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A Pseudo-Analyzer Approach to Formal Group Laws Not of Operad Type

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Formal group schemes, associated to affine group schemes or Lie groups by completion, can be described by classical formal group laws. More generally, cogroup objects in categories of complete algebras (e.g., associative) are described by group laws for operads or analyzers. M. Lazard has introduced analyzers to study formal group laws and group law chunks (truncated formal power series). A main example of a type of generalized formal group laws not given by an operad or analyzer are group laws corresponding to noncommutative complete Hopf algebras. To cover this case and other types of group laws, pseudo-analyzers are introduced. We point out differences to the (quadratic) operad case; e.g., there is no classification of group laws by Koszul duality. On the other hand we show how pseudo-analyzer cohomology can be used to describe extension of group law chunks. © 2001 Academic Press

Monomials, polynomials, and power series with variables from a (finite) set $X = \{x_1, \dots, x_m\}$ are defined after fixing a "type" or operad, e.g., associative commutative (type $\mathfrak{C}$ or $\mathfrak{C}om$, the classical case), associative noncommutative (type $\mathfrak{A}$), or nonassociative noncommutative (type $\mathfrak{N}$). We will be mostly interested in these, but we will also mention other examples.

Formal group laws over these operads were introduced (as group laws in analyzers) by Lazard (cf. [La1, La2]). They correspond to cogroup objects in the corresponding categories of complete algebras. Over a field K of characteristic 0, they are classified by Lazard–Lie theory, which can be described in terms of Koszul duality for operads (cf. [GK, Fr1]).

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Group laws corresponding to Hopf algebra structures on the free associative power series algebra $K\langle\langle X\rangle\rangle$ ($\hat{\otimes}$-cogroup structures), see [Ho1, Ho2], are not group laws in an operad or analyzer. Hopf algebras are not algebras over an operad; they are given by PROPs (cf. [ML1, FM]). Here types $\mathfrak{E}$ and $\mathfrak{C}$ are mixed in a certain way. The same is true for group laws corresponding to $\hat{\otimes}$-cogroup structures (and $\hat{*}$-cogroup structures) on free nonassociative power series algebras.

To cover these mixed types, we will give a modification of Lazard's theory. We introduce pseudo-analyzers and prove that the mixed types indicated above provide examples of pseudo-analyzers that are not analyzers. A main point is that Lazard's concept of describing extension of group law chunks by analyzer cohomology also applies to group laws over pseudo-analyzers. We show that the torsion theorem, the key to the classification of group laws over analyzers (in the rational case) is not true for pseudo-analyzers, and we describe some cohomology modules in the case corresponding to noncommutative complete Hopf algebras. We will then review the classification of group laws over analyzers in the rational case and show that structure constants providing formal group laws of two (operad) types, e.g., $\mathfrak{E}$ and $\mathfrak{C}$, also provide a (non-trivial) formal group law of the corresponding mixed type.

In Section 1 we recall the definition of an operad and define monomials and power series. Besides the "usual" power series, we will include "mixed types" like elements of $(K\langle\langle X\rangle\rangle)^{\hat{\otimes} p}$, $p \geq 2$.

The definitions of pseudo-analyzers and analyzers together with first examples of analyzers are given in Section 2. Lazard's cohomology modules are introduced for pseudo-analyzers.

In Section 3 we give examples for pseudo-analyzers that are not analyzers.

Section 4 contains the definition of group laws in pseudo-analyzers and the generalization of Lazard's criterion for extension of group law chunks. We also look at the type corresponding to complete (in general noncommutative) Hopf algebras.

Over a ground field of characteristic 0, we show that every finitely generated complete associative algebra with Hopf algebra structure is isomorphic to a free complete algebra $K\langle\langle X\rangle\rangle$ modulo an ideal I that is contained in the ideal generated by all commutators. For $i = 0$ such structures are nothing else but formal group laws in the corresponding pseudo-analyzer.

In Section 5 we prove that Lazard's torsion theorem does not hold for pseudo-analyzers. We describe cohomology modules in the analyzer and pseudo-analyzers cases.

The last section relates results from Section 5 to the concept of Koszul duality for operads and shows how the classification of group laws in rational analyzers provides examples for formal group laws not of operad type.

1. POWER SERIES OF DIFFERENT TYPES

We fix $m \in \mathbb{N}^*$. For all $p \in \mathbb{N}$, let $X^{\sqcup p}$ be the set $\{x_1^{(1)}, \dots, x_m^{(1)}, \dots, x_1^{(p)}, \dots, x_m^{(p)}\}$ of variables, ordered by $x_1^{(1)} < \dots < x_m^{(1)} < \dots < x_1^{(p)} < \dots < x_m^{(p)}$. $X^{\sqcup 1}$ is identified with $X = \{x_1, \dots, x_m\}$. K will denote a field of coefficients. (Later we will allow K to be a unitary commutative associative ring.)

Over X we will form monomials of the following types. The first case (classical case), type $\mathfrak{C}$, is the case of associative commutative variables. Type $\mathfrak{E}$ will denote the associative noncommutative case, type $\mathfrak{H}$ the nonassociative commutative case, and $\mathfrak{R}$ the nonassociative noncommutative case.

Definition 1.1. For $n \in \mathbb{N}^*$ let S_n be the symmetric group and S-Vect_K be the following category: Objects $\mathfrak{V}$ are sequences $(\mathfrak{V}(n))_{n \in \mathbb{N}}$ of vector spaces $\mathfrak{V}(n)$ with S_n-action. Morphisms are given by homomorphisms compatible with the S_n-action.

To every S-vector space $\mathfrak{V}$ there is associated an endofunctor on Vect_K given by $F_{\mathfrak{V}}(V) := \sum_{n=0}^{\infty} \mathfrak{V}(n) \otimes_{S_n} V^{\otimes n}$ (where S_n acts on $V^{\otimes n}$ by place permutation). S-Vect_K can be identified with the full subcategory of $\mathrm{End}(\mathrm{Vect}_K)$ consisting of functors of the form $F_{\mathfrak{V}}$. We note that $\mathrm{Id}_{\mathrm{Vect}_K}$ is given by the S-vector space $\mathfrak{I}$ with $\mathfrak{I}(1) = K$, $\mathfrak{I}(n) = 0$ for $n \neq 1$.

Then a (K-linear) *operad* $\mathfrak{P}$ is a monoid $(F_{\mathfrak{P}}, \mu : F_{\mathfrak{P}} \circ F_{\mathfrak{P}} \to F_{\mathfrak{P}}, 1 : \mathrm{Id} \to F_{\mathfrak{P}})$ with respect to the composition of functors in this subcategory (i.e., a monad, cf. [ML2, Chap. VI]), cf. [Fr1]. Clearly the unit is given by a homomorphism $K \to \mathfrak{P}(1)$. If we don't require operad-associativity (i.e., the associativity of $\mu : F_{\mathfrak{P}} \circ F_{\mathfrak{P}} \to F_{\mathfrak{P}}$), we get the notion of a *pseudo-operad*, cf. [HL].

For any operad $\mathfrak{P}$, $F_{\mathfrak{P}}(V)$ is called the *free* $\mathfrak{P}$*-algebra* generated by the space V.

Example 1.2. The functor $F_{\mathfrak{C}om}(V) := \sum_{n=1}^{\infty} (V^{\otimes n})_{S_n}$ (non-unitary symmetric algebra) defines the operad $\mathfrak{C}om$ given by $\mathfrak{C}om(n) = K \cdot x^{(1)} \cdot \ldots \cdot x^{(n)} \cong K$ (all $n \geq 1$); this is the classical case. $F_{\mathfrak{A}s}(V) := \sum_{n=1}^{\infty} V^{\otimes n}$ defines an operad where $\mathfrak{A}s(n)$ is the $n!$-dimensional K-space generated by the (type $\mathfrak{E}$)-monomials $x^{(\sigma(1))} \cdot \ldots \cdot x^{(\sigma(n))}$, $\sigma \in S_n$.

$\mathfrak{L}ie$ is an operad given by the $(n-1)!$-dimensional K-spaces generated by multilinear bracket monomials (with respect to e.g. the Jacobi identity); cf. [GK]. Types $\mathfrak{H}$ and $\mathfrak{R}$ correspond to free quadratic operads generated by one (resp. two) quadratic monomial(s) and the obvious S_n-actions; compare [GK].

Remark 1.3. We can regard *free* $\mathfrak{P}$*-algebras with* 1 generated by the vector space V, where V has X as basis. Clearly they are polynomial rings $K[X]$, $K\langle X\rangle$, if the operads are $\mathfrak{C}om$ (type $\mathfrak{C}$), $\mathfrak{A}s$ (type $\mathfrak{E}$). A K-basis is given in each case by the monomials of the corresponding type (together with 1).

We remark that $K\langle X^{\sqcup p}\rangle$ together with the p injections $i^{(n)}: K\langle X\rangle \to K\langle X^{\sqcup p}\rangle$, $x_i \mapsto x_i^{(n)}$ is the coproduct (or copower, cf. [ML2, p. 64]) $(K\langle X\rangle)^{*p} := K\langle X\rangle \sqcup_{\mathfrak{E}} \dots \sqcup_{\mathfrak{E}} K\langle X\rangle$ in the category of unitary associative K-algebras. The coproduct $\sqcup_{\mathfrak{C}}$ of commutative unitary K-algebras is the tensor product $\otimes = \otimes_K$. The commutative associative polynomial algebra $K[X^{\sqcup p}]$ corresponds to $(K[X])^{\otimes p} := K[X] \sqcup_{\mathfrak{C}} \dots \sqcup_{\mathfrak{C}} K[X]$ via $x_i^{(n)} \mapsto 1 \otimes \dots \otimes 1 \otimes x_i \otimes 1 \otimes \dots \otimes 1$ (x_i in the nth place).

Similar assertions hold for types $\mathfrak{K}$ and $\mathfrak{H}$. We denote the associated coproducts by $\sqcup_{\mathfrak{K}}$ and $\sqcup_{\mathfrak{H}}$. For all these copowers we can use monomials with variables from $X^{\sqcup p}$, $p \geq 2$, as a K-basis.

Now the coproduct-functors may have extensions to bigger categories. For example, one can consider the (non-free) associative algebra $K\langle X\rangle \otimes_K \dots \otimes_K K\langle X\rangle$ or form $K\{X\} *_K K\{X\}$ for the free nonassociative K-algebra $K\{X\}$.

This is the reason why we will make further distinctions when we speak of monomials with variables from $X^{\sqcup p}$, $p \geq 2$. We may allow relations (of commutativity or associativity) between variables $x_i^{(n)}, x_j^{(n')}$, $n \neq n'$, i, j arbitrary, by specifying the type of (categorical) coproduct, that induces the type of monomials over $X^{\sqcup p}$.

DEFINITION 1.4. Let $\mathfrak{A}$ be one of the types $\mathfrak{K}$, $\mathfrak{H}$, $\mathfrak{E}$, $\mathfrak{C}$ defined above and let either $\mathfrak{B} = \{\mathfrak{E}, \mathfrak{C}\}$ in case $\mathfrak{A} = \mathfrak{E}$, or $\mathfrak{B} \in \{\mathfrak{K}, \mathfrak{H}, \mathfrak{E}, \mathfrak{C}\}$ in case $\mathfrak{A} = \mathfrak{K}$, or $\mathfrak{B} = \{\mathfrak{H}, \mathfrak{C}\}$ in case $\mathfrak{A} = \mathfrak{H}$. Let A be the free algebra of type $\mathfrak{A}$ with variables from X. There is a (canonical) K-module basis consisting of homogeneous elements of

$$\underbrace{A \sqcup_{\mathfrak{B}} \dots \sqcup_{\mathfrak{B}} A}_{p}.$$

We call the basis elements ($\neq 1$) $\sqcup_{\mathfrak{B}} -{}^m\mathfrak{A}^{(p)}$*-monomials*.

We will sometimes abbreviate $\sqcup_{\mathfrak{B}} -{}^m\mathfrak{B}$ by ${}^m\mathfrak{B}$.

We count for every $n = 1, \dots, p$, how many variables $x_i^{(n)}$ occur in the monomial. This gives us a degree-tuple $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$. Then $|\alpha| := \sum_{n=1}^{p} \alpha_n$ is the total degree of the given monomial, while α_n is the degree with respect to the variables $x_1^{(n)}, \dots, x_m^{(n)}$. By $\sqcup_{\mathfrak{B}} -{}^m\mathfrak{A}_\alpha^{(p)}$-

monomials we mean $\sqcup_{\mathfrak{B}} -^m \mathfrak{A}^{(p)}$-monomials with multidegree $\alpha = (\alpha_1, \dots, \alpha_p)$.

Let $|\mathfrak{A}|_\alpha^p$ be the K-module freely generated by all $\mathfrak{A}_\alpha^{(p)}$-monomials.

Let $|\mathfrak{A}|_r^p = \oplus_{\{\alpha \in \mathbb{N}^p : |\alpha| = r\}} |\mathfrak{A}|_\alpha^p$, $|\mathfrak{A}|^p := \oplus_{r \in \mathbb{N}^*} |\mathfrak{A}|_r^p$, and $|\hat{\mathfrak{A}}|^p := \prod_{r \in \mathbb{N}^*} |\mathfrak{A}|_r^p$. Here $|\hat{\mathfrak{A}}|^p$ is considered as the completion of $|\mathfrak{A}|^p$ with respect to the topology given by the basis $(\oplus_{r \geq R} |\mathfrak{A}|_r^p)_{R \in \mathbb{N}^*}$ of 0-neighborhoods.

If $\alpha < \beta$ reverse lexicographically, i.e., if $(\alpha_p, \dots, \alpha_1)$ is lexicographically smaller than $(\beta_p, \dots, \beta_1)$, all monomials with multidegree $(\alpha_1, \dots, \alpha_p)$ are smaller than any monomial of multidegree $(\beta_1, \dots, \beta_p)$. Representatives of monomials of the same multidegree are compared reverse lexicographically (starting on the right) with respect to submonomials, where especially $w_1 < w_2$ if $\deg(w_1) < \deg(w_2)$. Monomials will be written as concatenation of increasing submonomials (recursively).

EXAMPLE 1.5. In the classical case (commutative associative, $\otimes$ as coproduct), we write the monomials as concatenation of increasing variables from $X^{\sqcup p}$.

Each $\mathfrak{C}_\alpha^{(p)}$-monomial is of the form $x_{i_1}^{(1)} \dots x_{i_{\alpha_1}}^{(1)} \dots x_{i_{r-\alpha_p+1}}^{(p)} \dots x_{i_r}^{(p)}$, where $r = |\alpha|$, with $1 \leq i_1 \leq \dots \leq i_{\alpha_1} \leq m, \dots, 1 \leq i_{r-\alpha_p+1} \leq \dots \leq i_r \leq m$.

Any $\otimes_K\text{-}^m \mathfrak{A}^{(p)}$-monomial can be viewed as a product $w^{(1)} \cdot \dots \cdot w^{(p)}$, of p separated (possibly empty) monomials $w^{(n)}$, each in m variables (for $n = 1, \dots, p$) $x_1^{(n)}, \dots, x_m^{(n)}$. In the $\otimes_K\text{-}^m \mathfrak{C}^{(p)}$-case, every monomial (with given degree-tuple α) is of the form $x_{i_1}^{(n_1)} \dots x_{i_{\alpha_1}}^{(n_1)} \cdot \dots \cdot x_{i_{r-\alpha_p+1}}^{(n_r)} \dots x_{i_r}^{(n_r)}$, $1 \leq n_1 \leq \dots \leq n_r \leq p$.

To combine $\mathfrak{K}$ with $\sqcup_{\mathfrak{C}}$ we proceed similarly and get expressions $w_1^{(n_1)} \cdot \dots \cdot w_r^{(n_r)}$, $n_i \neq n_{i+1}$. The representatives for the other types are chosen similarly.

Remark 1.6. In general the semigroup structure of the set of $\mathfrak{A}^{(p)}$-monomials induces the structure of a K-algebra with 1 on $K \oplus |\mathfrak{A}|^p$, without 1 on $|\mathfrak{A}|^p$.

$(K\langle X \rangle)^{*p}$ is identified with $K \oplus |*\text{-}^m \mathfrak{C}|^p$, $(K\langle X \rangle)^{\otimes p}$ with $K \oplus |\otimes\text{-}^m \mathfrak{C}|^p$ and $(K[X])^{\otimes p}$ with $K \oplus |^m \mathfrak{C}|^p$.

If $\hat{*}$ and $\hat{\otimes}$ denote the appropriate completions, we can identify $K \oplus \widehat{|*\text{-}^m \mathfrak{C}|^p}$, $K \oplus \widehat{|\otimes\text{-}^m \mathfrak{C}|^p}$, and $K \oplus \widehat{|^m \mathfrak{C}|^p}$ with the (complete) power series algebras $K\langle\langle X^{\sqcup p} \rangle\rangle = (K\langle\langle X \rangle\rangle)^{\hat{*} p}$, $(K\langle\langle X \rangle\rangle)^{\hat{\otimes} p}$, and $K[[X^{\sqcup p}]] = (K[[X]])^{\hat{\otimes} p}$, respectively. More generally, we have a structure of a topological K-algebra with 1 on $K \oplus |\hat{\mathfrak{A}}|^p$. There exist surjective continuous K-linear maps $\widehat{|*\text{-}^m \mathfrak{C}|^p} \to \widehat{|\otimes\text{-}^m \mathfrak{C}|^p}$ and $\widehat{|*\text{-}^m \mathfrak{C}|^p} \to \widehat{|^m \mathfrak{C}|^p}$ induced by the projections of the corresponding semigroups. In the nonassociative cases one has to regard magmas instead of semigroups.

For $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{K}$, we recover the usual notion of power series over the given operads, defined as follows.

DEFINITION 1.7. If $\mathfrak{P}(0) = 0$, $\mathfrak{P}(1) = K$, the *free complete $\mathfrak{P}$-algebra* (without 1) generated by V is defined by $\widehat{F_{\mathfrak{P}}}(V) := \prod_{n=0}^{\infty} \mathfrak{P}(n) \otimes_{S_n} V^{\otimes n}$. If $V = \langle x^{(1)}, \dots, x^{(n)} \rangle$ is the vector space with basis $x^{(1)}, \dots, x^{(n)}$, then the elements of $\widehat{F_{\mathfrak{P}}}(V)$ are called $\mathfrak{P}$-power series, cf. [Fr1, GK].

For A an algebra (of any type) with an augmentation $\varepsilon : A \to K$, $\overline{A}$ its augmentation ideal, we say that A is *complete* iff $\overline{A} = \overset{lim}{\leftarrow} \overline{A}/J_n$, $(J_n)_{n \in \mathbb{N}}$ a sequence of ideals s.t. all $\overline{A}/J_n$ are nilpotent.

2. ANALYZERS AND PSEUDO-ANALYZERS

The modern notion of an operad has Lazard's notion of an analyzer, see [La2], as a precursor. In our setting, this concept has some advantages. We want to modify the definition of (complete or incomplete) analyzers over K replacing two axioms by weaker axioms.

From now on, K denotes a commutative associative ring with 1.

Let $(\mathfrak{A}^n)_{n \in \mathbb{N}}$ be a sequence of K-modules, where in each $\mathfrak{A}^n$ there are n distinct elements $e_{n,1}, \dots, e_{n,n}$ selected. For every $p, q \in \mathbb{N}$ and all $g_1, \dots, g_p \in \mathfrak{A}^q$, let a K-linear map $\eta^{p,q}_{g_1, \dots, g_p} : \mathfrak{A}^p \to \mathfrak{A}^q$, called a composition map (or insertion map), be given.

If $f \in \mathfrak{A}^p$, we say that f has p arguments. For $1 \le i \le p$, we say f is neutral with respect to the ith argument, iff $\eta^{p,q}_{g_1, \dots, g_i, \dots, g_p}(f) = \eta^{p,q}_{g_1, \dots, g'_i, \dots, g_p}(f)$ for all $g_1, \dots, g_p$, g'_i. We denote by $\operatorname{supp}_{\{1, \dots, p\}}(f)$ the set $\{1, \dots, p\} - \{i : f \text{ is neutral w.r.t. } i\text{th argument}\}$. Two elements f and g of $\mathfrak{A}^p$ are said to be pairwise compatible if $\operatorname{supp}_{\{1, \dots, p\}}(f) \cap \operatorname{supp}_{\{1, \dots, p\}}(g) = \emptyset$.

DEFINITION 2.1. Let $(\mathfrak{A}^n)_{n \in \mathbb{N}}$, $e_{n,i}$, $\eta^{p,q}_{g_1, \dots, g_p}$ be given as above and assume

(C1) For all $p \in \mathbb{N}$, $f \in \mathfrak{A}^p : \eta^{p,p}_{e_{p,1}, \dots, e_{p,p}}(f) = f$.

For all $p, q, i \in \mathbb{N}$ with $1 \le i \le p$ and all $f_1, \dots, f_p \in \mathfrak{A}^q : \eta^{p,q}_{f_1, \dots, f_p}(e_{p,i}) = f_i$.

(C2a) For all $n, p, q \in \mathbb{N}$, all $f \in \mathfrak{A}^n$, all pairwise compatible $g_1, \dots, g_n \in \mathfrak{A}^p$, and all pairwise compatible $h_1, \dots, h_p \in \mathfrak{A}^q$ it holds that $\eta^{p,q}_{h_1, \dots, h_p}(\eta^{n,p}_{g_1, \dots, g_n}(f)) = \eta^{n,q}_{l_1, \dots, l_n}(f)$, where $l_i := \eta^{p,q}_{h_1, \dots, h_p}(g_i) \in \mathfrak{A}^q$ (all i).

(D1) $\mathfrak{A}^n$ is a multigraded K-module for every n: $\mathfrak{A}^n = \oplus_{r \in \mathbb{N}} \mathfrak{A}^n_r$ with $\mathfrak{A}^n_r = \oplus_{\{\alpha \in \mathbb{N}^n : |\alpha| = r\}} \mathfrak{A}^n_\alpha$, where $|\alpha| := \alpha_1 + \dots + \alpha_n$. As in [La1] we furthermore assume $\mathfrak{A}^n_0 = 0$ (all n).

(D2) The $e_{n,i}$ (all n, i) are multihomogeneous of multidegree $(0, \dots, 0, 1, 0, \dots, 0)$ (1 at the ith position).

(D3) For all $f \in \mathfrak{A}^p_\alpha$, $\alpha \in \mathbb{N}^p$, $g_i \in \mathfrak{A}^q$, $\lambda_i \in K$: $\eta^{p,q}_{\lambda_1 g_1, \dots, \lambda_n g_n}(f) = \lambda_1^{\alpha_1} \cdot \ldots \cdot \lambda_n^{\alpha_n} \cdot \eta^{n,p}_{g_1, \dots, g_n}(f)$, where $0^0 := 1$.

(D4) If $f \in \mathfrak{A}^n_{(\alpha_1, \dots, \alpha_n)}$ and (for all $i = 1, \dots, n$) $g_i \in \mathfrak{A}^p_{(\beta_{i,1}, \dots, \beta_{i,p})}$ then $\eta^{n,p}_{g_1, \dots, g_n}(f) \in \mathfrak{A}^p_{(\gamma_1, \dots, \gamma_n)}$ with $\gamma_j = \sum_{i=1}^n \alpha_i \beta_{i,j}$ (all $j = 1, \dots, p$).

(D5a) If, for $\beta \in \mathbb{N}^n$, P_β denotes the projection $\mathfrak{A}^n \to \mathfrak{A}^n_\beta$ onto the homogeneous component of multidegree β, we require for every $f \in \mathfrak{A}^p_\alpha$, $P_\beta(\eta^{p,p+1}_{(e_{p+1,1} + e_{p+1,2}), e_{p+1,3}, \dots, e_{p+1,p+1}}(f)) = 0$, if $\beta_1 + \beta_2 \neq \alpha_1$ or $\beta_i \neq \alpha_{i-1}$ for any $i = 3, \dots, p+1$.

Then $\mathfrak{A}$ will be called an *incomplete pseudo-analyzer* over K. The incomplete pseudo-analyzer $\mathfrak{A}$ is called an *incomplete analyzer* over K, if furthermore (C2b) and (D5b) hold:

(C2b) Axiom (C2) holds without the assumptions on compatibility.

(D5b) In case $\beta_1 + \beta_2 = \alpha_1$ and $\beta_i = \alpha_{i-1}$ one has

$$\eta^{p+1,p}_{e_{p,1}, e_{p,1}, e_{p,2}, \dots, e_{p,p}}\left(P_\beta\left(\eta^{p,p+1}_{(e_{p+1,1} + e_{p+1,2}), e_{p+1,3}, \dots, e_{p+1,p+1}}(f)\right)\right) = \frac{\alpha_1!}{\beta_1! \beta_2!} f.$$

Let $(\hat{\mathfrak{A}}^n)_{n \in \mathbb{N}}$ be a sequence of (multi)graded K-modules $\hat{\mathfrak{A}}^n = \prod_{r \in \mathbb{N}^*} \mathfrak{A}^n_r$ with $\mathfrak{A}^n_r = \bigoplus_{\{\alpha \in \mathbb{N}^n : |\alpha| = r\}} \mathfrak{A}^n_\alpha$. Assume $\mathfrak{A} := \bigcup_{n \in \mathbb{N}} \mathfrak{A}^n$ with $\mathfrak{A}^n = \bigoplus_{r \in \mathbb{N}} \mathfrak{A}^n_r$ is an incomplete (pseudo-)analyzer and that the composition maps of $\mathfrak{A}$ are continuously extended to $\hat{\mathfrak{A}} := \bigcup_{n \in \mathbb{N}} \hat{\mathfrak{A}}^n$ with respect to the topology given by the basis $(\bigoplus_{r \geq R} \mathfrak{A}^p_r)_{R \in \mathbb{N}^*}$ of 0-neighborhoods.

Then $\hat{\mathfrak{A}}$ (together with these maps) is called a *complete* (*pseudo*-) *analyzer*.

A *morphism* $\mathfrak{A} \to \mathfrak{A}'$ of pseudo-analyzers is a sequence φ of K-linear maps $\varphi_n : \mathfrak{A}^n \to (\mathfrak{A}')^n$ that respects the structure: We require $\varphi_n(e_{n,i}) = e'_{n,i}$, $\varphi_n(\mathfrak{A}^n_\alpha) \subseteq (\mathfrak{A}')^n_\alpha$, and $\eta^{\mathfrak{A}', n, p}_{g'_1, \dots, g'_n}(f') = \varphi_p(\eta^{\mathfrak{A}, n, p}_{g_1, \dots, g_n}(f))$ for every f, g_i with $\varphi_n(f) = f'$, $\varphi_p(g_i) = g'_i$.

EXAMPLE 2.2. The sequence $({}^1\mathfrak{C}^n)_{n \in \mathbb{N}^*}$ of K-modules $(K[x^{(1)}, \dots, x^{(n)}])_{n \in \mathbb{N}^*}$, i.e., the sequence $(|\mathfrak{C}|^n)_{n \in \mathbb{N}^*}$ for $m = 1$ of Section 1, together with the elements $e_{n,i} := x^{(i)}$, defines an incomplete analyzer, where the composition maps are (canonically) defined by insertion of polynomials. This analyzer ${}^1\mathfrak{C} := \bigcup_{n \in \mathbb{N}} {}^1\mathfrak{C}^n$ is called the classical analyzer; see [La1, p. 332, Example c]. ${}^1\hat{\mathfrak{C}}$ given by the sequence $(K[[x^{(1)}, \dots, x^{(n)}]])_{n \in \mathbb{N}^*}$ is the associated complete analyzer. The composition maps are given by insertion of power series without constant term.

Similarly defined are analyzers ${}^1\mathfrak{E}$ (occurring in [La1, p. 332, Example b]), ${}^1\mathfrak{H}$, ${}^1\mathfrak{R}$ [La1, Example a] and the corresponding complete analyzers.

${}^1\mathfrak{K}$ is called Kurosch's analyzer; cf. also [La1, p. 391]. There is an inclusion morphism from the Lie-analyzer [La1, p. 333, Example d] into the analyzer ${}^1\mathfrak{E}$.

We are going to look at pseudo-analyzers that are not analyzers in the next section.

Remark 2.3. For every operad $\mathfrak{P}$, the sequence $(\widehat{F_{\mathfrak{P}}}(\langle x^{(1)}, \ldots, x^{(n)}\rangle))_{n \in \mathbb{N}}$ forms an analyzer over K, cf. [Fr1], and we can get the operad back as a subspace (generated in each component by the multilinear monomials, i.e., the monomials having degree 1 in each variable). The corresponding analyzer of the $\mathfrak{C}om$-power series is ${}^1\hat{\mathfrak{C}}$, the analyzer of the $\mathfrak{A}s$-power series is ${}^1\hat{\mathfrak{E}}$, and so on.

There are analyzers that do not correspond to operads; cf. [La1, p. 333].

Note that, while associativity of composition is not required for pseudo-operads, we do require a partial associativity for pseudo-analyzers (C2a).

We will finish this section with some results from [La1] which hold for pseudo-analyzers.

DEFINITION 2.4. Let $\mathfrak{A}$ be a pseudo-analyzer.

For $R \in \mathbb{N}^*$ we will say that elements f, g from $\mathfrak{A}^p$, are equivalent up to degree R, write $f \equiv g \mod J^{R+1}$, if $\sum_{r=1}^{R} P_r f = \sum_{r=1}^{R} P_r g$, where P_r denotes the projection onto the homogeneous component of total degree r.

We can identify the elements of $\mathfrak{A}[R]^p := \bigoplus_{r=1}^{R} \mathfrak{A}_r^p$ with elements from $\mathfrak{A}^p \mod J^{R+1}$.

Remark 2.5. It is easy to see that $\mathfrak{A}[R] = \bigcup_{n \in \mathbb{N}} \mathfrak{A}[R]^n$ is also a pseudo-analyzer if $\mathfrak{A}$ is. $\mathfrak{A}[R]$ is both incomplete and complete. There is a canonically defined projection morphism from $\mathfrak{A}$ onto $\mathfrak{A}[R]$.

LEMMA 2.6 ("Composition Lemma" of [La2]). *Let $\mathfrak{A}$ be a pseudo-analyzer, $n \in \mathbb{N}^*$, $r \in \mathbb{N}$, and $F \in \mathfrak{A}^n$ with $F \equiv 0 \mod J^{r+1}$ be given.*

We regard, for $G^{(1)}, \ldots, G^{(n)} \in \hat{\mathfrak{A}}^p$ $(p \in \mathbb{N}^*)$, the element $\eta^{n,p}_{G^{(1)},\ldots,G^{(n)}}(F) \in \hat{\mathfrak{A}}^p$.

Then, given $L^{(i)} \equiv G^{(i)} \mod J^{s+1}$ $(s \in \mathbb{N})$, all $i = 1, \ldots, n$, we have

$$\eta^{n,p}_{G^{(1)},\ldots,G^{(n)}}(F) \equiv \eta^{n,p}_{L^{(1)},\ldots,L^{(n)}}(F) \qquad \mod J^{r+s+1}.$$

Proof. The proof is given in [La1] for analyzers. We note that we can replace in this proof the assumptions (C2) and (D5) by (C2a) and (D5a). Thus the same proof fits for pseudo-analyzers. ∎

DEFINITION 2.7. Let $\mathfrak{A}$ be a pseudo-analyzer, $n, i \in \mathbb{N}^*$, $i < n$.

We define the map $\partial_i : \mathfrak{A}^n \to \mathfrak{A}^{n+1}$ by $\eta^{\mathfrak{A},n,n+1}_{e^{(1)},\ldots,e^{(i-1)},(e^{(i)}+e^{(i+1)}),e^{(i+2)},\ldots,e^{(n+1)}} - \eta^{\mathfrak{A},n,n+1}_{e^{(1)},\ldots,e^{(i)},e^{(i+2)},\ldots,e^{(n+1)}} - \eta^{\mathfrak{A},n,n+1}_{e^{(1)},\ldots,e^{(i-1)},e^{(i+1)},\ldots,e^{(n+1)}}$ where $e_{n+1,i}$ is denoted by $e^{(i)}$ for short.

A differential $\delta = \delta_n : \mathfrak{A}^n \to \mathfrak{A}^{n+1}$ is defined by

$$
\begin{aligned}
\delta_n &= \sum_{i=1}^{n} (-1)^i \partial_i \\
&= \eta^{\mathfrak{A}, n, n+1}_{e^{(2)}, \dots, e^{(n+1)}} + \sum_{i=1}^{n} (-1)^i \eta^{\mathfrak{A}, n, n+1}_{e^{(1)}, \dots, (e^{(i)} + e^{(i+1)}), \dots, e^{(n+1)}} \\
&\quad + (-1)^{n+1} \eta^{\mathfrak{A}, n, n+1}_{e^{(1)}, \dots, e^{(n)}}.
\end{aligned}
$$

Remark 2.8. The same formula is given for analyzers in [La1, Sect. 8]. One checks $\delta_{n+1} \circ \delta_n = 0$ (all n) for pseudo-analyzers by the same direct computation as for analyzers, because all inserted elements are compatible.

Let us call $f \in \mathfrak{A}^n$ *pseudolinear with respect to the ith argument*, if $\partial_i f = 0$. f is called *pseudolinear*, if f is pseudolinear with respect to all arguments. f is called *multilinear* if f is homogeneous of degree $(1, \dots, 1)$, especially any such f is pseudolinear and no argument is neutral. Note that $f \in \mathfrak{A}^n$ is neutral with respect to the ith argument iff $\eta^{n,n}_{e_{n,1}, \dots, e_{n,i-1}, 0, e_{n,i+1}, \dots, e_{n,n}}(f) = f$.

$f \in \mathfrak{A}^n$ is called *symmetric* if $\sigma f = f$ for all $\sigma \in S_n$, where $\sigma f := \eta^{n,n}_{e_{n,\sigma(1)}, \dots, e_{n,\sigma(n)}}(f)$. f is called *anti-symmetric* if $\sigma f = (\operatorname{sign} \sigma) f$, all σ.

LEMMA 2.9. *It holds that* $\delta_n(\mathfrak{A}^n_r) \subseteq \mathfrak{A}^{n+1}_r$. *Thus* $\mathfrak{A} = \bigoplus_{n \in \mathbb{N}^*} \mathfrak{A}^n = \bigoplus_{n, r \in \mathbb{N}^*} \mathfrak{A}^n_r$ *together with* δ *is a (co) complex of K-modules and there are cohomology modules* $H^n(\mathfrak{A}) = \bigoplus_{r \in \mathbb{N}^*} H^n_r(\mathfrak{A})$, *where* $H^n_r(\mathfrak{A}) = \ker(\delta_n \mid \mathfrak{A}^n_r) / \operatorname{im}(\delta_{n-1} \mid \mathfrak{A}^{n-1}_r)$.

H is functorial on pseudo-analyzers.

Proof. To verify the last assertion, let φ be a morphism $\mathfrak{A} \to \mathfrak{A}'$. Then we have $\delta'_n \circ \varphi_n = \varphi_{n+1} \circ \delta_n$, because for all $f \in \mathfrak{A}^n$ (with $\varphi_n(e_{n,i}) = e'_{n,i}$ and denoting $\varphi_n(f)$ by f'),

$$
\begin{aligned}
\delta'_n(\varphi_n(f)) &= \eta^{\mathfrak{A}', n, n+1}_{e'_{n+1,2}, \dots, e'_{n+1,n+1}}(f') + \cdots + (-1)^{n+1} \eta^{\mathfrak{A}', n, n+1}_{e'_{n+1,1}, \dots, e'_{n+1,n}}(f') \\
&= \varphi_{n+1}(\delta_n(f)).
\end{aligned}
$$

∎

3. EXAMPLES OF PSEUDO-ANALYZERS

Let $p, r \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}^p$, let $\mathfrak{A}^{(p)}$ be one of the types defined in (1.4), i.e., $\mathfrak{A} \in \{{}^m\mathfrak{C}, \otimes\text{-}{}^m\mathfrak{E}, {}^m\mathfrak{E}, \otimes\text{-}{}^m\mathfrak{H}, {}^m\mathfrak{H}, \otimes\text{-}{}^m\mathfrak{R}, *\text{-}{}^m\mathfrak{R}, \sqcup_{\mathfrak{H}}\text{-}{}^m\mathfrak{R}, {}^m\mathfrak{R}\}$. For any K-module V, we denote by $(V)^m$ the K-module of m-tuples with entries from M (often written as column vectors). If φ is a homomorphism $V \to V$, then $(\varphi)^m$ denotes the induced homomorphism $(\varphi, \dots, \varphi): (V)^m \to (V)^m$.

Let $\mathfrak{A}_\alpha^p = (|\mathfrak{A}|_\alpha^p)^m$, $\mathfrak{A}_r^p = (|\mathfrak{A}|_r^p)^m$, and $\mathfrak{A}^p := \bigoplus_{r \in \mathbb{N}^*} \mathfrak{A}_r^p$, $\hat{\mathfrak{A}}^p := \prod_{r \in \mathbb{N}^*} \mathfrak{A}_r^p$.

Let $\mathfrak{A} = \bigcup_{n \in \mathbb{N}} \mathfrak{A}^n$, where $\hat{\mathfrak{A}}^0 = \mathfrak{A}^0 = 0$. By $e_{p,i} = e_{p,i}^{\mathfrak{A}}$ $(1 \le i \le p)$ we denote the elements $(x_1^{(i)}, \dots, x_m^{(i)}) \in \mathfrak{A}^p_{(0,\dots,0,1,0,\dots,0)}$ (1 in the ith place).

Forming direct products (see [La1, p. 323]) of the analyzers ${}^1\hat{\mathfrak{C}}$, ${}^1\hat{\mathfrak{E}}$, ${}^1\hat{\mathfrak{H}}$, and ${}^1\hat{\mathfrak{K}}$ given in (2.2) we can show that the analyzers we get as mth power $\prod^m$ are given by ${}^m\hat{\mathfrak{C}}$, ${}^m\hat{\mathfrak{E}}$, ${}^m\hat{\mathfrak{H}}$, and ${}^m\hat{\mathfrak{K}}$. Such a construction does not work for the "mixed" cases like $\otimes\text{-}{}^m\mathfrak{E}$, as, for example, $\otimes\text{-}{}^1\mathfrak{E} = {}^1\mathfrak{C}$. Moreover, there is an algebra homomorphism ("multiplication"), e.g. $\mu^{2,1}: K\langle X\rangle \sqcup_{\mathfrak{E}} K\langle X\rangle \to K\langle X\rangle$, defined by $x_i^{(n)} \mapsto x_i$, while it is well known that in the "mixed cases" we get only K-module homomorphisms (not algebra homomorphisms) like the map $K\langle X\rangle \sqcup_{\mathfrak{C}} K\langle X\rangle \to K\langle X\rangle$.

DEFINITION 3.1. For $\mathfrak{A} \in \{{}^m\mathfrak{C}, {}^m\mathfrak{E}, {}^m\mathfrak{H}, {}^m\mathfrak{K}\}$, we define a (unique, continuous) K-linear map $\mu^{\widehat{|\mathfrak{A}|}, p\cdot s, s}: |\hat{\mathfrak{A}}|^{p\cdot s} \to |\hat{\mathfrak{A}}|^s$ by

$$\mu^{\widehat{|\mathfrak{A}|}, p\cdot s, s}\big(x_i^{(n)}\big) = x_i^{(\bar{n})}, \qquad \text{where } \bar{n} \in \{1, \dots, s\} \text{ with } \bar{n} \bmod s = n \bmod s$$

$$\mu^{\widehat{|\mathfrak{A}|}, p\cdot s, s}(w) = \mu^{\widehat{|\mathfrak{A}|}, p\cdot s, s}(v_1) \cdot \mu^{\widehat{|\mathfrak{A}|}, p\cdot s, s}(v_2) \qquad \text{if } w = v_1 \cdot v_2.$$

Using the unique representatives as indicated in (1.5) and by inserting brackets as in $v_1(v_2(v_3 \dots) \dots)$ on the right (if necessary), we define $\mu^{|\hat{\mathfrak{B}}|, p\cdot s, s}$ also if $\mathfrak{B}$ is $\otimes\text{-}{}^m\hat{\mathfrak{H}}$, $\otimes\text{-}{}^m\hat{\mathfrak{K}}$, $*\text{-}{}^m\hat{\mathfrak{K}}$, $\sqcup_{\mathfrak{H}}\text{-}{}^m\hat{\mathfrak{K}}$, all m, or $\otimes\text{-}{}^m\hat{\mathfrak{E}}$, $m \ge 2$.

Here a $\otimes\text{-}{}^m\mathfrak{E}^{(p\cdot s)}$-monomial $w^{(1)} \dots w^{(p\cdot s)}$ will be identified with the corresponding $*\text{-}{}^m\mathfrak{E}^{(p\cdot s)}$-monomial and we make a similar identification in the nonassociative cases before applying one of the $\mu^{|\hat{\mathfrak{A}}|, p\cdot s, s}$ defined above. The last step is to re-order the letters to get the correct representative.

LEMMA 3.2. *Let $p, q \in \mathbb{N}^*$ and $\mathfrak{A}$ be as in* (1.4).

(i) *For $s, q \in \mathbb{N}$, $q \ge p + s$, there is an algebra homomorphism $\zeta_s^{|\hat{\mathfrak{A}}|, p, q}$ mapping $x_i^{(k)}$ to $x_i^{(k+s)}$ (all i, k).*

(ii) *Let $s \in \mathbb{N}^*$ and $G^{(k)} = (G_1^{(k)}, \dots, G_m^{(k)}) \in \hat{\mathfrak{A}}^s$ for $k = 1, \dots, p$ be given.*

There exists a unique K-linear map $\Delta^{\hat{\mathfrak{A}}, p, p\cdot s}_{G^{(1)}, \dots, G^{(p)}} = (\Delta^{|\hat{\mathfrak{A}}|, p, p\cdot s}_{G^{(1)}, \dots, G^{(p)}})^m : \hat{\mathfrak{A}}^p \to \hat{\mathfrak{A}}^{p\cdot s}$, where $\Delta^{|\hat{\mathfrak{A}}|, p, p\cdot s}_{G^{(1)}, \dots, G^{(p)}} : |\hat{\mathfrak{A}}|^p \to |\hat{\mathfrak{A}}|^{p\cdot s}$ is the algebra homomorphism given by

$$x_i^{(k)} \mapsto \zeta_{(k-1)s}^{|\hat{\mathfrak{A}}|, s, p\cdot s}\big(G_i^{(k)}\big) \qquad \text{for } i = 1, \dots, m \text{ and } k = 1, \dots, p.$$

Especially, $\Delta^{\hat{\mathfrak{A}},p,p}_{e_{1,1},\dots,e_{1,1}} = id_{\hat{\mathfrak{A}}^p}$, *and for* $1 \le k \le p$, $e_{p,k}$ *is mapped on* $(\zeta^{|\hat{\mathfrak{A}}|,s,p\cdot s}_{(k-1)s})^m(G^{(k)})$. ∎

DEFINITION 3.3. For $\mathfrak{A}, p, q$ as above and $G^{(1)},\dots,G^{(p)} \in \hat{\mathfrak{A}}^q$, let $\eta^{|\hat{\mathfrak{A}}|,p,q}_{G^{(1)},\dots,G^{(p)}}: |\hat{\mathfrak{A}}|^p \to |\hat{\mathfrak{A}}|^q$ be the (continuous) K-linear map given by $\mu^{|\hat{\mathfrak{A}}|,p\cdot q,q} \circ \Delta^{|\hat{\mathfrak{A}}|,p,p\cdot q}_{G^{(1)},\dots,G^{(p)}}$.

Let $\eta^{\hat{\mathfrak{A}},p,q}_{G^{(1)},\dots,G^{(p)}} := (\eta^{|\hat{\mathfrak{A}}|,p,q}_{G^{(1)},\dots,G^{(p)}})^m : \hat{\mathfrak{A}}^p \to \hat{\mathfrak{A}}^q$. Similarly defined are maps $\eta^{\mathfrak{A},p,q}_{G^{(1)},\dots,G^{(p)}}$ for the polynomial cases.

EXAMPLE 3.4. Let $m \ge 3$, $\mathfrak{A}$ as above, and let $L^{(1)} = e_{2,2}$, $L^{(2)} = L^{(3)} = e_{2,1} \in \mathfrak{A}^2$. Then $\eta^{|\mathfrak{A}|,3,2}_{L^{(1)},L^{(2)},L^{(3)}}(x_1^{(1)}(x_2^{(2)}x_3^{(3)})) = \mu^{|\mathfrak{A}|,6,2}(x_1^{(2)}(x_2^{(3)}x_3^{(5)}))$ is given by $x_1^{(2)}(x_2^{(1)}x_3^{(1)})$ if $\mathfrak{A} \in \{\mathfrak{E}, \mathfrak{K}, *\text{-}\mathfrak{K}\}$, and $x_2^{(1)}x_3^{(1)}x_1^{(2)}$ if $\mathfrak{A} \in \{\mathfrak{C}, \mathfrak{H}, \otimes\text{-}\mathfrak{E}, \otimes\text{-}\mathfrak{H}, \otimes\text{-}\mathfrak{K}\}$.

THEOREM 3.5. *Together with the selected elements* $e_{n,i} = (x_1^{(i)},\dots,x_m^{(i)})$ *and the composition maps from* (3.3), $\otimes\text{-}^m\hat{\mathfrak{H}}$, $\otimes\text{-}^m\hat{\mathfrak{K}}$, $*\text{-}^m\hat{\mathfrak{K}}$, $\sqcup_{\mathfrak{H}}\text{-}^m\hat{\mathfrak{K}}$, *all* m, *and* $\otimes\text{-}^m\hat{\mathfrak{E}}$ $(m \ge 2)$ *are complete pseudo-analyzers over* K *that are not analyzers*.

Proof. Let $\mathfrak{A}$ be as above or, more generally, as in (1.4). We are going to show that all $\mathfrak{A}$ are incomplete pseudo-analyzers. Then (using Lemma 2.6) it is clear that all $\hat{\mathfrak{A}}$ are complete pseudo-analyzers.

(1) From Lemma 3.2 one gets (all $1 \le i \le p$) $\eta^{\mathfrak{A},p,p}_{G^{(1)},\dots,G^{(p)}}(e_{p,i}) = G^{(i)}$ and $\eta^{\mathfrak{A},p,p}_{e_{p,1},\dots,e_{p,p}} = id_{\mathfrak{A}^p}$, as $\Delta^{\hat{\mathfrak{A}},p,p^2}_{e_{p,1},\dots,e_{p,p}}(e_{p,k}) = e_{p^2,(k-1)p+k}$. For $\lambda_1,\dots,\lambda_p \in K$ and $F \in \hat{\mathfrak{A}}^p_\alpha$, $\alpha = (\alpha_1,\dots,\alpha_p)$, $\Delta^{|\hat{\mathfrak{A}}|,p,p}_{\lambda_1 e_{1,1},\dots,\lambda_p e_{1,1}} = \lambda_1^{\alpha_1}\cdot\ldots\cdot\lambda_p^{\alpha_p}F$, where $0^0 = 1$. Similarly, $\eta^{\mathfrak{A},p,p}_{\lambda_1 G^{(1)},\dots,\lambda_p G^{(p)}}(F) = \lambda_1^{\alpha_1}\cdot\ldots\cdot\lambda_p^{\alpha_p}\eta^{\mathfrak{A},p,p}_{G^{(1)},\dots,G^{(p)}}(F)$ for all $G^{(i)} \in \mathfrak{A}^q$. For homogeneous $G^{(k)}$, $k = 1,\dots,p$, $G^{(k)} \in \mathfrak{A}^q_{(\beta_{k,1},\dots,\beta_{k,q})}$, it holds $\Delta^{\hat{\mathfrak{A}},p,p\cdot q}_{G^{(1)},\dots,G^{(p)}}(\hat{\mathfrak{A}}^p_{(\alpha_1,\dots,\alpha_p)}) \subseteq \hat{\mathfrak{A}}^{p\cdot q}_{(\alpha_1\beta_{1,1},\dots,\alpha_1\beta_{1,q},\dots,\alpha_p,\beta_{p,1},\dots,\alpha_p\beta_{p,q})}$. Hence $\eta^{\mathfrak{A},p,q}_{G^{(1)},\dots,G^{(p)}}(\mathfrak{A}^p_{(\alpha_1,\dots,\alpha_p)}) \subseteq \mathfrak{A}^q_{(\gamma_1,\dots,\gamma_q)}$, where $\gamma_j = \sum_{k=1}^p \alpha_k\,\beta_{k,j}$. We conclude that the conditions (C1), (D1)–(D4) are fulfilled.

(2) Assume $L^{(1)},\dots,L^{(n)} \in \mathfrak{A}^p$, $M^{(1)},\dots,M^{(p)} \in \mathfrak{A}^q$, $G^{(i)} = \eta^{\mathfrak{A},p,q}_{M^{(1)},\dots,M^{(p)}}(L^{(i)})$. For $\mathfrak{A} \notin \{{}^m\mathfrak{C}, {}^m\mathfrak{E}, {}^m\mathfrak{H}, {}^m\mathfrak{K}\}$ assume furthermore that $L^{(1)},\dots,L^{(n)}$ are pairwise compatible and also that $M^{(1)},\dots,M^{(p)}$ are pairwise compatible. Then $\eta^{\mathfrak{A},n,q}_{G^{(1)},\dots,G^{(n)}} = (\eta^{\mathfrak{A},p,q}_{M^{(1)},\dots,M^{(p)}})\circ(\eta^{\mathfrak{A},n,p}_{L^{(1)},\dots,L^{(n)}})$. In fact, as all maps are algebra homomorphisms if the assumptions on compatibility are fulfilled, it suffices to apply both sides to the $e_{n,i}$ and the equation is trivially fulfilled. Thus (C2a) holds for all $\mathfrak{A}$ and furthermore (C2b) holds for $\mathfrak{A} \in \{{}^m\mathfrak{C}, {}^m\mathfrak{E}, {}^m\mathfrak{H}, {}^m\mathfrak{K}\}$. Let $F \in \mathfrak{A}^p_\alpha$ and let $G = \eta^{p,p+1}_{L^{(1)},e_{p+1,3},\dots,e_{p+1,p+1}}(F)$, where $L^{(1)} := e_{p+1,1} + e_{p+1,2} = (x_1^{(1)} + x_1^{(2)},\dots, x_m^{(1)} + x_m^{(2)})$. Then in fact $G \in \bigoplus_{\{\beta_1,\beta_2 \in \mathbb{N}:\beta_1+\beta_2=\alpha_1\}}\mathfrak{A}^{p+1}_{\beta_1,\beta_2,\alpha_2,\dots,\alpha_p}$ (direct sum of $\alpha_1 + 1$ submodules), which shows (D5a) for all $\mathfrak{A}$.

(3) For a counterexample to (C2b) for $\mathfrak{A} = \otimes\text{-}\mathfrak{E}$, $m \ge 2$, let $p = 2$ and $L^{(1)} = e_{2,2}$, $L^{(2)} = e_{2,1} \in \mathfrak{A}^2$, $M^{(1)} = M^{(2)} = e_{1,1} \in \mathfrak{A}^1$. Then

$\eta^{|\mathfrak{A}|,2,2}_{L^{(1)},L^{(2)}}(x_1^{(1)}x_2^{(2)}) = x_2^{(1)}x_1^{(2)}$, which is mapped by $\eta^{|\mathfrak{A}|,2,1}_{M^{(1)},M^{(2)}}$ onto $x_2^{(1)}x_1^{(1)}$. Now $G^{(i)} := \eta^{\mathfrak{A},2,1}_{M^{(1)},M^{(2)}}(L^{(i)}) = e_{1,1}$ for $i = 1,2$, and $\eta^{|\mathfrak{A}|,2,1}_{e_{1,1},e_{1,1}}(x_1^{(1)}x_2^{(2)}) = x_1^{(1)}x_2^{(1)} \neq x_2^{(1)}x_1^{(1)}$. For $\mathfrak{A} \in \{\otimes\text{-}^m\mathfrak{H}, \ \otimes\text{-}^m\mathfrak{K}, \ *\text{-}^m\mathfrak{K}\}$, $m \geq 1$, similarly $\eta^{|\mathfrak{A}|,2,1}_{e_{1,1},e_{1,1}}(\eta^{|\mathfrak{A}|,3,2}_{e_{2,1},e_{2,1},e_{2,2}}(x_1^{(1)}x_1^{(2)}x_1^{(3)}x_1^{(3)})) = (x_1^{(1)}x_1^{(1)})(x_1^{(1)}x_1^{(1)}) \neq \eta^{|\mathfrak{A}|,3,1}_{e_{1,1},e_{1,1},e_{1,1}}(x_1^{(1)}x_1^{(2)}x_1^{(3)}x_1^{(3)})$, compare (3.1). If the representatives are chosen as indicated in (1.5) and $v \neq w$ are monomials of type $\mathfrak{K}$ over X, we use counter examples of the form $\eta^{|\mathfrak{A}|,2,1}_{e_{1,1},e_{1,1}}(\eta^{|\mathfrak{A}|,2,2}_{e_{2,2},e_{2,1}}(v^{(1)}w^{(2)})) = wv \neq \eta^{|\mathfrak{A}|,2,1}_{e_{1,1},e_{1,1}}(v^{(1)}w^{(2)})$ for $\sqcup_{\mathfrak{H}}\text{-}^m\mathfrak{K}$. ∎

Corollary 3.6. *For $R \in \mathbb{N}^*$, the assertions of* (3.5) *remain true for the corresponding truncations* $\mathfrak{A}[R]$.

Remark 3.7. From the proof above we can get a direct proof that ${}^m\hat{\mathfrak{C}}$, ${}^m\hat{\mathfrak{E}}$, ${}^m\hat{\mathfrak{H}}$, ${}^m\hat{\mathfrak{K}}$ are analyzers. We still have to verify that (D5b) holds. The main reason for this is that for the (operad) types $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{K}$ a simple support argument shows that we only have to check (D5b) in the case where $f = F$—see part (2) of the proof—is a single (normed) monomial. Then $\binom{\alpha_1}{\beta_1}F$, the right-hand side of (D5b) in (2.1), is obtained similarly to the fact that $P_{\beta_1,\alpha_1-\beta_1}((x_j^{(1)} + x_j^{(2)})^{\alpha_1})$ is given by $\sum_{I=\{i_1,\dots,i_{\beta_1}\}\subset\{1,2,\dots,\alpha_1\}} x_j^{(\delta_1)} \cdot \ldots \cdot x_j^{(\delta_{\alpha_1})}$, δ_s replaced by 1 if $s \in I$ and by 2 if $s \notin I$.

Remark 3.8. The projection $K\langle\langle X^{(n)}\rangle\rangle \to K[[S^{(n)}]]$ induces a surjective morphism of analyzers $\hat{\mathfrak{E}} \to \hat{\mathfrak{C}}$. The projection $(K\langle\langle X\rangle\rangle^{\otimes n} \to K[[S^{(n)}]]$ induces a surjective morphism of pseudo-analyzers $\otimes\text{-}\hat{\mathfrak{E}} \to \hat{\mathfrak{C}}$. The projection $\pi = (\pi^{(n)} : K\langle\langle X^{(n)}\rangle\rangle \to (K\langle\langle X\rangle\rangle^{\otimes n})_{n\in\mathbb{N}}$ does not induce a morphism of pseudo-analyzers: For $g_1 = g_2 = e_{1,1}$ and $f = x_2^{(2)}x_1^{(1)}$ we have $g_i' := \pi^{(1)}(g_i) = g_i$, $f' := \pi^{(2)}(f) = x_1^{(1)}x_2^{(2)}$, $\eta^{\mathfrak{E},2,1}_{g_1,g_2}(f) = x_2^{(1)}x_1^{(1)}$, and $\eta^{\otimes\text{-}\mathfrak{E},2,1}_{g_1',g_2'}(f') = x_1^{(1)}x_2^{(1)}$ is not given by $\pi^{(1)}(\eta^{\mathfrak{E},2,1}_{g_1,g_2}(f))$.

4. GROUP LAWS

Cogroups in categories of complete algebras can be described by formal group laws (for analyzers or operads). If we replace in the definition of a cogroup object in a category of algebras (of a given operad type $\mathfrak{A}$), see [ML2, BH], the categorical coproduct $\sqcup_{\mathfrak{A}}$ by some $\sqcup_{\mathfrak{B}}$ as in (1.4), we get the notion of a $\sqcup_{\mathfrak{B}}$*-cogroup* in the category of algebras of type $\mathfrak{A}$. This can be done similarly for complete augmented algebras, where we require the counit to be the augmentation map ε. Instead of elaborating the full definition, we define formal group laws in pseudo-analyzers and describe then how they are connected to $\widehat{\sqcup_{\mathfrak{B}}}$-cogroups.

Definition 4.1. Let $\mathfrak{A}$ be a complete pseudo-analyzer over the ring K and $F \in \mathfrak{A}^2$. We define $F^< := \eta^{\mathfrak{A},2,3}_{e_{3,1},e_{3,2}}(F)$, $F^> := \eta^{\mathfrak{A},2,3}_{e_{3,2},e_{3,3}}(F)$, and $\Gamma F :=$

$\eta^{\mathfrak{A},2,3}_{F^<,e_{3,3}}(F) - \eta^{\mathfrak{A},2,3}_{e_{3,1},F^>}(F) = \Delta^{\mathfrak{A},2,3}_{F,e_{1,1}}(F) - \Delta^{\mathfrak{A},2,3}_{e_{1,1},F}(F) \in \mathfrak{A}^3$. F is called *group law in* $\mathfrak{A}$, iff

(a) $\eta^{\mathfrak{A},2,1}_{e_{1,1},0}(F) = e_{1,1} = \eta^{\mathfrak{A},2,1}_{0,e_{1,1}}(F)$

(b) $\Gamma F = 0$.

Group laws in $\mathfrak{A}[R]$ are considered as R-chunks F (determined mod J^{R+1}) in $\mathfrak{A}$ by identifying an R-chunk with its canonical representative having zero (degree $\geq R+1$)-components.

The group laws in $\hat{\mathfrak{C}}$ are also called *classical formal group laws*.

Homomorphisms of group laws (or chunks) are defined as for classical group laws, cf. [Ha, La1]: $\varphi \in \mathfrak{A}^1$ is a *homomorphism* $F \to G$ if $\Theta\varphi := \eta^{\mathfrak{A},1,2}_F(\varphi) - \eta^{\mathfrak{A},2,2}_{\varphi^<,\varphi^>}(G) = 0$ in $\mathfrak{A}^2$, where $\varphi^< := \eta^{\mathfrak{A},1,2}_{e_{2,1}}(\varphi)$, $\varphi^> := \eta^{\mathfrak{A},1,2}_{e_{2,2}}(\varphi)$.

Such a homomorphism φ is called a *strict isomorphism*, if $\varphi \equiv e_{1,1}$ mod J^2.

EXAMPLE 4.2. For every pseudo-analyzer, $e_{2,1} + e_{2,2}$ is a group law and it is clearly the only possible 1-chunk.

The Campbell–Hausdorff series $x^{(1)} + x^{(2)} + \frac{1}{2}[x^{(1)}, x^{(2)}] + \dots$ is a group law in ${}^1\hat{\mathfrak{C}}$. It is also a formal group law over $\mathfrak{Lie}$. For operads $\mathfrak{P}$, see (1.1), group laws over $\mathfrak{P}$ are defined as group laws in the associated analyzer; cf. [Fr1, GK].

EXAMPLE 4.3. Let $m = n^2$ and write $X = \{x_{ij} : 1 \leq i, j \leq n\}$. Then the m-tuple (F_{ij}) with entries $F_{ij} = x^{(1)}_{ij} + x^{(2)}_{ij} + \sum_{l=1}^n x^{(1)}_{il} x^{(2)}_{lj}$ defines a group law in the pseudo-analyzer $\otimes$-$\hat{\mathfrak{C}}$; cf. [Ho2]. The algebra homomorphism given by $x_{ij} \mapsto F_{ij}$ defines a $\hat{\otimes}$-cogroup structure (i.e., structure of a complete Hopf algebra) on $K\langle\langle X \rangle\rangle$. It is the completion of the well-known bialgebra-structure on the coordinate ring on $n \times n$-matrices, or its Hopf envelope, which is also called the general linear quantum group.

We remark that the existence of the antipode is always given for complete bialgebras. It is known for complete $\mathfrak{P}$-algebras, $\mathfrak{P}$ an operad over a field K of characteristic 0, that they are isomorphic to a free complete $\mathfrak{P}$-algebra if they are endowed with a cogroup structure; see [Fr2].

The analogous result for $\otimes$-$\hat{\mathfrak{C}}$ is the following:

THEOREM 4.4. *Let A be a complete associative algebra, finitely generated, over a field K of characteristic* 0. *If A is a complete Hopf algebra, then A is as complete algebra isomorphic to the quotient of a free complete algebra $K\langle\langle x_1, \dots, x_m \rangle\rangle$ modulo a closed ideal I that is contained in the ideal generated by all commutators $[x_i, x_j]$.*

Proof. Since the ideal $[A, A]$ of all commutators in A is a Hopf ideal, $A/[A, A]$ is a complete commutative Hopf algebra or cogroup object, which is isomorphic to some $K[[X]]$ by the classical result for $\mathfrak{P} = \mathfrak{C}om$. For A we can obtain the form $K\langle\langle x_1, \dots, x_m \rangle\rangle / I$, I a closed ideal, for some m and we assume that m is minimal. I cannot contain elements of lower than quadratic order, because relations involving linear terms would allow the cancellation of a variable in contradiction to the minimality of m; see [GH, Sect. 6]. Thus $A/[A, A] = K[[x_1, \dots, x_m]]$ for the same m, and I must be contained in the ideal generated by the commutators. ∎

EXAMPLE 4.5. For every Hopf algebra, its completion with respect to the augmentation ideal $\ker \varepsilon$ is a complete Hopf algebra; see [Ho2]. In the case where $I = 0$, the Hopf algebra structures are exactly described by formal group laws of type $\otimes$-$\hat{\mathfrak{F}}$.

For the coordinate algebras of quantum groups defined by R-matrices, cf. [CP], like GL_q or O_q for $1 \neq q \in K^*$, the theorem shows that their relations must induce commutator relations in the completion. The relations, written in variables $x_{ij} \in \ker \varepsilon$, are for example $q(1 + x_{11})x_{12} = x_{12}(1 + x_{11})$. Since $(q - 1)x_{12}$ occurs as a linear term, x_{12} will be zero in the completion. Carrying this out one shows that the completions of all these Hopf algebras are commutative (see [Ho1]). This also holds for multi-parameter quantum deformations like the ones defined in [AST].

Remark 4.6. Let $R \in \mathbb{N}^*$. Every $F \in \mathfrak{F}[R]^2$ that fulfills condition (a) of Definition 4.1 is given by $\sum_{r=1}^{R}(2^r - 2)m^{r+1}$ coefficients from K. Similarly, $\{F \in \otimes\text{-}\mathfrak{F}[R]^2 : F \text{ fulfills (a)}\}$ is a free K-module of rank $\sum_{r=1}^{R}(r - 1)m^{r+1}$. In the classical case this number is $\sum_{r=1}^{R} m \cdot (\binom{2m-1+r}{2m-1} - 2\binom{m-1+r}{m-1}) = m(\binom{2m+R}{2m} - 2\binom{m+R}{m} + 1)$. Let $(c_r)_r$ denote the recursively defined sequence with $c_1 = 1$, $c_r = \sum_{s=1}^{r-1} c_s c_{r-s}$ for $r > 1$ (Catalan numbers for binary trees). Then the corresponding number of coefficients in the noncommutative nonassociative case $\mathfrak{R}[R]^2$ is $\sum_{r=1}^{R}(2^r - 2)m^{r+1}c_r$.

Condition (b) determines a system of equations for the coefficients, defining varieties (universal formal group law chunks) associated to the given type.

For a given $F \bmod J^{r+s}$, $s \geq 2$, which is $\bmod J^{r+1}$ an r-chunk, ΓF is of ord $\geq r + 1$ and we denote by $\Gamma_{r+1}F$ the (degree $r + 1$)-component of ΓF. Similarly, for a given homomorphism of r-chunks that actually are s-chunks with $s > r$, we denote by $\Theta_{r+1}\varphi$ the (degree $r + 1$)-component of $\Theta\varphi$.

The following proposition, proven by Lazard for analyzers, also holds for pseudo-analyzers.

PROPOSITION 4.7. *Let F be a group law r-chunk in a pseudo-analyzer $\mathfrak{A}$.*

(i) $\Gamma_{r+1}F \in \ker \delta_3$. *Thus $\Gamma_{r+1}F$ defines an element of H^3_{r+1}, called the obstruction of F.*

(ii) *Let F be a group law r-chunk and $F' := F + L$ for a (homogeneous) $L \in \mathfrak{A}^2_{r+1}$. Then $(\Gamma_{r+1}F - \Gamma_{r+1}F') = \delta_2(L)$. Thus there exists a (homogeneous) $L \in \mathfrak{A}^2_{r+1}$ such that $F + L$ is an $(r+1)$-chunk, if and only if the obstruction of F is* 0. *In this case, L is unique up to addition with an element of* $\ker(\delta_2)$.

(iii) *If F and G are $(r+1)$-chunks and $\varphi : F \to G$ is a homomorphism of the corresponding r-chunks, then $\Theta_{r+1}\varphi \in \ker(\delta_2)$. The residue class of $\Theta_{r+1}\varphi$ in H^2_{r+1} is called the obstruction of φ.*

(iv) *Let φ, F, and G be as in* (iii) *and $\varphi' := \varphi + \psi$ for a (homogeneous) $\psi \in \mathfrak{A}^1_{r+1}$. Then $(\Theta_{r+1}\varphi - \Theta_{r+1}\varphi') = \delta_1(\psi)$. Thus there exists a (homogeneous) $\psi \in \mathfrak{A}^1_{r+1}$ such that $\varphi + \psi$ is a homomorphism of $(r+1)$-chunks, if and only if the obstruction of φ is* 0. *In this case, ψ is unique up to addition with an element of* $\ker(\delta_1)$.

Proof. Since all inserted elements are compatible, the proofs of (i)–(iv) are the same for pseudo-analyzers as for analyzers; thus compare [La1] for the proofs. We give the proof of (iv), the other proofs are longer but similar. We use the composition lemma. $B' := G - e_{2,1} - e_{2,2}$ is $\equiv 0$ mod J^2. $\varphi' \equiv \varphi \mod J^{r+1}$. Thus $\eta^{\mathfrak{A},2,2}_{<\varphi',\varphi'>}(G) = \eta^{\mathfrak{A},1,2}_{e_{2,1}}(\varphi) + \eta^{\mathfrak{A},1,2}_{e_{2,1}}(\psi) + \eta^{\mathfrak{A},1,2}_{e_{2,2}}(\varphi) + \eta^{\mathfrak{A},1,2}_{e_{2,2}}(\psi) + \eta^{\mathfrak{A},2,2}_{<\varphi',\varphi'>}(B') \equiv \eta^{\mathfrak{A},1,2}_{e_{2,1}}(\psi) + \eta^{\mathfrak{A},1,2}_{e_{2,2}}(\psi) + \eta^{\mathfrak{A},2,2}_{<\varphi,\varphi>}(G)$ mod J^{r+2}. Similarly $\eta^{\mathfrak{A},1,2}_F(\varphi') = \eta^{\mathfrak{A},1,2}_F(\varphi) + \eta^{\mathfrak{A},1,2}_F(\psi) \equiv \eta^{\mathfrak{A},1,2}_F(\varphi) + \eta^{\mathfrak{A},1,2}_{e_{2,1}+e_{2,2}}(\psi) \mod J^{r+2}$. Subtraction yields assertion (iv). ∎

5. A SPECTRAL SEQUENCE

The torsion theorem (10.1′) of [La1] says that $H^n_r(\mathfrak{A}) = 0$ for all $n \neq r$ and all analyzers $\mathfrak{A}$ that are *rational* defined as follows:

DEFINITION 5.1. For $r \in N^*$, let $\mathbb{Q}_r := M^{-1}\mathbb{Z}$, where M is the multiplicative system $\{n \in \mathbb{N}^* : \text{if } p \text{ prime with } p \mid n \text{ then } p \leq r\}$. Then a complete or incomplete pseudo-analyzer over K is called *rational*, if $\mathfrak{A}^n_r$ is a $\mathbb{Q}_r$-module for all $r \in \mathbb{N}^*$.

For $\mathfrak{A}$ a pseudo-analyzer, denote by $\tilde{\mathfrak{A}}$ the subcomplex of elements $f \in \mathfrak{A}^n$ with $\eta^{n,n}_{e_{n,1},\ldots,e_{n,i-1},0,e_{n,i+1},\ldots,e_{n,n}}(f) = 0$ for all i.

Remark 5.2. To compute cohomology in the analyzer case, we can replace $\mathfrak{A}$ by $\tilde{\mathfrak{A}}$. Also in the pseudo-analyzer case, we will only be interested in the cohomology of $\tilde{\mathfrak{A}}$.

It is clear that $H^n_r(\tilde{\mathfrak{A}}) = 0$ for all $n > r$ (for all not necessarily rational pseudo-analyzers).

In [La1, p. 356] it is shown for analyzers $\mathfrak{A}$ that there is a canonical map φ_r from the submodule $\{f \in \tilde{\mathfrak{A}}^r_r : f \text{ anti-symmetric}\}$ of $\mathfrak{A}^r_r$ to $H^r_r(\tilde{\mathfrak{A}})$,

$r \in \mathbb{N}^*$. If $f \in \tilde{\mathfrak{A}}^r_r$ is an element of $\mathrm{im}(\delta_{r-1})$ then $\sum_{\sigma \in S_r} \mathrm{sign}(\sigma)\sigma f = 0$. Therefore any anti-symmetric F is in the kernel of φ_r iff $r!f = 0$.

The cokernel of φ_r is similarly described as additive torsion group (0 in the rational case); see [La1, Sect. 9]. (Even if $\bar{g} \in H^r_r(\tilde{\mathfrak{A}})$ is not represented by an anti-symmetric cocycle, the cohomology class of $r!g$ is.)

PROPOSITION 5.3. *Let* $\mathfrak{A} \in \{\mathfrak{E}, \mathfrak{H}, \mathfrak{K}\}$, $F \in \tilde{\mathfrak{A}}^3_3$.

For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *a permutation of* $(1,2,3)$, $i_1, i_2, i_3, n \in \{1, \dots, m\}$, *we denote the coefficient of* $x^{(\lambda_1)}_{i_1}(x^{(\lambda_2)}_{i_2}x^{(\lambda_3)}_{i_3})$ *in* F_n *by* $a(n; \lambda_1, \lambda_2, \lambda_3; i_1, i_2, i_3\}$, *where we furthermore assume* $\lambda_2 < \lambda_3$ *for* $\mathfrak{A} = \mathfrak{H}$. *The coefficient of* $(x^{(\lambda_1)}_{i_1}x^{(\lambda_2)}_{i_2})x^{(\lambda_3)}_{i_3}$ *in* F_n *for* $\mathfrak{A} = \mathfrak{K}$ *is denoted by* $a'(n; \lambda_1, \lambda_2, \lambda_3; i_1, i_2, i_3)$.

For $\mathfrak{A} \in \{\mathfrak{E}, \mathfrak{K}\}$, *F is a coboundary iff for all n, all* $\iota = (\iota_1, \iota_2, \iota_3) \in \{1, \dots, m\}^3$

$$\sum_\lambda sign(\lambda) a(n; \lambda; \iota) = 0,$$

and for $\mathfrak{A} = \mathfrak{K}$,

$$\sum_\lambda sign(\lambda) a'(n; \lambda; \iota) = 0.$$

For $\mathfrak{A} = \mathfrak{H}$, *the condition for F to be a coboundary is*

$$\sum_{\lambda \in S_3 : \lambda_2 < \lambda_3} sign(\lambda)\big(a(n; \lambda; \iota) - a(n; \lambda; \iota_1, \iota_3, \iota_2)\big) = 0,$$

$$\text{all } n, \iota \ (\iota_2 < \iota_3).$$

Proof. We have to show that F is of the form $\delta_2(L) = (\eta^{\mathfrak{A},2,3}_{e^{(1)},(e^{(2)}+e^{(3)})} - \eta^{\mathfrak{A},2,3}_{e^{(1)},e^{(2)}} - \eta^{\mathfrak{A},2,3}_{e^{(1)},e^{(3)}}) - (\eta^{\mathfrak{A},2,3}_{(e^{(1)}+e^{(2)}),e^{(3)}} - \eta^{\mathfrak{A},2,3}_{e^{(2)},e^{(3)}} - \eta^{\mathfrak{A},2,3}_{e^{(1)},e^{(3)}})$ iff the asserted equations hold.

Here $L \in \tilde{\mathfrak{A}}^2_3$ is given by $12m^4$ coefficients $\Phi(n; \lambda; \iota)$, $\Phi'(n; \lambda; \iota)$ in case $\mathfrak{A} = \mathfrak{K}$, $6m^4$ in case $\mathfrak{A} = \mathfrak{E}$, and $3m^4 + m^3$ in case $\mathfrak{A} = \mathfrak{H}$; as in all cases we can assume $1 \le \lambda_1, \lambda_2, \lambda_3 \le 2$, not all equal, and for $\mathfrak{H}$ furthermore $\lambda_2 \le \lambda_3$ and $i_2 \le i_3$ if $\lambda_2 = \lambda_3$.

Now $F = \delta_2(L)$ is equivalent to the system of equations

$$a(n; \lambda; \iota) = \Phi(n; \sqcap \lambda; \iota) - \Phi(n; \sqcup \lambda; \iota),$$

and for $\mathfrak{K}$,

$$a'(n; \lambda; \iota) = \Phi'(n; \sqcap \lambda; \iota) - \Phi'(n; \sqcap \lambda; \iota),$$

where $\sqcap : \{1,2,3\}^3 \to \{1,2\}^3$ fixes 1 and 2 and replaces 3 by 2, whereas $\sqcup$ fixes 1 and replaces 2 by 1 and 3 by 2. Thus for example $\sqcap(3,1,2) = (2,1,2)$ and $\sqcap(3,1,2) = (2,1,1)$. For $\mathfrak{A} = \mathfrak{H}$, when $(1,2,2)$ or $(2,1,1)$ occur, we have to replace ι by $(\iota_1, \iota_3, \iota_2)$ if $\iota_2 > \iota_3$.

The only-if direction of the assertion—already noted above—follows.

To complete the proof we regard the $2m^4$ coefficients $a(n;132;\iota)$ and $a'(n;132;\iota)$ (all n,ι) for $\mathfrak{K}$, m^4 coefficients $a(n;123;\iota)$ for $\mathfrak{E}$ and the $\frac{1}{2}(m^4-m^3)$ coefficients $a(n;123;\iota)$ with $\iota_2>\iota_3$ for $\mathfrak{H}$ as determined (via the system of equations of (5.3)) by the other coefficients a, a', which we assume to be freely chosen.

If we set all $\Phi^{(\prime)}(n;122;\iota)=0$ and thus only use $10m^4$, $5m^4$, or $3m^4 - \frac{1}{2}(m^4-m^3)$ coefficients describing L for $\mathfrak{K}$, $\mathfrak{E}$, $\mathfrak{H}$, respectively, it is easy to determine the coefficients of L such that $\delta(L)=F$. In fact, for $\mathfrak{E}$, $\mathfrak{K}$ we start with $-a^{(\prime)}(n;123;\iota)=\Phi^{(\prime)}(n;112;\iota)$ and use $a^{(\prime)}(n;\lambda;\iota)=\Phi^{(\prime)}(n;\sqcap\lambda;\iota)-\Phi^{(\prime)}(n;\sqcap\lambda;\iota)$ successively except for $a^{(\prime)}(n;132;\iota)$. For $\mathfrak{H}$ we use the same equations, but first for $\iota_2\le\iota_3$, then for $\iota_2>\iota_3$. For $\mathfrak{H}$, we note also that the m^3 equations with $\iota_2=\iota_3$ give empty conditions. ∎

Remark 5.4. Let us look at the proof for the torsion theorem given in [La1, Sect. 10].

The induction hypothesis is the following: For $r\le 2$ and all rational analyzers, $H_r^n(\tilde{\mathfrak{A}})=0$ holds for all $n\neq r$. To get this, it is essential to use (D5b), and (D5b) is not fulfilled for general pseudo-analyzers.

After fixing the total degree r, we are able to define an exhausting regular filtration $0\subseteq A_1\subseteq\ldots\subseteq A_r$ of $A_r=\oplus_n\tilde{\mathfrak{A}}_r^n$ for pseudo-analyzers as Lazard did for analyzers. For $f\in\mathfrak{A}^n$, f is an element of A_i iff the degree α_n with respect to (the last argument) $e_{n,n}$ is $\le i$ for every nonzero component of f.

We get a spectral sequence $(E^{p,n},d)$ with $E_0^{p,n}=A_p^n/A_{p-1}^n$ (zero if not $p+n-1\le r$, $n,p\in\mathbb{N}^*$) and d induced by δ as in Lazard's proof.

We note that $A_p^1=0$ if $p\neq r$ and that $E_0^{p,1}=A_p^1=E_1^{p,1}$. For $n\ge 2$ and all p, the complex $(E_0^{p,n},d_0:E_0^{p,n}\to E_0^{p,n+1})$ and its cohomology $E_1^{p,n}$ is determined by the complex for total degree $r-p$ and $n-1$ arguments: d_0 fixes all variables with superindex (n) and acts as δ_{n-1} with respect to the other $n-1$ arguments.

Let us look at the case $\mathfrak{A}=\otimes\text{-}^m\mathfrak{E}$ for small total degrees r. The case $r=1$ is trivial. Here $E_0^{1,1}=A_1^1=H_1^1$ is given by all m-tuples with entries from V, where V is the vector space generated by X.

By definition $H_r^1=\ker\delta_1\,|\,\mathfrak{A}_r^1$ is given by the pseudolinear elements of degree r. Since pseudolinearity can be checked componentwise, let us call $f\in|\hat{\mathfrak{A}}|^1$ pseudolinear, iff $(f,0,\ldots,0)$ is.

LEMMA 5.5. *For* $\widehat{|\otimes\text{-}^m\mathfrak{E}|}^1$, *the following elements* (*and their K-linear combinations*) *are pseudolinear*:

(i) *variables* $x_i\in X$

(ii) *the commutators* $[g,h]:=gh-hg$ *of pseudolinear* g *and* h.

Proof. We can restrict to the case where f is homogeneous of degree r. To be pseudolinear, f has to fulfill $P_{(s,r-s)}\eta_{e^{(1)}+e^{(2)}}(f) = 0$ for all $0 < s < \frac{r+1}{2}$ (this is equivalent to the definition). Part (i) is clear. For (ii), by assumption we only have to look at the case where s is the degree of g (and $r - s$ is the degree of h) and $\eta_{e^{(1)}}$ is applied to the factor g, $\eta_{e^{(2)}}$ to the factor h. Now clearly $[\eta_{e^{(1)}}(g), \eta_{e^{(2)}}(h)] = 0$. ∎

COROLLARY 5.6. *The torsion theorem* (10.1′) *of* [La1] *is not true for rational pseudo-analyzers.*

Proof. For $\mathfrak{A} = \otimes\text{-}^m\mathfrak{C}$, $m > 1$, $r > 1$, $H_r^1(\tilde{\mathfrak{A}})$ contains m-tuples of commutators given by (5.5). ∎

THEOREM 5.7. *For $r \in \{2, 3\}$ and the rational pseudo-analyzer given by $\mathfrak{A} = \otimes\text{-}^m\mathfrak{C}$, the nonzero modules of the cohomology in total degree r, $H_r^n(\tilde{\mathfrak{A}})$, are given by:*

(i) *For $r = 2$, $n = 1$, all m-tuples of linear combinations of $\{[x_h^{(1)}, x_i^{(1)}] : h < i\}$;*

for $n = 2$, all m-tuples of linear combinations of $\{[x_h^{(1)}, x_i^{(2)}] : h < i\}$.

(ii) *For $r = 3$, $n = 1$, spanned by all m-tuples with entries $[x_h^{(1)}, [x_i^{(1)}, x_j^{(1)}]]$;*

For $n = 2$, all m-tuples of linear combinations of $\{[x_h^{(1)}, x_i^{(1)}]x_j^{(2)} : h < i\}$ (or equivalently $\{x_h^{(1)}[x_i^{(2)}, x_j^{(2)}] : i < j\}$);

for $n = 3$, all anti-symmetric elements of $\tilde{\mathfrak{C}}_3^3$.

Proof. (1) We saw above that, for $r = 2$, only $(p, n) \in \{(2,1), (1,2)\}$ have to be considered (where p is the filtration degree). The only nontrivial map is $d_1 : A_2^1 = E_1^{2,1} \to E_1^{1,2} = A_1^2$. Its kernel $E_2^{2,1} = E_\infty^{2,1}$ is given by the pseudolinear elements of degree 2, i.e., tuples of commutators and their linear combinations. Since all elements of H_2^1 have degree 2 with respect to their last (and only) argument, $E_\infty^{2,1} = H_2^1$. Similarly the cokernel $E_2^{1,2}$ of this map, spanned by anti-symmetric $f \in \tilde{\mathfrak{A}}_2^2$, is H_2^1.

(2) Let now $r = 3$. For $n \geq 2$ the modules $E_1^{p,n}$ are determined by the results for lower total degrees (see above). Nonzero are $E_1^{2,2}$, spanned by the tuples with entries $x_h^{(1)}x_i^{(2)}x_j^{(2)}$, $E_1^{1,2}$, spanned by tuples with entries $x_h^{(1)}[x_i^{(2)}, x_j^{(2)}]$ and $E_1^{1,3}$, given by multilinear elements with entries $[x_h^{(1)}, x_i^{(2)}]x_j^{(3)}$. For $n = 1$, only $E_1^{3,1} = A_3^1$ occurs.

(3) On $E_1^{3,1}$, d_1 is the map $P_{(1,2)}\eta_{e^{(1)}+e^{(2)}} : A_3^1 \to A_2^2/A_1^2$. Thus its kernel $E_2^{3,1}$ is given by the pseudolinear elements of degree 3. We claim that for $|\otimes\text{-}^m\mathfrak{C}|_3^1$ the elements given in (5.5) are the only pseudolinear elements. Assume that f is pseudolinear and not a linear combination of

elements $[x_h^{(1)},[x_i^{(1)},x_j^{(1)}]]$. Since there are no pseudolinear elements of degree 3 for type $\mathfrak{C}$, f is contained in the ideal generated by commutators. It suffices to look at linear combinations of $[x_h^{(1)}x_i^{(1)}, x_j^{(1)}]$, $x_h^{(1)}[x_i^{(1)}, x_j^{(1)}]$, and $[x_i^{(1)}, x_j^{(1)}]x_h^{(1)}$. Aside from forming the commutator, we cannot combine the last two to have zero image under $P_{(1,2)}\eta_{e^{(1)}+e^{(2)}}$, as their images are $x_h^{(1)}[x_i^{(2)}, x_j^{(2)}]$. The image of $[x_h^{(1)}x_i^{(1)}, x_j^{(1)}]$ is $x_h^{(1)}[x_i^{(2)}, x_j^{(2)}] + x_i^{(1)}[x_h^{(2)}, x_j^{(2)}]$ and the only way to cancel this out is to form a combination like $[x_h^{(1)}x_i^{(1)}, x_j^{(1)}] - x_h^{(1)}[x_i^{(1)}, x_j^{(1)}] - x_i^{(1)}[x_h^{(1)}, x_j^{(1)}]$ which is $[x_i^{(1)},[x_j^{(1)}, x_h^{(1)}]]$.

(4) The map $d_1 : E_1^{2,2} \to E_1^{1,3}$ induced by δ_2 (i.e., induced by $\eta_{e^{(1)}, e^{(2)}+e^{(3)}}$) has the cokernel $E_2^{1,3} = E_\infty^{1,3} = H_3^3 = \{$anti-symmetric elements of $\tilde{\mathfrak{C}}_3^3\}$, because the quotient in question is the same quotient that occurs for type $\mathfrak{C}$. Note that $x_h^{(1)}x_i^{(2)}x_j^{(2)}$ and $x_h^{(1)}x_j^{(2)}x_i^{(2)}$ have the same image. Analogously, we get that $E_2^{2,2} = 0$: In comparison to the type $\mathfrak{C}$ case (where $E_2^{2,2} = 0$) the only new cocycles in $E_1^{2,2}$ are given by tuples with entries $x_h^{(1)}[x_i^{(2)}, x_j^{(2)}]$, which are in the image of $P_{(1,2)}\eta_{e^{(1)}+e^{(2)}}$.

(5) At last we look at the only possibly nonzero map $d_2 : E_2^{3,1} \to E_2^{1,2}$. Since $E_2^{3,1}$ consists of pseudolinear elements, this is also zero. Thus $E_2^{3,1} = E_\infty^{3,1}$ gives H_3^1 and $E_2^{1,2} = E_\infty^{1,2}$ gives the elements of H_3^2 with degree 1 with respect to the second argument. Since $E_2^{2,2} = 0$, they represent all elements of H_3^2. ∎

Remark 5.8. There is no reason why the case $r = 3$ of Theorem 5.7(ii) should not be typical. By Friedrich's Theorem, the description of pseudolinear elements is as simple for higher r as it is for $r \leq 3$.

We remark that for all n one can find cocycles $L \in \otimes\text{-}^m\tilde{\mathfrak{S}}^n$ that are not coboundaries, like the one given by $L_1 = x_1^{(1)} \cdot \ldots \cdot x_1^{(n-1)}(x_2^{(n)}x_1^{(n)} - x_1^{(n)}x_2^{(n)})$ and $L_i = 0$ $(i \geq 2)$.

We noted in part (4) of the proof above that the computation of $H_r^r(\sqcup_{\mathfrak{C}}\text{-}^m\tilde{\mathfrak{S}})$ is exactly the same as in type $\mathfrak{C}$, because we look at multilinear elements and multilinear elements obey the relations of $\mathfrak{C}$. It is easy to see that a similar result holds for the other mixed types of (3.5), and for all r. Thus we get:

PROPOSITION 5.9. *For the pseudo-analyzers* $\sqcup_{\mathfrak{B}}\text{-}^m\mathfrak{A}$ *given in* (3.5) *in the rational case*, *it holds* (*for all* r)

$$H_r^r\left(\sqcup_{\mathfrak{B}}\text{-}^m\tilde{\mathfrak{A}}\right) = H_r^r(^m\mathfrak{B}) = \left\{f \in {}^m\tilde{\mathfrak{B}}_r^r : f \text{ anti-symmetric}\right\}.$$

∎

6. INTERACTIONS BETWEEN DIFFERENT GROUP LAW TYPES

Our aim in this last section is to find solutions for the system of equations, given in (4.6), associated to universal formal group laws (or chunks) for a given type.

Proposition 4.7 and the torsion theorem (for the cases where it holds) will be our main tools. We can restrict to the subcomplex $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$, as $\Gamma\mathrm{F}$, $\Theta\varphi \in \tilde{\mathfrak{A}}$.

The first nontrivial extension problem is the extension from 2-chunks to 3-chunks (here the obstructions are in $H_3^3(\tilde{\mathfrak{A}})$). We denote the coefficient of $x_{i_1}^{(\lambda_1)} x_{i_2}^{(\lambda_2)}$ in F_n by $\Phi(n; \lambda_1, \lambda_2; i_1, i_2)$ with $i_1, i_2, n \in \{1, \dots, m\}$ and $\lambda_1 \neq \lambda_2 \in \{1, 2\}$, where $\mathfrak{A}$ is any of the pseudo-analyzers given in (3.5).

Remark 6.1. The group laws of types $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{K}$, and $\mathfrak{H}$, are group laws over operads. For the rational analyzer given by ${}^m\hat{\mathfrak{C}}$ it is well known that the extension of a 2-chunk to 3-chunk and group law is possible (and unique up to strict isomorphism) iff the elements $\Psi(n; i_1, i_2) := \Phi(n; 1, 2; i_1, i_2) - \Phi(n; 1, 2; i_2, i_1)$ of K are structure constants of an m-dimensional Lie algebra over K. This correspondence between classical group laws and Lie algebras defines an equivalence of categories.

The notion structure constants means that a structure of a (not necessarily unitary or associative) algebra is defined on the free K-module with basis $e_1, \dots, e_m$ by $e_{i_1} e_{i_2} := \sum_{n=1}^m \Psi(n; i_1, i_2) e_n$.

We have seen that $H_3^3(\otimes\text{-}\tilde{\mathfrak{E}}) = H_3^3(\mathfrak{C})$. This leads to the following proposition (of [Ho1], where a direct computation is given). For the nonuniqueness up to strict isomorphisms (see [Ho1, Sect. 6]) we only refer to the nontriviality of $H_3^2(\otimes\text{-}\tilde{\mathfrak{E}})$.

PROPOSITION 6.2. *A 2-chunk F in* $\otimes$-$\mathfrak{E}$ *is nonuniquely extendable to a* 3-*chunk, iff the* $\Psi_{h,i}^{(n)} := (\Phi(n; 1, 2; h, i) - \Phi(n; 1, 2; i, h))$, $h < i$, *are structure constants of a Lie algebra.* ∎

THEOREM 6.3. *Let F be a group law chunk of degree* 2 *in* $\mathfrak{K}$, $\mathfrak{H}$, *or* $\mathfrak{E}$ *given by its coefficients* $\Phi(n; \lambda_1, \lambda_2; i_1, i_2)$ *with* $i_1, i_2, n \in \{1, \dots, m\}$ *and* $\lambda_1, \lambda_2 \in \{1, 2\}$, *and let us denote* $\Phi(n; 1, 2; i_1, i_2) - \Phi(n; 2, 1; i_1, i_2)$ *by* $\Psi(n; i_1, i_2)$, *where* $\Phi(n; 2, 1; i_1, i_2) = \Phi(n; 1, 2; i_2, i_1)$ *for* $\mathfrak{A} = \mathfrak{H}$.

(i) *For* $\mathfrak{A} = \mathfrak{E}$, *the extension of F to a group law* 3-*chunk is possible if and only if* (*for all* n, ι_1, ι_2, ι_3),

$$\sum_{l=1}^{m} \Psi(n; l, \iota_3) \Psi(l; \iota_1, \iota_2) = \sum_{l=1}^{m} \Psi(n; \iota_1, l) \Psi(l; \iota_2, \iota_3).$$

In the rational case the condition means that the elements $\Psi(n; \iota_1, \iota_2)$ *form structure constants of an algebra* (*without unit*) *with associative multiplication.*

(ii) *For* $\mathfrak{A} = \mathfrak{K}$, *the extension of* F *to a group law* 3-*chunk is possible if and only if* (*for all* $n, \iota_1, \iota_2, \iota_3$) *both sides of the equation above equal zero. In the rational case the condition means that the elements* $\Psi(n; \iota_1, \iota_2)$ *form structure constants of an algebra* (*without unit*) *with cube-zero multiplication, i.e.*, $(ab)c = 0 = a(bc)$ *for all* a, b, c.

(iii) *For* $\mathfrak{A} = \mathfrak{H}$, $\Psi(n; \iota_2, \iota_1) = -\Psi(n; i_1, i_2)$ *and the condition for* F *to be extendable to a group law* 3-*chunk is given by*

$$0 = \sum_{l=1}^{m} \Psi(n; \iota_1, l)\Psi(l; \iota_2, \iota_3) \qquad (\iota_2 < \iota_3).$$

In the rational case the condition means that the elements $\Psi(n; \iota_1, \iota_2)$ *form structure constants of a* 3-*nilpotent Lie algebra* (*cube-zero bracket*).

Proof. We use the criterion (5.3) for $\Gamma_3 F$ to be a coboundary and collect summands associated to $(1,2,3), (3,1,2), (2,3,1)$ on the lefthand side and to $(1,3,2), (2,1,3), (3,2,1)$ on the righthand side (to omit negative signs). For $\mathfrak{K}$ we get

$$\begin{aligned} &-\sum_{l=1}^{m} \Phi(n; 1,2; \iota_1, l)\Phi(l; 1,2; \iota_2, \iota_3) \\ &\qquad + \sum_{l=1}^{m} \Phi(n; 2,1; \iota_1, l)\Phi(l; 1,2; \iota_2, \iota_3) + 0 \\ &\qquad = -\sum_{l=1}^{m} \Phi(n; 1,2; \iota_1, l)\Phi(l; 2,1; \iota_2, \iota_3) + 0 \\ &\qquad\qquad + \sum_{l=1}^{m} \Phi(n; 2,1; \iota_1, l)\Phi(l; 2,1; \iota_2, \iota_3) \end{aligned}$$

and

$$\begin{aligned} &\sum_{l=1}^{m} \Phi(n; 1,2; l, \iota_3)\Phi(l; 1,2; \iota_1, \iota_2) + 0 \\ &\qquad - \sum_{l=1}^{m} \Phi(n; 2,1; l, \iota_3)\Phi(l; 1,2; \iota_1, \iota_2) \\ &\qquad = 0 + \sum_{l=1}^{m} \Phi(n; 1,2; l, \iota_3)\Phi(l; 2,1; \iota_1, \iota_2) \\ &\qquad\qquad - \sum_{l=1}^{m} \Phi(n; 2,1; l; \iota_3)\Phi(l; 2,1; \iota_1, \iota_2). \end{aligned}$$

For $\mathfrak{E}$ (adding the equations) we get a similar system, or equivalently the system given in (i). For $\mathfrak{H}$ we get

$$-\sum_{l=1}^{m} \Phi(n;1,2;\iota_1,l)\big(\Phi(l;1,2;\iota_2,\iota_3) - \Phi(l;1,2;\iota_3,\iota_2)\big)$$
$$+\sum_{l=1}^{m} \Phi(n;1,2;l,\iota_1)\big(\Phi(l;1,2;\iota_2,\iota_3) - \Phi(l;1,2;\iota_3,\iota_2)\big) = 0.$$

∎

Remark 6.4. Over a field K of characteristic 0, Theorem 6.3 is a special example of the Koszul duality for quadratic operads given in [GK]. In fact, if we assume that the coefficients of the given 2-chunk can be divided by 2, then we can restrict to 2-chunks fulfilling the (canonical, cf. [La1]) property $\Phi(n;2,1;i_1,i_2) = -\Phi(n;1,2;i_1,i_2)$ by applying a strict isomorphism. Now we can replace the $\Psi(n;i_1,i_2)$ by $2\Phi(n;1,2;i_1,i_2)$ or just by $\Phi(n;1,2;i_1,i_2)$ in the equations of (6.3). The classification of formal group laws in ${}^m\hat{\mathfrak{E}}$ over a field K of characteristic 0, given in [GK, Sect. 2] follows from the torsion theorem (used both for existence and uniqueness of the extensions). A classification of formal group laws in ${}^m\hat{\mathfrak{H}}$ and ${}^m\hat{\mathfrak{K}}$ in the rational case is obtained similarly.

EXAMPLE 6.5. For ${}^1\mathfrak{K}$ there are no group law chunks which are not abelian (F is called abelian if $\eta_{e^{(2)},e^{(1)}}F = F$).

Consider ${}^2\mathfrak{K}$ over $K = \mathbb{Q}$, and let $\Phi(2;1,2;1,1) = -\Phi(2;2,1;1,1) = \frac{1}{2}$, all other coefficients 0. It is easy to show that a solution of the system of equations (ii) is given. Thus the corresponding 2-chunk $(F_1, F_2) = (x_1^{(1)} + x_1^{(2)}, x_2^{(1)} + x_2^{(2)} + \frac{1}{2}[x_1^{(1)}, x_1^{(2)}])$ is extendable to a 3-chunk and consequently to a group law. It corresponds to the 2-dimensional algebra $A = xK[x]/(x^3K[x])$ with basis $\bar{x}, \bar{x}^2$.

COROLLARY 6.6. *Let K be a field of characteristic 0 and V an m-dimensional vector space. Assume that elements $\Phi(n;\lambda_1,\lambda_2;i_1,i_2)$ of K for $i_1,i_2,n \in \{1,\dots,m\}$ and $\lambda_1 \neq \lambda_2 \in \{1,2\}$ are given such that*

(i) *$\Phi(n;1,2;i_1,i_2) - \Phi(n;2,1;i_1,i_2)$ are structure constants of a non-trivial associative algebra structure defined on V and*

(ii) *$\Phi(n;1,2;i_1,i_2) + \Phi(n;2,1;i_2,i_1) - \Phi(n;1,2;i_1,i_2) - \Phi(n;2,1;i_2,i_1)$ are structure constants of a nonabelian Lie algebra structure defined on V.*

Then there exist formal group laws of type ${}^m\mathfrak{E}$, ${}^m\mathfrak{C}$, and $\otimes$-${}^m\mathfrak{E}$, with 2-chunk defined by the coefficients $\Phi(n;\lambda_1,\lambda_2;i_1,i_2)$ (for ${}^m\mathfrak{E}$) or $\Phi(n;\lambda_1,\lambda_2;i_1,i_2) + \Phi(n;\lambda_2,\lambda_1;i_2,i_1)$ (for ${}^m\mathfrak{C}$ and $\otimes$-${}^m\mathfrak{E}$). All are non-trivial (i.e., not isomorphic to the trivial group law).

Proof. Theorem 6.3 shows that there exists the asserted formal group law in ${}^m\mathfrak{E}$. It is given by an m-tuple F with components from $K\langle\langle X^{(2)}\rangle\rangle$ and $0 = \Gamma F := \eta^{\mathfrak{A},2,3}_{F^{<}, e_{3,3}}(F) - \eta^{\mathfrak{A},2,3}_{e_{3,1}, F^{>}}(F)$ holds. Viewing the same m-tuple as elements of ${}^m\mathfrak{C}$ and $\otimes\text{-}{}^m\mathfrak{E}$, i.e., applying the module homomorphisms of (3.8), we get group laws, because the condition $0 = \Gamma F$ still holds. The new coefficients are sums $\Phi(n; i_1, i_2) := \Phi(n; \lambda_1, \lambda_2; i_1, i_2) + \Phi(n; \lambda_2, \lambda_1; i_2, i_1)$ of the old coefficients. We have to verify the assertion on non-triviality. Since the Lie algebra associated to the group law F in ${}^m\mathfrak{C}$ is nonabelian by assumption (ii), F itself is not isomorphic to the trivial group law; cf. (6.1). Now if the group law in $\otimes\text{-}{}^m\mathfrak{E}$ were trivial, then the group law in ${}^m\mathfrak{C}$ would be trivial, too. Thus all are non-trivial. ∎

EXAMPLE 6.7. The example corresponding to the completed general linear quantum group (4.3) is of this form.

The matrices $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ together with [,] form a 2-dimensional Lie algebra and it is closed under multiplication (of matrices). The theorem applies, and we get a non-abelian formal group law in $\otimes\text{-}{}^2\mathfrak{E}$ with 2-chunk defined by the $\Phi(1; 1, 2) = 1$, $\Phi(n; h, i) = 0$ otherwise.

Remark 6.8. It is easy to generalize (6.6) for other group laws of mixed type. It is not clear if the group laws constructed that way are the only examples. For R-chunks with $R < \infty$, there are much more.

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