Stability in terms of two measures of dynamic system on time scales

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By using a notion of upper quasi-monotone nondecreasing, this paper presents a new comparison principle which connects the solutions of two higher-dimensional dynamic systems on time scales. Then the stability criteria of a solution of a dynamic system in terms of two measures are obtained. Finally, two examples are provided to illustrate our results.

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1. Introduction

It is well known that dynamic systems on time scales is emerging as an important area of investigation since it demonstrates the interplay of the two different theories, namely, the theories of continuous and discrete dynamic systems [1–3]. The investigation of stability analysis of nonlinear systems has produced a vast body of important results, for example, Lyapunov stability, partial stability, eventual stability, practical stability, and so on [4–6]. The notion which unifies and includes those several known concepts of stability in a simple set up, is the stability in terms of two measures [7].

In the investigation of dynamic systems, the comparison principle is important to discuss the stability of solutions [8,9], and has been applied in dynamic systems on an arbitrary time scale. In the difference system, some stability criteria are obtained via quasi-difference inequality with the notion of upper quasi-monotone nondecreasing [10]. In this paper, the definition of upper quasi-monotone nondecreasing is extended to time scales and gives examples to illustrate the difference with quasi-monotone nondecreasing. On the basis of this study, this paper is developing a new comparison principle which connects the solutions of two higher-dimensions of such equations on time scales [1]. By using the vector Lyapunov functions together with the new comparison principle, criteria of stability for dynamic systems in terms of two measures, are obtained on time scales. In the end of the paper, two examples are provided to illustrate our results.

2. Preliminaries

Let $T$ be a time scale (an arbitrary nonempty closed subset of $\mathbb{R}$ with order and having the topology that it inherits from real numbers with the standard topology). The set $T^k$ is needed which is derived from the time scale $T$ as follows: if $T$ has a left scattered maximum $m$, then $T^k = T - \{m\}$. Otherwise, $T^k = T$. In summary,

$$T^k = \begin{cases} T \setminus \{\rho(\sup T), \sup T\}, & \sup T < \infty \\ T, & \sup T = \infty. \end{cases}$$

Other basic concepts on time scales can be found in [1–3].
Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space with norm $\| \cdot \|_M = \max_{1 \leq i \leq n} |x_i|$. 

**Definition 2.1.** The mapping $g: \mathbb{T} \rightarrow X$, where $X$ is a Banach space, is called $rd$-continuous if it is continuous in each right-dense $t \in \mathbb{T}$ and the left-sided limit $g(t^-)$ exists in each left-dense $t$.

**Definition 2.2.** For each $t \in \mathbb{T}_k$, let $U$ be a neighborhood of $t$. Defining the generalized derivative (or Dini derivative), $D^+u^A(t)$ means that, given $\epsilon > 0$, there exists a right neighborhood $U_\epsilon \subset U$ of $t$ such that

$$
\frac{u(\sigma(t)) - u(s)}{\mu(t, s)} < D^+u^A(t) + \epsilon,
$$

for all $s \in U_\epsilon$, $s > t$, where $\mu(t, s) \equiv \sigma(t) - s$. 

In case $t \in \mathbb{T}_k$ is right-scattered and $u(t)$ is continuous at $t$, we have

$$
D^+u^A(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)},
$$

where $\mu^*(t) = \sigma(t) - t$. 

**Definition 2.3.** A function $a(r)$ is said to be belong to class $\mathcal{K}$ if $a \in C[\mathbb{R}^+ \times \mathbb{R}^+_k]$, $a(r) = 0$ and $a(r)$ is strictly increasing in $r$.

3. **Comparison principle**

The following definitions are somewhat new and related with that of [1].

**Definition 3.1** (See [1]). A function $f(t, x): \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n (n \geq 1)$ is said to be quasi-monotone nondecreasing in $x$ if $x \leq y$ and $x_i = y_i$ for some $1 \leq i \leq n$ imply $f_i(x) \leq f_i(y)$.

**Definition 3.2.** A function $g(t, u): \mathbb{T} \times \mathbb{R}^n_+ \rightarrow \mathbb{R}^n (n \geq 1)$, is said to be upper quasi-monotone nondecreasing in $u$ if $u, w \in \mathbb{R}^n_+$ and $u \leq \max_{1 \leq i \leq n} w_i v$ imply $g(t, u) \leq \max_{1 \leq i \leq n} g(t, w) v$, in which $v = (v_1, v_2, \ldots, v_n)^T$, $v_i \equiv 1, i = 1, 2, \ldots, n$.

For Definition 3.2, we can also give another definition as follows.

**Definition 3.3.** A function $g(t, u): \mathbb{T} \times \mathbb{R}^n_+ \rightarrow \mathbb{R}^n (n \geq 1)$, is said to be upper quasi-monotone nondecreasing in $u$ if $u, w \in \mathbb{R}^n_+$ and $\|u\|_M \leq \|w\|_M$ imply $\|g(t, u)\|_M \leq \|g(t, w)\|_M$.

**Remark 3.1.** If $n = 1$, upper quasi-monotone nondecreasing and monotone nondecreasing are equivalent. If $n > 1$, they are not covered by each other. The following examples will simply illustrate that.

**Example 3.1.** If $n = 2$, assume that $g(t, y): \mathbb{T} \times \mathbb{R}^2_+ \rightarrow \mathbb{R}^2$, with

$$
g(t, y) = \begin{pmatrix} g_1(t, y) \\ g_2(t, y) \end{pmatrix} = \begin{pmatrix} y_1^2 \\ y_1 y_2 \end{pmatrix}
$$

and $y = (y_1, y_2)^T$. Clearly, $g(t, y)$ is quasi-monotone nondecreasing. On the other hand, choosing

$$
u = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad \omega = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}.
$$

Note that $u \leq \max_{1 \leq i \leq n} w_i v$, and

$$
g(t, u) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad g(t, \omega) = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}.
$$

which implies that $g(t, y)$ is not upper quasi-monotone nondecreasing.

Hence, quasi-monotone nondecreasing does not cover upper quasi-monotone nondecreasing in this case.

**Example 3.2.** If $n = 2$, assume that $g(t, y): \mathbb{T} \times \mathbb{R}^2_+ \rightarrow \mathbb{R}^2$, with

$$
g(t, y) = \begin{pmatrix} g_1(t, y) \\ g_2(t, y) \end{pmatrix} = \begin{pmatrix} (y_1 - y_2)^2 \\ 1/2 (y_1^2 + y_2^2 + (y_1 + y_2)|y_1 - y_2|) \end{pmatrix}
$$

and $y = (y_1, y_2)^T$. Clearly, $g(t, y)$ is quasi-monotone nondecreasing. On the other hand, choosing

$$
u = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad \omega = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.
$$

Note that $u \leq \max_{1 \leq i \leq n} w_i v$, and

$$
g(t, u) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad g(t, \omega) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.
$$

which implies that $g(t, y)$ is not upper quasi-monotone nondecreasing.

Hence, quasi-monotone nondecreasing does not cover upper quasi-monotone nondecreasing in this case.
and \( y = (y_1, y_2)^T \). Obversely, \( g(t, y) \) is upper quasi-monotone nondecreasing. On the other hand, choosing

\[
  u = \left( \frac{1}{2} \right), \quad \omega = \left( \frac{2}{5} \right).
\]

Note that \( u < \omega \), and

\[
  g(t, u) = \left( \frac{1}{4} \right), \quad g(t, \omega) = \left( \frac{1}{4} \frac{25}{4} \right)
\]

which implies that \( g(t, y) \) is not quasi-monotone nondecreasing. Hence, quasi-monotone nondecreasing is not covered by upper quasi-monotone nondecreasing in this case.

Consider the following dynamic system on time scales.

\[
  \begin{aligned}
    x^A = f(t, x), \\
    x(t_0) = x_0, \quad t_0 \geq 0,
  \end{aligned}
\]

where \( f \in C_d[\mathbb{T}^k \times \mathbb{R}^N, \mathbb{R}^N], f(t, 0) = 0 \) for each \( x \in \mathbb{R}^N \). We shall assume, for convenience, that the solution \( x(t) = x(t, t_0, x_0) \) of (3.1) exists and is unique for \( t \geq t_0 \).

On the basis of the definition, this paper develops a new comparison principle which connects the solutions of two higher-dimensions of such equations on time scales.

**Lemma 3.1.** Let \( v, \omega : \mathbb{T} \to \mathbb{R}^n \) be rd-continuous mappings that are differentiable for each \( t \in \mathbb{T} \) and satisfy

\[
  v^\Delta(t) \leq g(t, v), \quad \max_{1 \leq i \leq n} \omega_i^\Delta(t) v > \max_{1 \leq i \leq n} g_i(t, \omega) v, \quad t \in \mathbb{T},
\]

where \( g \in C_d[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n], g(t, u) \) and \( g(t, u) \mu^*(t) + u \) is upper quasi-monotone nondecreasing in \( u \) for each \( t \in \mathbb{T} \). Then

\[
  v(t_0) < \max_{1 \leq i \leq n} \omega_i(t_0) v,
\]

implies

\[
  v(t) < \max_{1 \leq i \leq n} \omega_i(t) v,
\]

where \( v = (v_1, v_2, \ldots, v_n)^T, v_i \equiv 1, i = 1, 2, \ldots, n \).

**Proof.** We apply the induction principle to prove the statement.

\( A(t) : v(t) < \max_{1 \leq i \leq n} \omega_i(t) v \).

(1) \( A(t_0) \) is clearly satisfied since \( v(t_0) < \max_{1 \leq i \leq n} \omega_i(t_0) v \).

(II) Let \( t \) be right-scattered and \( A(t) \) be true, we shall show that \( A(\sigma(t)) \) is true. Using the definition of the derivative for a right scattered point, we get

\[
  v(\sigma(t)) - \max_{1 \leq i \leq n} \omega_i(\sigma(t)) v = \left[ v^\Delta(t) - \max_{1 \leq i \leq n} \omega_i^\Delta(t) v \right] \mu^*(t) + v(t) - \max_{1 \leq i \leq n} \omega_i(t) v,
\]

which because of (3.2) and the fact that \( g(t, u) \mu^*(t) + u \) is upper quasi-monotone nondecreasing in \( u \) for each \( t \in \mathbb{T} \). Then

\[
  v(\sigma(t)) - \max_{1 \leq i \leq n} \omega_i(\sigma(t)) v < (g(t, v(t)) - \max_{1 \leq i \leq n} g_i(t, \omega(t))) v \mu^*(t) + v(t) - \max_{1 \leq i \leq n} \omega_i(t) v.
\]

Since \( A(t) \) is true, \( A(\sigma(t)) \) is also true.

(III) Let \( t \) be right-dense and \( U \) be a neighborhood of \( t \). Assume that \( A(t) \) is true, we need to show that \( A(s) \) is true for \( s \geq t, s \in U \).

If it is not true, because \( v(t) \) is continuous on \( s \geq t \), then there exists a \( s_0 \in U \), then

\[
  v(s_0) = \max_{1 \leq i \leq n} \omega_i(s_0) v, \quad v(s) \geq \max_{1 \leq i \leq n} \omega_i(s) v, \quad t \leq s_0 < s.
\]

\[
  v'(s_0) \leq g(t, v(s_0)) = \max_{1 \leq i \leq n} g_i(t, \omega(s_0)) v \leq \max_{1 \leq i \leq n} \omega_i^\prime(s_0) v,
\]

so \( v(s) \leq \max_{1 \leq i \leq n} \omega_i(s) v, t \leq s_0 < s \). This is a contradiction with (3.4), so \( A(s) \) is true.
(IV) Let \( t \) be left-dense such that \( A(s) \) is true for \( s < t \). We need to show that \( A(t) \) is true. By rd-continuity of \( \omega \) and \( v \), it follows that
\[
v(t) = \lim_{s \to t^-} v(s) < \lim_{s \to t^-} \max_{1 \leq i \leq n} \omega_i(t) \leq \max_{1 \leq i \leq n} \omega_i(t) v.
\]
Hence by the induction principle we conclude that
\[
v(t) \leq \max_{1 \leq i \leq n} \omega_i(t) v, \quad t \in T.
\]
This completes the proof of Lemma 3.1. \( \square \)

Consider another dynamic system on time scales.

\[
\begin{align*}
\dot{u}^A &= g(t, u), \\
u(t_0) &= u_0, \quad t_0 \geq 0
\end{align*}
\]

where
\[
g \in C_d(I \times \mathbb{R}^n), \quad \mathbb{R}_0 = ((\mathbb{I} \cap T) \times \mathbb{R}), \quad I = [t_0, t_0 + a] \cap T, \quad B = \{u \in \mathbb{R}^n : \|u - u_0\| \leq b\}, \quad \text{and } \|g(t, u)\| \leq M \text{ on } \mathbb{R}_0, \quad h = \min(a, \frac{b}{M}).
\]

Theorem 3.1. Let the assumptions of Lemma 3.1 hold, and \( m : I \to \mathbb{R}^n \) be a mapping that is differentiable for each \( t \in I \) satisfying
\[
m^A(t) \leq g(t, m(t)), \quad t \in I.
\]

Then, \( m(t_0) \leq \max_{1 \leq i \leq n} u_i(t_0) v \) implies that
\[
m(t) \leq \max_{1 \leq i \leq n} u_i(t) v, \quad t \in I.
\]

**Proof.** Let \( 0 < \epsilon < b/2 \) and consider the following initial value problem
\[
\begin{align*}
\dot{u}^A(t, \epsilon) &= g(t, u) + \epsilon, \quad u(t_0, \epsilon) = u_0 + \epsilon, \quad t_0 \geq 0.
\end{align*}
\]

Since \( g(t, u) + \epsilon \) is defined and rd-continuous on \( \mathbb{R}_e = \mathbb{I} \times \mathbb{B}_e \) where \( \mathbb{B}_e = \{u \in \mathbb{R}^n : \|u - (u_0 + \epsilon)\| \leq b/2\} \) and \( \mathbb{R}_e \subseteq \mathbb{R}_0 \), we conclude from the local existence theorem on time scales that (3.7) has a solution \( u(t, \epsilon) \) on the interval \( I \). For \( \epsilon > 0 \), we have
\[
\max_{1 \leq i \leq n} u_i(t) v = \max_{1 \leq i \leq n} g_i(t, u(t)) v + \epsilon > \max_{1 \leq i \leq n} g_i(t, u(t)) v,
\]
\[
m^A(t) \leq g(t, m(t)).
\]

And
\[
\max_{1 \leq i \leq n} u_i(t_0, \epsilon) v = \max_{1 \leq i \leq n} u_i(t_0) v + \epsilon > \max_{1 \leq i \leq n} u_i(t_0) v,
\]
\[
m(t_0) \leq \max_{1 \leq i \leq n} u_i(t_0) v < \max_{1 \leq i \leq n} u_i(t_0) v.
\]

Hence, by Lemma 3.1 it follows that
\[
m(t) \leq \max_{1 \leq i \leq n} u_i(t_0) v, \quad t \in I,
\]
and \( \lim_{\epsilon \to 0} \max_{1 \leq i \leq n} u_i(t_0, \epsilon) v = \max_{1 \leq i \leq n} u_i(t_0) v \) uniformly on \( I \). Hence,
\[
m(t) \leq \max_{1 \leq i \leq n} u_i(t_0) v, \quad \text{for } t \in I.
\]

The proof is complete. \( \square \)

4. Stability in terms of two measures

This section needs some class of functions and concepts. The class \( \mathcal{K} \) is defined in Section 2, so we give the others needed in the following.

\[\mathcal{L} = \left\{ \delta \in C[\mathbb{R}_+, \mathbb{R}_+] : \delta(u) \text{ is strictly decreasing in } u \text{ and } \lim_{u \to \infty} \delta(u) = 0 \right\},\]
\[\mathcal{C} \mathcal{K} = \{ a \in C_{ld}([T \times \mathbb{R}_+, \mathbb{R}_+] : a(t, s) \in \mathcal{K} \text{ for each } t \},\]
\[\mathcal{T} = \left\{ h \in C_{ld}([T \times \mathbb{R}^N, \mathbb{R}_+] : \inf_{(t, x)} h(t, x) = 0 \right\} .\]

We can define the stability concepts for the system (3.1) in terms of two measures as follow.
**Definition 4.1** (See [1]). The dynamic system (3.1) is said to be

(S1) \((h_0, h)-\text{equi-stable}\) if, for each \(\epsilon > 0\), \(t_0 \in \mathbb{T}\), there exists a positive function \(\delta = \delta(t_0, \epsilon)\) that is rd-continuous in \(t_0\) for each \(\epsilon\) such that \(h_0(t_0, x_0) < \delta\) implies \(h(t, x(t)) < \epsilon\), \(t \geq t_0\), where \(x(t) = x(t, t_0, x_0)\) is any solution of system (3.1);

(S2) \((h_0, h)-\text{uniformly stable}\) if the \(\delta\) in (S1) is independent of \(t_0\).

Other stability concepts for the system (3.1) are defined similarly.

**Definition 4.2** (See [1]). Let \(h_0, h \in \mathcal{T}\). It is said that

(S3) \(h_0\) is finer than \(h\) if there exist a \(\rho > 0\) and a function \(\phi \in \mathcal{C}K\) such that \(h_0(t, x) < \rho\) implies \(h(t, x) \leq \phi(t, h_0(t, x))\);

(S4) \(h_0\) is uniformly finer than \(h\) if \((S3)\) is independent of \(t_0\).

**Definition 4.3.** Let \(V \in C_d([\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n_+], \mathbb{R}^n_+)\). Then \(V\) is said to be

(S5) \(h\)-positive definite if there exist a \(\rho > 0\) and a function \(b \in \mathcal{K}\) such that \(h(b(h(t, x))) \leq \|V(t, x)\|_M\) whenever \(h(t, x) < \rho\);

(S6) \(h\)-decrescent if there exist a \(\rho > 0\) and a function \(a \in \mathcal{K}\) such that \(\|V(t, x)\|_M \leq a(h(t, x))\) whenever \(h(t, x) < \rho\);

(S7) \(h\)-weakly decrescent if there exists a \(\rho > 0\) and a function \(a \in \mathcal{C}K\) such that \(\|V(t, x)\|_M \leq a(t, h(t, x))\) whenever \(h(t, x) < \rho\).

We are now in a position to prove stability criteria in terms of two measures.

**Theorem 4.1.** Assume that

(A1) \(V \in C_d([\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}^n_+], \mathbb{R}^n_+)\), \(V(t, x)\) is locally Lipschitzian in \(x\) and \(h\)-positive;

(A2) \(
\mathcal{D}^TV^2(t, x) \leq 0, \quad \forall (t, x) \in S(h, \rho).
\)

Then \(S(h, \rho) = \{(t, x) \in \mathbb{T} \times \mathbb{R}^n : h(t, x) < \rho, \rho > 0\}\). Then

(a) if, in addition, \(h_0\) is finer than \(h\) and \(V(t, x)\) is \(h_0\)-weakly decrescent, then the system (3.1) is \((h_0, h)\)-equi-stable,

(b) if, in addition, \(h_0\) is uniformly finer than \(h\) and \(V(t, x)\) is \(h_0\)-decrescent, then the system (3.1) is \((h_0, h)\)-uniformly stable.

**Proof.** Let us first prove (a). Since \(V(t, x)\) is \(h_0\) weakly decrescent, then for \(t_0 \in \mathbb{T}^k, x_0 \in \mathbb{R}^n\), there exist a constant \(\rho_0 > 0\) and a function \(a \in \mathcal{C}K\) such that

\[
\|V(t_0, x_0)\|_M \leq a(t_0, h_0(t_0, x_0)) \quad \text{provided} \quad h_0(t_0, x_0) < \rho_0.
\]

(4.1)

The fact that \(V(t, x)\) is \(h\)-positive definite implies that there exist a constant \(\rho > 0\) and a function \(b \in \mathcal{K}\) such that

\[
b(h(t, x)) \leq \|V(t, x)\|_M \quad \text{whenever} \quad h(t, x) < \rho.
\]

(4.2)

Also, by the assumption that \(h_0\) is finer than \(h\), there exist a constant \(\rho_0 > 0\) and a function \(\phi \in \mathcal{C}K\) such that

\[
h(t_0, x_0) \leq \phi(t_0, h_0(t_0, x_0)) \quad \text{if} \quad h_0(t_0, x_0) < \rho_0
\]

(4.3)

where \(\rho_0\) is chosen so that \(\phi(t_0, \rho_0) < \rho\).

Let \(\epsilon \in (0, \rho)\) and \(t_0 \in \mathbb{T}^k\) be given. By the assumption on \(a\), there exists a \(\delta = \delta(t_0, \epsilon) > 0\) that is rd-continuous in \(t_0\) such that

\[
a(t_0, \delta) < b(\epsilon).
\]

(4.4)

Then \(h_0(t_0, x_0) < \delta\) implies, by (4.1)-(4.4), that

\[
b(h(t_0, x_0)) \leq \|V(t_0, x_0)\|_M \leq a(t_0, h_0(t_0, x_0)) < b(\epsilon),
\]

which in turn yields that \(h(t_0, x_0) < \epsilon\).

We now claim that for every solution \(x(t) = x(t, t_0, x_0)\) of (3.1) with \(h_0(t_0, x_0) < \delta\)

\[
h(t, x(t)) < \epsilon, \quad t \geq t_0.
\]

(4.5)

If this is not true, then there would exist a \(t_1 > t_0\) such that

\[
h(t_1, x(t_1)) \geq \epsilon \quad \text{and} \quad h(t, x(t)) < \epsilon, \quad t \in [t_0, t_1),
\]

(4.6)

for some solution \(x(t) = x(t, t_0, x_0)\) of (3.1).

Then, by Theorem 3.1 with \(g(t, u) = 0\), we get

\[
\|V(t, x)\|_M \leq \|V(t_0, x_0)\|_M, \quad t_0 \leq t \leq t_1,
\]

which, in view of (4.2), (4.4) and (4.6) give

\[
b(\epsilon) \leq b(h(t_1, x(t_1))) \leq \|V(t_1, x(t_1))\|_M \leq \|V(t_0, x_0)\|_M < b(\epsilon),
\]

which is a contradiction. Hence, (4.5) is true and the system (3.1) is \((h_0, h)\)-equi-stable.

To prove (b), note that if \(h_0\) is uniformly finer than \(h\) and \(V(t, x)\) is \(h_0\)-decrescent, the functions \(a, \phi\) in (4.3) and (4.5) are independent of \(t\). Consequently, it is easily seen that the constant \(\delta\) can be chosen to be independent of \(t_0\). Hence, the system (3.1) is \((h_0, h)\)-uniformly stable. The proof is complete. \(\Box\)
We next prove a result on \((h_0, h)\)-uniformly asymptotic stable.

**Theorem 4.2.** Assume that

(A1) \( h_0 \) is uniformly finer than \( h \);
(A2) \( V \in C_{C}([\mathbb{T}^n \times \mathbb{R}^N, \mathbb{R}_+^N]), V(t, x) \) is locally Lipschitzian in \( x \), \( h \)-positive definite and \( h_0 \)-decreasing;
(A3) \( D^+V^2(t, x) \leq -C(h_0(t, x))\psi(t, x) \), \( t, x \in S(h, \rho), \ C \in \mathcal{K} \).

Then the system (3.1) is \((h_0, h)\)-uniformly asymptotic stable.

**Proof.** Since \( V(t, x) \) is \( h \)-positive definite and \( h_0 \)-decreasing, there exist a constant \( 0 < \rho_0 < \rho \), and functions \( a, b \in \mathcal{K} \) such that

\[
 b(h(t, x)) \leq \|V(t, x)\|_M, \quad (t, x) \in S(h, \rho),
\]

and

\[
\|V(t_0, x_0)\|_M \leq a(h_0(t_0, x_0)), \quad \text{if} \ h_0(t, x) < \rho_0.
\]

\((h_0, h)\)-uniform stability of (3.1) follows from Theorem 4.1. Then, let \( \epsilon = \rho_0 > 0 \), and designate it by \( \delta_0 = \delta_0(\rho_0) > 0 \), so that, we have

\[
h_0(t_0, x_0) < \delta_0 \quad \text{implies} \quad h(t, x(t)) < \rho_0, \quad t \geq t_0,
\]

where \( x(t) \) is any solution of (3.1) with \( h_0(t_0, x_0) < \delta_0 \).

Let \( 0 < \epsilon < \rho \) and \( \delta = \delta(\epsilon) > 0 \) be the same \( \delta \) as in Definition 4.1 for \((h_0, h)\)-uniform stability.

Assume that \( h_0(t_0, x_0) < \delta_0 \). Choose \( T = T(\epsilon) = \frac{\delta_0}{C(h_0)} + 1 > 0 \). It is sufficient to show that there exists a \( t^* \in [t_0, t_0 + T] \), such that \( h_0(t^*, x(t^*)) < \delta \). If this is not true, there exists a solution of (3.1) with \( h_0(t_0, x_0) < \delta_0 \) such that

\[
h_0(t^*, x(t^*)) \geq \delta, \quad t \in [t_0, t_0 + T].
\]

Thus

\[
0 < V(t_0 + T, x(t_0 + T)) \leq V(t_0, x_0) - \int_{t_0}^{t_0 + T} C[h_0(s, x(s))\psi]\Delta s,
\]

which yields,

\[
\int_{t_0}^{t_0 + T} C[h_0(s, x(s))\psi]\Delta s \leq \|V(t_0, x_0)\|_M \leq a(h_0(t_0, x_0)) \leq a(\delta_0).
\]

On the other hand, from (4.9) and the definition of \( T(\epsilon) \), we obtain

\[
\int_{t_0}^{t_0 + T} C[(h_0(s, x(s)))\psi]\Delta s \geq C(\delta)T > a(\delta_0),
\]

which is a contradiction. The proof is complete. \( \square \)

5. Comparison result

In this section, we shall prove a comparison result via the new comparison principle set up in Section 3.

**Theorem 5.1.** Assume that

(A1) \( h_0, h \) are rd-continuous, and \( h_0 \) is uniformly finer than \( h \);
(A2) \( V \in C_{C}([\mathbb{T}^n \times \mathbb{R}^N, \mathbb{R}_+^N]), V(t, x) \) is locally Lipschitzian in \( x \), \( h \)-positive definite and \( h_0 \)-decreasing;
(A3) \( D^+V^2(t, x) \leq g(t, V(t, x)), g(t, 0) = 0, g \in C_{C}([\mathbb{T}^n \times \mathbb{R}_+^N, \mathbb{R}^N]), 1 < n \leq N, g(t, u) \) and \( g(t, u)\mu^*(t) + u \) is upper quasi-monotone nondecreasing in \( u \) for each \( t \in \mathbb{T}^n \).

Then the stability properties of the trivial solution of

\[
\begin{align*}
\dot{u}^* &= g(t, u), \\
u(t_0) &= u_0, \quad t_0 \geq 0
\end{align*}
\]

imply the corresponding \((h_0, h)\)-stability properties of (3.1).

**Proof.** We shall prove the \((h_0, h)\)-equi-asymptotic stability of (3.1). For this purpose, let us first prove \((h_0, h)\)-equi-stability. Since \( V(t, x) \) is \( h \)-positive definite, there exist a \( b \in \mathcal{K} \) and \( \rho > 0 \) such that

\[
b(h(t, x)) \leq \|V(t, x)\|_M, \quad \text{whenever} \ h(t, x) < \rho.
\]

\[
0 < V(t_0 + T, x(t_0 + T)) \leq V(t_0, x_0) - \int_{t_0}^{t_0 + T} C[h_0(s, x(s))\psi]\Delta s,
\]

which yields,

\[
\int_{t_0}^{t_0 + T} C[h_0(s, x(s))\psi]\Delta s \leq \|V(t_0, x_0)\|_M \leq a(h_0(t_0, x_0)) \leq a(\delta_0).
\]

On the other hand, from (4.9) and the definition of \( T(\epsilon) \), we obtain

\[
\int_{t_0}^{t_0 + T} C[(h_0(s, x(s)))\psi]\Delta s \geq C(\delta)T > a(\delta_0),
\]

which is a contradiction. The proof is complete. \( \square \)
Let $\epsilon > 0$ and $t_0 \in \mathbb{T}^{\Delta}$ be given. Suppose that the trivial solution of (5.1) is equi-stable. Then given $b(\epsilon) > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$, such that
\[
\| u_0 \|_M < \delta_1 \Rightarrow \| u(t) \|_M < a(\epsilon), \quad t \in \mathbb{T},
\] (5.3)
where $u(t) = u(t, t_0, u_0)$ is any solution of (5.1).
We choose $\| u_0 \|_M = \| V(t_0, x_0) \|_M$. Since $V(t, x)$ is $h_0$-decrecent and $h_0$ is finer than $h$, there exist a constant $\rho_0 > 0$ and functions $a, \phi \in \mathcal{K}$ such that
\[
\| V(t_0, x_0) \|_M \leq a(h_0(t_0, x_0)), \quad \text{if } h_0(t_0, x_0) < \rho_0, \quad \text{and } h_0(t_0, x_0) < \phi(h_0(t_0, x_0))
\] (5.4)
if $h_0(t_0, x_0) < \rho_0$, where $\rho_0$ is chosen to satisfy $\phi(\rho_0) < \rho$.
It follows from (5.2) that
\[
b(h(t_0, x_0)) \leq \| V(t_0, x_0) \|_M \leq a(h_0(t_0, x_0)), \quad \text{if } h_0(t_0, x_0) < \rho_0.
\] (5.5)
Choose $\delta = \delta(t_0, \epsilon) > 0$, such that $0 < \delta \leq \rho_0, a(\delta) < \delta_1$ and let $h_0(t_0, x_0) < \delta$. Then (5.3) and (5.4) show that
\[
b(h(t_0, x_0)) \leq \| V(t_0, x_0) \|_M \leq a(h_0(t_0, x_0)) < a(\delta) < b(\epsilon),
\]
i.e.
\[
h(t_0, x_0) < \epsilon.
\]
If $h_0(t_0, x_0) < \delta$, we claim that $h(t, x(t)) < \epsilon, \quad t \geq t_0$, where $x(t) = x(t, t_0, x_0)$ is the solution of (3.1) with $h_0(t_0, x_0) < \delta_0$. If this is not true, there exist a $t_1 \in \mathbb{T}, \quad t_1 > t_0$ and the solution of (3.1) satisfying
\[
h(t_1, x(t_1)) \geq \epsilon \quad \text{and} \quad h(t, x(t)) < \epsilon, \quad t_0 \leq t \leq t_1.
\] (5.6)
In view of the fact that if $h_0(t_0, x_0) < \delta, h(t_0, x_0) < \epsilon, \quad t \geq t_0$.
Setting $m(t) = V(t, x(t))$ for $t_0 \leq t \leq t_1$, by Theorem 3.1, we get
\[
V(t, x(t)) \leq \max_{1 \leq i \leq n} u_i(t), \quad t_0 \leq t \leq t_1.
\] (5.7)
Now the relations of (5.2), (5.3), (5.6) and (5.7) yield
\[
a(\epsilon) \leq \| V(t_1, x_1) \|_M \leq \| u(t_1) \|_M < a(\epsilon),
\]
which is a contradiction, proving the $(h_0, h)$-stability of (3.1).
Suppose next that the trivial solution of (5.1) is equi-attractive. Then we have that, given $b(\epsilon) > 0$ from $(h_0, h)$-stability, there exist positive numbers $\delta^*_0 = \delta^*_0(t_0)$ and $T = T(t_0, \epsilon) > 0$ such that
\[
\| u_0 \|_M < \delta^*_0 \implies \| u(t) \|_M < b(\epsilon), \quad t \in \mathbb{T}^{\Delta}.
\] (5.8)
Choosing $\| u_0 \|_M = \| V(t_0, x_0) \|_M$ as before, we find a $\delta^*_0 = \delta^*_0(t_0) > 0$ such that $0 < \delta^*_0 \leq \rho_0$ and $a(\delta^*_0) < \delta^*_1$. Let $h_0(t_0, x_0) < \delta_0$, the estimate (5.7) is valid for all $t \geq t_0$. We claim that $h(t, x(t)) < \epsilon$. If it is not true, we assume that there exists a sequence $\{t_k\}, \quad t_k \geq t_0 + T, \quad t_k \to \infty$ as $k \to \infty$ such that $h(t_k, x(t_k)) \geq \epsilon$. This leads to a contradiction
\[
b(\epsilon) \leq \| V(t_k, x_k) \|_M \leq \| u(t_k) \|_M < b(\epsilon).
\]
Hence, the system (3.1) is $(h_0, h)$-equi-asymptotically stable and the proof is complete. □

6. Examples

Two examples in this section are provided to illustrate our results.

**Example 6.1.** Consider the dynamic system on a time scale $T = \mathbb{R}$.
\[
\begin{align*}
x_1^\Delta &= f_1(t, x) = -\frac{\sin^2 x_1}{x_1^3} - x_1x_2^2, \quad x_1, x_2 \neq 0 \\
x_2^\Delta &= f_2(t, x) = -\frac{\sin^2 x_2}{x_2^3} - x_2x_1^2, \quad x_1, x_2 \neq 0 \\
x_1(t_0) &= x_{10}, \quad x_2(t_0) = x_{20},
\end{align*}
\] (6.1)
where $f(t, x) = (f_1(t, x), f_2(t, x))^T, f(t, 0) = 0$.
We choose $V(t, x) = (V_1, V_2)^T$, where $V_1 = x_1^2, V_2 = x_2^2$, and $h_0(t, x) = h(t, x) = \| x \| = (x_1^2 + x_2^2)^{\frac{1}{2}}$. It can easily be checked that
\[
b(h(t, x)) \leq \| V(t, x) \|_M \leq a(h(t, x)), \quad h(t, x) \leq \varphi(h_0(t, x)),
\]
where \( b(r) = \frac{1}{2} r^2 \), \( a(r) = r^2 \), \( \varphi(r) = r \), and
\[
D^+ V_1^A = -2 \frac{\sin^2 x_1^2}{x_1^2} - 2x_1^2 x_2^2 \leq -2 \frac{\sin^2 V_1}{V_1},
\]
\[
D^+ V_2^A = -2 \frac{\sin^2 x_2^2}{x_2^2} - 2x_1^2 x_2^2 \leq -2 \frac{\sin^2 V_2}{V_2}.
\]

Therefore, we choose
\[
\begin{aligned}
&u_1^A = g_1(t, u) = -2 \frac{\sin^2 u_1}{u_1}, \quad u_1 \neq 0 \\
&u_2^A = g_2(t, u) = -2 \frac{\sin^2 u_2}{u_2}, \quad u_2 \neq 0 \\
&u_1(t_0) = u_{10}, \quad u_2(t_0) = u_{20}
\end{aligned}
\]

where \( g(t, u) = (g_1(t, u), g_2(t, u))^T \in C_{df}[T \times \mathbb{R}^2, \mathbb{R}^2], \) \( g(t, 0) = 0, \) \( 0 \leq u_1, u_2 \leq \frac{\pi}{2} \), and notice that, for \( u, \omega \in \mathbb{R}^+, \) \( \|u\|_M \leq \|\omega\|_M \), we have
\[
\|g(t, u)\|_M \leq \|g(t, \omega)\|_M.
\]

Hence, \( g(t, u) \) is upper quasi-monotone in \( u \). And the zero solution of (6.2) is uniformly stable. Thus from Theorem 5.1, the system (6.1) is \((h_0, h)\)-uniformly stable.

The function \( g(t, u) \) of above system satisfies quasi-monotone nondecreasing. Of course, we can also obtain the same result by the known Theorem 2.4.1 [1]. However, in the following example, the function \( g(t, u) \) violates the quasi-monotone nondecreasing condition, the stability properties of the original system are obtained by our new result also.

**Example 6.2.** Consider the dynamic system on an arbitrary time scale \( T \)
\[
\begin{aligned}
x_1^A &= f_1(t, x) = \frac{1}{2} \left[ \left( -\frac{1}{x_1} - x_2 \right) \sin^2 x_1 + \left( -\frac{1}{x_2} - x_1 \right) \sin^2 x_2 \\
&\quad + \left( -\frac{1}{x_1} - x_2 \right) \sin^2 x_1 + \left( \frac{1}{x_2} + x_1 \right) \sin^2 x_2 \right] - e_\alpha(t, 0)x_1^2, \quad x_1, x_2 > 0,
\end{aligned}
\]
\[
\begin{aligned}
x_2^A &= f_2(t, x) = \frac{1}{2} \left[ \left( -\frac{\sin^2 x_1}{x_1} - \frac{\sin^2 x_2}{x_2} \right) + \left( -\frac{\sin^2 x_1}{x_1} + \frac{\sin^2 x_2}{x_2} \right) \right] - e_\alpha(t, 0)x_2^2, \quad x_1, x_2 \neq 0
\end{aligned}
\]
\[
x_1(t_0) = x_{10}, \quad x_2(t_0) = x_{20}.
\]

where \( f(t, x) = (f_1(t, x), f_2(t, x))^T, f(t, 0) = 0 , \) and \( \alpha \in \mathbb{R}^+ \) is a positive regressive function.

We choose \( V(t, x) = (V_1, V_2)^T \), where \( V_1 = |x_1|, \) \( V_2 = |x_2|, \) and \( h_0(t, x) = h(t, x) = \|x\| = (x_1^2 + x_2^2)^{1/2} \). It can be checked easily that
\[
b(h(t, x)) \leq \|V(t, x)\|_M \leq a(h(t, x)), \quad h(t, x) \leq \varphi(h_0(t, x)),
\]

where \( b(r) = \frac{2}{2} r^2, \) \( a(r) = r, \) \( \varphi(r) = r \), and
\[
\begin{aligned}
D^+ V_1^A &\leq \frac{1}{2} \left[ \left( \frac{-1}{V_1} - V_2 \right) \sin^2 V_1 + \left( -\frac{1}{V_2} - V_1 \right) \sin^2 V_2 + \left( \frac{-1}{V_1} - V_2 \right) \sin^2 V_1 + \left( \frac{1}{V_2} + V_1 \right) \sin^2 V_2 \right],
\end{aligned}
\]
\[
\begin{aligned}
D^+ V_2^A &\leq \frac{1}{2} \left[ \left( -\frac{\sin^2 V_1}{V_1} - \frac{\sin^2 V_2}{V_2} \right) + \left( -\frac{\sin^2 V_1}{V_1} + \frac{\sin^2 V_2}{V_2} \right) \right].
\end{aligned}
\]

Therefore, we choose
\[
\begin{aligned}
&u_1^A = g_1(t, u) = \frac{1}{2} \left[ \left( -\frac{1}{u_1} - u_2 \right) \sin^2 u_1 + \left( \frac{1}{u_2} + u_1 \right) \sin^2 u_2 \right] + \left( -\frac{1}{u_1} - u_2 \right) \sin^2 u_1 + \left( \frac{1}{u_2} + u_1 \right) \sin^2 u_2, \quad u_1, u_2 \neq 0
\end{aligned}
\]
\[
\begin{aligned}
&u_2^A = g_2(t, u) = \frac{1}{2} \left[ \left( -\frac{\sin^2 u_1}{u_1} - \frac{\sin^2 u_2}{u_2} \right) + \left( -\frac{\sin^2 u_1}{u_1} + \frac{\sin^2 u_2}{u_2} \right) \right], \quad u_1, u_2 \neq 0
\end{aligned}
\]
\[
\begin{aligned}
&u_1(t_0) = u_{10}, \quad u_2(t_0) = u_{20}.
\end{aligned}
\]
where $g(t, u) = (g_1(t, u), g_2(t, u))^T \in C_{d}[T \times \mathbb{R}_+^2, \mathbb{R}^2]$, $g(t, 0) = 0$, $0 \leq u_1, u_2 \leq \frac{\pi}{2}$, and notice that the function $g(t, u)$ violates the quasi-monotone nondecreasing condition. Hence, we cannot deduce the stability properties of (6.3). But $g(t, u)$ is upper quasi-monotone nondecreasing in $u$, for $u, \omega \in \mathbb{R}_+^2$, $\|u\|_M \leq \|\omega\|_M$, we have

$$\|g(t, u)\|_M \leq \|g(t, \omega)\|_M.$$ 

And the zero solution of (6.4) is stable. Thus from Theorem 5.1, the system (6.3) is $(h_0, h)$-stable.

References