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Extremal presentations for classical Lie algebras

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ABSTRACT

The long-root elements in Lie algebras of Chevalley type have been well studied and can be characterized as extremal elements, that is, elements x such that the image of $(\operatorname{ad} x)^2$ lies in the subspace spanned by x. In this paper, assuming an algebraically closed base field of characteristic not 2, we find presentations of the Lie algebras of classical Chevalley type by means of minimal sets of extremal generators. The relations are described by simple graphs on the sets. For example, for C_n the graph is a path of length $2n$, and for A_n the graph is the triangle connected to a path of length $n-3$.

1. Introduction

A nonzero element x of a Lie algebra $\mathcal{L}$ over a field $\mathbb{F}$ of characteristic not 2 is called *extremal* if $[x, [x, \mathcal{L}]] \subseteq \mathbb{F}x$. Extremal elements are a well-studied class of elements in simple finite-dimensional Lie algebras of Chevalley type: they are the long root elements. In [4], Cohen, Steinbach, Ushirobira and Wales have studied Lie algebras generated by extremal elements, in particular those of Chevalley type. The authors also find the minimum size of a set of generating extremal elements for the Lie algebras of Chevalley type and find such minimal generating sets of extremal elements explicitly. In the present paper, we also find such minimal generating sets of extremal elements explicitly for the four classical families of Lie algebras: those of type A_n, B_n, C_n and D_n. We will do this in a more geometrical setting and will find criteria for sets of extremal elements to generate Lie algebras of this type.

By Lemma 2.2, each Lie algebra generated by a pair of linearly independent extremal elements is in one of only three isomorphism classes: either the two-dimensional commutative Lie algebra, or the so-called Heisenberg Lie algebra $\mathfrak{h}$, or $\mathfrak{sl}_2$. Given a generating set S of extremal elements,

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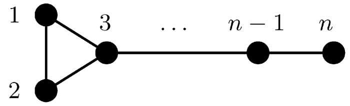

Fig. 1.1. The graph $\Gamma_{A;n}$ for $\mathfrak{sl}_n$.

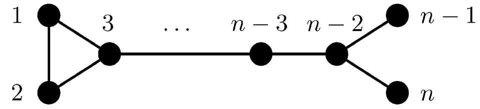

Fig. 1.2. The graph $\Gamma_{B;n}$ for $\mathfrak{o}_{2n-1}$.

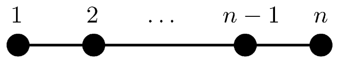

Fig. 1.3. The graph $\Gamma_{C;n}$ for $\mathfrak{sp}_n$.

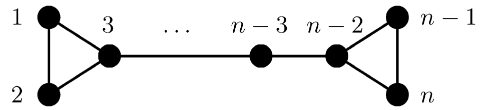

Fig. 1.4. The graph $\Gamma_{D;n}$ for $\mathfrak{o}_{2n}$.

we examine the subalgebras generated by pairs of these elements. These give rise to graphs: the vertices correspond to the elements of S, and two vertices are adjacent if the corresponding extremal elements generate a three-dimensional algebra and nonadjacent if they commute. We will say that the Lie algebra generated by S *realizes* this graph.

Following experiments using the GAP computer algebra system [6] and the GBNP package [7] we conjectured one such graph for *each* Lie algebra of classical Chevalley type, depicted in Figs. 1.1 up to 1.4.

In this paper we show that if a Lie algebra realizes one of the graphs in Figs. 1.1 up to 1.4, then in the generic case it is isomorphic to the Lie algebra of the corresponding Chevalley type, in the following sense. Given a graph Γ, we define a vector space $\mathcal{V}(\Gamma)$ parametrizing the Lie algebras that realize Γ. Let $X(\Gamma)$ be the subset of $\mathcal{V}(\Gamma)$ of values $\mathfrak{f}$ for which the associated Lie algebra $\mathcal{L}(\Gamma, \mathfrak{f})$ has maximal dimension among such algebras for the same graph Γ. We will see in Lemma 3.5 that $X(\Gamma)$ carries the structure of an affine variety. The following theorem and analogues for the three other families of Chevalley type Lie algebras will be our main results.

Theorem 1.1. *Let $n \geqslant 5$. Let $\Gamma_{D;n}$ be the graph of Fig. 1.4. There is an open dense subset S of $X(\Gamma_{D;n})$ such that, if $\mathfrak{f} \in S$, then $\mathcal{L}(\Gamma_{D;n}, \mathfrak{f})$ is isomorphic to the Lie algebra of type D_n.*

The graphs $\Gamma_{A;n}$, $\Gamma_{B;n}$, and $\Gamma_{C;n}$ are subgraphs of $\Gamma_{D;n}$, and as such induce extra commuting relations. Therefore, the D_n case is the most complicated one, from which we will deduce the conclusions for the other cases.

1.1. Contents and strategy

In the rest of this section we will introduce some conventions and notation that we will use in this paper. In Section 2 we review some of the underlying theory. In Section 3 we show how to work with abstract Lie algebras that realize a given graph. We apply this to four proposed graphs $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$ and $\Gamma_{D;n}$ for the classical Chevalley types in Section 4. In Section 5, we extend this study to the parameter space $\mathcal{V}$ referred to above. In Theorems 6.10, 6.11, 6.18 and 6.19, we give concrete realizations of the Lie algebras of types A_n, B_n, C_n and D_n corresponding to the graphs from Figs. 1.1 up to 1.4.

Finally, in Section 7 we prove the main results of this paper: we show that a Lie algebra $\mathcal{L}$ realizing one of the graphs $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$ and $\Gamma_{D;n}$ is in the generic case a quotient of the realization $\mathcal{M}$ found in Section 6. Since $\mathcal{M}$ is simple in most cases, it will follow that $\mathcal{L}$ and $\mathcal{M}$ are isomorphic. The only exception is A_n if $p \mid n+1$.

This paper was inspired by the Masters thesis of the third author [12] and reported on more extensively in the second author's Ph.D. thesis [11].

1.2. Conventions and notation

For the rest of this paper, $\mathbb{F}$ will be an algebraically closed field of characteristic not 2 and $\mathcal{L}$ will be a Lie algebra over $\mathbb{F}$.

Since we approach the matter from the angle of the generating sets of abstract extremal elements, we let n be the number of generating extremal elements. In Theorems Section 6, this will mean that we study, for example, the Lie algebra of type $C_{n/2}$, defined over a vector space of dimension n, where we have to assume that n is even. In that section, it would be more convenient to study the Lie algebra of type C_n instead, but we choose consistency over convenience and keep the meaning of n as the number of extremal generators.

If no confusion is possible, we write xy for $[x, y]$, and xyz for $[x, [y, z]]$; we will write $(xy)z$ for $[[x, y], z]$. So, anticommutativity and the Jacobi identity will be written as

$$xx = 0 \tag{AC}$$

and

$$xyz + yzx + zxy = 0. \tag{J}$$

We will often work with long products of indexed elements. We use the following notation to make these products somewhat manageable. The general idea is that we put two numbers in the subscript with an operator consisting of one or two arrows in between, such as $x_{5\uparrow\Downarrow 2}$; the first factor in the product is then indexed by the first number, after which we iterate adding (for up arrows) or subtracting (for down arrows) one (for single stroke arrows) or two (for double stroke arrows) to the index until we encounter the last number, where every step gives the next factor for this product. So the previous example $x_{5\uparrow\Downarrow 2}$ is short for $x_5x_6x_4x_5x_3x_4x_2$.

In particular, there are four operators that we use, defined more precisely as follows. If $i \leqslant j$, the notation $x_{i\uparrow j}$ will mean $x_ix_{i+1}x_{i+2}\cdots x_{j-1}x_j$, and $x_{j\downarrow i}$ will mean $x_jx_{j-1}x_{j-2}\cdots x_{i+1}x_i$. Furthermore, $x_{j\uparrow\Downarrow i}$ will mean $x_jx_{j+1}x_{j-1}x_jx_{j-2}x_{j-1}\cdots x_{i+1}x_{i+2}x_i$, and similarly, $x_{i\downarrow\Uparrow j}$ will mean $x_ix_{i-1}x_{i+1}x_ix_{i+2}x_{i+1}x_{i+3}\cdots x_{j-1}x_{j-2}x_j$.

We will also use constructions such as $x_{3\uparrow 6}x_{4\uparrow\Downarrow 1}$, which will mean $x_3x_4x_5x_6x_4x_5x_3x_4x_2x_3x_1$. Occasionally, it will be convenient to include in a set of monomials of the form, say, $x_{j\downarrow i}x_{i-2}$ the case $j = i-1$; this monomial will then simply be x_{i-2}. So in this case $x_{j\downarrow i}$ cannot be seen as a separate monomial.

We extend the notation to cover the case where we have a sequence $i_1, \ldots, i_k$ of indices: then we write $x_{i_{k\downarrow 1}}$ for $x_{i_k}x_{i_{k-1}}\cdots x_{i_2}x_{i_1}$.

We say that a set of Lie algebra elements $\{x_i \mid i \in V\}$ *realizes* a given graph $\Gamma = (V, E)$ if:

- each x_i is an extremal element of $\langle x_j \mid j \in I\rangle_{\mathrm{Lie}}$;
- vertices i and j are connected if and only if x_i and x_j do not commute.

We will sometimes also say that the Lie algebra $\langle x_i\rangle_{\mathrm{Lie}}$ realizes Γ. Later in this paper it will be essential that each x_i is nonzero, which is implied by it being extremal.

1.3. Related results

In [5] Lie algebras realized by simply laced affine Dynkin diagrams were considered. There it was shown that in the generic case the Lie algebra is of the corresponding finite type. The diagram for A_n given there can be transformed into Fig. 1.1 using a procedure similar to that described in Lemma 7.2. The diagram for D_n given in [5] is related in a less straightforward manner, since Fig. 1.4 has n vertices whereas its affine Dynkin diagram has $n+1$ vertices.

Although the generators arising from our graphs and some of the Chevalley generators [2] are similar in the sense that they both correspond to long root elements, no direct relation is apparent.

2. Preliminaries

In this section, we will introduce a bilinear form defined on all Lie algebras generated by extremal elements, and recall some of its properties. None of these results are new; most can be found in e.g. [4] and thus we will omit most of the proofs. We will start by introducing a related family of linear functionals.

For extremal x, let $f_x : \mathcal{L} \to \mathbb{F}$ be the linear map defined by $xxy = f_x(y)x$. Since $[\cdot,\cdot]$ is bilinear, this is indeed a linear map. We call f_x the *extremal functional* on x.

Lemma 2.1. *$f_x(y) = f_y(x)$ for all extremal $x, y \in \mathcal{L}$.*

Lemma 2.2. *Let $\mathcal{L} = \langle x, y\rangle_{\mathrm{Lie}}$ with x and y extremal and linearly independent. Then $\mathcal{L}$ is isomorphic to the two-dimensional commutative Lie algebra, the Heisenberg algebra, or $\mathfrak{sl}_2$.*

Lemma 2.3. *If $\mathcal{L}$ is generated by extremal elements, then it is linearly spanned by extremal elements.*

Lemma 2.4. *If $\mathcal{L}$ is generated by extremal elements, the definition of $f_x(y)$ can be extended to a unique bilinear form $f(x, y)$ on $\mathcal{L}$ with $f(x, y) = f_x(y)$ if x is an extremal element. This bilinear form is associative and symmetric:*

$$\forall x, y, z: \quad f(x, yz) = f(xy, z), \tag{AS}$$

$$\forall x, y: \quad f(x, y) = f(y, x). \tag{SM}$$

We call f the *extremal form*. We will use the following identities involving the extremal form, the first two of which go back to Premet and were first used in [3]:

Lemma 2.5. *If $x, y, z \in \mathcal{L}$ and x extremal, then*

$$2(xy)xz = f(x, yz)x + f(x, z)xy - f(x, y)xz, \tag{P1}$$

$$2xyxz = f(x, yz)x - f(x, z)xy - f(x, y)xz, \tag{P2}$$

$$f(x, yxz) = -f(x, z)f(x, y). \tag{P3}$$

Proof. By the Jacobi identity,

$$(xy)xz \overset{(\mathrm{J})}{=} \big((xy)x\big)z + x(xy)z = -f(x, y)xz + x(xy)z,$$

and similarly,

$$(xy)xz \overset{(\mathrm{AC})}{=} -(xz)xy \overset{(\mathrm{J})}{=} -\big((xz)x\big)y - x(xz)y \overset{(\mathrm{J})}{=} f(x, z)xy - xxzy - x(xy)z.$$

Adding these two equations and applying anti-commutativity a few times, we obtain Eq. (P1). Then we find Eq. (P2) as follows:

$$2xyxz \overset{(\mathrm{J})}{=} 2(xy)xz + 2yxxz \overset{(P1)}{=} f(x, yz)x + f(x, z)xy - f(x, y)xz - 2f(x, z)xy.$$

For Eq. (P3), we need the next lemma. The equation is then easily obtained as follows:

$$f(x, yxz) \overset{(\mathrm{AS})}{=} f(xy, xz) \overset{(\mathrm{AC})}{=} -f(yx, xz) \overset{(\mathrm{AS})}{=} -f(y, xxz) = -f(x, y)f(x, z). \qquad \square$$

3. The general framework

In this section, we will establish a framework for dealing with Lie algebras generated by extremal elements where we prescribe the values of the extremal form. In the end, we prove Theorem 3.5 which shows that a certain parameter space for these prescribed values is an algebraic variety. The main objective of this section is to introduce the techniques for proving that theorem. In Section 5 we will use those techniques to prove similar theorems for a substantially smaller parameter space, but then for specific Lie algebra families.

Let $n \in \mathbb{N}_+$ be fixed. Let Γ be a graph on n numbered vertices. Let $\mathcal{F}$ be the free Lie algebra over $\mathbb{F}$ on n generators $x_1, \ldots, x_n$ with the standard grading. We will construct a quotient of $\mathcal{F}$ where the projections of x_i in the quotient are extremal generators. Let

$$\mathcal{F}_\Gamma = \mathcal{F}/\big\langle x_i x_j \bigm| \{i, j\} \notin E(\Gamma)\big\rangle_{\mathrm{Idl}}.$$

$\mathcal{F}_\Gamma$ inherits the grading of $\mathcal{F}$; this is possible because the ideal that is divided out is *homogeneous* with respect to the grading of $\mathcal{F}$, in the sense that it is spanned by its intersections with the homogeneous components of $\mathcal{F}$.

Let $\mathfrak{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ be an element of $\mathcal{V}(\Gamma) := (\mathcal{F}_\Gamma{}^*)^n$, so it consists of n functionals in the dual of $\mathcal{F}_\Gamma$; we will make sure that $\mathfrak{f}_i$ is the extremal functional f_{x_i} in the Lie algebra we will construct. To that end, define the ideal

$$I_{\Gamma,\mathfrak{f}} = \big\langle x_i x_i y - \mathfrak{f}_i(y) x_i \bigm| y \in \mathcal{F}_\Gamma, 1 \leqslant i \leqslant n\big\rangle_{\mathrm{Idl}}.$$

When taking $\mathfrak{f} = 0$, we see that $I_{\Gamma,0}$ is homogeneous with respect to the standard grading of $\mathcal{F}_\Gamma$. Let $\mathcal{L}(\Gamma, \mathfrak{f}) = \mathcal{F}_\Gamma / I_{\Gamma,\mathfrak{f}}$ and let $\xi_\mathfrak{f} : \mathcal{F}_\Gamma \to \mathcal{L}(\Gamma, \mathfrak{f})$ be the natural projection. We will sometimes omit $\xi_\mathfrak{f}$ if that does not stand in the way of clarity. Clearly each x_i is either an extremal element of $\mathcal{L}(\Gamma, \mathfrak{f})$ or zero, and $\mathfrak{f}_i(y) = f(x_i, y)$. We find the following slight extension of Lemma 4.3 of [4]:

Lemma 3.1. *There is a finite list $\mathcal{M}_\Gamma$ of monomials in $x_1, \ldots, x_n$ satisfying the following properties*:

(1) $\xi_0(\mathcal{M}_\Gamma)$ *is a basis of* $\mathcal{L}(\Gamma, 0)$,
(2) *if* $x_i m \in \mathcal{M}_\Gamma$, *then* $m \in \mathcal{M}_\Gamma$,
(3) $\mathcal{M}_\Gamma$ *contains all generators* x_i, *and*
(4) $\mathcal{L}(\Gamma, \mathfrak{f}) = \langle \xi_\mathfrak{f}(\mathcal{M}_\Gamma)\rangle_\mathbb{F}$ *for all* $\mathfrak{f} \in \mathcal{V}(\Gamma)$.

Proof. $\mathcal{L}(\Gamma, 0)$ is a quotient of $\mathcal{L}_n := \mathcal{L}(K_n, 0)$, where K_n is the complete graph. By Theorem 1 of Zel'manov [16] (or for characteristic 3, by Theorem 1 of Zel'manov and Kostrikin [17]), we know that $\mathcal{L}_n$ is finite-dimensional. Hence we can find a finite set $\mathcal{M}_\Gamma$ of monomials in $\mathcal{F}_\Gamma$ satisfying conditions (1), (2) and (3), by the following procedure. We start by setting $\mathcal{M}_\Gamma$ equal to $\{x_1, \ldots, x_n\}$. This set is linearly independent because the free Abelian Lie algebra on n generators is a quotient of $\mathcal{L}(\Gamma, 0)$, and the images of the x_i in it are linearly independent. Then we perform a number of rounds as follows. In each round we form the monomials $x_i m$, where x_i iterates over the generators of $\mathcal{F}_\Gamma$ and m iterates over the longest monomials in $\mathcal{M}_\Gamma$ so far. We select a subset of these such that its images under ξ_0 in $\mathcal{L}(\Gamma, 0)$ are linearly independent of each other and of the images under ξ_0 of the elements in $\mathcal{M}_\Gamma$ so far, and add it to $\mathcal{M}_\Gamma$. Then we continue with the next round if we have added any new monomials this round. Since $\mathcal{L}(\Gamma, 0)$ is finite-dimensional, this procedure terminates after finitely many steps.

Let $U = \langle \mathcal{M}_\Gamma \rangle_\mathbb{F} \subset \mathcal{F}_\Gamma$. We now prove condition (4) by showing that $\xi_\mathfrak{f}(U) = \mathcal{L}(\Gamma, \mathfrak{f})$ for all $\mathfrak{f} \in \mathcal{V}(\Gamma)$. Note that $I_{\Gamma,0}$ is spanned by elements of the form $x_{i_{k\downarrow 1}} x_{i_1} r$, with r a monomial in $\mathcal{F}_\Gamma$.

Clearly $\xi_\mathfrak{f}(U) \subset \mathcal{L}(\Gamma, \mathfrak{f})$. Suppose that it is a proper subset; then there are monomials $s \in \mathcal{F}_\Gamma$ such that $\xi_\mathfrak{f}(s) \notin \xi_\mathfrak{f}(U)$, whence $s \notin U$. Let s be such a monomial of lowest degree. Since $\mathcal{F}_\Gamma = U + I_{\Gamma,0}$, it is possible to express s as a linear combination of monomials in $\mathcal{M}_\Gamma$ and monomials of the form

$x_{i_{k\downarrow 1}}x_{i_1}r$. All these monomials have the same degree, because only the Jacobi identity and anticommutativity can be used for rewriting, in addition to homogeneous elements being 0. Let t be a monomial of the form $x_{i_{k\downarrow 1}}x_{i_1}r$ such that $\xi_{\mathfrak{f}}(t) \notin \xi_{\mathfrak{f}}(U)$ and let $t_0 = x_{i_{k\downarrow 1}}$. Then

$$\mathfrak{f}_{i_1}(r)\xi_{\mathfrak{f}}(t_0) = \xi_{\mathfrak{f}}(t) \notin \xi_{\mathfrak{f}}(U),$$

so $\mathfrak{f}_{i_1}(r) \neq 0$ and $t_0 \notin U$. Since $\deg t_0 < \deg t = \deg s$, we have a contradiction. Hence $\xi_{\mathfrak{f}}(U) = \mathcal{L}(\Gamma, \mathfrak{f})$. □

Define $U = \langle \mathcal{M}_\Gamma \rangle_{\mathbb{F}}$ as in the preceding proof. Note that $\mathcal{F}_\Gamma = U + I_{\Gamma,\mathfrak{f}}$ for all $\mathfrak{f}$, not just for $\mathfrak{f} = 0$. Define I and m_i by letting $\mathcal{M}_\Gamma = \{m_i \mid i \in I\}$.

Lemma 3.2. *For every monomial $m \in \mathcal{F}_\Gamma$, there exists a map $n_m : \mathcal{V}(\Gamma) \to U$, such that $n_m(\mathfrak{f}) = m \pmod{I_{\Gamma,\mathfrak{f}}}$ for all $\mathfrak{f} \in \mathcal{V}(\Gamma)$ and the following property holds. If $n_m(\mathfrak{f}) = \sum_{i\in I} \alpha_{m,i,\mathfrak{f}} m_i$, then $\alpha_{m,i,\mathfrak{f}}$, when regarded as a function in i and $\mathfrak{f}$, is a polynomial function in the values of $\mathfrak{f}$ at monomials of degree less than $\deg m$.*

Proof. Let $m = x_{i_{k\downarrow 1}}$ be a monomial in $\mathcal{F}_\Gamma$ of degree k. If $k = 1$, we put $n_m(\mathfrak{f}) = m$. We proceed by induction on $\deg m$. Since $\mathcal{F}_\Gamma = U + I_{\Gamma,0}$, we can write m as the sum of an element u of U and an element w of $I_{\Gamma,0}$; all monomials involved have the same degree, because only the Jacobi identity and anticommutativity can be used for rewriting, in addition to homogeneous elements being 0. If we prove that $n_w(\mathfrak{f}) = w \pmod{I_{\Gamma,\mathfrak{f}}}$ and that its coefficients $\alpha_{m,i,\mathfrak{f}}$ satisfy the polynomiality condition, then setting

$$n_m(\mathfrak{f}) = u + n_w(\mathfrak{f}) \tag{3.1}$$

will be sufficient to show that the lemma holds for m.

We may assume that w is a single monomial. So the proof obligation reduces to the case where $m \in I_{\Gamma,0}$. Then there exist $r, h \in \mathbb{N}_+$ such that $i_r = i_{r-1} = h$ and m is thus of the form $x_{i_{k\downarrow r+1}}x_h x_h x_{i_{r-2\downarrow 1}}$. Hence, by the induction hypothesis,

$$m = \mathfrak{f}_h(x_{i_{r-2\downarrow 1}})x_{i_{k\downarrow r}} = \mathfrak{f}_h(x_{i_{r-2\downarrow 1}})n_{x_{i_{k\downarrow r}}}(\mathfrak{f}) \pmod{I_{\Gamma,\mathfrak{f}}}.$$

We choose

$$n_m(\mathfrak{f}) = \mathfrak{f}_h(x_{i_{r-2\downarrow 1}})n_{x_{i_{k\downarrow r}}}(\mathfrak{f}), \tag{3.2}$$

so that

$$\alpha_{m,j,\mathfrak{f}} = \mathfrak{f}_h(x_{i_{r-2\downarrow 1}})\alpha_{x_{i_{k\downarrow r}},j,\mathfrak{f}}.$$

The coefficients $\alpha_{x_{i_{k\downarrow r}},j,\mathfrak{f}}$ are, by the induction hypothesis, polynomials in the values of $\mathfrak{f}$ at monomials of degree less than $k - r + 1 < k = \deg m$, so equations (3.1) and (3.2) define a map satisfying the conditions in the lemma. □

Note that we do not claim that n_m is uniquely determined by these conditions. We choose a map n_* as above and extend it to general elements of $\mathcal{F}_\Gamma$ by linearity.

Let $X(\Gamma) = \{\mathfrak{f} \mid \dim \mathcal{L}(\Gamma, \mathfrak{f}) = |\mathcal{M}_\Gamma|\}$ and let $R : X(\Gamma) \to (U^*)^n$ be the map that restricts a functional to U.

Lemma 3.3. *The restriction map R is injective.*

Proof. Let $\mathfrak{f} \in X(\Gamma)$. Then all $m_i \in \mathcal{M}_\Gamma$ are linearly independent in $\mathcal{L}(\Gamma, \mathfrak{f})$, so $x_i \notin I_{\Gamma,\mathfrak{f}}$. Let m be a monomial in $\mathcal{F}_\Gamma$. We will show that $\mathfrak{f}_i(m)$ can be expressed in the values of $\mathfrak{f}_i$ on monomials in M. If $m \in M$, there is nothing to prove, so assume $m \notin M$. Since

$$x_i x_i m = \mathfrak{f}_i(m) x_i \pmod{I_{\Gamma,\mathfrak{f}}},$$

and also

$$x_i x_i m = x_i x_i n_m(\mathfrak{f}) = \mathfrak{f}_i\big(n_m(\mathfrak{f})\big) x_i \pmod{I_{\Gamma,\mathfrak{f}}},$$

we find that $\mathfrak{f}_i(m) = \mathfrak{f}_i(n_m(\mathfrak{f}))$. Since $n_m(\mathfrak{f})$ only depends on monomials of lower degree than m, we see that $\mathfrak{f}_i(m)$ can be expressed in the values of $\mathfrak{f}_i$ at monomials of lower degree than m. By induction on the degree of m, it can therefore be expressed ultimately in the values of $\mathfrak{f}_i$ on M, as we set out to prove.

Let $\mathfrak{f}, \mathfrak{f}' \in X(\Gamma)$ with $R(\mathfrak{f}) = R(\mathfrak{f}')$. Then $\mathfrak{f}_i$ and $\mathfrak{f}'_i$ agree on U, and thus on $\mathcal{F}_\Gamma$, for all i. Hence $\mathfrak{f} = \mathfrak{f}'$. □

Lemma 3.4. *$R(X(\Gamma))$ is a closed subset of $(U^*)^n$.*

Proof. For all $\mathfrak{f} \in \mathcal{V}(\Gamma)$, let the bilinear anticommutative map $[\cdot,\cdot]_\mathfrak{f} : U \times U \to U$ be determined by

$$[v, w]_\mathfrak{f} = n_{[v,w]}(\mathfrak{f}).$$

If $\mathfrak{f} \in X(\Gamma)$, then

(1) $[\cdot,\cdot]_\mathfrak{f}$ is a Lie multiplication (i.e. it satisfies the Jacobi identity),
(2) $[x_i, [x_i, v]_\mathfrak{f}]_\mathfrak{f} = \mathfrak{f}_i(v) x_i$ for all $v \in U$ and all i,
(3) $[x_i, x_j]_\mathfrak{f} = 0$ if nodes i and j are not connected by a line,
(4) the Lie algebra $(U, [\cdot,\cdot]_\mathfrak{f})$ is generated by $x_1, \ldots, x_n$.

On the other hand, if all of the above conditions hold for a multiplication map μ, then (U, μ) is a quotient of $\mathcal{L}(\Gamma, \mathfrak{f})$ of the same dimension, and hence isomorphic to $\mathcal{L}(\Gamma, \mathfrak{f})$. But these conditions are all polynomial in the values of $\mathfrak{f}$ on U: the Jacobi identity and conditions (2) and (3) are polynomial in a straightforward way, and condition (4) is always satisfied: for every $w = x_{i_{k\downarrow 1}} \in \mathcal{M}_\Gamma$, we have $w = [x_{i_k}, [x_{i_{k-1}}, \ldots, [x_{i_2}, x_{i_1}]_\mathfrak{f} \ldots]_\mathfrak{f}]_\mathfrak{f}$, since $n_w(\mathfrak{f}) = w$. So $R(X(\Gamma))$ is given as the zero set of a set of polynomial equations; thus it is closed. □

Theorem 3.5. *$X(\Gamma)$ carries a natural structure of an affine variety.*

Proof. The restriction map R is a continuous bijection of $X(\Gamma)$ with a Zariski closed subset of $(U^*)^n$. Clearly the restriction map R is continuous. The preceding two lemmas show that it is injective and that its image is closed. □

4. The monomials

In Figs. 1.1 up to 1.4 we defined four graphs, $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$ and $\Gamma_{D;n}$, to be used for Γ in the framework of Section 3. In this section, we will construct the basis $\mathcal{M}_\Gamma$ of U explicitly for each such Γ, resulting in Theorems 4.2–4.5.

Clearly, each of the algebras $\mathcal{F}_\Gamma$ is defined by a subset of the four following relations.

$$x_i x_j = 0 \quad \text{for all } i, j \text{ with } |i-j| > 1,\ \{1,3\} \neq \{i,j\} \neq \{n-2,n\}, \tag{R1}$$

$$x_1 x_3 = 0, \tag{R2}$$

$$x_{n-2} x_n = 0, \tag{R3}$$

$$x_{n-1} x_n = 0. \tag{R4}$$

We will use the following technical lemma:

Lemma 4.1. *Let $a, b, n \in \mathbb{N}_+$, $i, j, k, \ell, m, i_1, \ldots, i_a, j_1, \ldots, j_b \in \{1, \ldots, n\}$. Let Γ be one of the graphs $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$ and $\Gamma_{D;n}$, let $\mathfrak{f} \in \mathcal{V}(\Gamma)$, let x_i be the standard generators of $\mathcal{L}(\Gamma, \mathfrak{f})$, and let $t, u \in \mathcal{L}(\Gamma, \mathfrak{f})$. Furthermore, let x_{i_p} commute with x_{j_q} for all p and q and let x_i commute with x_j. For Eq.* (Q2) *only, assume that $i < n-2$. Then*:

$$x_j x_i t = x_i x_j t, \tag{Q1}$$

$$x_i x_{i+1} x_{i+2} x_i t = \frac{1}{2}\big(f(x_i, x_{i+1}x_{i+2}t)x_i - f(x_i, x_{i+2}t)x_i x_{i+1} - f(x_i, x_{i+1})x_i x_{i+2} t\big), \tag{Q2}$$

$$\begin{aligned} x_k x_\ell x_m x_k t = \frac{1}{2}\big(&f(x_k, x_m t)x_\ell x_k + f(x_k, t)x_\ell x_k x_m - f(x_k, x_m)x_\ell x_k t + f(x_k, x_\ell x_m t)x_k \\ &- f(x_k, x_m t)x_k x_\ell - f(x_k, x_\ell)x_k x_m t - f(x_k, x_\ell x_m)x_k t + f(x_k, x_m)x_k x_\ell t \\ &+ f(x_k, x_\ell)x_k x_m t - f(x_k, x_m x_\ell t)x_k + f(x_k, x_\ell t)x_k x_m + f(x_k, x_m)x_k x_\ell t \\ &+ f(x_k, x_\ell t)x_m x_k - f(x_k, t)x_m x_k x_\ell - f(x_k, x_\ell)x_m x_k t\big) + x_k x_m x_\ell x_k t, \end{aligned} \tag{Q3}$$

$$\begin{aligned} f(u, x_k x_\ell x_m x_k t) = \frac{1}{2}\big(&f(x_k, x_m t)f(u, x_\ell x_k) + f(x_k, t)f(u, x_\ell x_k x_m) - f(x_k, x_m)f(u, x_\ell x_k t) \\ &+ f(x_k, x_\ell x_m t)f(u, x_k) - f(x_k, x_m t)f(u, x_k x_\ell) - f(x_k, x_\ell)f(u, x_k x_m t) \\ &- f(x_k, x_\ell x_m)f(u, x_k t) + f(x_k, x_m)f(u, x_k x_\ell t) + f(x_k, x_\ell)f(u, x_k x_m t) \\ &- f(x_k, x_m x_\ell t)f(u, x_k) + f(x_k, x_\ell t)f(u, x_k x_m) + f(x_k, x_m)f(u, x_k x_\ell t) \\ &+ f(x_k, x_\ell t)f(u, x_m x_k) - f(x_k, t)f(u, x_m x_k x_\ell) - f(x_k, x_\ell)f(u, x_m x_k t)\big) \\ &+ f(u, x_k x_m x_\ell x_k t), \end{aligned} \tag{Q4}$$

$$x_{i_1} x_{j_1} x_{j_2} \cdots x_{j_b} = 0. \tag{Q5}$$

The proof of this lemma is straightforward, using the identities from Lemma 2.5 and the Jacobi identity.

4.1. Monomials for $\Gamma_{D;n}$

In the rest of this section, we will let $\mathfrak{f} \in \mathcal{V}(\Gamma_{D;n})$ be given and consider $\mathcal{L}(\Gamma_{D;n}, \mathfrak{f})$, that is, only the relations in (R1) from page 302 are divided out. In Theorem 4.2, we will give a list of $2n^2 - n$ monomials in $x_1, \ldots, x_n$ and prove that they span $\mathcal{L}(\Gamma_{D;n}, \mathfrak{f})$ linearly.

Note that the precise contents of this list is of little general importance. For example, the number of classes may be reduced using a more clever notation. However, one needs to fix such a list for the proof of Theorem 7.4, and the one presented below suffices.

Theorem 4.2. *Let $\mathcal{M}_{\Gamma_{D;n}}$ be the set consisting of the following monomials:*

$$\begin{aligned}
y^1_{k,m} &= x_{k\downarrow m}, \quad n \geqslant k \geqslant m \geqslant 1,\\
y^2_{k,m} &= x_{k\uparrow n-2} x_{n\downarrow m}, \quad n-2 \geqslant k > m \geqslant 1,\\
y^3_{k,m} &= x_{k\downarrow m+1} x_{m-1\Uparrow\!\Downarrow 1}, \quad n \geqslant k \geqslant m \geqslant 3,\\
y^4_{k,m} &= x_{k\uparrow n-2} x_{n\downarrow m+1} x_{m-1\Uparrow\!\Downarrow 1}, \quad n-2 \geqslant k \geqslant m \geqslant 3,\\
y^5_m &= x_n x_{n-2\downarrow m}, \quad n-2 \geqslant m \geqslant 1,\\
y^6_m &= x_{n-1} x_n x_{n-2\downarrow m}, \quad n-2 \geqslant m \geqslant 1,\\
y^7_k &= x_{k\downarrow 3} x_1, \quad n \geqslant k \geqslant 3,\\
y^8_k &= x_{k\uparrow n-2} x_{n\downarrow 3} x_1, \quad n-2 \geqslant k \geqslant 2,\\
y^9_m &= x_n x_{n-2\downarrow m+1} x_{m-1\Uparrow\!\Downarrow 1}, \quad n-2 \geqslant m \geqslant 3,\\
y^{10}_m &= x_{n-1} x_n x_{n-2\downarrow m+1} x_{m-1\Uparrow\!\Downarrow 1}, \quad n-2 \geqslant m \geqslant 3,\\
y^{11} &= x_1 x_{3\uparrow n-2} x_{n\downarrow 1},\\
y^{12} &= x_1 x_{3\uparrow n-2} x_{n\downarrow 2},\\
y^{13} &= x_n x_{n-2\downarrow 3} x_1,\\
y^{14} &= x_{n-1} x_n x_{n-2\downarrow 3} x_1,\\
y^{15} &= x_{n-2} x_n x_{n-3\Uparrow\!\Downarrow 1},\\
y^{16} &= x_{n-1} x_{n-2} x_n x_{n-3\Uparrow\!\Downarrow 1},\\
y^{17} &= x_n x_{n-1} x_{n-2} x_n x_{n-3\Uparrow\!\Downarrow 1}.
\end{aligned}$$

Let x be a monomial in $x_1, \ldots, x_n$ of length ℓ. Then x is a linear combination of monomials $y \in \mathcal{M}_{\Gamma_{D;n}}$ with $\operatorname{length}(y) \leqslant \ell$.

For convenience in the proof of Theorem 4.2, we extend the above definition: we define $y^2_{n-1,m} = y^1_{n,m}$ and $y^4_{n-1,m} = y^3_{n,m}$, for $m \leqslant n-1$; and $y^3_{2,2} = x_1$, $y^7_2 = x_1$, and $y^8_{n-1} = y^7_n$.

Proof. Let x be a monomial in $x_1, \ldots, x_n$ of length ℓ. If $\ell = 1$ then $x \in \mathcal{M}_{\Gamma_{D;n}}$. If $\ell = 2$, then either $x = 0$ or $x \in \mathcal{M}_{\Gamma_{D;n}}$ or $-x \in \mathcal{M}_{\Gamma_{D;n}}$ (using Eqs. (AC) and (R1)). We use induction on ℓ and may assume $\ell > 2$.

We have an $i \in \{1, \ldots, n\}$ and a monomial y of length $\ell - 1$ such that $x = x_i y$. Moreover, because of the induction hypothesis we can assume $y \in \mathcal{M}_{\Gamma_{D;n}}$. We will consider $x = x_i y^j_*$ for each of the seventeen classes in $\mathcal{M}_{\Gamma_{D;n}}$ separately. In each case we will write x as a linear combination of monomials of length at most ℓ, where all monomials in the linear combination of length ℓ are members of $\mathcal{M}_{\Gamma_{D;n}}$. By the induction hypothesis, this suffices to prove the theorem.

We will work modulo monomials of length at most $\ell - 1$, so because of extremality of x_j,

$$x_j x_j t = 0 \tag{XT}$$

whenever the left-hand side occurs in a monomial of length ℓ.

We show the cases $j = 1$ and $j = 3$ as an example, the other cases are very similar. For the complete proof, we refer to [11].

Case 1: $j=1$, $k \in \{1,\dots,n\}$ and $m \in \{1,\dots,k\}$. Since $\ell > 2$, we know that $m < k$. We distinguish the following sub-cases:

- If $i > k+1$ and $k = n-2$, then $x = y^5_m$.
- If $i > k+1$ and $k \neq n-2$, then $x = 0$ by Eq. (Q5).
- If $i = k+1$, then $x = y^1_{k+1,m}$.
- If $i = k$, then extremality of x_i shows that $x = 0$.
- If $i = k-1$, then:

$$x = \begin{cases} x_i x_{i+1} x_i \overset{\text{(AC)}}{=} -x_i x_i x_{i+1} \overset{\text{(XT)}}{=} 0, & \text{if } m = k-1, \\ x_i x_{i+1} x_i y^1_{i-1,m} \overset{\text{(P2)}}{=} 0, & \text{otherwise.} \end{cases}$$

- If $i < k-1$ and $i = n-2$, then $x = y^2_{n-2,m}$.
- If $i < k-1$ and $i \neq n-2$, then $x = x_i x_{k\downarrow m}$. Applying Eq. (Q1) a sufficient number of times leads to one of the following situations:
  - If $i > m$, then $x = x_{k\downarrow i+2} x_i x_{i+1} x_{i\downarrow m} \overset{\text{(P2)}}{=} 0$.
  - If $i = m$ and $i \neq 1$, then $x = x_{k\downarrow m+2} x_m x_{m+1} x_m \overset{\text{(AC), (XT)}}{=} 0$.
  - If $i = m$ and $i = 1$, then $x = x_{k\downarrow 4} x_1 x_3 x_2 x_1 \overset{\text{(AC)}}{=} -x_{k\downarrow 4} x_1 x_3 x_1 x_2 \overset{\text{(P2)}}{=} 0$.
  - If $i = m-1$ and $i \neq 1$, then $x = x_{k\downarrow m+1} x_{m-1} x_m \overset{\text{(AC)}}{=} -y^1_{k,m-1}$.
  - If $i = m-1$ and $i = 1$, then $x = x_{k\downarrow 4} x_1 x_3 x_2 \overset{\text{(J)}}{=} y^3_{k,3} - y^1_{k,1}$.
  - If $i < m-1$ and $m \neq 3$, then $x = x_{k\downarrow m+1} x_i x_m \overset{\text{(R1)}}{=} 0$.
  - If $i < m-1$ and $m = 3$, then $x = x_{k\downarrow 4} x_1 x_3 \overset{\text{(AC)}}{=} -y^7_k$.

Case 2: $j=3$, $k \in \{3,\dots,n\}$ and $m \in \{3,\dots,k\}$.

- If $i > k+1$ and $k = n-2$, then $x = y^9_m$.
- If $i > k+1$ and $k \neq n-2$, then $x = x_i x_{k\downarrow m+1} x_{m-1\uparrow\Downarrow 1} \overset{\text{(Q5)}}{=} 0$.
- If $i = k+1$, then $x = y^3_{i,m}$.
- If $i = k$, then

$$x = x_i y^3_{i,m} = \begin{cases} x_i x_{i-1} x_i y^3_{i-1,i-1} \overset{\text{(P2)}}{=} 0, & \text{if } k = m, \\ x_i x_i y^3_{i-1,m} \overset{\text{(XT)}}{=} 0, & \text{otherwise.} \end{cases}$$

- If $i = k-1$, then we have

$$x = x_i y^3_{i+1,m} = \begin{cases} x_i x_i y^3_{i+1,i} \overset{\text{(XT)}}{=} 0, & \text{if } i+1 = m, \\ x_i x_{i+1} y^3_{i,i} = y^3_{i+1,i+1}, & \text{if } i+1 = m+1, \\ x_i x_{i+1} x_i y^3_{i-1,m} \overset{\text{(P2)}}{=} 0, & \text{otherwise.} \end{cases}$$

- If $i < k-1$ and $i = n-2$, then $x = y^4_{n-2,m}$.
- If $i < k-1$ and $i \neq n-2$, then $x = x_i x_{k\downarrow m+1} x_{m-1\uparrow\Downarrow 1}$. Repeated application of Eq. (Q1) leads to one of the following situations:
  - If $i > m$, then $x = x_{k\downarrow i+2} x_i x_{i+1} x_i y^3_{i-1,m} \overset{\text{(P2)}}{=} 0$.
  - If $i = m$, then $x = x_{k\downarrow m+2} x_m x_{m+1} x_{m-1\uparrow\Downarrow 1} = y^3_{k,m+1}$.
  - If $i = m-1$, then $x = x_{k\downarrow m+1} x_{m-1} x_{m-1} y^3_{m,m-1} \overset{\text{(XT)}}{=} 0$.

– If $i < m-1$, $m \neq 3$ and $i \neq 1$, then $x = x_{k\downarrow m+1} x_{m-1\uparrow\Downarrow i+2} x_{i+3} x_i x_{i+1} x_{i+2} x_i y^3_{i+1,i} \overset{(Q2)}{=} 0$.
– If $i < m-1$, $m \neq 3$ and $i = 1$, then $x = x_{k\downarrow m+1} x_{m-1\uparrow\Downarrow 4} x_5 x_1 x_3 x_4 x_2 x_3 x_1 = 0$. This last identity follows from:

$$x_1x_3x_4x_2x_3x_1 \overset{(J)}{=} x_1x_3x_4x_1x_3x_2 + x_1x_3x_4x_3x_2x_1 \overset{(Q3),(P2)}{=} x_1x_4x_3x_1x_3x_2 \overset{(P2)}{=} 0.$$

– If $i < m-1$ and $m = 3$, then $x = x_{k\downarrow 4} x_1 x_2 x_3 x_1 \overset{(AC)}{=} -x_{k\downarrow 4} x_1 x_2 x_1 x_3 \overset{(P2)}{=} 0$. □

4.2. *Monomials for $\Gamma_{B;n}$, $\Gamma_{A;n}$ and $\Gamma_{C;n}$*

Recall that in $\mathcal{F}_{\Gamma_{B;n}}$, we divide out relations (R1) and (R3) from page 302. In Theorem 4.3, we will give a list of $2n^2 - 3n + 1$ monomials in $x_1, \ldots, x_n$ and show that these *monomials* span $\mathcal{F}_{\Gamma_{B;n}}$ linearly. We will prove this using Theorem 4.2.

Theorem 4.3. *Let $\mathcal{M}_{\Gamma_{B;n}}$ be the set $\mathcal{M}_{\Gamma_{D;n}}$ without the following monomials*:

(1) $y^1_{n,n-1} = x_n x_{n-1}$,
(2) $y^3_{n,n} = x_{n-1\uparrow\Downarrow 1}$,
(3) $y^6_m = x_{n-1} x_n x_{n-2\downarrow m}$ *with* $n-2 \geqslant m \geqslant 1$,
(4) $y^{10}_m = x_{n-1} x_n x_{n-2\downarrow m+1} x_{m-1\uparrow\Downarrow 1}$ *with* $n-2 \geqslant m \geqslant 3$,
(5) $y^{12} = x_1 x_{3\uparrow n-2} x_{n\downarrow 3} x_1$,
(6) $y^{14} = x_{n-1} x_n x_{n-2\downarrow 3} x_1$,
(7) $y^{17} = x_n x_{n-1} x_{n-2} x_n x_{n-3\uparrow\Downarrow 1}$.

Let x be a monomial in $x_1, \ldots, x_n$ of length ℓ. Then x is a linear combination of monomials $y \in \mathcal{M}_{\Gamma_{B;n}}$ with $\text{length}(y) \leqslant \ell$.

Proof. Because of Theorem 4.2 it suffices to prove that the seven given classes of monomials can be written as a linear combination of shorter monomials and monomials from $\mathcal{M}_{\Gamma_{B;n}}$. We will consider these classes separately and work modulo shorter monomials.

Case 1: $y = y^1_{n,n-1} = x_n x_{n-1} \overset{(R3)}{=} 0$.

Case 2: $y = y^3_{n,n}$. We find $y = x_{n-1} x_n x_{n-2\uparrow\Downarrow 1} \overset{(Q1),(R3)}{=} x_n x_{n-1} x_{n-2} x_{n-1} x_{n-3\uparrow\Downarrow 1} \overset{(P2)}{=} 0$.

Case 3: $y = y^6_m$. We find $y = x_{n-1} x_n x_{n-2\downarrow m} \overset{(J)}{=} x_n x_{n-1} y^1_{n-2,m} + (x_{n-1} x_n) y^1_{n-2,m} \overset{(R3)}{=} y^1_{n,m}$.

Case 4: $y = y^{10}_m$. We find $y = x_{n-1} x_n x_{n-2\downarrow m+1} x_{m-1\uparrow\Downarrow 1} \overset{(J)}{=} x_n x_{n-1} y^3_{n-2,m} + (x_{n-1} x_n) y^3_{n-2,m} \overset{(R3)}{=} y^3_{n,m}$.

Case 5: $y = y^{12}$. y^{12} can be written as a linear combination of the following monomials:

$$y^2_{3,1},\ y^3_{n,n-1},\ y^4_{3,3},\ y^4_{4,4},\ \ldots,\ y^4_{n-2,n-2},\ y^8_2,\ y^{12},\ y^{16} \quad \text{and} \quad x_{n-2} x_{n-1} x_n x_{n-3\uparrow\Downarrow 1} =: t.$$

All of these except t are in $\mathcal{M}_{\Gamma_{B;n}}$, so we only need to analyze t.

$$t = x_{n-2} x_{n-1} x_n x_{n-3\uparrow\Downarrow 1} \overset{(Q1)}{=} x_{n-2} x_n x_{n-1} x_{n-3\uparrow\Downarrow 1} = y^4_{n-2,n-2}.$$

Case 6: $y = y^{14}$. We find $y = x_{n-1}x_nx_{n-2\downarrow 3}x_1 \overset{(Q1)}{=} x_{n\downarrow 3}x_1 = y_n^7$.

Case 7: $y = y^{17}$. We find $y = x_nx_{n-1}x_{n-2}x_nx_{n-3\uparrow\Downarrow 1} \overset{(Q1)}{=} x_{n-1}x_nx_{n-2}x_nx_{n-3\uparrow\Downarrow 1} \overset{(P2)}{=} 0$. □

The following two theorems can be proved in exactly the same manner.

Theorem 4.4. *Let $\mathcal{M}_{\Gamma_{A;n}}$ be the set consisting of the following monomials:*

$$y_{k,m}^1 = x_{k\downarrow m}, \quad n \geqslant k \geqslant m \geqslant 1,$$
$$y_{k,m}^3 = x_{k\downarrow m+1}x_{m-1\uparrow\Downarrow 1}, \quad n \geqslant k \geqslant m \geqslant 3,$$
$$y_k^7 = x_{k\downarrow 3}x_1, \quad n \geqslant k \geqslant 3.$$

Let x be a monomial in $x_1, \dots, x_n$ of length ℓ. Then x is a linear combination of monomials $y \in \mathcal{M}_{\Gamma_{A;n}}$ with $\text{length}(y) \leqslant \ell$.

Theorem 4.5. *Let*

$$\mathcal{M}_{\Gamma_{C;n}} = \{y_{k,m}^1 = x_{k\downarrow m} \mid n \geqslant k \geqslant m \geqslant 1\}.$$

Let x be a monomial in $x_1, \dots, x_n$ of length ℓ. Then x is a linear combination of monomials $y \in \mathcal{M}_{\Gamma_{C;n}}$ with $\text{length}(y) \leqslant \ell$.

5. The parameter space

Recall that $X(\Gamma) = \{\mathfrak{f} \mid \dim \mathcal{L}(\Gamma, \mathfrak{f}) = |\mathcal{M}_\Gamma|\}$. For $\Gamma \in \{\Gamma_{A;n}, \Gamma_{B;n}, \Gamma_{C;n}, \Gamma_{D;n}\}$, we will find bijections ψ_Γ from $X(\Gamma)$ to a vector space, such that $\mathcal{L}(\Gamma, \mathfrak{f})$ is isomorphic to the Lie algebra of the corresponding Chevalley type if $\psi_\Gamma(\mathfrak{f})$ is in a certain open dense subset of that vector space. In this section, we will be constructing the bijection. It will take some work to show that the map is injective; this will be the content of Lemmas 5.1–5.4. We will find the open dense subset in Section 7.

5.1. Parameters for $\Gamma_{D;n}$

Lemma 5.1. *Let*

$$\psi_{\Gamma_{D;n}} : X(\Gamma_{D;n}) \to \mathbb{F}^{n+4},$$
$$\mathfrak{f} \mapsto \big(\mathfrak{f}_1(x_2), \mathfrak{f}_2(x_3), \dots, \mathfrak{f}_{n-1}(x_n);\ \mathfrak{f}_1(x_3), \mathfrak{f}_1(x_2x_3), \mathfrak{f}_{n-2}(x_n), \mathfrak{f}_{n-2}(x_{n-1}x_n), \mathfrak{f}_1(x_{3\uparrow n-2}x_{n\downarrow 2})\big).$$

Then $\psi_{\Gamma_{D;n}}$ is injective.

Proof. The values of all $\mathfrak{f}_i$ together determine the values of the extremal bilinear form on all of $\mathcal{F}_{\Gamma_{D;n}}$, since $f(x_iy, z) = f(x_i, yz) = \mathfrak{f}_i(yz)$. We will show that each value $\mathfrak{f}_i(y)$ for $y \in \mathcal{M}_{\Gamma_{D;n}}$ can be expressed in the values $\mathfrak{f}_j(z)$ in the theorem. To make this notationally convenient, let $\mathbb{F}_{D;n}$ be the rational function field obtained from $\mathbb{F}$ by extending it with $n+4$ symbols as follows. For every $\mathfrak{f}_j(z)$ in the lemma, we extend $\mathbb{F}$ with the symbol $\mathfrak{f}_j(z)$ and assume that evaluating $\mathfrak{f}_j$ at z yields this symbol $\mathfrak{f}_j(z) \in \mathbb{F}_{D;n}$. We will show that each value $\mathfrak{f}_i(y)$ is an element of $\mathbb{F}_{D;n}$ for *all* i and *all* $y \in \mathcal{M}_{\Gamma_{D;n}}$.

Let $y = y^j_* \in \mathcal{M}_{\Gamma_{D;n}}$ of length ℓ and $i \in \{1, \dots, n\}$. We will use induction on ℓ. We consider the seventeen classes of monomials in $\mathcal{M}_{\Gamma_{D;n}}$ separately, and give cases $j = 1$ and $j = 3$ as an example. For the complete proof, we refer to [11].

Let $x = x_i x_i y = f(x_i, y)x_i$. It is sufficient to prove that $x = 0$ or $f(x_i, y) = 0$.

Case 1: $j = 1$, $k \in \{1, \dots, n\}$, $m \in \{1, \dots, k\}$. Note that $y^1_{k,m} = x_{k\downarrow m}$.

If $m = k$ then y has length 1 and $f(x_i, y)$ is either 0 or in the list of values in the theorem. So assume that $k > m$.

- If $i > k + 1$ and $k = n - 2$, then:

$$x \overset{\text{(AC)}}{=} -x_n x_n x_{n-2\downarrow m+2} x_m x_{m+1} \overset{\text{(Q1)}}{=} -x_m x_n x_n x_{n-2\downarrow m+1} \overset{\text{(XT)}}{=} -f\big(x_n, y^1_{n-2,m+1}\big) x_m x_n \overset{\text{(R1)}}{=} 0.$$

- If $i > k + 1$ and $k < n - 2$, then

$$x \overset{\text{(Q1)}}{=} x_k x_i x_i y^1_{k-1,m} \overset{\text{(XT)}}{=} f\big(x_i, y^1_{k-1,m}\big) x_k x_i \overset{\text{(R1)}}{=} 0.$$

- If $i = k + 1$, $m = k - 1$ and $i = n$, then $f(x_i, y) = f(x_n, x_{n-1}x_{n-2}) \in \mathbb{F}_{D;n}$.
- If $i = k + 1$, $m = k - 1$ and $3 < i < n$, then

$$f(x_i, y) = f(x_i, x_{i-1}x_{i-2}) \overset{\text{(AS)}}{=} -f(x_{i-1}, x_i x_{i-2}) \overset{\text{(R1)}}{=} 0.$$

- If $i = k + 1$, $m = k - 1$ and $i = 3$, then $f(x_i, y) = f(x_3, x_2 x_1) \in \mathbb{F}_{D;n}$.
- If $i = k + 1$, $m < k - 1$ and $m > 1$, then

$$x \overset{\text{(AC)}}{=} -x_i x_{i\downarrow m+2} x_m x_{m+1} \overset{\text{(Q1)}}{=} -x_m x_i x_{i\downarrow m+1} \overset{(XT)}{=} -f\big(x_i, y^1_{i-1,m+1}\big) x_m x_i \overset{(R1)}{=} 0.$$

- If $i = k + 1$, $m < k - 1$ and $m = 1$, then

$$\begin{aligned} x &\overset{\text{(J)}}{=} x_i x_{i\downarrow 4} x_2 x_3 x_1 + x_i x_{i\downarrow 4} x_1 x_2 x_3 \overset{\text{(Q1)}}{=} x_2 x_i x_{i\downarrow 3} x_1 + x_1 x_i x_{i\downarrow 4} x_2 x_3 \\ &\overset{\text{(XT)}}{=} f(x_i, x_{i-1\downarrow 3} x_1) x_2 x_i + f(x_i, x_{i-1\downarrow 4} x_2 x_3) x_1 x_i \overset{\text{(R1)}}{=} 0. \end{aligned}$$

- If $i = k$, then $x = x_i x_i x_{i\downarrow m} = 0$.
- If $i = k - 1$ and $i > m$, then

$$f(x_i, y) = f\big(x_i, x_{i+1} x_i y^1_{i-1,m}\big) \overset{\text{(P3)}}{=} -f(x_i, x_{i+1}) f\big(x_i, y^1_{i-1,m}\big) \in \mathbb{F}_{D;n}.$$

  Here we use the induction hypothesis for $f(x_i, y^1_{i-1,m})$.
- If $i = k - 1$ and $i = m$, then

$$x = x_i x_i x_{i+1} x_i \overset{\text{(AC)}}{=} -x_i x_i x_i x_{i+1} = 0.$$

- If $i < k - 1$, $i = n - 2$ and $m = n - 1$, then

$$f(x_i, y) = f(x_{n-2}, x_n x_{n-1}) \overset{\text{(AS), (SM)}}{=} f(x_n, x_{n-1} x_{n-2}) \in \mathbb{F}_{D;n}.$$

- If $i < k-1$, $i = n-2$ and $m = n-2$, then

$$f(x_i, y) = f(x_{n-2}, x_n x_{n-1} x_{n-2}) \overset{\text{(AC)}}{=} -f(x_{n-2}, x_n x_{n-2} x_{n-1}) \overset{\text{(P3)}}{=} f(x_{n-2}, x_n) f(x_{n-2}, x_{n-1}).$$

- If $i < k-1$, $i = n-2$ and $m \leqslant n-3$, then

$$\begin{aligned} f(x_i, y) &= f(x_{n-2}, x_{n\downarrow m}) \overset{\text{(AS)}}{=} f(x_{n-2} x_n, x_{n-1\downarrow m}) \overset{\text{(AC)}}{=} -f(x_n x_{n-2}, x_{n-1\downarrow m}) \\ &\overset{\text{(AS)}}{=} -f(x_n, x_{n-2} x_{n-1} x_{n-2\downarrow m}) \overset{\text{(P2)}}{\in} \mathbb{F}_{D;n}. \end{aligned}$$

- If $i < k-1$, $i < n-2$ and $k > 3$, then

$$x = x_i x_i x_{k\downarrow m} \overset{\text{(Q1)}}{=} x_k x_i x_i x_{k-1\downarrow m} \overset{\text{(XT)}}{=} f\big(x_i, y^1_{k-1,m}\big) x_k x_i \overset{\text{(R1)}}{=} 0.$$

- If $i < k-1$, $i < n-2$, $k = 3$ and $m = 2$, then $f(x_i, y) = f(x_1, x_3 x_2) \in \mathbb{F}_{D;n}$.
- If $i < k-1$, $i < n-2$, $k = 3$ and $m = 1$, then we have:

$$f(x_i, y) = f(x_1, x_3 x_2 x_1) \overset{\text{(AS)}}{=} f(x_1 x_3, x_2 x_1) \overset{\text{(AC)}}{=} f(x_3 x_1, x_1 x_2) \overset{\text{(AS)}}{=} f(x_3, x_1 x_1 x_2) \in \mathbb{F}_{D;n}.$$

Case 2: $j = 3$, $k \in \{3, \dots, n\}$, $m \in \{3, \dots, k\}$. Remember that $y^3_{k,m} = x_{k\downarrow m+1} x_{m-1\uparrow\Downarrow 1}$.

- If $i > k+1$ and $m = k$, then

$$x = x_i x_i x_{k-1} y^3_{k,k-1} \overset{\text{(Q1)}}{=} x_{k-1} x_i x_i y^3_{k,k-1} \overset{\text{(XT)}}{=} f\big(x_i, y^3_{k,k-1}\big) x_{k-1} x_i \overset{\text{(R1)}}{=} 0.$$

- If $i > k+1$ and $m < k$, then

$$x = x_i x_i x_{k\downarrow m+1} x_{m-1} x_m x_{m-2\uparrow\Downarrow 1} \overset{\text{(Q1)}}{=} x_{m-1} x_i x_i x_{k\downarrow m+1} x_m x_{m-2\uparrow\Downarrow 1} \in \mathbb{F}_{D;n} x_{m-1} x_i \overset{\text{(R1)}}{=} \{0\}.$$

- If $i = k+1$, $m = k$ and $i = n$, then

$$\begin{aligned} f(x_i, y) &= f(x_n, x_{n-2\uparrow\Downarrow 1}) \overset{\text{(AS)}}{=} f(x_n x_{n-2}, x_{n-1} x_{n-3\uparrow\Downarrow 1}) \overset{\text{(AC)}}{=} -f(x_{n-2} x_n, x_{n-1} x_{n-3\uparrow\Downarrow 1}) \\ &\overset{\text{(AS)}}{=} -f(x_{n-2}, x_n x_{n-1} x_{n-3\uparrow\Downarrow 1}) \overset{\text{(Q1)}}{=} -f(x_{n-2}, x_{n-3} x_n x_{n-1} x_{n-2} x_{n-4\uparrow\Downarrow 1}) \\ &\overset{\text{(AS), (AC)}}{=} f(x_{n-3}, x_{n-2} x_n x_{n-1} x_{n-2} x_{n-4\uparrow\Downarrow 1}). \end{aligned}$$

Now we repeat the steps in the last line $n-6$ times to obtain

$$\begin{aligned} f(x_i, y) &= (-1)^n f(x_3, x_{4\uparrow n-2} x_{n\downarrow 4} x_2 x_3 x_1) \\ &\overset{\text{(J)}}{=} (-1)^n \big(f\big(x_3, x_{4\uparrow n-2} x_{n\downarrow 4} (x_2 x_3) x_1\big) + f(x_3, x_{4\uparrow n-2} x_{n\downarrow 4} x_3 x_2 x_1)\big). \end{aligned}$$

We will treat these terms separately.

$$\begin{aligned} f\big(x_3, x_{4\uparrow n-2} x_{n\downarrow 4} (x_2 x_3) x_1\big) &\overset{\text{(AC)}}{=} f(x_3, x_{4\uparrow n-2} x_{n\downarrow 4} x_1 x_3 x_2) \overset{\text{(Q1)}}{=} f(x_3, x_1 x_{4\uparrow n-2} x_{n\downarrow 2}) \\ &\overset{\text{(AS), (AC)}}{=} -f(x_1, x_{3\uparrow n-2} x_{n\downarrow 2}) \in \mathbb{F}_{D;n}, \end{aligned}$$

and

$$f(x_3, x_{4\uparrow n-2}x_{n\downarrow 1}) \overset{\text{(AS), (AC)}}{=} -f(x_4, x_3x_{5\uparrow n-2}x_{n\downarrow 1}) \overset{\text{(Q1)}}{=} -f(x_4, x_{5\uparrow n-2}x_{n\downarrow 5}x_3x_{4\downarrow 1}) \overset{\text{(P1)}}{\in} \mathbb{F}_{D;n}.$$

- If $i = k+1$, $m = k$ and $i < n$, then

$$x = x_ix_ix_{i-2}y^3_{i-1,i-2} \overset{\text{(Q1)}}{=} x_{i-2}x_ix_iy^3_{i-1,i-2} \overset{(XT)}{\in} \mathbb{F}_{D;n}x_{i-2}x_i \overset{\text{(R1)}}{=} 0.$$

- If $i = k+1$ and $m < k$, then

$$x = x_ix_{i\downarrow m+1}x_{m-1}y^3_{m,m-1} \overset{\text{(Q1)}}{=} x_{m-1}x_ix_{i\downarrow m+1}y^3_{m,m-1} \overset{\text{(XT)}}{\in} \mathbb{F}_{D;n}x_{m-1}x_i \overset{\text{(R1)}}{=} 0.$$

- If $i = k$ and $m = k$, then

$$f(x_i, y) = f\left(x_i, x_{i-1}x_iy^3_{i-1,i-1}\right) \overset{\text{(P3)}}{\in} \mathbb{F}_{D;n}.$$

- If $i = k$ and $m < k$, then

$$f(x_i, y) = f\left(x_i, x_iy^3_{i-1,m}\right) \overset{\text{(AS)}}{=} f\left(x_ix_i, y^3_{i-1,m}\right) \overset{\text{(AC)}}{=} 0.$$

- If $i = k-1$ and $m = k$, then

$$f(x_i, y) = f\left(x_i, x_iy^3_{i+1,i}\right) \overset{\text{(AS)}}{=} f\left(x_ix_i, y^3_{i+1,i}\right) \overset{\text{(AC)}}{=} 0.$$

- If $i = k-1$ and $m = k-1$, then

$$\begin{aligned} f(x_i, y) &= f(x_i, x_{i+1}x_{i-1}x_ix_{i-2\uparrow\Downarrow 1}) \overset{\text{(AS)}}{=} f(x_ix_{i+1}, x_{i-1}x_ix_{i-2\uparrow\Downarrow 1}) \\ &\overset{\text{(AC)}}{=} -f(x_{i+1}x_i, x_{i-1}x_ix_{i-2\uparrow\Downarrow 1}) \overset{\text{(AS)}}{=} -f(x_{i+1}, x_ix_{i-1}x_ix_{i-2\uparrow\Downarrow 1}) \overset{\text{(P2)}}{\in} \mathbb{F}_{D;n} \end{aligned}$$

- If $i = k-1$ and $m < k-1$, then

$$f(x_i, y) = f\left(x_i, x_{i+1}x_iy^3_{i-1,m}\right) \overset{\text{(P3)}}{\in} \mathbb{F}_{D;n}.$$

- If $i = k-2$, $m = k$ and $i > 1$, then

$$f(x_i, y) = f(x_i, x_{i+1}x_{i+2}x_ix_{i+1}x_{i-1\uparrow\Downarrow 1}) \overset{\text{(Q4)}}{\in} \mathbb{F}_{D;n} + f(x_i, x_{i+1}x_ix_{i+2}x_{i+1}x_{i-1\uparrow\Downarrow 1}) \overset{\text{(P3)}}{=} \mathbb{F}_{D;n}.$$

- If $i = k-2$, $m = k$ and $i = 1$, then

$$f(x_1, y) = f(x_1, x_2x_3x_1) \overset{\text{(AC)}}{=} -f(x_1, x_2x_1x_3) \overset{\text{(P3)}}{\in} \mathbb{F}_{D;n}.$$

- If $i = k-3$, $m = k$ and $i > 1$, then

$$x = x_ix_ix_{k-1}y^3_{k,k-1} \overset{\text{(Q1)}}{=} x_{k-1}x_ix_iy^3_{k,k-1} \overset{\text{(XT)}}{=} f\left(x_i, y^3_{k,k-1}\right)x_{k-1}x_i \overset{\text{(R1)}}{=} 0.$$

- If $i = k - 3$, $m = k$ and $i = 1$, then

$$\begin{aligned} f(x_i, y) &= f(x_1, x_3x_4x_2x_3x_1) \overset{\text{(J)}}{=} f(x_1, x_3x_4x_3x_2x_1) + f(x_1, x_3x_4x_1x_3x_2) \\ &\overset{\text{(Q1)}}{=} f(x_1, x_3x_4x_3x_2x_1) + f(x_1, x_3x_1x_4x_3x_2) \overset{\text{(P2), (P3)}}{\in} \mathbb{F}_{D;n}. \end{aligned}$$

- If $i < k - 3$ and $m = k$, then

$$x = x_ix_ix_{k-1}y^3_{k,k-1} \overset{\text{(Q1)}}{=} x_{k-1}x_ix_iy^3_{k,k-1} \overset{\text{(XT)}}{=} f\big(x_i, y^3_{k,k-1}\big)x_{k-1}x_i \overset{\text{(R1)}}{=} 0.$$

- If $i < k - 1$, $m < k$, $i = n - 2$ and $m = n - 1$, then

$$f(x_i, y) = f\big(x_{n-2}, x_nx_{n-2}y^3_{n-1,n-2}\big) \overset{\text{(P3)}}{\in} \mathbb{F}_{D;n}.$$

- If $i < k - 1$, $m < k$, $i = n - 2$ and $m = n - 2$, then

$$\begin{aligned} f(x_i, y) &= f(x_{n-2}, x_nx_{n-1}x_{n-3\uparrow\Downarrow 1}) \overset{\text{(AS), (AC)}}{=} -f(x_n, x_{n-2}x_{n-1}x_{n-3\uparrow\Downarrow 1}) = -f\big(x_n, y^3_{n-1,n-1}\big) \\ &\in \mathbb{F}_{D;n}, \end{aligned}$$

  as proven earlier.
- If $i < k - 1$, $m < k$, $i = n - 2$ and $m < n - 2$, then

$$\begin{aligned} x &= x_{n-2}x_{n-2}x_{n\downarrow m+1}x_{m-1}y^3_{m,m-1} \overset{\text{(Q1)}}{=} x_{m-1}x_{n-2}x_{n-2}x_{n\downarrow m+1}y^3_{m,m-1} \\ &\overset{\text{(XT)}}{=} f\big(x_{n-2}, y^3_{n,m-1}\big)x_{m-1}x_{n-2} \overset{\text{(R1)}}{=} 0. \end{aligned}$$

- If $i < k - 1$, $m < k$ and $i < n - 2$, then

$$x = x_ix_ix_ky^3_{k-1,m} \overset{\text{(Q1)}}{=} x_kx_ix_iy^3_{k-1,m} \overset{\text{(XT)}}{=} f\big(x_i, y^3_{k-1,m}\big)x_kx_i \overset{\text{(R1)}}{=} 0. \qquad \square$$

5.2. Parameters for $\Gamma_{B;n}$

Lemma 5.2. *Let*

$$\begin{aligned} &\psi_{\Gamma_{B;n}} : X(\Gamma_{B;n}) \to \mathbb{F}^{n+2}, \\ &\qquad \mathfrak{f} \mapsto \big(\mathfrak{f}_1(x_2), \mathfrak{f}_2(x_3), \ldots, \mathfrak{f}_{n-2}(x_{n-1});\ \mathfrak{f}_1(x_3), \mathfrak{f}_1(x_2x_3), \mathfrak{f}_{n-2}(x_n), \mathfrak{f}_1(x_{3\uparrow n-2}x_{n\downarrow 2})\big). \end{aligned}$$

Then $\psi_{\Gamma_{B;n}}$ is injective.

Proof. Remember that $\mathcal{F}_{\Gamma_{B;n}}$ is a quotient of $\mathcal{F}_{\Gamma_{D;n}}$. Hence relations between values of $\mathfrak{f}_i$ that hold in $\mathcal{F}_{\Gamma_{D;n}}$, hold in $\mathcal{F}_{\Gamma_{B;n}}$ as well. This allows us to express $\mathfrak{f}_i(y)$ in the values of Lemma 5.1. It then suffices to prove that $f(x_n, x_{n-1})$ and $f(x_n, x_{n-1}x_{n-2})$ are zero:

- $f(x_n, x_{n-1}) = 0$, because $x_nx_nx_{n-1} \overset{\text{(R3)}}{=} 0$,
- $f(x_n, x_{n-1}x_{n-2}) = f(x_n, x_{n-1}x_{n-2}) \overset{\text{(AS)}}{=} f(x_nx_{n-1}, x_{n-2}) \overset{\text{(R3)}}{=} 0.$ $\square$

5.3. Parameters for $\Gamma_{A;n}$

Lemma 5.3. *Let*

$$\psi_{\Gamma_{A;n}} : X(\Gamma_{A;n}) \to \mathbb{F}^{n+1}, \qquad \mathfrak{f} \mapsto \big(\mathfrak{f}_1(x_2), \mathfrak{f}_2(x_3), \dots, \mathfrak{f}_{n-1}(x_n);\ \mathfrak{f}_1(x_3), \mathfrak{f}_1(x_2x_3)\big).$$

Then $\psi_{\Gamma_{A;n}}$ is injective.

Proof. $\mathcal{F}_{\Gamma_{A;n}}$ is a quotient of $\mathcal{F}_{\Gamma_{D;n}}$, so by Lemma 5.1 and a reasoning similar to the one in the proof of Lemma 5.2, it suffices to show that $f(x_n, x_{n-2})$, $f(x_n, x_{n-1}x_{n-2})$ and $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})$ are zero:

- $f(x_n, x_{n-2}) = 0$, because $x_nx_nx_{n-2} \overset{\text{(R4)}}{=} 0$,
- $f(x_n, x_{n-1}x_{n-2}) \overset{\text{(AC)}}{=} -f(x_n, x_{n-2}x_{n-1}) \overset{\text{(AS)}}{=} -f(x_nx_{n-2}, x_{n-1}) \overset{\text{(R4)}}{=} 0$,
- $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) \overset{\text{(AS)}}{=} f(x_1x_3, y_{4,2}^2) \overset{\text{(R2)}}{=} 0$. □

5.4. Parameters for $\Gamma_{C;n}$

Lemma 5.4. *Let*

$$\psi_{\Gamma_{C;n}} : X(\Gamma_{C;n}) \to \mathbb{F}^{n-1}, \qquad \mathfrak{f} \mapsto \big(\mathfrak{f}_1(x_2), \mathfrak{f}_2(x_3), \dots, \mathfrak{f}_{n-1}(x_n)\big).$$

Then $\psi_{\Gamma_{C;n}}$ is injective.

Proof. $\mathcal{F}_{\Gamma_{C;n}}$ is a quotient of $\mathcal{F}_{\Gamma_{A;n}}$, so by Lemma 5.3 and a reasoning similar to the one in the proof of Lemma 5.2, it suffices to show that $f(x_3, x_1)$ and $f(x_3, x_2x_1)$ are zero:

- $f(x_3, x_1) = 0$, because $x_3x_3x_1 \overset{\text{(R2)}}{=} 0$,
- $f(x_3, x_2x_1) \overset{\text{(AC)}}{=} -f(x_3, x_1x_2) \overset{\text{(AS)}}{=} -f(x_3x_1, x_2) \overset{\text{(R2)}}{=} 0$. □

Corollary 5.5. *$\psi_\Gamma(X(\Gamma))$ is an algebraic variety for all $\Gamma \in \{\Gamma_{A;n}, \Gamma_{B;n}, \Gamma_{C;n}, \Gamma_{D;n}\}$.*

6. Realizations of the four classical families

In this section, we will find generating sets of extremal elements for the four classical families of Lie algebras, where these generating sets realize the graphs $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$ and $\Gamma_{D;n}$. We keep n as the number of extremal generators and will see that these graphs correspond to the Lie algebras of type A_{n-1}, B_{n-1}, $C_{n/2}$ and D_n, respectively. In particular, the objective of this section will be the formulating and proving of Theorems 6.10, 6.11, 6.18 and 6.19, giving these explicit generators.

The extremal elements in Lie algebras of type A_{n-1} or $C_{n/2}$ correspond to *transvections*, which will be discussed in Section 6.1. In the orthogonal Lie algebras, the extremal elements correspond to the *Siegel transvections* or Siegel transformations. These are examined in Section 6.2. In both sections, we first explore these elements in a general setting, and then discuss generators for the two series of Lie algebras specifically.

6.1. Transvections

Let $n \in \mathbb{N}_+$, $x \in V = \mathbb{F}^n$, $h \in V^*$, and fix a basis $e_1, \dots, e_n$ of V and a corresponding dual basis $f_1, \dots, f_n$ of V^*. We will see that the linear transformation $x \otimes h : v \mapsto h(v)\, x$ is an extremal element of $\mathfrak{sl}(V)$ if $h(x) = 0$ and x, h nonzero. A *transvection* is a linear transformation of the form $1 + x \otimes h$ where $h(x) = 0$ and x, h nonzero. We call x the *centre* of the transvection and h the *axis*. We then call $x \otimes h$ an *infinitesimal transvection*.

A *transvection group* is a group $\{1+tx\otimes h \mid t\in\mathbb{F}\}$. The Lie algebra of a transvection group consists of the transvections $tx\otimes h$. We will use a result of McLaughlin [10] which classifies groups generated by transvection subgroups. This is a weaker version of a reformulation by Cameron and Hall, Theorem 2 from [1]:

Theorem 6.1. *Let G be a nontrivial group of linear transformations of the finite-dimensional $\mathbb{F}$-vector space V, which is generated by $\mathbb{F}$-transvection subgroups. If V is spanned by a G-orbit on centres of these transvection subgroups, and V^* is spanned by the axes, then one of the following holds*:

(1) $G=\mathrm{SL}(V)$;
(2) $G=\mathrm{Sp}(V,B)$ *for some symplectic form B.*

We will need a tool to distinguish between $\mathrm{SL}(V)$ and $\mathrm{Sp}(V,B)$. This tool will be provided by analysis of the occurrence of Heisenberg subalgebras generated by pairs of extremal elements, further detailed in Lemma 6.5 and Corollary 6.9.

The following lemmas are easily seen to be true.

Lemma 6.2. $\mathfrak{sl}(V)$ *is the span of the infinitesimal transvections in* $\mathfrak{gl}(V)$.

Lemma 6.3. *Infinitesimal transvections are extremal elements of* $\mathfrak{sl}(V)$.

Lemma 6.4. *All extremal elements in* $\mathfrak{sl}(V)$ *are infinitesimal transvections.*

Lemma 6.5. *Let $x\otimes h$ and $y\otimes k$ be two infinitesimal transvections and let $\mathcal{L}=\langle x\otimes h, y\otimes k\rangle_{\mathrm{Lie}}$. Then the isomorphism class of $\mathcal{L}$ depends on the geometrical configuration of $\mathbb{F}x$, $\mathbb{F}y$, $\mathrm{Ker}\,h$ and $\mathrm{Ker}\,k$, as follows*:

- *If $\mathbb{F}x=\mathbb{F}y$ and $\mathrm{Ker}\,h=\mathrm{Ker}\,k$, then $\mathcal{L}$ is one-dimensional.*
- *If either $\mathbb{F}x=\mathbb{F}y$ or $\mathrm{Ker}\,h=\mathrm{Ker}\,k$ but not both, then $\mathcal{L}$ is two-dimensional.*

Assume for the other cases that $\mathbb{F}x\neq\mathbb{F}y$ and $\mathrm{Ker}\,h\neq\mathrm{Ker}\,k$.

- *If $\mathbb{F}x\subset\mathrm{Ker}\,k$ and $\mathbb{F}y\subset\mathrm{Ker}\,h$, then $\mathcal{L}$ is two-dimensional as in the preceding case.*
- *If either $\mathbb{F}x\subset\mathrm{Ker}\,k$ or $\mathbb{F}y\subset\mathrm{Ker}\,h$ but not both, then $\mathcal{L}$ is isomorphic to the Heisenberg algebra.*
- *If $\mathbb{F}x\not\subset\mathrm{Ker}\,k$ and $\mathbb{F}y\not\subset\mathrm{Ker}\,h$, then $\mathcal{L}$ is isomorphic to $\mathfrak{sl}_2$.*

If $x\otimes h$ and $y\otimes k$ generate a Heisenberg algebra, we say that they form an *extraspecial pair*.

We will also realize $\mathfrak{sp}(V)$, the Lie algebra of type $C_{n/2}$, using infinitesimal transvections. In order to do this, let us assume that n is even and that we have a nondegenerate symplectic form B. We will denote the matrix of B by B as well. For $y\in V$, we write

$$u(y):=y\otimes\big(v\mapsto B(y,v)\big):V\to V,\quad v\mapsto B(y,v)y.$$

Hence $u(y)$ is an infinitesimal transvection. The following lemma is easily proven:

Lemma 6.6. *The infinitesimal transvections in the Lie algebra of type $C_{n/2}$ can all be written as $u(y)$ for some y.*

Since $u(y)$ is an extremal element in $\mathfrak{sl}(V)$, it is also an extremal element in $\mathfrak{sp}(V)$. The following two lemmas are equivalents of Lemmas 6.2 and 6.4 for $\mathfrak{sp}(V)$.

Lemma 6.7. $\mathfrak{sp}(V)$ *is spanned by its infinitesimal transvections.*

Lemma 6.8. *All extremal elements in $\mathfrak{sp}(V)$ are infinitesimal transvections.*

Proof sketch. We may assume that $B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Let M be the matrix of an extremal element of $\mathfrak{sp}(V)$, then M can be written as a block matrix $\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & -M_{11}^T \end{pmatrix}$, where M_{12} and M_{21} are symmetric. For every matrix A also in $\mathfrak{sp}(V)$, we have that

$$\bigl[M, [M, A]\bigr] = M^2A - 2MAM + AM^2 \in \mathbb{F}M. \tag{6.1}$$

We will mostly take for A a block matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with A_{12} and A_{21} symmetric and $A_{22} = -A_{11}^T$. However, substituting the $2n \times 2n$ identity matrix into the linear equation (6.1) yields a tautology, so we can add multiples of the $n \times n$ identity matrices to A_{11} and A_{22} independently. We take the following values for A.

- Take $A_{11} = A_{12} = A_{22} = 0$ and for A_{21} all matrices E_{ij} that has a one in position (i, j) and zeroes elsewhere. This shows that we can find a vector $y_0 \in \mathbb{F}^n$ such that $M_{12} = y_0 y_0^T$. Similarly there is an $y_1 \in \mathbb{F}^n$ with $M_{21} = -y_1 y_1^T$.
- Take $A_{11} = I$ and $A_{12} = A_{21} = A_{22} = 0$, then we find that

$$M' := \begin{pmatrix} 2M_{12}M_{21} & -M_{11}M_{12} - M_{12}M_{11}^T \\ -M_{21}M_{11} - M_{11}^T M_{21} & -2M_{21}M_{12} \end{pmatrix} \quad \text{is a multiple of } M. \tag{6.2}$$

  If M' is nonzero, it follows fairly easily that $M_{11} = -y_0 y_1^T$, and we are done. We continue with the case $M' = 0$, where one sees fairly easily that the products of pairs of distinct elements of $\{M_{11}, M_{12}, M_{21}\}$ are zero.
- Take $A_{12} = A_{21} = 0$ and $A_{22} = -A_{11}^T$. Using the fact that the products mentioned above are zero, we obtain that

$$M'' := \begin{pmatrix} [M_{11}, [M_{11}, A_{11}]] + 2M_{12}A_{11}^T M_{21} & -2(M_{11}A_{11}M_{12} + M_{12}A_{11}^T M_{11}^T) \\ -2(M_{21}A_{11}M_{11} + M_{11}^T A_{11}^T M_{21}) & -[M_{11}^T, [M_{11}^T, A_{11}^T]] - 2M_{21}A_{11}M_{12} \end{pmatrix}$$
$$\text{is a multiple of } M. \tag{6.3}$$

  We take for A_{11} the matrices E_{ij}. The rest of the proof is easy if one distinguishes the case where $y_0 = y_1 = 0$ from the other case. □

Corollary 6.9. *$\mathfrak{sp}(V)$ does not contain an extraspecial pair.*

Proof. Let $x \otimes h$ and $y \otimes k$ form an extraspecial pair. Then we may assume that $h(y) = 0 \neq k(x)$. But $h(y) = B(x, y) = -B(y, x) = -k(x)$. □

We now proceed with the theorems giving the generating extremal elements.

Theorem 6.10. *Suppose that n is even. Let B be the nondegenerate symplectic form determined by the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The transformations*

$$u(x_i) : v \mapsto B(x_i, v)x_i, \quad i = 1, \ldots, n,$$

realize the graph $\Gamma_{C;n}$ for $n \geqslant 2$ if we take these values for x_i:

$$x_{2\ell-1} = e_\ell \quad \textit{for } 1 \leqslant \ell \leqslant n/2,$$
$$x_{2\ell} = e_{\ell+n/2} + e_{\ell+n/2+1} \quad \textit{for } 1 \leqslant \ell < n/2,$$
$$x_n = e_n.$$

The Lie algebra $\mathcal{L} = \langle u(x_i)\rangle_{\mathrm{Lie}}$ *is* $\mathfrak{sp}_n$.

Proof. It is easy to check that the transformations $u(x_i)$ realize the graph. In order to prove that they generate $\mathfrak{sp}_n$, consider the group $G = \langle 1 + tu(x_i) \mid t \in \mathbb{F},\ 1 \leqslant i \leqslant n\rangle_{\mathrm{Gp}}$, of which $\mathcal{L}$ is the Lie algebra. The action of G on $\mathcal{L}$ is such that

$$u(x)^{1+tu(y)} = \big(1 + tu(y)\big)u(x)\big(1 - tu(y)\big) = u\big(x^{1+tu(y)}\big),$$

as can be seen immediately by inspection. So if $B(x, y) \neq 0$, then

$$\big(1 + B(x, y)^{-1}u(y)\big)\big(1 + B(x, y)^{-1}u(x)\big)(y) = x.$$

This shows that the orbit of G on x_1 spans V. Hence, using Theorem 6.1, either $G = \mathrm{Sp}(V, B)$ or $G = \mathrm{SL}(V)$. Thus $\mathcal{L}$ is either $\mathfrak{sp}_n$ or $\mathfrak{sl}_n$. But all given transformations are in $\mathfrak{sp}_n$. This can be verified by examining the matrices $A_i^T M + MA_i$, where M is the matrix of B and A_i is the matrix of $u(x_i)$: if we view elements of V as column vectors, then $A_i = x_i x_i^T M$, so $A_i^T = -Mx_i x_i^T$. Thus $A_i^T M + MA_i = 0$, so $\mathcal{L} = \mathfrak{sp}_n$. □

Theorem 6.11. *The transformations $x_i \otimes h_i$ realize the graph $\Gamma_{A;n}$ for $n \geqslant 2$ if we take these values for x and h:*

$$x_1 = e_1 - e_2, \qquad h_1 = f_1 + f_2,$$
$$x_i = e_{i-1} + e_i, \qquad h_i = f_{i-1} - f_i \quad \textit{for } 1 < i \leqslant n.$$

The Lie algebra $\mathcal{L} = \langle x_i \otimes h_i\rangle_{\mathrm{Lie}}$ *is* $\mathfrak{sl}_n$.

Proof. It is easy to check that the transformations $x_i \otimes h_i$ realize the graph. In order to prove that they generate $\mathfrak{sl}_n$, consider the group $G = \langle 1 + t(x_i \otimes h_i) \mid t \in \mathbb{F},\ 1 \leqslant i \leqslant n\rangle_{\mathrm{Gp}}$, of which $\mathcal{L}$ is the Lie algebra. It is clear that the orbit of G on x_1 spans V. Hence, using Theorem 6.1, either $G = \mathrm{Sp}(V, B)$ or $G = \mathrm{SL}(V)$. Hence $\mathcal{L}$ is either $\mathfrak{sp}_n$ or $\mathfrak{sl}_n$. Now consider $\exp(2\,\mathrm{ad}\, x_3 \otimes h_3)(x_1 \otimes h_1) = (x_1 - 2x_3) \otimes (h_1 - 2h_3)$. It forms an extraspecial pair with $x_2 \otimes h_2$, because $(h_1 - 2h_3)(x_2) = 0 \neq h_2(x_1 - 2x_3)$. By Corollary 6.9, $\mathcal{L} = \mathfrak{sl}_n$. □

6.2. Siegel transvections

In order to realize the two orthogonal types of algebras we use Siegel transvections. As an equivalent of Theorem 6.1 we will use the main theorem of Steinbach's paper [13], dealing with Siegel transvection groups in a similar way to how Theorem 6.1 deals with transvection groups. Steinbach's main theorem is reprinted here in a weaker form as Theorem 6.16. In this form, Theorem 6.16 also follows from the work by Liebeck and Seitz [9].

Let the dimension of V be $2n$ or $2n-1$. Let B be a nondegenerate orthogonal bilinear form on V; we will denote the corresponding matrix by B as well. We may assume that B is either $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right)$. Let $u, v \in V$ be linearly independent with $B(u, u) = B(u, v) = B(v, v) = 0$; in other words, u and v span an *isotropic line*. When we speak of lines in an orthogonal space we mean isotropic lines. Then

$$S_{u,v}: V \to V, \qquad x \mapsto x + B(u,x)v - B(v,x)u$$

is known as the *Siegel transvection* determined by u and v. If $S_{u,v}$ is a Siegel transvection, then we will call the map $T_{u,v} = S_{u,v} - 1$ an *infinitesimal Siegel transvection*. Note that $T_{u,v}$ is determined up to scalar multiples by the projective line containing u and v. We call the group $\langle 1 + tT_{u,v} \mid t \in \mathbb{F}\rangle_{\mathrm{Gp}}$ a *Siegel transvection group*.

Lemma 6.12. *$\mathfrak{o}(V)$ is spanned by infinitesimal Siegel transvections.*

Lemma 6.13. *Infinitesimal Siegel transvections are extremal elements of $\mathfrak{o}(V)$.*

Lemma 6.14. *For all $t \in \mathbb{F}$ and $u, v, w, x \in V$, we have $\exp(t \operatorname{ad} T_{u,v})T_{w,x} = T_{w+tT_{u,v}w, x+tT_{u,v}x}$.*

In the following lemma, we will let "point" refer to projective points and "line" to projective isotropic lines.

Lemma 6.15. *Let $\mathcal{L} = \langle T_{u,v}, T_{w,x}\rangle_{\mathrm{Lie}}$, where $T_{u,v}$ and $T_{w,x}$ are two infinitesimal Siegel transvections. Then the isomorphism class of $\mathcal{L}$ depends on the geometrical configuration of the lines $\ell = \langle u, v\rangle_{\mathbb{F}}$ and $m = \langle w, x\rangle_{\mathbb{F}}$, as follows:*

- *If $\ell = m$, then $\mathcal{L}$ is one-dimensional.*
- *If $\ell \cap m$ is a point, then $\mathcal{L}$ is two-dimensional.*
- *If each point on ℓ is collinear to each point on m, then $\mathcal{L}$ is two-dimensional.*
- *If exactly one point on ℓ is collinear with all of m and exactly one point on m is collinear with all of ℓ, then $\mathcal{L}$ is isomorphic to the Heisenberg algebra.*
- *If every point on ℓ is collinear with exactly one point of m and every point on m is collinear with exactly one point of ℓ, then $\mathcal{L}$ is isomorphic to $\mathfrak{sl}_2$.*

These are all cases.

In the case where $\mathcal{L}$ is isomorphic to the Heisenberg algebra, we say that ℓ and m form an *extraspecial pair*, just as in the previous section.

We focus our attention on the Siegel transvection groups now. Since the theorem of Steinbach is somewhat more involved than Theorem 6.1, we need to introduce some notation and terminology first.

Let $Y = \Omega(V,B)$ be the commutator subgroup of the orthogonal group $O(V,B)$. Then by [13, Section 1.1] Y is generated by all Siegel transvection subgroups of $O(V,B)$. Let $G \neq 1$ be a subgroup of Y generated by some of the Siegel transvection groups. For each Siegel transvection group $A \subseteq Y$, let $A^0 = A \cap G$; then either $A^0 = 1$ or $A^0 = A$—this can easily be seen in the Lie algebra. This situation is a special case of the situation in the main theorem of [13], reprinted here in a weaker version as Theorem 6.16, which tells us to which isomorphism classes such a group G can belong. Following [13], we use the following abbreviations for different situations:

- (O): G is an orthogonal group over the same vector space, viewed as a vector space over a smaller field. Also, if B has maximal Witt index over a vector space of even dimension, then it contains the orthogonal group over a vector space of one dimension less where B has a one-dimensional radical.
- (E): G is a special case related to triality and $\dim V = 8$.
- (ND): G is the special linear group on a vector space V' such that V has double the dimension of V' and B has maximal Witt index. V is the direct sum of the space V' where G acts naturally and a space where it acts dually.
- (I): G is a unitary group. The unitary space is then regarded as orthogonal space over the fixed field of the involutory automorphism.

(IQ): G is a unitary group over a quaternion division ring instead of a field.
(KC): G is a special case related to the Klein correspondence and $\dim V \in \{5, 6\}$.

Theorem 6.16. *Let $G \neq 1$ be a subgroup of $Y = \Omega(V, q)$ generated by Siegel transvection groups S_{u_i,v_i}. Suppose these conditions are satisfied:*

(H1′) *If A, B are Siegel transvection subgroups of Y that intersect G nontrivially, and $\langle A, B\rangle_{\mathrm{Gp}} = SL_2(\mathbb{F})$, then $\langle A^0, B^0\rangle_{\mathrm{Gp}} = SL_2(\mathbb{F})$.*
(H2) *All nilpotent normal subgroups of G are contained in $Z(G)$ and in the commutator group G', and it is not possible to decompose the set Σ of Siegel transvection subgroups into two nonempty parts Σ_1, Σ_2 such that all groups from Σ_1 commute with all groups from Σ_2.*
(H3) *There are three pairwise distinct commuting Siegel transvection subgroups of G such that there exist Siegel transvection subgroups T of G commuting with exactly two of these three subgroups.*

Then one of the situations (O), (E), (ND), (I), (IQ) *and* (KC) *holds.*

Theorem 6.17. *Suppose all conditions from Theorem* 6.16 *hold and the following extra conditions hold as well.*

(X1) *The dimension of V is greater than* 8.
(X2) *There are extraspecial pairs in G.*
(X3) *There is no G-invariant decomposition of V into two equal-dimensional subspaces, such that each of these subspaces intersects every line corresponding to a Siegel transvection subgroup nontrivially.*
(X4) *One of the Siegel transvection subgroups is isomorphic to $\mathbb{F}^+$.*
(X5) $\langle u_i, v_i\rangle_{\mathbb{F}} = V$.

Then $G = Y$.

Proof. Extra condition (X1) shows that we are not in situations (E) or (KC). By extra condition (X2), we must be in situation (ND) or (O). By extra condition (X3), we are not in situation (ND). Hence, we are in situation (O): G is an orthogonal group. Then condition (X4) shows that the field is all of $\mathbb{F}$. Finally, because of extra condition (X5), G is all of Y. □

We define the basis of V in an order corresponding to the matrix of B, as follows. Let $k = n$ for D_n and $k = n - 1$ for B_{n-1}. The basis of V consists first of vectors $\{e_i\}_{i=1}^k$ spanning a maximal isotropic subspace, then of vectors $\{f_i\}_{i=1}^k$ spanning a maximal isotropic complement to $\langle e_i\rangle_{\mathbb{F}}$ and with $B(e_i, f_j) = \delta_{i,j}$, and if n is odd, finally a vector g with $B(g, g) = 2$. We can interpret the vectors and linear functionals from Section 6.1, and in particular from Theorem 6.11, in this context as well: $\{f_i\}$ is still the dual basis of $\{e_i\}$. So we find the well-known isomorphic copy of $\mathfrak{sl}(W)$ in $\mathfrak{o}(V)$, where W is a maximal isotropic subspace; the isomorphism is determined by sending $x \otimes h \in \mathfrak{sl}(W)$ to $T_{x,h} \in \mathfrak{o}(V)$. We use the generating elements of $\mathfrak{sl}(W)$ in finding those for $\mathfrak{o}(V)$, but with extra parameters for which the necessity will become apparent in Section 7.

Theorem 6.18. *Let $\alpha, \beta \in \mathbb{F}$ and write κ for $\sqrt{1+\beta}$. Let*

$$\lambda = \begin{cases} \frac{\alpha}{\alpha+2}, & \text{if } n \text{ is odd,} \\ -\frac{\alpha\kappa}{(\alpha+2)(1+\beta+\kappa)}, & \text{if } n \text{ is even.} \end{cases}$$

Suppose that $(\alpha+2)\beta(\beta+1) \neq 0$ and that $\lambda(2 - \beta + \lambda\beta) \neq 1$. The transformations T_{u_i,v_i} for $i \leqslant n$ realize the graph $\Gamma_{D;n}$ for $n \geqslant 5$ if we take these values for u and v:

$$u_1 = e_1 - e_2, \qquad v_1 = f_1 + f_2 + \alpha\tilde{f},$$

$$u_i = e_{i-1} + e_i, \qquad v_i = f_{i-1} - f_i \quad \text{for } 1 < i < n,$$

$$u_n = e_{n-2} + \beta f_{n-1} + e_n, \qquad v_n = f_{n-2} + e_{n-1} - (1+\beta) f_n,$$

where

$$\tilde{f} = (0, 0, 1, 1, \ldots, 1, 0 \mid 0, 0, 1, -1, \ldots, -1, -1-\beta), \quad \text{if } n \text{ is odd},$$
$$\tilde{f} = \frac{1}{1+\beta+\kappa}\big(0, 0, -\kappa, -\kappa, \ldots, -\kappa, 1 \mid$$
$$0, 0, 1+\beta+\kappa, -1-\beta-\kappa, \ldots, 1+\beta+\kappa, (\beta+1)(\kappa+1)\big), \quad \text{if } n \text{ is even};$$

where we first write the coefficients of e_i, then a bar (|), then those of f_i and finally, if n is odd, the coefficient of g. Then $\mathcal{L} = \langle T_{u_i,v_i}\rangle_{\mathrm{Lie}}$ is $\mathfrak{o}_{2n}$.

Note that $B(\tilde{f}, \tilde{f}) = 0$, and

$$B(\tilde{f}, u_i) = \begin{cases} 0, & \text{if } i \neq 3, \\ 1, & \text{if } i = 3; \end{cases}$$
$$B(\tilde{f}, v_i) = \begin{cases} 0, & \text{if } i \neq 3, \\ -1, & \text{if } i = 3 \text{ and } n \text{ is odd}, \\ \frac{\kappa}{1+\beta+\kappa}, & \text{if } i = 3 \text{ and } n \text{ is even}. \end{cases}$$

Hence $\lambda = -\frac{\alpha}{\alpha+2} B(\tilde{f}, v_3)$.

We will prove this theorem using Theorem 6.17. We will first state a similar theorem for the Lie algebra of type B_{n-1}, Theorem 6.19, then state and prove some additional lemmas necessary to show that the conditions of Theorem 6.17 hold, and finally prove Theorems 6.18 and 6.19 on page 319.

Theorem 6.19. *Let $\gamma \in \mathbb{F}$ be such that $\gamma(\gamma+1) \neq 0$. The transformations T_{u_i,v_i} for $i \in I = \{1, \ldots, n\}$ realize the graph $\Gamma_{B;n}$ for $n \geqslant 5$ if we take these values for u and v:*

$$u_1 = e_1 - e_2, \qquad v_1 = f_1 + f_2,$$
$$u_i = e_{i-1} + e_i, \qquad v_i = f_{i-1} - f_i \quad \text{for } 1 < i < n,$$
$$u_n = \gamma e_{n-2} + f_{n-2} + \gamma e_{n-1} - f_{n-1}, \qquad v_n = e_{n-2} - f_{n-2} + (1-\gamma) e_{n-1} + g.$$

Then $\mathcal{L} = \langle T_{u_i,v_i}\rangle_{\mathrm{Lie}}$ is $\mathfrak{o}_{2n-1}$.

The proof will be similar for D_n and B_{n-1}, so let $\mathcal{L}$ be one of the two algebras defined in Theorems 6.18 and 6.19 and let u_i and v_i be the corresponding vectors. Let G be the group generated by the Siegel transvection groups S_{u_i,v_i}. We denote by Σ the orbit of G on the lines $\langle u_i, v_i\rangle_{\mathbb{F}}$, or alternatively, on the projective infinitesimal Siegel transvections $\mathbb{F}T_{u_i,v_i}$, or alternatively, on the Siegel transvection subgroups $\langle 1 + \mu T_{u_i,v_i} \mid \mu \in \mathbb{F}\rangle_{\mathrm{Gp}}$; bijections between these three sets are given in Lemma 6.20. Before we start checking the conditions of Theorem 6.17, we will first verify that the action of G on these three classes of objects is equivalent and that G is transitive on Σ. This is asserted by the following two lemmas, which are readily proven.

Lemma 6.20. *The action of $1 + \mu T_{w,x}$ on $\mathfrak{o}(V)$ by conjugation from the left is the same as the natural action of $\exp(\mu \operatorname{ad} T_{w,x})$. Furthermore, the bijections sending $\mathbb{F}T_{u,v} \in \Sigma$ to $\langle u, v\rangle_{\mathbb{F}}$ and $\langle 1 + \mu T_{u,v} \mid \mu \in \mathbb{F}\rangle_{\mathrm{Gp}}$ commute with the action of G.*

Lemma 6.21. *All T_{u_i,v_i} are in one orbit under G.*

Lemma 6.22. *Σ contains an extraspecial pair.*

Proof. We take $\langle u_2, v_2\rangle_{\mathbb{F}}$ as the first line. For D_n, the second line is (the line corresponding to)

$$\exp(2\,\mathrm{ad}\,T_{u_3,v_3})T_{u_1,v_1} = T_{u_1+2u_3,v_1+2(1+\alpha)v_3+2(\alpha+2)\lambda u_3};$$

for B_{n-1}, we specialize from these values by setting $\alpha = \lambda = 0$ and obtain as second line (the line corresponding to)

$$\exp(2\,\mathrm{ad}\,T_{u_3,v_3})T_{u_1,v_1} = T_{u_1+2u_3,v_1+2v_3}. \qquad \square$$

Lemma 6.23. *Σ contains a copy of the set of infinitesimal transvections in $\mathfrak{sl}_{n-1}$.*

Proof. We will exhibit two sets of generators of different isomorphic copies of $\mathfrak{sl}_{n-1}$. Let U be the vector space spanned by e_i for $i < n$. If we are studying D_n, then define

$$v_i' = \begin{cases} v_i - \lambda u_i, & i \in \{1, n\} \\ v_i + \lambda u_i, & 1 < i < n; \end{cases} \qquad h_i : U \to \mathbb{F}, x \mapsto -B(v_i, x); \tag{6.4}$$

for B_{n-1}, substitute $\lambda = 0$ (whence $v_i' = v_i$), but let $v_n' = u_n - \gamma v_n$. Additionally, for both B_{n-1} and D_n define $u_i' = u_i$, except that $u_n' = \frac{1}{1+\gamma}(u_n + v_n)$ for B_{n-1}. Then $B(u_i', u_j') = B(v_i', v_j') = 0$ for all $\{i, j\} \neq \{n-1, n\}$. Furthermore, $u_i \otimes h_i = T_{u_i,v_i} = T_{u_i',v_i'}$ as linear transformations, and $U^* := \langle h_i \mid i < n\rangle_{\mathbb{F}}$ is a vector space dual to U where the duality is provided by the form B. The infinitesimal transvections $u_i \otimes h_i$ generate $\mathfrak{sl}(U)$ because of the same arguments that prove Theorem 6.11: the centres and axes of the transvections span U and its dual, respectively, and there is an extraspecial pair (both lines from Lemma 6.22 are in $\mathfrak{sl}(U)$). The group generated by all of the corresponding transvections is a subgroup of G, and it is transitive on all transvections in $\mathfrak{sl}(U)$. So in particular, the group of Siegel transvections is transitive on all infinitesimal Siegel transvections in $\mathfrak{sl}(U)$.

Similarly, we can define $\tilde{U} = \langle u_1', u_2', \ldots, u_{n-2}', u_n'\rangle_{\mathbb{F}}$ and $\tilde{U}^* = \langle v_1', v_2', \ldots, v_{n-2}', v_n'\rangle_{\mathbb{F}}$, on which the transvections generate $\mathfrak{sl}(\tilde{U})$. $\square$

For B_{n-1}, there is a nontrivial intersection between $U + \tilde{U}$ and $U^* + \tilde{U}^*$: since

$$\gamma(1+\gamma)u_n' + v_n' = (1+\gamma)u_n = (1+\gamma)(\gamma u_{n-1} + v_{n-1}) = \gamma(1+\gamma)u_{n-1}' + (1+\gamma)v_{n-1}',$$

the intersection is spanned by $u_n' - u_{n-1}'$, which is $\gamma(1+\gamma)$ times $(1+\gamma)v_{n-1}' - v_n'$.

By the previous lemma, Σ contains an isomorphic copy of $\mathfrak{sl}_4$, which proves the following lemma.

Lemma 6.24. *Σ contains a 4-tuple (T_a, T_b, T_c, T_d) of projectively distinct infinitesimal Siegel transvections such that T_c and T_d do not commute, but every other pair does.*

Lemma 6.25. *Let $T_a = T_{u_a,v_a} \in \Sigma$ and $T_b = T_{u_b,v_b} \in \Sigma$ satisfy $C_\Sigma(T_a) = C_\Sigma(T_b)$. Then $T_a = T_b$.*

Proof. Since G is transitive on Σ, we may assume that $T_a = T_{u_2,v_2} = T_{u_2',v_2'}$. Let us denote the subspace of V perpendicular with respect to the bilinear form B to a vector $u \in V$ by $u^\perp$, and similarly, let us denote the subspace of V perpendicular to a subspace S of V by $S^\perp$.

Recall from Lemma 6.23 the definitions of U and U^*. Pick a nonzero vector $u \in U$ which is perpendicular to v_2', and let nonzero $v \in U^*$ be perpendicular to u and to u_2'. Then the infinitesimal transvection $u \otimes v = T_{u,v}$ is in $\mathfrak{sl}(U)$, hence in Σ, and it commutes with $T_{u_2',v_2'}$. So T_b should also commute with it. Then $\langle u_b, v_b\rangle_{\mathbb{F}}$ either intersects $\langle u, v\rangle_{\mathbb{F}}$, or u_b and v_b are both perpendicular to u

and v. So if $\langle u_b, v_b\rangle_{\mathbb{F}}$ does not intersect all $\langle u, v\rangle_{\mathbb{F}}$ for fixed u and all nonzero $v \in S := u^{\perp} \cap u_2'^{\perp} \cap U^*$, then u_b and v_b are perpendicular to u. Let us consider the case where $\langle u_b, v_b\rangle_{\mathbb{F}}$ intersects every such $\langle u, v\rangle_{\mathbb{F}}$. We will show that u_b and u_v are perpendicular to u in this case as well. S has codimension 1 or 2 in U^* of dimension $n-1$, so its dimension is at least 2. If $\dim S > 2$, then $\langle u_b, v_b\rangle_{\mathbb{F}}$ must contain u to intersect every $\langle u, v\rangle_{\mathbb{F}}$. In that case, u_b and v_b are certainly perpendicular to u. Hence assume $\dim S = 2$ and $u \notin \langle u_b, v_b\rangle_{\mathbb{F}}$. Then for different lines $\langle u, v\rangle_{\mathbb{F}}$, the intersection with $\langle u_b, v_b\rangle_{\mathbb{F}}$ is different. So the intersections span all of $\langle u_b, v_b\rangle_{\mathbb{F}}$. In particular, u_b and v_b themselves are on lines $\langle u, v\rangle_{\mathbb{F}}$. Thus both are perpendicular to u.

We see that u_b and v_b are perpendicular to all of $v_2'^{\perp} \cap U$. Similarly, they are perpendicular to $v_2'^{\perp} \cap \tilde{U}$; that is, they are perpendicular to $S_v := v_2'^{\perp} \cap (U + \tilde{U})$. Dually, we see that u_b and v_b are perpendicular to $S_u := u_2'^{\perp} \cap (U^* + \tilde{U}^*)$.

For B_{n-1}, the intersection of $U + \tilde{U}$ and $U^* + \tilde{U}^*$ is spanned by $u_n' - u_{n-1}'$, which is perpendicular to both u_2' and v_2'; for D_n, the intersection of $U + \tilde{U}$ with $U^* + \tilde{U}^*$ is empty. So $S_u + S_v$ is a $(2n-2)$-dimensional space for D_n and to a $(2n-3)$-dimensional space for B_{n-1}. Since the form is nondegenerate, there is in both cases only a 2-dimensional space of vectors perpendicular to $S_u + S_v$. This space is $\langle u_2', v_2'\rangle_{\mathbb{F}}$. Hence u_b and v_b are in $\langle u_2', v_2'\rangle_{\mathbb{F}}$. □

Lemma 6.26. *The graph $F(\Sigma)$ with vertex set Σ and where two infinitesimal Siegel transvections are adjacent if they generate an algebra isomorphic to $\mathfrak{sl}_2$, is connected.*

Proof. According to Lemma 2.13 of [15], if $|\Sigma| > 1$, then $F(\Sigma)$ is connected if and only if Σ is a conjugacy class in G and $F(\Sigma)$ has an edge. Both of these conditions are fulfilled. □

Lemma 6.27. *G is a quasisimple group.*

Proof. We use Lemma 2.14 of [15]. A weaker version of it states that if the following conditions are satisfied:

- $|\Sigma| > 1$;
- the graph $F(\Sigma)$ from Lemma 6.26 is connected;
- there exists no pair $A \neq C \in \Sigma$ with $C_\Sigma(A) = C_\Sigma(B)$;
- Σ contains an extraspecial pair;
- the elements of Σ correspond to extremal Lie algebra elements;

then G is a quasisimple group. Lemmas 6.22, 6.25 and 6.26 show that the three nontrivial conditions are fulfilled. □

The technical proof of the following lemma uses the previous lemmas and has been omitted here for brevity. It can be found in [11].

Lemma 6.28. *G satisfies conditions* (H2) *from Theorem* 6.16 *and* (X3) *from Theorem* 6.17.

Proof of Theorems 6.18 and 6.19. We intend to apply Theorem 6.17, so we will need to show that its conditions hold.

- Condition (H1′) follows from the fact that for every Siegel transvection subgroup A, we have $A^0 = 1$ or $A^0 = A$.
- Condition (H2) follows from Lemma 6.28.
- Condition (H3) follows from Lemma 6.24.
- Condition (X1) is clearly satisfied.

- Condition (X2) follows from Lemma 6.22.
- Condition (X3) follows from Lemma 6.28.
- Condition (X4) is true for every Siegel transvection subgroup.
- Condition (X5) is clearly satisfied. □

7. The algebras nearly always correspond to these realizations

In this section, we show that a Lie algebra $\mathcal{L}$ realizing one of the graphs $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$ and $\Gamma_{D;n}$, is in the generic case a quotient of the realization $\mathcal{M}$ we found in the previous section. In order to see this, we find different sets of generators. For types B_{n-1} and D_n, we will need the parameters α, β and γ from Theorems 6.18 and 6.19 to have sufficient degrees of freedom to be able to make the sets of generators of $\mathcal{L}$ and $\mathcal{M}$ match up.

Since $\mathcal{M}$ is simple in most cases, it will follow that $\mathcal{L}$ and $\mathcal{M}$ are isomorphic. The only exception is A_{n-1} if $p \mid n$, as is well known from literature (see e.g. [8] and [14]).

Theorem 7.1. *Let $\mathcal{L} = \langle x_1, \dots, x_n \rangle_{\mathrm{Lie}}$ be any Lie algebra realizing the graph $\Gamma_{C;n}$ from Fig.* 1.3. *Suppose that n is even and that $f(x_i, x_{i+1}) \neq 0$ for all i. Then $\mathcal{L}$ is isomorphic to $\mathfrak{sp}_n$.*

Proof. Let $\mathcal{M}$ be the realization from Theorem 6.10. By that Theorem, $\mathcal{M} = \mathfrak{sp}_n$. Denote the generators of $\mathcal{M}$ realizing $\Gamma_{C;n}$ by z_i. Scale x_i such that the extremal form is identical on both sets of generators. Then the map $\sigma : \mathcal{M} \to \mathcal{L}$ mapping each of the monomials $y_{k,m}$ in z_i to the same monomial in x_i, is a Lie algebra homomorphism by Lemma 5.4. Hence $\mathcal{L}$ is a quotient of $\mathcal{M}$. But since $\mathcal{M}$ is simple, $\mathcal{L} \cong \mathcal{M}$. □

Already for Lie algebras of type A_{n-1}, we need more degrees of freedom than can be obtained by just scaling the generators. The following lemma, which is based on Section 5 of [4], will be sufficient.

Lemma 7.2. *Let $\pi, \rho, \sigma \in \mathbb{F}$ all be nonzero. Let x, y, z be extremal elements of $\mathcal{L}$ such that $f(x, yz)^2 \neq 2f(x, y)f(x, z)f(y, z)$ and $f(x, y) \neq 0 \neq f(y, z)$, and such that x and y commute with a set S of elements of $\mathcal{L}$. We can find extremal elements x', y' and z' with the following properties*:

- $\langle x, y, z \rangle_{\mathrm{Lie}} = \langle x', y', z' \rangle_{\mathrm{Lie}}$,
- *x' and y' commute with S,*
- $(f(x', y'), f(x', z'), f(y', z'), f(x', y'z')) = (\pi, \rho, \sigma, 0)$.

Proof. Let $s = f(x, yz)/(f(x, y)f(y, z))$. Let $\hat{x} = \exp(s \operatorname{ad} y)(x)$. Then $\langle x, y, z \rangle_{\mathrm{Lie}} = \langle \hat{x}, y, z \rangle_{\mathrm{Lie}}$ (since $\exp(-s \operatorname{ad} y)(\hat{x}) = x$) and

$$
\begin{aligned}
f(\hat{x}, y) &= f(x, y), \\
f(\hat{x}, z) &= f(x, z) - \frac{f(x, yz)^2}{2f(x, y)f(y, z)}, \\
f(\hat{x}, yz) &= 0.
\end{aligned}
$$

Note that $f(\hat{x}, z) \neq 0$. We drop the circumflex from now on and use x to denote $\hat{x}$. Scale x, y and z to obtain $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$:

$$
\tilde{x} = \sqrt{\frac{\pi \rho f(y, z)}{\sigma f(x, y) f(x, z)}}\, x,
$$

$$\tilde{y} = \sqrt{\frac{\pi\sigma f(x,z)}{\rho f(x,y) f(y,z)}}\, y,$$

$$\tilde{z} = \sqrt{\frac{\rho\sigma f(x,y)}{\pi f(x,z) f(y,z)}}\, z.$$

Now all values of f are as intended, possibly up to a factor of -1, and $f(\tilde{x},\tilde{y})f(\tilde{x},\tilde{z})f(\tilde{y},\tilde{z}) = \pi\rho\sigma$. So either all values of f are exactly as intended (including sign), in which case we are done, or exactly two of them need their sign changed; say $f(\tilde{x},\tilde{y})$ and $f(\tilde{x},\tilde{z})$. Then let $x' = -\tilde{x}$, $y' = \tilde{y}$ and $z' = \tilde{z}$.

y' commutes with S, since it is merely a scaled version of y. By the Jacobi identity, $[x, y]$ commutes with S, as well; hence $x' \in \langle x, [x, y], y\rangle_{\mathbb{F}}$ commutes with S. □

Theorem 7.3. *Let $\mathcal{L} = \langle x_1, \ldots, x_n\rangle_{\mathrm{Lie}}$ be a realization of the graph $\Gamma_{A;n}$ in Fig.* 1.1. *Suppose that the following Zariski-open conditions on the extremal form hold*:

- $f(x_1, x_2x_3)^2 \neq 2f(x_1, x_2)f(x_1, x_3)f(x_2, x_3)$,
- $f(x_i, x_{i+1}) \neq 0$ *for all i.*

Then:

- *if* $\operatorname{char}\mathbb{F} = p > 0$ *and* $p \mid n$, *then $\mathcal{L}$ is isomorphic to either $\mathfrak{sl}_n$ or its simple subalgebra of codimension* 1;
- *otherwise, $\mathcal{L}$ is isomorphic to $\mathfrak{sl}_n$.*

Proof. Let $\mathcal{M}$ be the realization from Theorem 6.11. By that theorem, $\mathcal{M} = \mathfrak{sl}_n$. Denote the generators of $\mathcal{M}$ realizing $\Gamma_{A;n}$ by y_i. We will exhibit a Lie algebra homomorphism from $\mathcal{M}$ to $\mathcal{L}$, showing that $\mathcal{L}$ is a quotient of $\mathcal{M}$. Since $\mathcal{L}$ cannot be one-dimensional, we then have the desired result.

Apply Lemma 7.2 with $(\pi, \rho, \sigma) = (1, 1, 1)$ and $(x, y, z) = (x_1, x_2, x_3)$. We obtain a new set of generators x'_i of $\mathcal{L}$ that still realize $\Gamma_{A;n}$. Also apply Lemma 7.2 with $(x, y, z) = (y_1, y_2, y_3)$ and with the same values for π, ρ and σ, obtaining new generators y'_i. Now for any pair of monomials in y'_1, y'_2 and y'_3, the extremal form on that pair is equal to the extremal form on the corresponding pair of monomials in x'_1, x'_2 and x'_3. For $i > 3$, inductively define $x'_i = f(y_{i-1}, y_i)f(x'_{i-1}, x_i)^{-1}x_i$ and $y'_i = y_i$. Now the extremal form is equal on all pairs of monomials given by $\psi_{\Gamma_{A;n}}(\mathfrak{f})$, so the extremal form is identical. Then the desired Lie algebra homomorphism can be obtained as mapping x'_i to y'_i. □

Theorem 7.4. *Let $\mathcal{L} = \langle x_1, \ldots, x_n\rangle_{\mathrm{Lie}}$ be a realization of the graph $\Gamma_{D;n}$ in Fig.* 1.4. *Then $\mathcal{L}$ is isomorphic to $\mathfrak{o}_{2n}$ if the values of the extremal form satisfy a number of Zariski-open conditions.*

These open conditions can be found as follows. If n is odd,

- we have the condition $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) \neq 8$,
- furthermore, we define α by Eq. (7.5) and β by Eq. (7.4);

if n is even,

- we define α by Eq. (7.8) and β by Eq. (7.7);

then the (other) conditions are

- $f(x_1, x_2x_3)^2 \neq 2f(x_1, x_2)f(x_1, x_3)f(x_2, x_3)$,
- $f(x_n, x_{n-1}x_{n-2})^2 \neq 2f(x_n, x_{n-1})f(x_n, x_{n-2})f(x_{n-1}, x_{n-2})$,

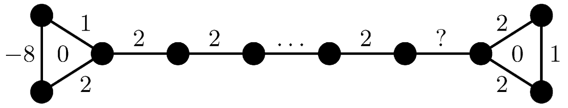

Fig. 7.1. The values of f on $\mathcal{L}$. The numbering of the nodes is the same as in Fig. 1.4.

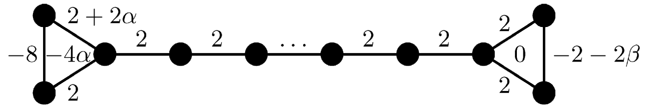

Fig. 7.2. The values of f on $\mathcal{M}$, prior to any changes. The numbering of the nodes is the same as in Fig. 1.4.

- $f(x_i, x_{i+1}) \neq 0$ for all i,
- $(\alpha+2)\beta(\beta+1) \neq 0$,
- $\lambda(2-\beta+\lambda\beta) \neq 1$.

Note that if n is odd and $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) = 8$, then $\mathcal{L}$ is also isomorphic to $\mathfrak{o}_{2n}$, but under slightly different conditions. This can be seen from the proof below.

Proof. Let $\mathcal{M}$ be the realization we defined in Theorem 6.18, for some values of α and β which we will choose later. We will make some changes to the sets of generating elements that do not change the algebra generated by these elements and then claim that f is identical on the two. This shows that $\mathcal{L}$ is isomorphic to a quotient of D_n. Since D_n is simple, this quotient is D_n itself.

Let x_i be the ith extremal generator of $\mathcal{L}$ (so the value of x_i will change during the rest of this proof). First, we perform the procedure of Lemma 7.2 above on x_1, x_2 and x_3 with $(\pi, \rho, \sigma) = (-8, 1, 2)$. Then we do the same on x_{n-2}, x_{n-1} and x_n, this time with $(\pi, \rho, \sigma) = (2, 2, 1)$. Now there are $n-5$ pairs of elements left (on the "connecting line between the two triangles"), on which the value of f has not yet been adjusted, and additionally $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})$. We assume that $f(x_{i-1}, x_i) \neq 0$ for $4 \leqslant i \leqslant n-3$ and scale x_4 up to x_{n-3} such that $f(x_{i-1}, x_i) = 2$. This leaves $f(x_{n-3}, x_{n-2})$ and $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})$. The values of f other than $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})$ are as given in Fig. 7.1.

We now perform a similar procedure on $\mathcal{M}$. Call the ith extremal generator $z_i = T_{u_i, v_i}$. The values of f prior to any changes are as given in Fig. 7.2. We perform the procedure of Lemma 7.2 on z_1, z_2 and z_3, with $(\pi, \rho, \sigma) = (-8, 1, 2)$. Note that $s = \alpha/4$. In the first step, z_1 becomes

$$z_1 - \frac{\alpha}{4}[z_1, z_2] - \frac{\alpha^2}{4} z_2 = T_{u_1 - \frac{\alpha}{2}u_2, v_1 + \frac{\alpha}{2}v_2}.$$

Now

$$f(z_1, z_2) = -8, \qquad f(z_1, z_3) = \frac{(\alpha+2)^2}{2},$$
$$f(z_2, z_3) = 2, \qquad f(z_1, z_2 z_3) = 0,$$

so z_1 and z_3 are divided by $(\alpha+2)/\sqrt{2}$ and z_2 is multiplied with that same constant.

At the other end, we find that applying the procedure of Lemma 7.2 only entails scaling the generators by a factor; in particular, for $(\pi, \rho, \sigma) = (2, 2, 1)$, we divide z_{n-1} and z_n by $\sqrt{-2-2\beta}$ and multiply z_{n-2} by the same factor. Finally, to obtain a 2 for the value of $f(z_{i-1}, z_i)$ where $4 \leqslant i \leqslant n-3$, we multiply z_i by $(\alpha+2)/\sqrt{2}$ for even i and divide it by that factor for odd i. Hence

$$f(z_{n-3}, z_{n-2}) = (2\alpha+4)^{(-1)^{n+1}} \sqrt{-1-\beta}. \tag{7.1}$$

Now consider $z = z_{3\uparrow n-2} z_{n\downarrow 2}$. The generators involved have been scaled, but not changed otherwise. For the generators that occur twice, viz. $z_3, z_4, \ldots, z_{n-2}$, their scaling factor affects z twice; the scaling factors of the other three (z_2, z_{n-1}, and z_n) have an effect only once. In total, z was multiplied by $(\alpha+2)/\sqrt{2}$ by all these scalings, if n is odd, and divided by that constant if n is even. We will now compute the value of z explicitly. This is easier if we temporarily forget all the scaling that occurred; so until further notice, we will use the values before scaling of the z_i.

With induction it is easy to see that $z_{k\downarrow 2} = (-1)^k (T_{u_k, v_2} + T_{u_2, v_k})$ for $3 \leqslant k < n$. Then

$$\begin{aligned} z_{n\downarrow 2} &= (-1)^{n-1}\big([z_n, T_{u_{n-1}, v_2}] + [z_n, T_{u_2, v_{n-1}}]\big) \\ &= (-1)^n (T_{u_n, v_2} - \beta T_{v_n, v_2} - T_{u_2, v_n} + T_{u_n, u_2}) \\ &= (-1)^n (T_{u_n, u_2+v_2} - T_{u_2-\beta v_2, v_n}). \end{aligned}$$

We can again use induction to see that $z_{k\uparrow n-2} z_{n\downarrow 2} = (-1)^n (T_{u_k, u_2+v_2} - T_{u_2-\beta v_2, v_k})$ for $4 \leqslant k \leqslant n-2$. Finally, we compute

$$\begin{aligned} z = z_{3\uparrow n-2} z_{n\downarrow 2} &= (-1)^n \big([z_3, T_{u_4, u_2+v_2}] - [z_3, T_{u_2-\beta v_2, v_4}]\big) \\ &= (-1)^n (-T_{u_4, v_3} + T_{u_3, u_2+v_2} + T_{u_3, u_4} - T_{u_2-\beta v_2, v_3} + T_{u_3, v_4} - \beta T_{v_3, v_4}) \\ &= (-1)^n (T_{u_3, u_2+v_2+u_4+v_4} - T_{u_2-\beta v_2+u_4-\beta v_4, v_3}). \end{aligned}$$

We re-remember the scaling factors and find that

$$\begin{aligned} f(z_1, z) z_1 &= \big[z_1, [z_1, z]\big] \\ &= \frac{2(-1)^n}{(\alpha+2)^2} \left(\frac{\alpha+2}{\sqrt{2}}\right)^{(-1)^{n+1}} \\ &\quad \times \big[T_{u_1-\frac{\alpha}{2}u_2, v_1+\frac{\alpha}{2}v_2}, [T_{u_1-\frac{\alpha}{2}u_2, v_1+\frac{\alpha}{2}v_2}, T_{u_3, u_2+v_2+u_4+v_4} - T_{u_2-\beta v_2+u_4-\beta v_4, v_3}]\big] \\ &= \begin{cases} f(T_{u_1-\frac{\alpha}{2}u_2, v_1+\frac{\alpha}{2}v_2}, T_{u_2-\beta v_2+u_4-\beta v_4, v_3} - T_{u_3, u_2+v_2+u_4+v_4}) z_1, & \text{if } n \text{ is odd,} \\ \frac{2}{(\alpha+2)^2} f(T_{u_1-\frac{\alpha}{2}u_2, v_1+\frac{\alpha}{2}v_2}, T_{u_3, u_2+v_2+u_4+v_4} - T_{u_2-\beta v_2+u_4-\beta v_4, v_3}) z_1, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

A straightforward computation shows that

$$f(T_{u_1-\frac{\alpha}{2}u_2, v_1+\frac{\alpha}{2}v_2}, T_{u_2-\beta v_2+u_4-\beta v_4, v_3} - T_{u_3, u_2+v_2+u_4+v_4}) = 4\alpha(1+\beta) + 8.$$

We finish the proof by case distinction.

Case 1: Suppose that n is odd. Taking the previous equations together with Eq. (7.1), we need to choose α and β such that

$$f(x_1, x_{3\uparrow n-2} x_{n\downarrow 2}) = 4\alpha(1+\beta) + 8 \tag{7.2}$$

and

$$f(x_{n-3}, x_{n-2}) = (2\alpha+4)\sqrt{-1-\beta}. \tag{7.3}$$

If $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) = 8$, then we choose $\alpha = 0$ and $\beta = -1 - f(x_{n-3}, x_{n-2})^2/16$, otherwise we use Eq. (7.2) to obtain

$$1 + \beta = \frac{f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) - 8}{4\alpha}; \tag{7.4}$$

substituting this into Eq. (7.3), we obtain

$$(\alpha + 2)\sqrt{\frac{8 - f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})}{\alpha}} = f(x_{n-3}, x_{n-2})$$

or

$$\sqrt{\alpha} + \frac{2}{\sqrt{\alpha}} = \frac{f(x_{n-3}, x_{n-2})}{\sqrt{8 - f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})}}.$$

This is a quadratic equation in $\sqrt{\alpha}$ that can easily be solved to

$$\alpha = \pm\frac{(f(x_{n-3}, x_{n-2}) \pm \sqrt{f(x_{n-3}, x_{n-2})^2 + 8f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) - 64}\,)^2}{32 - 4f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})}, \tag{7.5}$$

four solutions (some of which may coincide). Then β can be found from Eq. (7.4). Now all values of f are the same and hence $\mathcal{L}$ is a quotient of $\mathcal{M}$. Since $\mathcal{M}$ is simple, $\mathcal{L} = \mathcal{M}$.

Case 2: Suppose that n is even. We need to choose α and β such that

$$f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) = -\frac{8\alpha(1 + \beta) + 16}{(\alpha + 2)^2} \tag{7.6}$$

and

$$f(x_{n-3}, x_{n-2}) = (2\alpha + 4)\sqrt{-1 - \beta}.$$

The last equation can be written as

$$\beta = -\frac{f(x_{n-3}, x_{n-2})^2}{2(\alpha + 2)} - 1. \tag{7.7}$$

If we substitute this into Eq. (7.6), we get

$$f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})(\alpha + 2)^3 = 8\alpha f(x_{n-3}, x_{n-2})^2 - 16\alpha - 32, \tag{7.8}$$

which we can solve for α and thus find an explicit value for β. We find an isomorphism again. □

Theorem 7.5. *Let $\mathcal{L} = \langle x_1, \ldots, x_n\rangle_{\text{Lie}}$ be a realization of the graph $\Gamma_{B;n}$ in Fig. 1.2. Then $\mathcal{L}$ is isomorphic to $\mathfrak{o}_{2n-1}$ if the values of the extremal form satisfy these Zariski-open conditions*:

- $f(x_1, x_2x_3)^2 \neq 2f(x_1, x_2)f(x_1, x_3)f(x_2, x_3)$,
- $f(x_i, x_{i+1}) \neq 0$ *for* $i \leqslant n - 2$,
- $f(x_{n-2}, x_n) \neq 0$,

- $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) \neq 0$,
- $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) \neq 8 \cdot (-1)^{n+1}$.

Proof. Let $\mathcal{M}$ be the realization defined in Theorem 6.19; we will specify the value of γ later. Let $z_i = T_{u_i,v_i}$ be the ith extremal generator of $\mathcal{M}$. In the same manner as in the proof of Theorem 7.4, we change x_i such that the values of $f(x_i, x_j)$ are equal to those of $f(z_i, z_j)$ for $i, j \leqslant 3$, and such that $f(x_1, x_2x_3) = f(z_1, z_2z_3)$. By scaling x_i we can assure that $f(x_{i-1}, x_i) = f(z_{i-1}, z_i)$ for $i < n$, and by scaling x_n we can assure that $f(x_{n-2}, x_n) = f(z_{n-2}, z_n)$. Then what remains is assuring that $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2}) = f(z_1, z_{3\uparrow n-2}z_{n\downarrow 2})$.

Like in the proof of Theorem 7.4, we will do this by choosing the value of γ appropriately. This requires explicitly constructing $z_{3\uparrow n-2}z_{n\downarrow 2}$. Again like before, for $3 \leqslant k < n$ we find with induction that $z_{k\downarrow 2} = (-1)^k(T_{u_k,v_2} + T_{u_2,v_k})$. Then

$$z_{n\downarrow 2} = (-1)^{n-1}\big([z_n, T_{u_{n-1},v_2}] + [z_n, T_{u_2,v_{n-1}}]\big) = (-1)^{n-1}T_{u_n,\gamma u_2+v_2}.$$

With induction we see that $z_{k\uparrow n-2}z_{n\downarrow 2} = (-1)^{n-1}T_{\gamma u_k+v_k,\gamma u_2+v_2}$ for $3 < k \leqslant n-2$. Then

$$z_{3\uparrow n-2}z_{n\downarrow 2} = (-1)^{n-1}[T_{u_3,v_3}, T_{\gamma u_4+v_4,\gamma u_2+v_2}] = (-1)^{n-1}T_{\gamma u_3+v_3,\gamma u_2+v_2+\gamma u_4+v_4}.$$

Then

$$\begin{aligned} f(z_1, z_{3\uparrow n-2}z_{n\downarrow 2}) &= 2\big(B(u_1, \gamma u_3+v_3)B(v_1, \gamma u_2+v_2+\gamma u_4+v_4) \\ &\qquad - B(u_1, \gamma u_2+v_2+\gamma u_4+v_4)B(v_1, \gamma u_3+v_3)\big) \\ &= (-1)^n 8\gamma. \end{aligned}$$

So we choose γ to be $(-1)^n/8$ times the value of $f(x_1, x_{3\uparrow n-2}x_{n\downarrow 2})$. Then f is identical on $\mathcal{L}$ and $\mathcal{M}$, so $\mathcal{L}$ is a quotient of $\mathcal{M}$; since $\mathcal{M}$ is simple, they are isomorphic. □

For each of the four infinite families of Chevalley type Lie algebras, we have given a family of graphs such that a generic Lie algebra generated by a set of extremal elements realizing such a graph, is isomorphic to the corresponding Lie algebra.

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