\lambda_3}\binom{q-r}{\lambda_4}$$

$$=F_{m+2}^{q-r}F_{m+3}^{r}, \tag{3}$$

where the second summation is over all λ_i such that $\lambda_1\geqslant 0$, $\lambda_2\geqslant 0$, $0\leqslant\lambda_3\leqslant r$, $0\leqslant\lambda_4\leqslant q-r$, and $\lambda_1+\lambda_2+\lambda_3(m+3)+\lambda_4(m+2)=k$, if $0\leqslant k\leqslant(n+q)/2$; otherwise, $f_q(n,k)=0$.

Proof. The explicit formula for $f_q(n,k)$ was obtained by Prodinger [5]. Summing over all k-subsets without q-separation we obtain the combinatorial identity (3).

COROLLARY 4. *Let $f_q(n_1, n_2, ..., n_q, k)$ denote the number of ways of selecting k objects from q lines of lengths $n_1, n_2, ..., n_q$ without two selected objects being consecutive. Also, let $n_1=n_2=\cdots=n_r=m+1$ and $n_{r+1}=n_{r+2}=\cdots=n_q=m$. Then, for $0\leqslant k_i\leqslant n_i$ $(1\leqslant i\leqslant q)$,*

$$\sum_{k\geqslant 0} f_q(n_1, n_2, ..., n_q, k)=F_{m+2}^{q-r}F_{m+3}^{r}$$

and

$$\sum_{k\geqslant 0}\ \sum_{k_1+k_2+\cdots+k_q=k}\ \prod_{i=1}^{q}\binom{n_i+1-k_i}{k_i}=F_{m+2}^{q-r}F_{m+3}^{r}. \tag{4}$$

Proof. $f_q(n_1, n_2, ..., n_q, k)$ was explicitly determined by Hwang, Korner, and Wei [1]. Identity (4) follows from the theorem and Kaplansky's result after summing over all k-subsets not containing two consecutive integers.

$T(n,q)$ can also be determined by computing the number of binary (0, 1) sequences of length n without q-separation (i.e., no two ones are separated by $q-1$ bits). We state without proof the following result:

$$T(n,q)=T(mq+r,q)=\sum_{i_0=0}^{r}\binom{r}{i_0}\sum_{i_1=0}^{q-i_0}\binom{q-i_0}{i_1}\cdots\sum_{i_m=0}^{q-i_{m-1}}\binom{q-i_{m-1}}{i_m}. \tag{5}$$

Combining this result with Corollary 4 we obtain the following combinatorial identity:

COROLLARY 5. *If* $n_1 = n_2 = \cdots = n_r = m+1$ *and* $n_{r+1} = \cdots = n_q = m$, *then for* $0 \leqslant k_i \leqslant n_i$ $(1 \leqslant i \leqslant q)$,

$$\sum_{k \geqslant 0} \sum_{k_1 + k_2 + \cdots + k_q = k} \prod_{i=1}^{q} \binom{n_i + 1 - k_i}{k_i}$$
$$= \sum_{i_0=0}^{r} \sum_{i_1=0}^{q-i_0} \cdots \sum_{i_m=0}^{q-i_{m-1}} \binom{r}{i_0} \prod_{j=1}^{m} \binom{q - i_{j-1}}{i_j}.$$

$T(n, q)$ can also be expressed in terms of a recurrence relation. Using an inductive argument it can be shown that if $n = mq + r$, where $1 \leqslant r \leqslant q$, then $T(n, q)$ satisfies the recurrence relation:

$$T(n, q) = \sum_{i=0}^{r} \binom{r}{i} T(n - r - i, q), \tag{6}$$

with the boundary conditions $T(n, q) = 2^n = F_3^n$ for $0 \leqslant n \leqslant q$.

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