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Brauer Groups and Character Groups of Function Fields

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An element y of a p-primary abelian torsion group is said to be of infinite height if for all n the equation $p^n x = y$ has a solution in the group. Let E be a field which is finitely generated over its prime field. In this paper we investigate the subgroup consisting of the elements of infinite height in the p-primary component of $B(E)$, the Brauer group of E, and $X(E)$, the continuous character group of E.

Before stating our results we introduce some notation. We denote the p-primary component of an abelian torsion group G by G_p. The Ulm subgroups of G_p are defined inductively for any ordinal λ by: $G_p(0) = G_p$, $G_p(\lambda + 1) = pG_p(\lambda)$, and for λ a limit ordinal, $G_p(\lambda) = \bigcap_{\beta<\lambda} G_p(\beta)$. If ω denotes the first infinite ordinal, then $G_p(\omega)$ represents simply the elements of infinite height in G_p. $G_p(\omega 2)$ is the subgroup of $G_p(\omega)$ consisting of the elements of $G_p(\omega)$ of infinite height in $G_p(\omega)$. The least ordinal λ such that $G_p(\lambda) = G_p(\lambda + 1)$ is called the Ulm length of G_p and denoted by $l_p(G)$. For $\lambda = l_p(G)$, $G_p(\lambda) = DG_p$, the maximal divisible subgroup of G_p. Since DG_p is a divisible group it is a direct summand of G_p. We denote the Brauer group of a field E by $B(E)$. Let E_{ab} be the maximal abelian extension of E in some algebraic closure of E. Then $\mathrm{Gal}(E_{ab}/E)$ is the Galois group of E_{ab} over E, and $X(E)$ is the continuous character group $\hom(\mathrm{Gal}(E_{ab}/E), Q/Z)$. We denote the characteristic of E by char E. For any subfield F of E, t.d. E/F is the transcendence degree of E over F. By a global field we mean either an algebraic number field or an algebraic function field in one variable over a finite constant field. p will always stand for a prime number.

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The completion of a field K at a nonarchimedean valuation π is written K_π; $\bar{K}_\pi$ stands for the residue class field of K_π at π. The fixed field of the kernel of $\sigma \in X(K)$ is denoted by K^σ; K^σ is a cyclic extension of K of degree equal to the order of σ in $X(E)$. Conversely, if L is a cyclic extension of K, then there are a finite number of $\sigma \in X(K)$ with $L = K^\sigma$.

With this notation established, we can now state our main results.

THEOREM 1. *Let E be a field finitely generated over its prime field and $p \neq \operatorname{char} E$. Then $X(E)_p(\omega) \simeq H \oplus DX(E)_p$ where H is a finite group and $DX(E)_p$ is isomorphic to the direct sum of a finite number of copies of $Z(p^\infty)$. In particular, the Ulm length of $X(E)_p$ is $< \omega 2$.*

As a consequence of Theorem 1 and the results of [3] and [4] we obtain information about certain Brauer groups:

THEOREM 2. *Let E be a purely transcendental extension of a global field F with $1 \leqslant \text{t.d. } E/F < \infty$ and $p \neq \operatorname{char} E$. Then $B(E)_p(\omega) \simeq G \oplus DB(E)_p$ where G is countable and isomorphic to a direct sum of cyclic p-groups of unbounded exponent and $DB(E)_p$ is isomorphic to the direct sum of ω copies of $Z(p^\infty)$. In particular, the Ulm length of $B(E)_p$ equals $\omega 2$. Moreover, if G has a cyclic direct summand of order p^n, then G has a direct summand isomorphic to the direct sum of ω copies of Z/p^nZ.*

We begin the proofs of the theorems with two lemmas. We have made no attempt to prove the lemmas in the greatest generality possible but have stated them only in the form in which they will be used. A special case of the second lemma appears implicitly in the proof of Theorem 4 of [3].

LEMMA 3. *Let E be a field finitely generated over its prime field, π a discrete rank one valuation of E, $p \neq \operatorname{char}(\bar{E}_\pi)$, and let m be a fixed natural number. Then there is an integer $n = n(E, p, \pi, m)$ which is minimal so that for any finite extension L_π of E_π with $[L_\pi : E_\pi]$ dividing m, neither L_π nor $\bar{L}_\pi$ contains a primitive p^nth root of unity.*

Proof. By Hensel's lemma it is enough to prove the result for $\bar{L}_\pi$. Since $[\bar{L}_\pi : \bar{E}_\pi]$ divides m it is clearly enough to show that $\bar{E}_\pi$ is finitely generated over its prime field. We prove this by induction on t.d. E/F where F is the prime field of E. By induction we may suppose that $E = K(t)$ where t.d. $K/F + 1 =$ t.d. E/F and where $\bar{K}_\pi$ is finitely generated over $\bar{F}_\pi$; here K_π and F_π denote the completions of K and F under the valuations obtained by restriction of π.

Let V_E and V_K denote the valuation rings of E and K with respect to π and $\pi|_K$, respectively, and let P_E and P_K denote the maximal ideals of these valuation rings. Since t or $1/t$ is in V_E, we may assume that $t \in V_E$. If $t \in P_E$, then $\bar{E}_\pi \simeq \bar{K}_\pi$ and we are finished. If $t \notin P_E$, then $\bar{E}_\pi \simeq \bar{K}_\pi(t + P_E)$ and again we conclude that $\bar{E}_\pi$ is finitely generated over its prime field. This proves our first lemma.

LEMMA 4. *Let E be finitely generated over its prime field. If $\sigma \in X(E)_p(\omega)$, then E^σ/E is unramified at all discrete rank one valuations π of E with* char $\bar{E}_\pi \neq p$.

Proof. Suppose $\sigma \in X(E)_p(\omega)$ and let π be a discrete rank one valuation of E with char $\bar{E}_\pi \neq p$. Then $E_\pi E^\sigma$ is a cyclic tamely ramified p-extension of E_π. Let T be the maximal unramified extension of $E_\pi E^\sigma$ and suppose that $T \neq E_\pi E^\sigma$. By Lemma 3 with $m = 1$, there is an s such that T does not contain a primitive p^sth root of unity. Since $\sigma \in X(E)_p(\omega)$, there exists a $\delta \in X(E)_p$ with $p^s\delta = \sigma$. Then $p^s = [E^\delta : E^\sigma]$. Let L be the completion of E^δ at a valuation of E^δ extending π. Since E^δ is a cyclic p-extension of E and $p \neq$ char $\bar{E}_\pi$, $\bar{L}$ is separable over $\bar{E}_\pi$ and so Hilbert theory applies. Since the intermediate fields between E^δ and E are linearly ordered and since $T \neq E_\pi E^\sigma$, it follows that the decomposition field for π from E to E^δ is a subfield of E^σ [12, Proposition 4.10.8]. Thus L is a cyclic totally and tamely ramified extension of T and $p^t = [L : T] > [E^\delta : E^\sigma] = p^s$. By [9, Lemma 11, p. 74] T contains a primitive p^tth root of unity, contradicting our choice of s. Thus $T = E_\pi E^\sigma$ and so π is unramified from E to E^σ, as desired.

We now turn to the proof of Theorem 1. Consider the following assertion:

(*) Let L be a field finitely generated over its prime field and let $p \neq$ char L. Then there exist only finitely many $\sigma \in X(L)_p(\omega)$ with σ of order p.

Assume that we have proved (*) and let E be a field finitely generated over its prime field, char $E \neq p$. Let H be a reduced subgroup of $X(E)_p(\omega)$ so that $X(E)_p(\omega) \cong H \oplus DX(E)_p(\omega)$. Finally let Γ be the directed graph whose vertices are elements of H and such that δ is connected to τ if $p\tau = \delta$.

It follows from (*) that each vertex of Γ has finite order. If Γ were finite, the Konig infinity lemma would imply that Γ contains an infinite path. An infinite path in Γ corresponds, however, to a divisible subgroup of H, which is impossible since H is reduced. We conclude that Γ is finite and so H is a finite group. $DX(E)_p(\omega)$ is a direct sum of copies of $Z(p^\infty)$, and (*) implies that there are only finitely many summands. Thus, to prove Theorem 1, it is enough to prove (*).

Assume now that L is finitely generated over its prime field and $p \neq$ char L. Since (*) is obvious if L is a finite field, we may assume that L contains a global field F. We proceed by induction on t.d. L/F.

Suppose first that t.d. $L/F = 0$. Then L is itself a global field. Let $\sigma \in X(L)_p(\omega)$, σ of order p, $p \neq$ char L. By Lemma 4, L^σ/L is unramified at all discrete rank one valuations π of L with char $\bar{L}_\pi \neq p$. By [7, Satz 11.8] there are only finitely many possibilities for L and so there are only finitely many possible σ's.

Assume next that $r =$ t.d. $L/F \geqslant 1$. By [5, p. 166] there is a global field $F_1 \subset L$ and a separating transcendence base $t_1, \ldots, r_\gamma$ for L over F_1 such that L is a finite separable extension of $F_1(t_1, \ldots, t_\gamma)$. Let $K_0 = F_1(t_1, \ldots, t_{\gamma-1})$, let K be the algebraic closure of K_0 in L, and let $t = t_r$. Then L is a finite separable extension of $K(t)$, K is algebraically closed in L, and t.d. $L/F =$ t.d. $K/F_1 + 1$. Let $\sigma \in X(L)_p(\omega)$, σ of order p. By Lemma 4, L^σ/L is unramified at all discrete rank one valuations π of L with char $L_\pi \neq p$.

Suppose $L^\sigma = LW$ where W is algebraic over K. Then W is separable over K since LW is separable over L and K is algebraically closed in L. Let W_1 be the normal closure of W in some algebraic closure of L. By [2, Corollary 1, p. 90], $[W_1 : K] = [LW_1 : L]$. But L^σ/L is Galois so $LW_1 = L^\sigma$ and so $p = [L^\sigma : L] = [W_1 : K]$. Thus $W = W_1$ and so W is a cyclic extension of K of degree p.

Let π be the discrete rank one valuation of $K(t)$ trivial on K and having t as uniformizing parameter. The residue class field of $K(t)_\pi$ is then K and, since $p \neq \operatorname{char} L$ by assumption, it follows that $p \neq \operatorname{char} \bar{L}_\gamma$ for any extension γ of π to L. Fix some extension of π to L and denote this extended valuation by π. For simplicity, we write $\bar{L}$ rather than $\bar{L}_\pi$.

Assume $W \not\subset \bar{L}$. Then $W\bar{L}$ is a cyclic extension of $\bar{L}$ of degree p and so there is a $\delta \in X(\bar{L})_p$ with $\bar{L}^\delta = W\bar{L}$. We claim that $\delta \in X(\bar{L})_p(\omega)$. Suppose not. Then there is an m so that there is no solution $X(\bar{L})_p$ of the equation $p^m x = \delta$. Let $n = n(\bar{L}, p, \pi, p^m)$ be as in Lemma 3 and let $s = n + m$. Since $\sigma \in X(L)_p(\omega)$, there is a $\theta \in X(L)_p$ with $p^{s-1}\theta = \sigma$. Let $Y = L^\theta$ so $[Y : L] = p^s$. Since $L^\sigma = LW$ and $W \not\subset \bar{L}$, π has a unique extension to L^σ and this extension is unramified. Since Y is a cyclic p-extension of L, the intermediate fields are linearly ordered. Since char $\bar{L} \neq p$, L is the decomposition field for π from L to Y and π has a unique extension to Y [12, sects. 4–10]. Denoting the extension of π to Y by π and setting $\bar{Y}_\pi$ equal to $\bar{Y}$, we see that $\bar{Y}$ is a cyclic p-extension of $\bar{L}$ containing $W\bar{L} = \bar{L}^\delta$. By our choice of m, $[\bar{Y} : \bar{L}]$ divides p^m. By Lemma 3 $\bar{Y}$ does not contain a primitive p^nth root of unity. Let R_π be the maximal unramified extension of L_π in Y_π. Since $[Y : L] = p^s = [Y_\pi : L_\pi] = [Y_\pi : R_\pi][R_\pi : L_\pi]$ by [12, sects. 4–10] and $\bar{Y} = \bar{R}_\pi$, we have $[Y_\pi : R_\pi] \geqslant p^{s-m} = p^n$. But $p \neq$ char $\bar{Y}_\pi = \operatorname{char} \bar{L}$ so Y_π is a cyclic totally and tamely ramified extension of R_π of degree $\geqslant p^n$. By [9, Lemma 11, p. 74], $\bar{R}_\pi$ contains a primitive p^nth root of unity and so $\bar{R}_\pi = \bar{Y}$ does also. This is a contradiction and shows that $\delta \in X(\bar{L})_p(\omega)$. Since $\bar{L}$ is finitely generated over its prime field, $p \neq \operatorname{char} \bar{L}$, and t.d. $\bar{L}/F = r - 1$, we conclude by our inductive hypothesis that $X[\bar{L})_p(\omega)$ contains only finitely many elements of order p. This implies that there are only finitely many possible W's as above with $W \not\subset \bar{L}$ and WL unramified over L at π. Since there are only finitely many cyclic extensions W of K with $W \subset \bar{L}$ we conclude that there can be only finitely many $\sigma \in X(L)_p(\omega)$, σ of order p, such that $L^\sigma = LW$ with W algebraic over K.

Now suppose that $\sigma \in X(L)_p(\omega)$, σ of order p, and assume that there does not exist a W algebraic over K with $L^\sigma = LW$. Let $\tilde{K}$ be an algebraic closure of K in some field containing L as a subfield. Since L^σ is not a constant field extension by assumption, $\tilde{K}L^\sigma$ is a cyclic extension of $\tilde{K}L$ of degree p. Let π be an arbitrary discrete rank one valuation of $\tilde{K}L$ which is trivial on $\tilde{K}$. Then the residue class field of $\tilde{K}L_\pi$ contains $\tilde{K}$ and so has characteristic different from p since $p \neq \operatorname{char} L$ by assumption. Since L^σ/L is unramified at all discrete rank one valuations γ of L with $p \neq \operatorname{char} L_\gamma$, it follows that L^σ/L is unramified at $\pi \mid L$. Thus $\tilde{K}L^\sigma/\tilde{K}L$ is unramified at all discrete rank one valuations of $\tilde{K}L$ which are trivial on $\tilde{K}$. Corresponding to the extension $\tilde{K}L/\tilde{K}(t)$ there is a uniquely determined algebraic curve A which is a complete nonsingular irreducible variety [10, p. 18]. The extension $\tilde{K}L^\sigma/\tilde{K}L$ then corresponds to a cyclic unramified cover of degree p of A. By a theorem of Lang and Serre [8, Theorem 4, p. 327] (see also [10, pp. 127–128] and [11, pp. 373 and 381]) there exist only finitely many such covers. We are thus reduced to proving that there are only finitely many $\tau \in X(L)_p(\omega)$, τ of order p, such that $\tilde{K}L^\sigma = \tilde{K}L^\tau$.

Suppose then that $\tau \in X(L)_p(\omega)$, τ of order p, $\tilde{K}L^\sigma = \tilde{K}L^\tau$, and $L^\sigma \neq L^\tau$. Since $\tilde{K}L^\sigma = \tilde{K}L^\tau$, there is a finite extension V of K such that $VL^\sigma = VL^\tau$. Since $[VL^\sigma L^\tau : VL] = p$, it follows that $[VL \cap L^\sigma L^\tau : L] = p$. Let $U = VL \cap L^\sigma L^\tau \cap V$. If $U = K$, then $VL \cap L^\sigma L^\tau$ is an algebraic function field with K as field of constants. Since $L^\sigma L^\tau$ is separable over L and L is separable over K, $L^\sigma L^\tau \cap VL$ is separable over K. Since $L^\sigma L^\tau \cap VL\langle V\rangle = VL$ we have $[VL : L^\sigma L^\tau \cap VL] = [V : K]$ by [2, Corollary 1, p. 90]. But V is the field of constants of VL by [2, Theorem 2, p. 90] so $[V : K] = [VL : L]$. Thus $[L^\sigma L^\tau \cap VL : L] = 1$, a contradiction. We conclude that $U \neq K$ and so $VL \cap L^\sigma L^\tau = UL$. Let $\gamma \in X(L)_p$ with $L^\gamma = UL$. We claim that $\gamma \in X(L)_p(\omega)$. Let n be arbitrary and let L^α and L^β be, respectively, cyclic extensions of L^σ and L^τ of degree p^n where $\alpha, \beta \in X(L)_p$. α and β exist since $\sigma, \tau \in X(L)_p(\omega)$. Since $UL \subset L^\sigma L^\tau$, $UL \subset L^\alpha L^\beta$. Let $\langle\theta\rangle = \operatorname{Gal}(L^\alpha L^\beta/L^\alpha)$, $\langle\varphi\rangle = \operatorname{Gal}(L^\alpha L^\beta/L^\beta)$. Since both θ and φ have order p^{n+1} neither induces the identity automorphism when restricted to UL. Without loss of generality we may assume that $\theta \mid UL = \varphi \mid UL$. Then $\theta^{-1}\varphi$ has order p^{n+1} and its fixed field L_1 is a cyclic extension of L containing UL. Thus there is a $\delta \in X(L)_p$ with $p^n\delta = \gamma$ and $L^\delta = L_1$. We conclude that $\gamma \in X(L)_p(\omega)$ as desired. Since L^γ is a constant field extension LU, there are only finitely many choices for γ by the preceding arguments. Thus there are only finitely many possibilities for L^γ. Since $L^\tau \subset L^\sigma L^\gamma$, there are only finitely many possibilities for τ. This completes the proof of Theorem 1.

Before proceeding to the proof of Theorem 2 we take this opportunity to digress for a moment and point out an interesting technical point that arises during the proof of Theorem 1. Suppose (maintaining the context of the proof) that $\sigma \in X(L)_p(\omega)$ where $L^\sigma = LW$ with W a cyclic extension of K of degree p. Then $W = K^\zeta$ for some $\zeta \in X(K)_p$. It is natural to conjecture that $\zeta \in X(K)_p(\omega)$, but we have not been able to prove this except when $p \nmid [L : K(t)]$. As the argument in the proof shows, this would also follow if there exists a discrete rank one valuation π of L which is trivial on K and such that $\bar{L}_\pi = K$. Such valuations need not, however, exist.

We next turn to the proof of Theorem 2. Let E be a purely transcendental extension of a global field F with $1 \leqslant \text{t.d. } E/F < \infty$ and suppose $p \neq \operatorname{char}(E)$.

We proceed by induction on t.d. E/F and so we may assume that the result is true for subfields of E of lower transcendence degree over F. The cases where transcendence degree equals 1 were handled in [3] and [4]. Let $E = K(t)$ where t is transcendental over K and t.d. $E/F =$ t.d. $K/F + 1$. By a basic result of Auslander and Brumer [1, Proposition 4.1] (see also [3, Proposition 1]) there is a split exact sequence

$$0 \to B(K)_p \to B(E)_p \to \bigoplus_f X(K_f) \to 0$$

where f runs through all monic irreducible polynomials in $K[t]$ and $K_f = K[t]/(f(t))$. (We take this opportunity to correct a minor error in [3]. The exact sequence above is misstated in Proposition 1 as:

$$0 \to B(K)_p \to B(E)_p \to \bigoplus_V X(V) \to 0$$

where V runs through all finite extensions of E in some algebraic closure of E. This misstatement does not, however, affect any of the results of [3].)

If t.d. $E/F = 1$, then the Ulm length $l_p(B(E))$ of $B(E)_p$ equals $\omega 2$ by Theorem 1 of [4]. If t.d. $E/F > 1$, then $l_p(B(K)) = \omega 2$ by induction. Since $l_p(X(K_f)) < \omega 2$ by Theorem 1 we conclude that $l_p(B(E)) = \omega 2$. $B(E)_p$ is countable since E is. By induction $DB(K)_p$ is a direct sum of ω copies of $Z(p^\infty)$ if t.d. $E/F > 1$; this holds also for t.d. $E/F = 1$ by the classical theory of Brauer groups over global fields. Thus we conclude from the Auslander–Brumer exact sequence that $DB(E)_p$ is a direct sum of ω copies of $Z(p^\infty)$. Let G be a subgroup of $B(E)_p(\omega)$ so that $B(E)_p(\omega) \cong G \oplus DB(E)_p$. Since $l_p(B(E)) = \omega 2$, $DB(E)_p = B(E)_p(\omega 2)$. Thus G is isomorphic to $B(E)_p(\omega)/B(E)_p(\omega 2)$, and is then a countable p-primary abelian group with no elements of infinite height. Thus G is a direct sum of cyclic groups [6, Theorem 11, p. 23]. Finally, the last assertion of Theorem 2 follows from the result [4, Theorem 3] that the Ulm invariants of $B(E)_p$ only can have the values 0 or ω. This completes the proof of Theorem 2.

We remark that some additional information about the structure of $B(E)_p$ can be found in [4].

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