BEHAVIOURS OF CONCURRENT SYSTEMS

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Abstract. Concurrent systems and their behaviours are investigated. The behaviour of a system is understood as the set of processes which the system is capable to realize. The processes may be elementary (indivisible) or may consist of some components. Two ways of composing processes are considered: sequentially (one component is a continuation of another) and in parallel (the components are concurrent, i.e. independent).

The behaviour of a system is defined as a set of processes which can be obtained by composing certain elementary processes. All information on the existing independence is reflected so that the system is completely determined by its behaviour.

It is explained which sets of processes are the behaviours of concurrent systems.

1. Introduction

We consider systems like those discussed by Petri [6], where certain processes may run concurrently (independently). Such systems are represented by Petri nets with distinguished cases. The net representing a system describes the relationships between the conditions which may hold and the events which may occur in the system. The cases are the maximal sets of conditions which may hold concurrently. The set of cases is usually assumed to be closed under the processes corresponding to events. In our considerations we distinguish not only cases but also other (not necessarily maximal) configurations of conditions that may hold concurrently.

We are interested in the systems describable by finite Petri nets whose events have some pre- and postconditions. Cases are assumed to be safe (cf. [5]), which is not an essential restriction (introducing negations of conditions one can always come to safe cases), and leads to proper concurrent schemes of Mazurkiewicz [4].

Our purpose is to characterize the behaviours of systems, i.e. the sets of processes generated by particular systems. We want to have a characterization such that every system is uniquely determined by its behaviour. This requires reflecting the concurrency of conditions and events in the processes or, in other words, representing appropriately non-sequential processes. Holt and Commoner [2] and Petri [7], suggested to describe the processes in a system by unfoldings of the corresponding Petri net. A similar approach has been developed by Mazurkiewicz [3, 4] and
Winkowski [8]. We follow this approach and represent non-sequential processes by casually ordered sets of holdings of conditions. The idea of Winkowski [8] is also exploited to apply the operations of composing processes sequentially and in parallel as a tool to describe the behaviours of systems. In this way we come to certain algebraic criteria allowing us to answer whether a set of processes is the behaviour of a system and how to find such a system.

2. Concurrent systems

We shall consider systems in which concurrent (independent) processes may occur but there are constraints on the concurrence or precedence of these occurrences. Such systems (that are said to be concurrent) will be specified using Petri nets.

Given a binary relation \( R \subseteq X \times Y \) and \( A \subseteq X, \ B \subseteq Y, \ a \in X, \ b \in Y \), we use denotations:

\[
AR := \{ y \in Y : xRy \text{ for some } x \in A \}, \quad aR := \{ a \} R, \\
RB := \{ x \in X : xRy \text{ for some } y \in B \}, \quad Rb := R\{ b \}.
\]

**Definition 2.1.** A *Petri net* is a triple \( N = (B, E, F) \) such that:

1. \( B \cap E = \emptyset \),
2. \( F \subseteq B \times E \cup E \times B \),
3. \( \text{domain} (F) \cup \text{range} (F) = B \cup E \neq \emptyset \).

Each \( b \in B \) (resp. \( e \in E \)) is called a state element (resp. transition element) of \( N \), and \( F \) is called the flow relation of \( N \). Each \( b \in B \) such that \( bFe \) (resp. \( eFb \)) is called an input element (resp. output element) of \( e \). Every subset of state elements is called a constellation. We say that a constellation \( l \) is reachable in one step from a constellation \( k \) iff there exists a non-empty subset \( U \) of transition elements such that: \((Fu \cup uF) \cap (Fv \cup vF) = \emptyset \) for distinct, \( u, v \in U, \ FU \subseteq k, \ UF \subseteq l, \) and \( k-FU = l-UF \). Then we write

\[
k \rightarrow_U l
\]

and call this triple a reachability step. We say that a constellation \( l \) is reachable from a constellation \( k \), and write \( k \Rightarrow l \), iff \( l = k \Rightarrow \) there is a finite sequence of reachability steps:

\[
k = k_0 \Rightarrow_U k_1 \Rightarrow_U \cdots \Rightarrow_U k_n \Rightarrow_U l.
\]

An example of a Petri net is shown in Fig. 1 (as usual, state elements are represented by circles, transition elements by bars, and the flow relation by directed edges). Examples of reachability steps are shown in Fig. 2.
A system will be specified by a Petri net with a set of distinguished constellations.

**Definition 2.2** A (finite, concurrent) system is a quadruple \( S = (B, E, F, C) \) such that:

1. \( N = (B, E, F) \) is a finite Petri net,
2. \( Fe \neq \emptyset \) and \( eF \neq \emptyset \) for every \( e \in E \),
3. \( Fu = Fv \) and \( uF = vF \) implies \( u = v \) for every \( u, v \in E \),
4. \( C \) is a set of constellations of \( N \),
5. \( c \in C \) and \( d \subseteq c \) implies \( d \in C \),
6. \( Fe \in C \) and \( eF \in C \) for every \( e \in E \),
7. for every constellation \( r \) and every \( e \in E \):
   \[(Fe \cap r = \emptyset \text{ and } Fe \cup r \in C) \iff (eF \cap r = \emptyset \text{ and } eF \cup r \in C).\]

The constellations belonging to \( C \) are called **configurations**. The configurations which are maximal (are not proper parts of other configurations) are called **cases**. Thus \( C \) is the set of parts of cases.

The state elements, called conditions, will represent certain atomic situations which may hold or not – depending on the processes which occur in the system. The transition elements, called events, will represent elementary processes which may occur in the system. Each occurrence of an event \( e \) ends a holding of the input conditions (preconditions) of \( e \) and begins a holding of the output conditions (postconditions) of \( e \). A holding of a condition \( b \) begins with an occurrence of exactly one event \( u \in Fb \). Such a holding ends with an occurrence of exactly one event \( v \in bF \). The constellations from the set \( C \) (configurations) represent the sets of conditions which may hold concurrently in the system.

According to (S2), every transition element has input and output elements. (S3) means that the net is simple in the sense of Petri [6]. According to (S5), the set \( C \) of configurations is closed with respect to taking subsets. (S6) ensures that the sets of input elements and the sets of output elements of transition elements are configurations so that the corresponding events can occur (forward and backward). (S7) means that the constraints which are imposed on the occurrences of events have a local character, and implies the safety of configurations regarded as zero-one markings (cf. [5]).

**Example 2.1.** Consider a system consisting of a resource and of two parts \( A \) and \( B \), each using the resource in a phase of its activity. It is assumed that the resource can not be used by the two parts simultaneously. Such a system can be specified by the net shown in Fig. 1 and the following set of cases:

\[
\{\{1, 2, 5\}, \{1, 2, 7\}, \{1, 4, 5\}, \{1, 4, 7\}, \{2, 6\}, \{3, 5\}, \{3, 7\}, \{4, 6\}\}.
\]
The meaning of the elements is as follows:
1: the resource is available,
2 (resp. 5): A (resp. B) needs the resource,
3 (resp. 6): A (resp. B) uses the resource,
4 (resp. 7): A (resp. B) does not use the resource and does not need it,
a (resp. d): A (resp. B) takes the resource,
b (resp. e): A (resp. B) releases the resource,
c (resp. f): A (resp. B) passes a phase in which the resource is not needed.
The specified set of cases is closed with respect to the reachability relation and its inverse (see Fig. 2).

**Example 2.2.** Four balls are moving along a circle clockwise and counter-clockwise (see Fig. 3). Because of collisions the directions of the motions of the balls change. For instance, ball 1 moving clockwise will start to move counter-clockwise after the collision with ball 2 moving counter-clockwise, whereas ball 2 will start to move
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Fig. 3.

clockwise. The system can be specified by the net shown in Fig. 4 and the following set of cases:

\[
\{(1, 2, 3, 4), (1, 2, 3, 4), (1, 2, 3, 4), (1, 2, 3, 4), (1, 2, 3, 4), (1, 2, 3, 4)\}
\]

The meaning of the elements is:

1 (resp. 2, 3, 4): ball 1 (resp. 2, 3, 4) is moving clockwise,
\(\bar{1}\) (resp. \(\bar{2}, \bar{3}, \bar{4}\)): ball 1 (resp. 2, 3, 4) is moving counter-clockwise,
\(12\) (resp. \(23, 34, 41\)): a collision between ball 1 (resp. 2, 3, 4) moving clockwise and ball 2 (resp. 3, 4, 1) moving counter-clockwise.

The set of cases is closed with respect to the reachability relation and its inverse (see fig. 5).

Fig. 4.

Fig. 5.
It is interesting that the system can also be described by the net shown in Fig. 6 with the following set of cases:

\[ \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2\} \].

However, this representation is wrong in the sense of our definition because (S7) is not satisfied (the corresponding marked nets are not safe).

The following facts are simple consequences of our definition of concurrent systems.

**Proposition 2.1.** Let \( N = (B, E, F) \) be a net satisfying (S1)–(S3) and \( C \) a set satisfying (S4)–(S6). Then \( S = (B, E, F, C) \) is a system iff the following condition is satisfied:

\[ (S7') \quad \text{for every configuration } d \text{ and every } e \in E:\]

\[ (Fe \cap d = \emptyset \text{ and } Fe \cup d \in C) \iff (eF \cap d = \emptyset \text{ and } eF \cup d \in C). \]

**Proof.** If either side of the equivalence holds true for a constellation, then this is a configuration.

**Proposition 2.2.** If a constellation \( l \) is reachable from a constellation \( k \) and one of the constellations is a configuration, then the other is also a configuration.

**Proof.** It is sufficient to consider a reachability step \( k \xrightarrow{U} l \) with \( U = \{e_1, \ldots, e_n\} \), to decompose such a step into \( n \) steps:

\[ k = k_0 \xrightarrow{e_1} k_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} k_n = l, \]

and to apply (S7) to each of the obtained steps.

3. Histories

Processes in systems will be described by specifying their histories. Histories will be considered as partially ordered sets of occurrences of situations rather than
sequences of actions. They will be represented by labelled partially ordered sets satisfying appropriate conditions.

Given a partially ordered set \((X, \leq)\), by a chain (resp. anti-chain) we mean a set of mutually comparable (resp. incomparable) elements of \(X\). Given a maximal anti-chain \(Y \subseteq X\), we use denotations:

\[ Y^- := \{ x \in X : x \leq y \text{ for some } y \in Y \} \quad \text{and} \quad Y^+ := \{ x \in X : y \leq x \text{ for some } y \in Y \}. \]

**Definition 3.1.** A (finite) history (over a certain set \(L\) of atomic situations) is a triple \(H = (X, \leq, l)\) such that:

1. \((X, \leq)\) is a finite partially ordered set,
2. \(l : X \to L\) is a mapping (a labelling) assigning elements of \(L\) (labels) to the elements of \(X\),
3. \(l(x) = l(y)\) implies \(x \leq y\) or \(y \leq x\),
4. given a maximal antichain \(Y \subseteq X\) and a maximal chain \(Z \subseteq X\), the intersection \(Y \cap Z\) is non-empty (see Fig. 7).

One may think that such a history \(H\) consists of causally connected occurrences of certain atomic situations belonging to \(L\). The nature of the situations and their occurrences is irrelevant. In particular, the situations may be conditions in a system. Then occurrences may be considered as holdings of such conditions.

The situations are represented by the elements of \(L\). Particular occurrences of situations are represented by the elements of \(X\). The occurrences of a particular situation \(s\) are represented by the elements with the label \(s\). No assumption is made on the presence of a time scale. The ordering \(\leq\) reflects only the causal relation between particular occurrences. Namely, an occurrence \(x\) of \(s = l(x)\) is considered to be a consequence of all occurrences \(y\) such that \(y \leq x\) and \(y \neq x\). Maximal antichains represent global configurations of occurrences that could possibly arise in the history. Maximal chains represent what we may call signal lines throughout the history. A completeness of the causal relation is assumed such that all the occurrences of a
particular situation are causally connected (H3), and all the signal lines are repre-
sented in all the glob configurations by the presence of an occurrence of a situation
(H4).

A process may be a part of another. This can be described by saying that the
histories of the first process occur in the histories of the second. To express that
formally we shall use the concept of occurrence.

**Definition 3.2.** An occurrence $f : H \rightarrow H'$ of a history $H = (X, \leq, l)$ in a history
$H' = (X', \leq', ')$ is an injection $f : X \rightarrow X'$ such that:

1. $x \leq y$ iff $f(x) \leq' f(y)$,
2. $l(x) = l'(f(x))$ for every $x \in X$,
3. $f(x) \leq' z \leq' f(y)$ implies $z' = f(z)$ for some $z \in X$,
4. there exists an antichain $Y' \subseteq X'$ such that for every maximal antichain
$Y \subseteq X$ the sets $f(Y)$ and $Y'$ are disjoint and $f(Y) \cup Y'$ is a maximal
antichain.

If $f : X \rightarrow X'$ is a bijection, then (03) and (04) can be omitted (they follow from (01))
and $f^{-1} : X' \rightarrow X$ is an occurrence. Then $f : H \rightarrow H'$ is called an isomorphism and we
say that $H$ and $H'$ are isomorphic.

A history has an internal structure which is determined by maximal antichains.

**Definitions 3.3.** The restriction of a history $H = (X, \leq, l)$ to a maximal antichain
$Y \subseteq X$ is a history called a cut of $H$. Such a cut $c$ determines two histories:

- **head** ($H$, $c$) := $(Y^-, \leq | Y^-, | Y^-)$ and **tail** ($H$, $c$) := $(Y^+, \leq | Y^+, | Y^+)$.

The set of cuts of $H$ with the ordering

$c \leq d$ iff $c$ is a cut of head($H$, $d$)

is called the cut structure of $H$. To the set of minimal elements of $X$ there corre-
sponds the least cut of $H$. This cut is called the origin of $H$ and is denoted by origin($H$).
Similarly, to the set of maximal elements of $X$ there corresponds the greatest cut of
$H$. This cut is called the end of $H$ and is denoted by end($H$).

**Proposition 3.1.** The cut structure of a history $H$ is a lattice with the least element
origin($H$) and the greatest element end($H$).

**Proof.** Let $H = (X, \leq, l)$. Given two cuts $c$ and $d$ corresponding to maximal
antichains $Y$ and $Z$, respectively, we define

$c \cap d := (Y \cap Z, \leq | Y \cap Z, l | Y \cap Z)$ and
$c \cup d := (Y \cup Z, \leq | Y \cup Z, l | Y \cup Z)$,
where
\[ Y \cap Z := (Y \cap Z^-) \cup (Z \cap Y^-) \quad \text{and} \quad Y \cup Z := (Y \cap Z^+) \cup (Z \cap Y^+). \]

It suffices to prove that \( Y \cap Z \) is a maximal antichain (for \( Y \cup Z \) the proof is similar).

1. \( Y \cap Z \) is an antichain. Suppose the contrary. Then there are \( x \leq y \) in \( Y \cap Z \) and we have one of the following two cases: \( y \notin Y \) or \( y \notin Z \). In the first case \( x \in Y \) so that \( x \in Z \). Then \( y \notin Z \) and there must be \( z \in Z \) such that \( y \leq z \). Thus \( x \leq y \leq z \) or \( x, z \in Z \), which is impossible. In the second case \( x \notin Z \) so that \( x \in Y \). Then \( y \notin Y \) and there must be \( t \in Y \) such that \( y \leq t \). Thus \( x \leq y \leq t \) for \( x, t \in Y \), which is impossible.

2. \( Y \cap Z \) is a maximal antichain. This property can be proved due to (H4). Suppose the contrary. Then there is \( x \) which is incomparable with the elements of \( Y \cap Z \). It suffices to consider the case \( x \in Y^- \cap Z^- \) (the case \( x \in Y^+ \cap Z^+ \) is similar and the other cases are trivial).

Since \( x \) is incomparable with the elements of \( Y \cap Z \), there exist \( y \in Y \) and \( z \in Z \) not in \( Y \cap Z \) such that \( x \leq y, x \leq z, y' \leq z \) for some \( y' \in Y \), and \( z' \leq y \) for some \( z' \in Z \). Due to (H4) we can take a maximal chain containing \( x \) and \( y \) and find \( z'' \in Z \) such that \( x \leq z'' \leq y \). Then \( z'' \in Y \cap Z \) and \( x \) is comparable with \( z'' \). Thus we obtain a contradiction with our assumption.

Another internal structure of a history can be derived by considering suitable partitions of the corresponding partially ordered set.

**Definition 3.4.** Given a history \( H = (X, \leq, I) \), a pair \( s = (U, V) \) of disjoint subsets of \( X \) such that \( U \cup V = X \) and \( u \) is incomparable with \( v \) for every \( u \in U, v \in V \) is called a splitting of \( H \). Such a splitting \( s \) determines two histories:
\[
\text{left}(H, s) := (U, \leq | U, l | U) \quad \text{and} \quad \text{right}(H, s) := (V, \leq | V, l | V).
\]

The set of splittings of \( H \) with the ordering
\[(U, V) \preceq (U', V') \quad \text{iff} \quad U \subseteq U' \]
is called the splitting structure of \( H \). The pair \((\emptyset, X)\) (resp. \((X, \emptyset)\)) is the least (resp. the greatest) splitting of \( H \).

**Proposition 3.2.** The splitting structure of a history is a Boolean Algebra.

**Proof.** Given two splittings \( s = (U, V) \) and \( t = (U', V') \), we can define:
\[ s \cap t := (U \cap U', V \cup (U \cap V')) \quad \text{and} \quad s \cup t := (U \cup (V \cap U'), V \cap V'). \]
The complement of a splitting \( s = (U, V) \) can be defined as \( s' := (V, U) \).

4. Processes

Processes will be represented by isomorphism classes of histories.
Definition 4.1. A (finite) process is an isomorphism class of (finite) histories. The process containing a given history \( H \) will be denoted by \([H]\).

A process \( P \) can be understood as a pattern that shows which atomic situations and according to which causal ordering should occur. When realized, such a process gives isomorphic histories \( H \in P \). To every realization there corresponds a particular history with particular occurrences of atomic situations. For instance, the process in which atomic situations \( a, b, c \) occur such that the occurrence of \( c \) is a direct consequence of the occurrences of \( a \) and \( b \) may be represented by the 'history with unnamed occurrences of situations' shown in Fig. 8. A particular history corresponding to the realization of this process with particular occurrences \( x, y, z \) of \( a, b, c \), resp., is shown in Fig. 9.

![Fig. 8.](image)

![Fig. 9.](image)

That a process is a part of other one can be described by a suitable concept of occurrence. Such a concept can be introduced by means of occurrences of histories.

Definition 4.2. Given histories \( H, I, J, K \) and occurrences \( f : H \rightarrow I, g : J \rightarrow K \), we say that such occurrences are equivalent iff there are isomorphisms \( h : H \rightarrow J, i : I \rightarrow K \) such that \( fi = hg \), i.e. the diagram in Fig. 10 commutes. An occurrence \( U : P \rightarrow Q \) of a process \( P \) in a process \( Q \) is an equivalence class of occurrences of histories of \( P \) in histories of \( Q \). If \( f : H \rightarrow I \) is one of the occurrences belonging to this class, then \( U : P \rightarrow Q \) is written as \([f] : [H] \rightarrow [I] \). Given processes \( P, Q, R \), we define the composition \( W : P \rightarrow R \) of two occurrences \( U : P \rightarrow Q, V : Q \rightarrow R \) by choosing histories \( H \in P, I \in Q, J \in Q, K \in R \), some occurrences \( f : H \rightarrow I, g : J \rightarrow K \) such that \( f \in U \) and \( g \in V \), and an isomorphism \( i : I \rightarrow J \), and by taking \( W = [gif] \), where \( gif \) is the usual

![Fig. 10.](image)
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4. Operations on processes

Compositions of processes will be represented by corresponding operations on processes, called the sequential composition and the parallel composition, respectively. Such (partial, binary) operations play a role similar to that of the concatenation of the usual sequences.

We shall start by defining two auxiliary unary operations.

Definition 5.1. To every cut \( c \) of a history \( H \) of a process \( P \) there corresponds the process \([c]\) which can be identified with the set of labels of elements of \( c \). Such a process (set) will be called a state of \( P \). The process (state) \([\text{origin}(H)]\) depends on \( P \) only. This process (state) will be called the initial state or the domain of \( P \) and will be denoted by \( \text{dom}(P) \). Similarly, the process (state) \([\text{end}(H)]\), depending on \( P \) only, will be called the final state or the codomain of \( P \) and will be denoted by \( \text{cod}(P) \).

Every state of a process may be interpreted as a possible snapshot of this process. Observe that \( \text{dom}(P) = \text{cod}(P) = \text{id}(P) \) for every process \( P \) which is a state of a process.

The sequential composition of processes can be defined as follows.

Definition 5.2. Given two processes \( P \) and \( Q \), due to (H4) in Definition 3.1 there may be at most one process \( R \) with a history \( J \) and a cut \( c \) of \( J \) such that \( \text{head}(J, c) \in P \) and \( \text{tail}(J, c) \in Q \). Such a process \( R \) is called the sequential composition of \( P \) and \( Q \) and is denoted by \( P \cdot Q \). The occurrences corresponding to the inclusions of \( \text{head}(J, c) \) and \( \text{tail}(J, c) \) in \( J \) are called the canonical occurrences of \( P \) and \( Q \) in \( P \cdot Q \), respectively.

Proposition 5.1. Given two processes \( P \) and \( Q \), the sequential composition \( P \cdot Q \) exists iff the following two conditions are satisfied:

1. \( \text{cod}(P) = \text{dom}(Q) \),
2. for every \( H = (X_1, \leq_1, l_1) \in P, I = (X_2, \leq_2, l_2) \in Q \) and every \( x \in X_1, y \in X_2 \) with \( l_1(x) = l_2(y) \) there exist a maximal \( z \in X_1 \) and a minimal \( t \in X_2 \) such that \( x \leq_1 z, t \leq_2 y, \) and \( l_1(z) = l_2(t) \).

Proof. Suppose that the conditions (1) and (2) are satisfied. Due to \( \text{cod}(P) = \text{dom}(Q) \) there exists a one-to-one correspondence between the maximal elements of every \( H \in P \) and the minimal elements of every \( I \in Q \) such that the corresponding elements
have the same labels. Thus there exist histories \( H = (X_1, \preceq_1, l_1) \in P, I = (X_2, \preceq_2, l_2) \in Q \) such that: \( X_0 := X_1 \cap X_2 \) is exactly the set of maximal elements of \( H \), the same \( X_0 \) is exactly the set of minimal elements of \( I \), and \( l_1 | X_0 = l_2 | X_0 \). Having such \( H \) and \( I \) we define \( J := (X, \preceq, l) \), where \( X = X_1 \cup X_2, l = l_1 \cup l_2 \), and \( \preceq \) is the weakest ordering such that \( x \preceq_1 y \) or \( x \preceq_2 y \) implies \( x \preceq y \). It remains to prove that \( J \) is a history.

It follows from the condition (1) and from the definition of the ordering \( \preceq \) that (H3) is satisfied. We shall prove that (H4) is also satisfied.

Let \( Y \subseteq X \) be a maximal antichain. We shall prove that \( U := (Y - X_2) \cup (X_0 \cap Y^+) \) is a maximal antichain.

That \( U \) is an antichain can be shown as in the proof of Proposition 3.1. Suppose that \( U \) is not a maximal antichain. Then we have \( x \), say in \( X_1 \), such that \( x \) is incomparable with the elements of \( U \). There exists \( y \in Y \) which is comparable with \( x \) and such \( y \) must belong to \( Y - X_1 \). Thus \( y \in Y \cap X_2 \) and \( y \notin X_0 \). By the definition of the ordering \( \preceq \), there exists \( z \in X_0 \) such that \( x \preceq z \preceq y \). Such \( z \) must belong to \( X_0 \cap Y^+ \) (otherwise \( z \in X_0 \cap Y^+ \) and \( y \) would be comparable with another element of \( Y \)). Then \( x \) is comparable with an element of \( X_0 \cap Y^- \). This contradicts to our assumption. For similar reasons we can not have \( x \in X_2 \) which is incomparable with the elements of \( U \).

Analogously, we can prove that \( V := (Y - X_1) \cup (X_0 \cap Y^-) \) is a maximal antichain.

Now we shall prove that the intersection of \( Y \) with every maximal chain \( Z \subseteq X \) is non-empty. The chain \( Z \cap X_1 \) is maximal in \( X_1 \) and the chain \( Z \cap X_2 \) is maximal in \( X_2 \). Thus \( Z \cap X_1 \cap U \neq \emptyset \) and \( Z \cap X_2 \cap V \neq \emptyset \). If \( (Z \cap X_1) \cap (Y - X_2) = \emptyset \) then \( (Z \cap X_1) \cap (X_0 \cap Y^-) \neq \emptyset \) and \( (Z \cap X_1) \cap Y \neq \emptyset \) or \( (Z \cap X_1) \cap Y^+ = \emptyset \). In the first case we have \( Z \cap Y \neq \emptyset \). In the second case we have \( (Z \cap X_2) \cap V = (Z \cap X_2) \cap ((Y - X_1) \cup (X_0 \cap Y^+)) \) with \( (Z \cap X_2) \cap (X_0 \cap Y^+) = \emptyset \), so that \( (Z \cap X_2) \cap (Y - X_1) \neq \emptyset \), i.e. \( Z \cap Y \neq \emptyset \).

Thus (H4) is satisfied and \( J \) is a history such that \( \text{head}(J, c) \in P \) and \( \text{tail}(J, c) \in Q \) for the cut \( c \) corresponding to the maximal antichain \( X_0 \).

Suppose that \( P \cdot Q \) exists. Then \( \text{cod}(P) = \text{dom}(Q) \) by the definition of the sequential composition and the condition (2) is satisfied due to (H3) and (H4).

Intuitively, \( P \cdot Q \) is obtained by 'glueing' every maximal element of \( P \) with the minimal element of \( Q \) that has the same label. An example is shown in Fig. 11 (the edges resulting from the transitivity of the ordering are omitted).

\[
\begin{bmatrix}
4 & 1 \\
5 & 3
\end{bmatrix} \quad \begin{bmatrix}
2 & 6 \\
4 & 1 \\
5 & 5
\end{bmatrix} = \begin{bmatrix}
2 & 6 \\
4 & 1 \\
5 & 5
\end{bmatrix}
\]

Fig. 11.
The sequential composition $P \cdot Q$ represents the process of executing the processes $P$ and $Q$ one after another. The final state $\text{dom}(P)$ of the first process must be exactly the initial state $\text{cod}(Q)$ of the second.

The following result is a direct consequence of Proposition 5.1.

**Proposition 5.2.** The sequential composition is associative ($(P \cdot Q) \cdot R = P \cdot (Q \cdot R)$ whenever either side is defined), $\text{dom}(P \cdot Q) = \text{dom}(P)$ and $\text{cod}(P \cdot Q) = \text{cod}(Q)$ whenever $P \cdot Q$ is defined, and $\text{dom}(P) \cdot P = P \cdot \text{cod}(P) = P$ for every process $P$.

The parallel composition of processes can be defined as follows.

**Definition 5.3.** Given two processes $P$ and $Q$, there may be at most one process $R$ with a history $J$ and a splitting $s$ of $J$ such that $\text{left}(J, s) \subseteq P$ and $\text{right}(J, s) \subseteq Q$. Such a process $R$ is called the parallel composition of $P$ and $Q$ and is denoted by $P + Q$. The occurrences corresponding to the inclusions of $\text{left}(J, s)$ and $\text{right}(J, s)$ in $J$ are called the canonical occurrences of $P$ and $Q$ in $P + Q$, respectively.

**Proposition 5.3.** Given two processes $P$ and $Q$, the parallel composition $P + Q$ exists iff the sets of labels occurring in $P$ and $Q$ are disjoint.

**Proof.** If the sets of labels occurring in $P$ and $Q$ are disjoint, then there are histories $H = (X_1, \leq_1, l_1) \subseteq P$ and $I = (X_2, \leq_2, l_2) \subseteq Q$ with $X_1 \cap X_2 = \emptyset$, $\leq_1 \cap \leq_2 = \emptyset$, and $l_1 \cap l_2 = \emptyset$. Thus we can define $J = (X, \leq, l)$, where $X = X_1 \cup X_2$, $\leq = \leq_1 \cup \leq_2$, and $l = l_1 \cup l_2$. That $J$ is a history follows directly from the fact that the maximal antichains of $(X, \leq)$ are exactly disjoint unions of the maximal antichains of $(X_1, \leq_1)$ and $(X_2, \leq_2)$ and that every chain of $(X, \leq)$ is contained either in $X_1$ or in $X_2$.

That the sets of labels occurring in $P$ and $Q$ are disjoint if $P + Q$ exists follows from (H3).

Intuitively, $P + Q$ is obtained by taking a process which consists of two independent parts $P$ and $Q$. An example is shown in Fig. 12.

![Fig. 12](image-url)

The parallel composition $P + Q$ represents the process of executing concurrently (independently) the processes $P$ and $Q$. Such a process exists iff the components are independent in the sense that they do not contain a common atomic situation.

The following properties of the parallel composition are immediate.

**Proposition 5.4.** The parallel composition is associative $((P + Q) + R = P + (Q + R)$ whenever either side is defined), commutative $(P + Q = Q + P$ whenever either side is
defined), and has a neutral element \((0 := [(\emptyset, \emptyset, \emptyset)])\) such that \(O + P = P + O = P\) for every process \(P\). If \(P + Q\) is defined, then \(P + \text{dom}(Q)\) and \(P + \text{cod}(Q)\) are also defined. In particular, if \(P + Q\) is defined, then \(\text{dom}(P) + \text{dom}(Q)\) and \(\text{cod}(P) + \text{cod}(Q)\) are defined. Moreover, we have \(\text{dom}(P) + \text{dom}(Q) = \text{dom}(P + Q)\) and \(\text{cod}(P) + \text{cod}(Q) = \text{cod}(P + Q)\) whenever \(P + Q\) is defined.

**Proposition 5.5.** Given two processes \(P\) and \(Q\), the parallel composition \(P + Q\) exists iff for every state \(s\) of \(P\) and every state \(t\) of \(Q\) there exists the parallel composition \(s + t\).

**Proof.** The parallel composition of processes which are states is defined iff such processes are disjoint as sets of labels and such a composition reduces to the usual set theoretic union. The existence of the parallel compositions \(s + t\) for every state \(s\) of \(P\) and every state \(t\) of \(Q\) means thus that the sets of labels occurring in \(P\) and \(Q\) are disjoint.

**Proposition 5.6.** Given processes \(P\), \(Q\), \(R\), \(S\), if \(P \cdot Q\), \(R \cdot S\), \(P + R\), \(P + S\), \(Q + R\), \(Q + S\) exist, then \((P \cdot Q) + (R \cdot S)\) and \((P + R) \cdot (Q + S)\) exist and are identical.

**Proof.** The existence of \((P \cdot Q) + (R \cdot S)\) and \((P + R) \cdot (Q + S)\) follows directly from Propositions 5.1, 5.3 and 5.4. For every \(H \in (P \cdot Q) + (R \cdot S)\) there are: a splitting \(s\) of \(H\) such that \(\text{left}(H, s) \in P \cdot Q\) and \(\text{right}(H, s) \in R \cdot S\), a cut \(c\) of \(\text{left}(H, s)\) such that \(\text{head}(\text{left}(H, s), c) \in P\) and \(\text{tail}(\text{left}(H, s), c) \in Q\), an\(\cdot\) a cut \(d\) of \(\text{right}(H, s)\) such that \(\text{head}(\text{right}(H, s), d) \in R\) and \(\text{tail}(\text{right}(H, s), d) \in S\). Taking the cut \(e\) of \(H\) consisting of \(c\) and \(d\) we can decompose \(s\) into a splitting \(t\) of \(\text{head}(H, e)\) and a splitting \(u\) of \(\text{tail}(H, e)\) such that \(\text{left}(\text{head}(H, e), t) \in P\) and \(\text{right}(\text{head}(H, e), t) \in R\) and \(\text{left}(\text{tail}(H, e), u) \in Q\) and \(\text{right}(\text{tail}(H, e), u) \in S\). Thus \((P \cdot Q) + (R \cdot S) = (P + R) \cdot (Q + S)\).

**Proposition 5.7.** Given processes \(P\), \(Q\), \(R\), \(S\), if \((P \cdot Q) + (R \cdot S)\) exists, then \(P + R\), \(P + S\), \(Q + R\), \(Q + S\), \((P + R) \cdot (Q + S)\) also exist and \((P \cdot Q) + (R \cdot S) = (P + R) \cdot (Q + S)\).

**Proof.** It suffices to apply Propositions 5.5 and 5.6.

**Propositions 5.8.** Every process \(P\) which can be obtained by composing sequentially and in parallel given processes \(P_1, \ldots, P_r\) and their compositions can be represented in the following "sequential" form:

\[
P = (P_{11} + \cdots + P_{1n_1} + P_{i_1}) \cdots \cdots (P_{m_1} + \cdots + P_{mm_m} + P_{i_m}),
\]

where \(P_{i_1}, \ldots, P_{i_m}\) is a sequence of \(P_1, \ldots, P_r\) and \(P_{11}, \ldots, P_{1n_1}, \ldots, P_{mn_m}\) are of the form \(\text{dom}(P_k)\) or \(\text{cod}(P_k)\) for some \(k \in \{1, \ldots, r\}\).

**Proof.** We can represent \(P\) as a finite binary tree with some of \(P_1, \ldots, P_r\) at the
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terminal nodes and with the symbol · or + at the non-terminal nodes. Due to Propositions 5.2, 5.4, and 5.7, in this tree we can perform replacements as in Fig. 13 without changing the represented process. This allows us to 'move down' the symbols + and to 'move up' the symbols · in the tree. After a finite number of such 'moves' we obtain the needed representation.

![Fig. 13.](image)

That there is an occurrence of a process in other one can be characterized as follows.

**Proposition 5.9.** The existence of an occurrence $E : S \rightarrow R$ of a process $S$ in a process $R$ is equivalent to the existence of processes $U, V, W$ such that $V = \text{dom}(V)$ and $R = U \cdot (S + V) \cdot W$.

**Proof.** That $E : S \rightarrow R$ exists if $R = U \cdot (S + V) \cdot W$ is trivial. Let $E : S \rightarrow R$ be an occurrence. Then there are: $H = (X, \leq, l) \in S, H' = (X', \leq', l') \in R$, and an antichain $Y' \subseteq X'$ such that: $H$ is the restriction of $H'$ to $X, x, y \in X$ and $x \leq z \leq' y$ implies $z \in X, Y' \cap X = \emptyset$, and $Y \cup Y'$ is a maximal antichain of $(X', \leq')$ for every maximal antichain $Y$ of $(X, \leq)$. Let $I$ be the restriction of $H'$ to $Y'$, let $c$ be the cut of $H'$ consisting of $\text{origin}(H)$ and $I$, and let $d$ be the cut of $H'$ consisting of $\text{end}(H)$ and $I$. It suffices to take the processes $U := [\text{head}(H', c)], V := [I], W := [\text{tail}(H', d)]$.

There are processes which can not be decomposed in a nontrivial way. Such 'atomic' processes and their occurrences will play an important role in what follows.

**Definition 5.4.** Let $P$ be a process with a history $H = (X, \leq, l)$. We say that $P$ is a one-element process iff $X$ contains exactly one element (such a process can be identified with the label $l(x)$ of the unique $x \in X$). We say that $P$ is a prime process iff $X$ contains at least two elements, all elements of $X$ are minimal or maximal, and every minimal element is comparable with every maximal element and vice-versa. One-element and prime processes are said to be elementary.

Occurrences of elementary processes in compound ones have an important property.
Proposition 5.10. Let a process $R$ be the sequential or the parallel composition of two processes $P$ and $Q$ with the canonical occurrences $F : P \rightarrow R$ and $G : Q \rightarrow R$. Then every occurrence $E : S \rightarrow R$ of an elementary process $S$ in $R$ has a unique factorization $S \rightarrow P \rightarrow R$ or a unique factorization $S \rightarrow Q \rightarrow R$. If $R = P + Q$, or $R = P \cdot Q$ and $S$ is prime or has no occurrence in $\text{cod}(P) = \text{dom}(Q)$, then only one of the two factorizations exists.

Proof. The only non-trivial case is $R = P \cdot Q$ with prime $S$. Then there are $H = (X, \leq, l) \in S, H' = (X', \leq', l') \in R,$ and a maximal antichain $Y'$ of $(X', \leq')$ such that: $H$ is the restriction of $H'$ to $X$, $x, y \notin X$ and $x \leq' z \leq' y$ implies $z = x$ or $z = y$, and $\text{head}(H', c) \in P, \text{tail}(H', c) \in Q$ for the cut $c$ of $H'$ that corresponds to $Y'$. It remains to prove that $X \subseteq (Y')^-$ or $X \subseteq (Y')^+$. Suppose the contrary. Then $X - (Y')^+ \neq \emptyset$ and $X - (Y')^- \neq \emptyset$. Thus in $X - (Y')^+$ there may be only minimal elements of $X$. Similarly, in $X - (Y')^-$ there may be only maximal elements of $X$. So, if we take $p \in X - (Y')^+$ and $q \in X - (Y')^-$, then $p \leq q$ and $p \neq q$. By (H4) of Definition 3.1 there exists $r$ such that $p \leq' r \leq' q$ and it must be $r \neq p, r \neq q$. On the other hand, there are no elements of $X'$ between a minimal element of $X$ and a maximal one. Thus we obtain a contradiction.

An important consequence of Proposition 5.10 is that the number of occurrences of prime processes in a process $R$ which can be decomposed into $P \cdot Q$ or $P - Q$ is the sum of the numbers of occurrences in $P$ and $Q$. Thus the proofs on processes can be carried out by induction on the number of occurrences of prime processes in a considered process.

It is interesting that all processes can be decomposed into elementary ones.

Proposition 5.11. Every process can be decomposed into elementary processes.

Proof. Let $P$ be a process with a history $H = (X, \leq, l)$. There exists a finite maximal chain of cuts:

$$\text{origin}(H) = c_0 \subseteq c_1 \subseteq \cdots \subseteq c_m = \text{end}(H)$$

and $c_{i-1} \neq c_i$ for $i \in \{1, \ldots, m\}$.

For every $i \in \{1, \ldots, m\}$ we take the maximal antichains $Y_{i-1}, Y_i$ corresponding to $c_{i-1}, c_i$, define $H_i$ as the restriction of $H$ to $(Y_{i-1} - Y_i) \cup (Y_i - Y_{i-1})$, define $H_{ij}$ as the restrictions of $H$ to the one-element sets $\{x_{ij}\} \subseteq \{x_{i1}, \ldots, x_{im}\} = Y_{i-1} \cap Y_i$.

Since $Y_{i-1}$ and $Y_i$ are different maximal antichains, there are $x \in Y_{i-1} - Y_i$ and $y \in Y_i - Y_{i-1}$. If such elements were incomparable, then there would be a maximal antichain $Z$ containing $x$ and $y$. Considering the cut $d$ corresponding to $Z$ and the cut $e = (c_{i-1} \sqcup d) \cap c_i$ we would have $c_{i-1} \subseteq e \subseteq c_i$ with $x, y \in e$, and thus $e \neq c_{i-1}, e \neq c_i$, so that the chain $c_0, c_1, \ldots, c_m$ could not be maximal. As a consequence, $P_i := [H_i]$ is a prime process.
Taking the prime processes $P_i$ and the one-element processes $P_{ij} := [H_{ij}]$ we obtain:

$$P = (P_{11} + \cdots + P_{1n_1} + P_1) \cdots (P_{m1} + \cdots + P_{mn_m} + P_m).$$

6. Sets of processes generated by systems

The behaviour of a concurrent system will be specified as the set of processes which can occur in the system. This will be done by assigning one-element processes to state elements and prime processes to transition elements and by taking suitable compositions of the assigned processes. The obtained set of processes will correspond to the behaviour in the sense that the reachability via the processes will be exactly the usual reachability. We shall show that the behaviours have certain algebraic properties which are characteristic in the sense that every set of processes having such properties corresponds exactly to the behaviour of a system.

**Definition 6.1.** Given a concurrent system $S = (B, E, F, C)$, we define a process $P(b)$ corresponding to a state element $b$ of $S$ as the one-element process with the label $b$ (such a process can be identified with the set $\{b\}$), and a process $P(e)$ corresponding to a transition element $e$ of $S$ as the prime process whose domain is $Fe$ and whose codomain is $eF$ (according to our convention, the domain and codomain may be regarded as the sets $Fe$ and $eF$, resp.). Given a subset $A \subseteq B \cup E$, by $P(A)$ we denote the set of processes corresponding to the elements of $A$. Given any set $X$ of processes, by closure$(X)$ we denote the closure of $X$ with respect to the sequential and parallel compositions of processes. By a process of the system $S$ we mean every process $P \in \text{closure}(P(B \cup E))$ satisfying the condition:

\[(P1) \quad \text{for every states } u \text{ and } v \text{ of } P \text{ and every } r \in C \text{ we have } u, v \in C \text{ and } (u \cap r = \emptyset \text{ and } u \cup r \in C) \iff (v \cap r = \emptyset \text{ and } v \cup r \in C).\]

The set of processes of $S$ will be called the behaviour of $S$ will be denoted by $\text{processes}(S)$.

**Example 6.1.** The prime processes in Fig. 14 correspond to the transition elements of the system in Example 2.1 (Fig. 1). The process $(P(a) \cdot P(b) \cdot \{5\}) \cdot (P(c) + P(d))$ (shown in Fig. 15) is a process of the system. The process $P(a) + \{6\}$ is not a process of the system ($\text{cod}(P(a) + \{6\}) = \{3, 6\}$ is not a configuration).

**Example 6.2.** For the transition elements of the system in Example 2.2 (Fig 4) we have the prime processes shown in Fig. 16. Taking $(P(12) + P(34)) \cdot ((\bar{1}, 4) + P(23))$ (shown in Fig. 17) we obtain a process of the system.
Let us consider a fixed system $S = (B, E, F, C)$. The properties of the behaviour of $S$ are described in the following series of propositions.

**Proposition 6.1.** $C \cup P(E) \subseteq \text{processes}(S)$.

**Proof.** (P1) is trivially true for every $P \in C$. Due to (S6) and (S7), (P1) is satisfied by $P(e)$ for every $e \in E$. 
Proposition 6.2. If $P \in \text{closure}(P(B \cup E))$ and $\text{dom}(P) \subseteq C$ or $\text{cod}(P) \subseteq C$, then $P \in \text{processes}(S)$.

**Proof.** We shall prove the proposition by induction on the number of occurrences of prime processes in $P$ (cf. Proposition 5.10).

If not more than one prime process occurs in $P$, then it suffices to apply Proposition 6.1 or (S7). Suppose that the number of occurrences of prime processes in $P$ is $n + 1$. Then $P = Q \cdot (R + c)$ with $n$ prime processes occurring in $Q$, $R = P(e)$ for some $e \in E$, and $c \in C$. If $\text{dom}(Q) = \text{dom}(P) \subseteq C$ and we assume that $Q$ is a process of $S$, then $\text{dom}(R + c) = \text{cod}(Q) \subseteq C$ and by (S7) we have: $\text{cod}(R + c) \subseteq C$ and for every $r \in C$: $\text{dom}(R + c) \cap r = \emptyset$ and $\text{dom}(R + c) \cup r \subseteq C$ iff $\text{cod}(R + c) \cap r = \emptyset$ and $\text{cod}(R + c) \cup r \subseteq C$. Thus $P$ is a process of $S$. In the case $\text{cod}(P) \subseteq C$ we obtain the same by considering a decomposition $P = (R + c) \cdot Q$.

Proposition 6.3. If $P$ and $Q$ are any processes such that $P \cdot Q$ exists, then $P \cdot Q \in \text{processes}(S)$ iff $P \in \text{processes}(S)$ and $Q \in \text{processes}(S)$.

**Proof.** If $P$ and $Q$ are processes of $S$ and $P \cdot Q$ exists, then, by Proposition 6.2, also $P \cdot Q$ is a process of $S$. If $P \cdot Q$ exists and is a process of $S$ then, by Proposition 5.8, we have $P = P_1 \cdot \cdots \cdot P_m$ and $Q = Q_1 \cdot \cdots \cdot Q_n$ for some processes $P_1, \ldots, P_m$, $Q_1, \ldots, Q_n$ of $S$. Thus $P$ and $Q$ are also processes of $S$.

Proposition 6.4. The parallel composition of two processes $P$ and $Q$ of $S$ exists and is a process of $S$ iff there exist a state $c$ of $P$ and a state $d$ of $Q$ such that $c \cap d = \emptyset$ and $c \cup d \subseteq C$.

**Proof.** If $P + Q$ exists and is a process of $S$, then the existence of the needed states is a direct consequence of the property (P1). By induction on the total number of occurrences of prime processes in both of processes $P$ and $Q$ we shall prove that the existence of appropriate states ensures the existence of the parallel composition $P + Q$ and that $P + Q$ is a process of $S$.

If not more than one prime process occurs in both of $P$ and $Q$, then it suffices to apply (S7). Suppose that the proposition holds true for not more than $n$ prime processes occurring in $P$ and $Q$. Suppose that $P$ and $Q$ are processes of $S$ such that $n + 1$ prime processes occur in $P$ and $Q$. Then one of the processes, say $Q$, is of the form $R \cdot (U + r)$, where $R$ is a process of $S$, $U$ is a prime process of $S$, $r \in C$, and $U + r$ is a process of $S$. Let $c$ be a state of $P$ and $d$ a state of $Q$ such that $c \cap d = \emptyset$ and $c \cup d \subseteq C$. Suppose that $d$ is a state of $R$, and that $P + R$ exists and is a process of $S$. Then $\text{cod}(R) \cap e = \emptyset$ and $\text{cod}(R) \cup e \subseteq C$ for every state $e$ of $P$. By Proposition 2.1 this implies $\text{cod}(U + r) \cap e = \emptyset$. As a consequence, $P + Q$ exists. On the other hand, every state of $P + Q$ is a disjoint union of states $f_1$ and $f_2$ of $P$ and $Q$, respectively. Besides, due to (S7) and Proposition 5.8, for every $p \subseteq B$...
we have: \( \text{cod}(P + Q) \cap p = \emptyset \) and \( \text{cod}(P + Q) \cup p \in C \) iff \((f_1 \cup f_2) \cap p = \emptyset \) and \((f_1 \cup f_2) \cup p \in C\). Taking \( f_1 = c \) and \( f_2 = d \) we conclude that \( \text{cod}(P + Q) \in C \), i.e., that \( P + Q \) is a process of \( S \). If \( d \) is not a state of \( R \) it suffices to consider \( d = \text{cod}(U + r) \). Then, due to (S7), we have \( c \cap \text{cod}(R) = \emptyset \) and \( c \cup \text{cod}(R) \in C \). Thus we can replace \( d \) by \( \text{cod}(R) \) and come to the previous situation.

**Proposition 6.5.** If \( P \) and \( Q \) are processes of \( S \) such that \( \text{cod}(P) = \text{dom}(Q) \), then the sequential composition \( P \cdot Q \) exists and is a process of \( S \).

**Proof.** If \( \text{cod}(P) = \text{dom}(Q) \), then we have histories \( H = (X_1, <_1, l_1) \in P \) and \( I = (X_2, <_2, l_2) \in Q \) such that \( \text{end}(H) \) is the restriction of \( H \) to \( X_0 := X_1 \cap X_2 \) and \( \text{origin}(I) \) is the restriction of \( I \) to the same \( X_0 \). According to proposition 5.1, it suffices to prove that for every \( p \in X_1 \) and \( q \in X_2 \) with \( l_1(p) = l_2(q) \) there exists \( r \in X_0 \) such that \( p \leq_1 r \) and \( r \leq_2 q \).

Suppose the contrary. Then we have \( p \in X_1 \) and \( q \in X_2 \) satisfying \( l_1(p) = l_2(q) \) and such that \( X_p := \{x \in X_0: p \leq_1 x\} \) and \( X_q := \{x \in X_0: x \leq_2 q\} \) are disjoint.

Let \( c = \text{end}(H) = \text{origin}(I) \). Then there exist a cut \( d \) of \( H \) that contains \( p \) and \( X_0 - X_p \) and a cut \( e \) of \( I \) that contains \( q \) and \( X_0 - X_q \). Thus we have processes \( P_1 \) and \( Q_1 \) of \( S \) and configurations \( s, t, u, v \) such that \( P_1 + u = [\text{tail}(H, d)] \), \( O_1 + t = [\text{head}(I, e)] \), \( t + u = [c] \), \( s + u = [d] \), \( t + v = [e] \), \( s \) contains \( l_1(p) \), \( v \) contains \( l_2(q) \), and \( t, u \) do not contain \( l_1(p) = l_2(q) \). Since \( t + u \) is a configuration and \( t = \text{cod}(P_1) \) and \( u = \text{dom}(Q_1) \), there must exist the parallel composition \( P_1 + Q_1 \). This is however impossible because \( P_1 \) contains the label \( l_1(p) \) and \( Q_1 \) contains the label \( l_2(q) = l_1(p) \).

The following proposition shows that our concept of behaviour is adequate.

**Proposition 6.6.** A configuration \( d \in C \) is reachable from a configuration \( c \in C \) if there exists process \( P \) of \( S \) such that \( \text{dom}(P) = c \) and \( \text{cod}(P) = d \).

**Proof.** (a) By induction on the number of reachability steps we shall prove that \( c \Rightarrow d \) implies the existence of a process \( P \) and \( S \) with \( \text{dom}(P) = c \) and \( \text{cod}(P) = d \).

If \( c \Rightarrow e \) in \( n \) steps with a process \( P \) of \( S \) satisfying \( \text{dom}(P) = c \) and \( \text{cod}(P) = e \), and \( e \Rightarrow d \) for \( U = \{u_1, \ldots, u_m\} \), then, due to Proposition 6.4, \( Q := P(u_1) + \cdots + P(u_m) + (c - FU) \) is a process of \( S \) with \( \text{dom}(Q) = e \) and \( \text{cod}(Q) = d \). By Proposition 6.5, \( P \cdot Q \) is a process of \( S \) satisfying \( \text{dom}(P \cdot Q) = c \) and \( \text{cod}(P \cdot Q) = d \).

(b) Let \( P \) be a process of \( S \) such that \( \text{dom}(P) = c \) and \( \text{cod}(P) = d \). We can represent \( P \) in the (sequential) form:

\[
P = (P_{11} + \cdots + P_{1n_1} + P_1) \cdots \cdots (P_{m1} + \cdots + P_{mn_m} + P_m),
\]

where \( P_i \) are of the form \( P(e_i) \) for some \( e_i \in E \) and \( P_{ij} \) are in \( C \). In this manner we obtain a finite sequence of reachability steps:

\[
c = c_0 \xrightarrow{|e_1|} c_1 \xrightarrow{|e_2|} \cdots \xrightarrow{|e_m|} c_m = d.
\]
Now we shall answer the question which properties of sets of processes are characteristic for the behaviours of concurrent systems. We shall introduce the concept of regular sets of processes and show that the regular sets are exactly the behaviours of concurrent systems.

**Definition 6.2.** A set $U$ of processes is said to be **regular** iff it has the following properties:

1. **(R1)** the labels of all $P \in U$ are from a finite set $L$,
2. **(R2)** if $P \in U$ and $P = Q \cdot R$ or $P = Q + R$, then $Q \in U$ and $R \in U$,
3. **(R3)** if $P \in U$ and $Q \in U$ and $P \cdot Q$ exists, then $P \cdot Q \in U$,
4. **(R4)** given $P \in U$ and $Q \in U$, the parallel composition $P + Q$ exists and belongs to $U$ iff there exist a state $c$ of $P$ and a state $d$ of $Q$ such that $c + d$ exists and belongs to $U$.

Our main result is the following proposition.

**Proposition 6.7.** A set of processes is the behaviour of a concurrent system iff it is regular.

**Proof.** It follows from Propositions 6.1–6.4 that the behaviour of a system is regular.

Let $U$ be a regular set of processes. It follows from (R1) and Proposition 5.11 that there is a finite set $B$ of one-element processes and a finite set $E$ of prime processes such that the labels of the processes from $U$ belong to $B$ and $U \subseteq \text{closure}(B \cup E)$.

Defining $xFy$ as $y \in E$ and $x \in \text{dom}(y)$ or $x \in E$ and $y \in \text{cod}(x)$ we obtain a Petri net $N = (B, E, F)$ satisfying the requirements (S1)–(S3) of Definition 2.2.

Let $C = \{c \in U : c \subseteq \text{dom}(P)$ for some $P \in U\}$. We shall prove that $S = (B, E, F, C)$ is a concurrent system. To this end we have to prove (S7).

Let $e \in E$ and $r \in B$ be such that $Fe \cap r = \emptyset$ and $Fe \cup r \in C$. Then $Fe = \text{dom}(e)$, $Fe \in C$, $r \in C$, and, by (R4), $e + r$ exists and belongs to $U$. Thus $eF \cap r = \emptyset$ and $eF \cup r \in C$. Similarly, $eF \cap r = \emptyset$ and $eF \cup r \in C$ implies $Fe \cap r = \emptyset$ and $Fe \cup r \in C$. In this way we have proved (S7).

It remains to prove that $U = \text{processes}(S)$.

Let $P \in \text{processes}(S)$. Then

$$P = (P_{11} + \cdots + P_{1n_1} + P_1) \cdots (P_{m1} + \cdots + P_{mn_m} + P_m)$$

with some prime $P_i \in U$ and one-element $P_{ij} \in U$ such that $P_{i1} + \cdots + P_{in_i} + \text{dom}(P_i) \subseteq \text{dom}(Q_i)$ for some $Q_i \in U$. Due to (R2) we have $P_{i1} + \cdots + P_{in_i} + \text{dom}(P_i) \subseteq U$. Thus, by (R4), $P_{i1} + \cdots + P_{in_i} + P_i \in U$. As a consequence, due to (R3), we obtain $P \in U$.

Let $P \in U$. By the definition of $B$ and $E$ we have $U \subseteq \text{closure}(P(B \cup E))$. It remains to prove that $P$ enjoys the property (P1).

For every cut $d$ of $H \in P$ we have $[d] = \text{cod}([\text{head}(H, d)])$. If $u$ and $v$ are any states of $P$ and $u \cap r = \emptyset$ and $u \cup r \in C$, then $P + r$ is defined and belongs to $U$. Thus $v + r$ is
also defined and belongs to $U$, i.e., $v \cap r = \emptyset$ and $v \cup r \in C$. Thus we have proved that $P$ is a process of $S$.

Finally, we have $U = \text{processes}(S)$.

As a simple consequence of Propositions 6.5 and 6.7 we obtain that in every regular set $U$ of processes the equality $\text{cod}(P) = \text{dom}(Q)$ implies the existence of the sequential composition of $P$ and $Q$ and $P \cdot Q \in U$. Together with (R4) this yields very simple criteria of sequential and parallel composability of processes in regular sets.

7. Comments

Our characterization of the behaviours of concurrent systems corresponds to that of the behaviours of finite automata but there are also some differences. We consider sets of processes and their sequential and parallel compositions instead of sets of sequences of actions and their concatenations. We take regular sets of processes instead of regular languages of sequences of actions. Such an approach is motivated by the need to describe the behaviours of concurrent systems so as to reflect the independence existing in such systems. The obtained result shows that we indeed come to an adequate description that ensures the possibility to recover a system from its behaviour.

There is a real need to apply means stronger than formal languages. Representing the behaviours with tools like Petri net languages (cf. [1, 5]) we would lose an information on the considered systems. In particular, it would be difficult to identify a concurrent system by investigating its language only.

References