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On the generalised Selberg integral of Richards and Zheng ☆

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Abstract

In a recent paper Richards and Zheng compute the determinant of a matrix whose entries are given by beta-type integrals, thereby generalising an earlier result by Dixon and Varchenko. They then use their result to obtain a generalisation of the famous Selberg integral.

In this note we point out that the Selberg-generalisation of Richards and Zheng is a special case of an integral over Jack polynomials due to Kadell. We then show how an integral formula for Jack polynomials of Okounkov and Olshanski may be applied to prove Kadell's integral along the lines of Richards and Zheng.

MSC: 05E05; 33B15

Keywords: Selberg integrals; Jack polynomials

Recently, Richards and Zheng established the following theorem [8].

Theorem 1 *(Richards & Zheng). Let* r *be a nonnegative integer,* $x_1, \dots, x_n \in \mathbb{R}$ *and* $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ *such that* $\mathrm{Re}(\alpha_i) > 0$ *for all* $1 \leqslant i \leqslant n$*. If*

$$a_{ij} = \int_{x_i}^{x_{i+1}} y^{j+r-1} \prod_{l=1}^{n} (y - x_l)^{\alpha_l - 1}\, \mathrm{d}y$$

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then

$$\det_{1\leqslant i,j\leqslant n-1}(a_{ij}) = \prod_{1\leqslant i<j\leqslant n}(x_j - x_i)\prod_{\substack{i,j=1\\ i\neq j}}^{n}(x_j - x_i)^{\alpha_i - 1}$$
$$\times \sum_{|\nu|=r}\binom{r}{\nu}\frac{\Gamma(\alpha_1+\nu_1)\cdots\Gamma(\alpha_n+\nu_n)}{\Gamma(\alpha_1+\cdots+\alpha_n+r)}x^{r-\nu}.$$

In the above we use the following notation. The sum on the right is over compositions $\nu = (\nu_1,\dots,\nu_n)$ of r, i.e., $\nu_1,\dots,\nu_n$ are nonnegative integers such that $|\nu| := \nu_1+\cdots+\nu_n = r$. The $\binom{r}{\nu}$ is a multinomial coefficient:

$$\binom{r}{\nu} = \frac{r!}{(\nu_1)!\cdots(\nu_n)!} \quad \text{for } |\nu| = r,$$

and $x^{r-\nu} = x_1^{r-\nu_1}\cdots x_n^{r-\nu_n}$. Finally we note that the principal branch of each term of the form $x^{\alpha-1}$ is fixed by $-\pi/2 < x < 3\pi/2$.

For $r = 0$ Theorem 1 is due to Dixon [2] and Varchenko [10].

Richards and Zheng use Theorem 1 to prove a generalisation of the celebrated Selberg integral. Let $(a)_k$ be a Pochhammer symbol:

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

and define the symmetric polynomial

$$f_r(x;\gamma) = \sum_{|\nu|=r}\binom{r}{\nu}(\gamma)_{\nu_1}\cdots(\gamma)_{\nu_n}x^{r-\nu}.$$

Theorem 2 *(Richards & Zheng). For $\alpha,\beta,\gamma\in\mathbb{C}$ such that*

$$\mathrm{Re}(\alpha) > 0, \quad \mathrm{Re}(\beta) > 0, \quad \mathrm{Re}(\gamma) > -\min\bigl\{1/n, \mathrm{Re}(\alpha+r)/(n-1), \mathrm{Re}(\beta)/(n-1)\bigr\}$$

and r a nonnegative integer,

$$\int_{[0,1]^n} f_r(x;\gamma)\prod_{1\leqslant i<j\leqslant n}|x_i - x_j|^{2\gamma}\prod_{i=1}^{n}x_i^{\alpha-1}(1-x_i)^{\beta-1}\,\mathrm{d}x$$
$$= \frac{\Gamma(\alpha)\Gamma(\alpha+\beta+(n-1)\gamma+r)}{\Gamma(\alpha+r)\Gamma(\alpha+\beta+(n-1)\gamma)}$$
$$\times\prod_{i=1}^{n}\frac{\Gamma(\alpha+(i-1)\gamma+r)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(i+n-2)\gamma+r)\Gamma(\gamma+1)}.$$

Since $f_0(x;\gamma)=1$ this yields the Selberg integral [9] when $r=0$.

Richards and Zheng seem to have been unaware that the polynomial $f_r(x;\gamma)$ is in fact a Jack polynomial. Let $g_r^{(\alpha)}(x)$ be the symmetric polynomial defined on p. 378 of Macdonald's monograph on symmetric functions [5]:

$$\sum_{r=0}^{\infty} g_r^{(\alpha)}(x)\, t^r = \prod_{i=1}^{n} (1-tx_i)^{-1/\alpha}.$$

Using the binomial theorem and the notation $x^{-1}=(x_1^{-1},\dots,x_n^{-1})$ we also have

$$\sum_{r=0}^{\infty} \frac{f_r(x^{-1};\gamma)(tx_1\cdots x_n)^r}{r!} = \prod_{i=1}^{n}\left(\sum_{\nu_i=0}^{\infty} \frac{(\gamma)_{\nu_i}(tx_i)^{\nu_i}}{(\nu_i)!}\right) = \prod_{i=1}^{n}(1-tx_i)^{-\gamma}.$$

Comparing the above two results we are led to conclude that

$$f_r(x;\gamma) = r!\,(x_1\cdots x_n)^r\, g_r^{(1/\gamma)}\big(x^{-1}\big).$$

Now let $P_\lambda^{(\alpha)}(x)$ be the Jack polynomial labelled by the partition λ. On p. 380 of [5] we find that

$$P_{(r)}^{(\alpha)}(x) = \frac{r!\, g_r^{(\alpha)}(x)}{(1/\alpha)_r}.$$

Hence

$$f_r(x;\gamma) = (\gamma)_r (x_1\cdots x_n)^r\, P_{(r)}^{(1/\gamma)}\big(x^{-1}\big).$$

Finally we use the well-known fact that for k a positive integer not exceeding n,

$$P_{(r^k)}^{(\alpha)}(x) = (x_1\cdots x_n)^r\, P_{(r^{n-k})}^{(\alpha)}\big(x^{-1}\big)$$

to arrive at

$$f_r(x;\gamma) = (\gamma)_r\, P_{(r^{n-1})}^{(1/\gamma)}(x).$$

This result shows that Theorem 2 is in fact the $\lambda=(r^{n-1})$ case of Kadell's integral [4] (see also [5, pp. 385–386]).

Theorem 3 *(Kadell). For λ a partition of at most n parts and $\alpha,\beta,\gamma\in\mathbb{C}$ such that*

$$\mathrm{Re}(\alpha) > -\lambda_n,\quad \mathrm{Re}(\beta)>0,\quad \mathrm{Re}(\gamma) > -\min\big\{1/n, \mathrm{Re}(\alpha+\lambda_i)/(n-i), \mathrm{Re}(\beta)/(n-1)\big\}$$

there holds

$$\frac{1}{n!}\int\limits_{[0,1]^n} P_\lambda^{(1/\gamma)}(x) \prod_{1\leqslant i<j\leqslant n} |x_i-x_j|^{2\gamma} \prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1}\,\mathrm{d}x$$
$$= \prod_{1\leqslant i<j\leqslant n} \frac{\Gamma((j-i+1)\gamma+\lambda_i-\lambda_j)}{\Gamma((j-i)\gamma+\lambda_i-\lambda_j)}$$
$$\times \prod_{i=1}^n \frac{\Gamma(\alpha+(n-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)}{\Gamma(\alpha+\beta+(2n-i-1)\gamma+\lambda_i)}.$$

Given that the Richards–Zheng integral is included in Kadell's integral it is a natural question to ask for a proof of the latter along the lines of Richards–Zheng. We address this below, giving a very short proof of Theorem 3. Key is the next theorem, which follows by taking a limit in the q-integral formula for Macdonald polynomials due to Okounkov [6] (Okounkov attributes this limiting case to Olshanski).

Theorem 4 *(Okounkov–Olshanski). Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_{n-1})$ and denote*

$$x_1<y_1<x_2<\cdots<x_{n-1}<y_{n-1}<x_n$$

by $y\prec x$. For λ a partition of at most $n-1$ parts

$$P_\lambda^{(1/\gamma)}(x) = \prod_{i=1}^{n-1} \frac{\Gamma(\lambda_i+(n-i+1)\gamma)}{\Gamma(\lambda_i+(n-i)\gamma)\Gamma(\gamma)} \prod_{1\leqslant i<j\leqslant n} (x_j-x_i)^{1-2\gamma}$$
$$\times \int\limits_{y\prec x} P_\lambda^{(1/\gamma)}(y) \prod_{1\leqslant i<j\leqslant n-1} (y_j-y_i) \prod_{i=1}^{n-1}\prod_{j=1}^{n} |y_i-x_j|^{\gamma-1}\,\mathrm{d}y.$$

As remarked in [6], for $\gamma=1$ this is equivalent to the standard definition of the Schur function as a ratio of determinants.

Proof of Theorem 3. First observe that we may assume without loss of generality that λ has at most $n-1$ parts, i.e., $\lambda_n=0$. Indeed, if we define $\mu=(\lambda_1-\lambda_n,\dots,\lambda_{n-1}-\lambda_n,0)$ and denote Kadell's integral by $I_\lambda(\alpha,\beta,\gamma)$, then

$$I_\lambda(\alpha,\beta,\gamma)=I_\mu(\alpha+\lambda_n,\beta,\gamma)$$

thanks to

$$P_\lambda^{(1/\gamma)}(x)=(x_1\cdots x_n)^{\lambda_n}P_\mu^{(1/\gamma)}(x).$$

We now proceed to prove the theorem by induction on n.

For $n=1$ (so that $\lambda=0$ since $\lambda_n=0$) the integral is nothing but the Euler beta integral [3]

$$\int\limits_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Now assume that $n > 1$ and use the symmetry of the integrand to write Kadell's integral in the form

$$\begin{aligned}
&\int\limits_{0<x_1<x_2\cdots<x_n<1} P_\lambda^{(1/\gamma)}(x) \prod_{1\leqslant i<j\leqslant n} (x_j - x_i)^{2\gamma} \prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1}\,\mathrm{d}x \\
&\quad = \prod_{1\leqslant i<j\leqslant n} \frac{\Gamma((j-i+1)\gamma+\mu_i-\mu_j)}{\Gamma((j-i)\gamma+\mu_i-\mu_j)} \\
&\quad\quad \times \prod_{i=1}^n \frac{\Gamma(\alpha+(n-i)\gamma+\mu_i)\Gamma(\beta+(i-1)\gamma)}{\Gamma(\alpha+\beta+(2n-i-1)\gamma+\mu_i)}.
\end{aligned}$$

Exhibiting the n-dependence we denote the integral on the left by $I_{\lambda;n}(\alpha,\beta,\gamma)$.

Since λ has at most $n-1$ parts we may eliminate the Jack polynomial in the integrand using Theorem 4. Also interchanging the order of the x- and y-integrals then gives

$$\begin{aligned}
I_{\lambda;n}(\alpha,\beta,\gamma) &= \prod_{i=1}^{n-1} \frac{\Gamma(\lambda_i+(n-i+1)\gamma)}{\Gamma(\lambda_i+(n-i)\gamma)\Gamma(\gamma)} \\
&\quad \times \int\limits_{0<y_1<\cdots<y_{n-1}<1} P_\lambda^{(1/\gamma)}(y)\, J_n(y;\alpha,\beta,\gamma) \prod_{1\leqslant i<j\leqslant n-1} (y_j-y_i)\,\mathrm{d}y,
\end{aligned}$$

where

$$\begin{aligned}
J_n(y;\alpha,\beta,\gamma) &= \int\limits_{0<x_1<y_1<\cdots<y_{n-1}<x_n<1} \prod_{1\leqslant i<j\leqslant n} (x_j-x_i) \\
&\quad \times \prod_{i=1}^n \Biggl[x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{j=1}^{n-1} |x_i-y_j|^{\gamma-1} \Biggr] \mathrm{d}x.
\end{aligned}$$

As shown in [7], an immediate consequence of the $r=0$ instance of Theorem 1 is the integral

$$\begin{aligned}
&\int\limits_{y\prec x} \prod_{1\leqslant i<j\leqslant n-1} (y_j-y_i) \prod_{i=1}^{n-1}\prod_{j=1}^{n} |y_i-x_j|^{\alpha_j-1}\,\mathrm{d}y \\
&\quad = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)} \prod_{1\leqslant i<j\leqslant n} (x_j-x_i)^{\alpha_i+\alpha_j-1}.
\end{aligned}$$

(Peter Forrester pointed out to me that this is also implicit in Anderson's proof of the Selberg integral [1].) For $\alpha_1=\cdots=\alpha_n=\gamma$ this is of course the $\lambda=0$ case of Theorem 4.

The integral $J_n(y;\alpha,\beta,\gamma)$ is precisely of the above form, with $n\to n+1$ and

$$(x_1,\dots,x_n;y_1,\dots,y_{n-1};\alpha_1,\dots,\alpha_{n+1}) \to (0,y_1,\dots,y_{n-1},1;x_1,\dots,x_n;\alpha,\underbrace{\gamma,\dots,\gamma}_{n-1\text{ times}},\beta).$$

Therefore

$$J_n(y;\alpha,\beta,\gamma)=\frac{\Gamma^{n-1}(\gamma)\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+(n-1)\gamma)} \times \prod_{1\leqslant i<j\leqslant n-1}(y_j-y_i)^{2\gamma-1}\prod_{i=1}^{n-1}y_i^{\alpha+\gamma-1}(1-y_i)^{\beta+\gamma-1}$$

and, consequently,

$$I_{\lambda;n}(\alpha,\beta,\gamma) =\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+(n-1)\gamma)}\prod_{i=1}^{n-1}\frac{\Gamma(\lambda_i+(n-i+1)\gamma)}{\Gamma(\lambda_i+(n-i)\gamma)} \times \int\limits_{0<y_1<\dots<y_{n-1}<1} P_\lambda^{(1/\gamma)}(y)\prod_{1\leqslant i<j\leqslant n-1}(y_j-y_i)^{2\gamma}\prod_{i=1}^{n-1}y_i^{\alpha+\gamma-1}(1-y_i)^{\beta+\gamma-1}\,\mathrm{d}y.$$

Since the right-hand side is Kadell's integral with n replaced by $n-1$ we have

$$\begin{aligned} I_{\lambda;n}(\alpha,\beta,\gamma) &= I_{\lambda;n-1}(\alpha+\gamma,\beta+\gamma,\gamma) \times \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+(n-1)\gamma)}\prod_{i=1}^{n-1}\frac{\Gamma(\lambda_i+(n-i+1)\gamma)}{\Gamma(\lambda_i+(n-i)\gamma)} \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+(n-1)\gamma)}\prod_{1\leqslant i<j\leqslant n-1}\frac{\Gamma((j-i+1)\gamma+\lambda_i-\lambda_j)}{\Gamma((j-i)\gamma+\lambda_i-\lambda_j)} \\ &\quad\times\prod_{i=1}^{n-1}\frac{\Gamma(\alpha+(n-i)\gamma+\lambda_i)\Gamma(\beta+i\gamma)}{\Gamma(\alpha+\beta+(2n-i-1)\gamma+\lambda_i)}\,\frac{\Gamma(\lambda_i+(n-i+1)\gamma)}{\Gamma(\lambda_i+(n-i)\gamma)}, \end{aligned}$$

where the second equality follows from the usual induction hypothesis. It is readily checked that the final expression on the right is in accordance with the $\lambda_n=0$ case of Theorem 3, thus completing the proof. □

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