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On Existence of Compound Perfect Squared Squares of Small Order*

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A compound perfect squared square must contain at least 22 subsquares. The proof utilizes elementary combinatoric and graph theoretic arguments and an extensive computer search.

1. INTRODUCTION

A *squared* rectangle is a rectangle subdivided into a finite number of squares. A squaring is *perfect* if no two component squares are congruent. A *simple* squared rectangle is one that properly contains no squared rectangle consisting of more than one square. A squaring that is not simple is *compound*. The *order* of a squared rectangle is the number of its component squares.

Duijvestijn [3] showed there are no perfect simple squared squares of order less than 20. We complement Duijvestijn's results (in Theorem 5) by proving the nonexistence of a compound perfect squared square of order less than 22. The perfect squared square of least order known has order 24 and is compound. It was found by Willcocks [11, 12] in 1948. A compound perfect squared square was first discovered by Sprague [6]. Tutte [7, 9], Willcocks [12], and Federico [4, 5] found many interesting compound squared squares and rectangles, Federico by computer. Federico used the method of substituting an unknown resistance in one or more wires in the electrical network corresponding to a net (see Section 2) and solving that network by Kirchhoff's Laws, subject to the constraint that the resultant squared rectangle be a square. We use a graph theory approach

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to search for compound squared squares. Both methods ultimately depend upon a knowledge of simple squared rectangles.

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2. Graphs Corresponding to Perfect Squared Rectangles

Throughout this paper we deal only with finite graphs having no loops, and we consider them as point sets. Two graphs are considered as identical if their vertex adjacency matrices are the same or are similar via a permutation matrix. The *order* of a graph is the number of edges it contains. A *net* is a connected planar graph with positive order. If two vertices on the same face of a net are designated as poles, the net is a *polar* net. If A is a polar net, $A \subset B$, and A meets the closure of $B - A$ only at the poles of A, then A is a *polar subnet* of B. If S is a polar net with poles v and w, the completion S^+ of S is given by $S^+ = S \cup \{\text{edge } vw\}$.

Let S be a net. If there exists a vertex v of S such that $S - v$ is not connected, then S is *1-connected.* If S is not 1-connected, H and K are subsets of S each containing at least two edges, and v and w are vertices of S such that $S = H \cup K$ and $H \cap K = v \cup w$, then S is *2-connected.* If S is neither 1-connected nor 2-connected, then S is *3-connected.* As usual, $\overline{M}$ denotes the closure of a point set M.

We begin our proof of nonexistence of a compound perfect squared square of order less than 22 by stating a theorem for rectangles that is analogous to a theorem of Tutte's [8, Section 2.2] for triangles:

Theorem 1. *For any squaring of a rectangle R, with component squares S_j, there exist closed line segments p_i^σ ($\sigma = h, v$; $i = 1, 2,..., m_\sigma$) where m_h and m_v are positive integers, such that*:

(a) *The union of the p_i^σ is the union of the sides of the S_j, each side of each S_j being contained in some p_i^σ.*

(b) *p_i^σ is horizontal or vertical as $\sigma = h$ or $\sigma = v$.*

(c) *Two distinct segments have at most one point in common.*

(d) *If w is a vertex of some S_j and not a vertex of R, then w is an interior point of just one of the segments p_i^σ. If such a vertex w is common to four of the squares $\{S_j\}$, then w is an interior point of some p_i^v.*

A polar net $P = P(R)$ corresponding to R may be defined as in [1, p. 314] or [10] with vertices and edges corresponding to the p_i^h and S_j, respectively. (If the squaring has no four squares meeting at a point, P is identical to the *normal* polar net of [1, p. 314].) By part (d) of Theorem 1, $P(R)$ is well-defined for a given orientation of R. If we consider P as an electrical network with unit resistance in each edge, a voltage across the poles induces currents in the edges which are proportional to the sides of the squares they represent. The currents are determined by Kirchhoff's Laws.

We define a class of polar nets which includes all nets associated with perfect squared rectangles according to the above correspondence:

DEFINITION 1. For $n \geqslant 5$, let $\mathscr{L}_n$ denote the set of all planar graphs S such that:

(a) S is a polar net.

(b) S has n edges.

(c) S^+ is 2-connected or 3-connected.

(d) No two edges of S have the same pair of end-points.

(e) Each vertex of S that is not a pole is an end-point of at least three edges.

The set $\mathscr{L}_n$ is nonempty for each $n \geqslant 5$:

LEMMA 1. *If A is a polar net having $n \geqslant 5$ edges and A^+ is 3-connected, then A belongs to $\mathscr{L}_n$.*

This easy lemma ensures that $\mathscr{L}_n$ is nonempty for $n = 5$ and all $n \geqslant 7$ since there exist 3-connected nets of all orders 8 and above and of order 6. The complete graph on 4 vertices (with two vertices designated as poles) is a polar net in $\mathscr{L}_6$.

LEMMA 2. *Let R be a perfect squared rectangle of order n. Then $P = P(R) \in \mathscr{L}_n$.*

Proof. The net P is polar by construction, its poles corresponding to the p_i^h at the top and bottom of R. It has more than 5 edges because a perfect squared rectangle must contain at least nine component squares [1, p. 324]. To establish the third property it is sufficient to show that for any vertex v of P^+ there is a circuit containing v and the poles of P. Such a circuit may be found by tracing a path from v to each pole via the corresponding squares in R, and including the edge $P^+ - P$. Finally, if either of the last two properties of membership in $\mathscr{L}_n$ were violated by P, then two edges of P would carry the same (nonzero) current in the electrical model. This is not possible because R is perfect. ∎

We are now able to characterize elements of $\mathscr{L}_n$. By Lemma 2, $\bigcup_5^\infty \mathscr{L}_n$ contains those polar nets we must study to find perfect squared squares.

THEOREM 2. *Each element S of $\mathscr{L}_n$ satisfies exactly one of the following*:

(1) *S^+ is 3-connected.*

(2) *$S = X \cup x$, where $X \in \mathscr{L}_{n-1}$ and x is an edge added to a pole p of X in such a way that x connects p to one pole of S and $X \cap x = p$. The second pole of X is the second pole of S.*

(3) *$S = Y \cup y$, where $Y \in \mathscr{L}_{n-1}$, the poles of Y are the poles of S, and y is an edge joining the poles of Y.*

(4) *There exist integers m and k with $m, k \geqslant 5$ and polar nets A in $\mathscr{L}_m$ and B in $\mathscr{L}_k$ such that A^+ is 3-connected, and S^+ is formed by joining A and B at their poles.*

Remark. One can always choose A in Conclusion 4 so that $A \subset S$, but other choices for A in Conclusion 4 with $A \subset S^+$ but $A \not\subset S$ may exist.

We use three lemmas in proving Theorem 2:

LEMMA 3. *Let T be a 2-connected or 3-connected net, and let U be a polar subnet of T. Then U^+ is 2-connected or 3-connected.*

The proofs of this and the following lemma are straightforward and are omitted.

LEMMA 4. *Let C be a polar net with more than one edge. If no two edges of C have the same end-points, and if each vertex of C that is not a pole is the end-point of at least three edges of C, then C has at least five edges.*

LEMMA 5. *If $S \in \mathscr{L}_n$, then there exists a polar subnet A of S such that A^+ is 3-connected and A has at least five edges.*

Proof. Given $S \in \mathscr{L}_n$, let $\mathscr{D}$ be the set of all polar subnets of S having more than one edge. The set $\mathscr{D}$ is nonempty because $S \in \mathscr{D}$. Let A be an element of $\mathscr{D}$ with minimum order. By Lemma 3, A^+ is either 2-connected or 3-connected since S^+ is. If A^+ is 2-connected, then there exist vertices v and w separating A^+ into two polar subnets U and W with common poles v and w, and each of U and W contains more than one edge. We may assume $U \subset A$. Then U is *à fortiori* a polar subnet of S. But W contains at least two edges, one of which belongs to A (since $A^+ - A$ is just one edge). Thus U is an element of $\mathscr{D}$ having fewer edges than A, contradicting the choice of A.

Since A^+ is neither 1-connected nor 2-connected, it is 3-connected. As a member of $\mathscr{D}$, A has more than one edge, so that, by Lemma 4, A has at least five edges. ∎

Proof of Theorem 2. We first prove that Conclusions (1)–(4) of the theorem are mutually exclusive. Suppose S satisfies Conclusion (2). Then one pole of S is the end-point of exactly one edge of S, so that S cannot also satisfy Conclusion (3). Let $t = \overline{S^+ - S}$, and let $H = t \cup x$. Then $H \cap X$ contains only the poles of X, each of H and X contains at least two edges, and $H \cup X = S^+$. Therefore S^+ is 2-connected so that Conclusions (2) and (1) are mutually exclusive.

If S satisfies Conclusions (2) and (4), the net B in Conclusion (4) contains both x and $t = \overline{S^+ - S}$, because A^+ is 3-connected. But then the vertex $t \cap x$ meets only t and x, so that B does not belong to $\mathscr{L}_k$. Thus Conclusions (2) and (4) are mutually exclusive. Arguments similar to these show that Conclusion (3) cannot hold simultaneously with either of Conclusions (1) and (4). Finally, Conclusions (1) and (4) are disjoint possibilities because S^+ cannot be both 3-connected and 2-connected.

For the remainder of the proof we assume S satisfies none of Conclusions (1), (2), and (3) of the theorem. We must show S satisfies Conclusion (4). By Lemma 5, there exists a polar subnet A of S such that A^+ is 3-connected and A has at least five edges. (We only use the relation $A \subset S^+$ in the remaining argument.) Let m be the order of A. By Lemma 1, $A \in \mathscr{L}_m$. Let $B = \overline{S^+ - A}$, and let the poles v and w of A be the poles of B. We show that B is a polar net, S^+ is formed by joining A and B at their poles, and $B \in \mathscr{L}_k$ for some $k \geqslant 5$.

The set $B \cap A$ consists only of v and w. If B is not connected, then $B = H \cup K$, where $H \cap \overline{K}$ and $\overline{H} \cap K$ are both empty while neither of H and K is. The vertex v is not an isolated point of B, and we may assume $v \in H$. Let $H' = H - v$, and let $K' = (K \cup A) - v$. Then both H' and K' are nonempty, each of $\overline{H'} \cap K'$ and $H' \cap \overline{K'}$ is empty, and $H' \cup K' = S^+ - v$. Thus S^+ is 1-connected, which contradicts the hypothesis that $S \in \mathscr{L}_n$. Therefore B is connected. Since A is also connected, v and w cannot be separated by a circuit (simple closed curve) in B. Consequently B is a polar net, and $S^+ = A \cup B$, with B and A joined together only at their poles.

It remains to prove that $B \in \mathscr{L}_k$ for some $k \geqslant 5$. Let k be the order of B. Suppose two edges of B have the same end-points. If one such edge is $\overline{S^+ - S}$, the other also has the poles of S as its end-points, which contradicts the assumption that S does not satisfy Conclusion (3). If both these edges are in S, the hypothesis that $S \in \mathscr{L}_n$ is contradicted. Thus no two edges of B have the same end-points. Further, if a vertex r of B is a pole of S but not a pole of B, and r is the end-point of exactly one edge of S, then S satisfies Conclusion (2) of the theorem, contrary to our assumption. Since $S \in \mathscr{L}_n$, any other vertex of B is an end-point of at least three edges of S, hence of B. Lemma 3 and the conclusions of this paragraph imply $B \in \mathscr{L}_k$ if we can prove $k \geqslant 5$. We proceed to do so.

Since B is connected, $k \geqslant 1$. If $k = 1$, then v and w are connected by exactly one edge in B, so that $S^+ = A^+$ and S^+ is 3-connected. This violates our assumption that S does not satisfy Conclusion (1) of the theorem. Therefore $k > 1$. We may now apply Lemma 4 to conclude that B has at least five edges. Therefore $B \in \mathscr{L}_k$ for some $k \geqslant 5$, and S does satisfy Conclusion (4) of the theorem. ∎

It is convenient to refer to the nets in $\mathscr{L}_n$ by the types which Theorem 2 assigns to them. We therefore make the following definition:

DEFINITION 2. A net S in $\mathscr{L}_n$ is a T_i *net* ($i = 1, 2, 3, 4$) if S satisfies the i-th conclusion of Theorem 2.

Although we are searching for a compound perfect square of order 21 or less, we need not examine all elements of $\mathscr{L}_n$ for each n ($5 \leqslant n \leqslant 21$). For example, most T_1 nets do not correspond to compound perfect squarings of rectangles. But even those which do can be avoided:

THEOREM 3. *Let R be a compound perfect squared rectangle, and let $P = P(R)$. Then P^+ is 2-connected.*

Proof. By Lemma 2, $P \in \mathscr{L}_n$ for some n. Therefore P^+ is 2-connected or 3-connected. Since R is compound, it properly contains a perfect squared subrectangle R_1. Let $P_1 = P(R_1)$. By Theorem 1(d), a vertex of P_1 that is not a pole of P_1 is incident only with edges corresponding to squares of R_1. Thus P_1 is a polar subnet of P, and P^+ is 2-connected. ∎

3. Nets Corresponding to Compound Perfect Squared Squares

In this section we show that the search for nets derived from compound perfect squared squares may be restricted to T_4 nets.

LEMMA 6. *If Q is a nontrivial (compound or simple) perfect squared square, and $P = P(Q)$, then any two meshes of P^+ have at most one edge in common.*

Proof. Let $t = P^+ - P$. Suppose there exist two meshes of P^+ with more than one edge in common. If t is one such edge, then all current flowing between the poles of P in the electrical analog of P must pass through another such edge r. Then r corresponds to a square whose width equals that of Q, which is impossible since Q is nontrivial.

On the other hand, if r and s are edges common to these meshes with $r \neq t$ and $s \neq t$, then the current in r equals the current in s in the electrical analog of P because each current is the algebraic sum of the mesh currents in the meshes common to r and s; see Figure 1. Therefore Q cannot be perfect, unless r and s carry zero current. But r and s correspond to squares in Q and hence have nonzero current. ∎

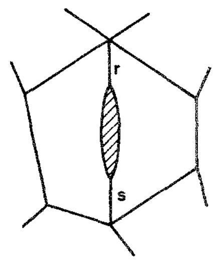

FIG. 1. Meshes with common edges r and s.

THEOREM 4. *If Q is a compound perfect squared square of order n, then $P = P(Q)$ is a T_4 net of order n.*

Proof. By Lemma 2 and Theorem 2, P is a T_i net for some i, $1 \leqslant i \leqslant 4$. Theorem 3 and Lemma 6 imply i is neither 1 nor 2. Suppose P is a T_3 net. Then the voltage across the poles of the electrical analog of P is the voltage across a single edge y joining the poles, which forces one square in Q to have a height equal to the height of Q.

We conclude P is a T_4 net. ∎

4. GNOMONS

We have narrowed our search for compound perfect squared squares to the collection of squarings of rectangles derived from T_4 nets. When we dissect a T_4 net using Kirchhoff's Laws, we first complete it and put a battery in the new edge.

DEFINITION 3. The completion of a T_4 net of $n - 1$ edges is a *gnomon* of order n.

Thus to organize an exhaustive search for compound perfect squared squares it is sufficient to create a hierarchal list of gnomons. Theorem 2 provides the means for doing this. Conclusion (4) of Theorem 2 describes the compound structure of gnomons, and all the conclusions describe the basic parts of gnomons. In this section we present an algorithm for creating a complete, hierarchal list of gnomons. We also state a number of lemmas which help to eliminate certain portions of this list from consideration. After applying these lemmas wherever feasible, we generated the remainder of the nets in the list having 22 or fewer edges by electronic computer and dissected them one at a time, also by computer. The computer program we used to perform the dissections is a modification of Duijvestijn's program [3] that was written by James Reeds III. We emphasize that we dissect a gnomon of order n in all possible ways; that is, we solve the electrical networks determined by placing a battery in turn in each edge of the gnomon and unit resistances in each of the other $n - 1$ edges. The resultant n squared rectangles may be all different or there may be several alike. Each may be perfect or imperfect.

We describe one method of constructing gnomons. Consider a T_4 net $S = S_0$ in $\mathscr{L}_n$, and let A, B, m, and k be as in Conclusion (4) of Theorem 2. Since B belongs to $\mathscr{L}_k$, it is a T_i net for some i. If B is a T_2 or a T_3 net, remove the edge that corresponds to the edge x or y of the theorem to obtain a net B_1 in $\mathscr{L}_{k-1}$. Repeat the procedure, if possible, to obtain $B_0 = B \in \mathscr{L}_k$, $B_1 \in \mathscr{L}_{k-1}, \ldots, B_j \in \mathscr{L}_{k-j}, \ldots$. The procedure terminates at some $S_1 = B_l$, where S_1 is a T_1 or a T_4 net. Then the gnomon S^+ can be realized as the union of S_1, a T_1 net $A_0 = A$, and $i_1 = l$ edges of the types of x and y in Theorem 2. If S_1 is a T_1 net, no further decompositions are needed; otherwise, S_1 is a T_4 net, and S_1 is decomposed in the way S was decomposed. Repeat the procedure for each S_j $(j = 1, 2, \ldots)$ until a T_1 net, say S_r, is reached. At this final stage the original T_4 net S is described by:

(1) a sequence of T_1 nets $A_0, \ldots, A_{r-1}$, $A_r = S_r$,

(2) a sequence of T_4 nets $S_0, \ldots, S_{r-1}$, and

(3) $q = i_1 + \cdots + i_r$ edges of the types of x and y in Theorem 2,

such that, for each $j < r$, the gnomon S_j^+ is the union of A_j, S_{j+1} and i_{j+1} edges of the types of x and y; see Figure 2 for an example.

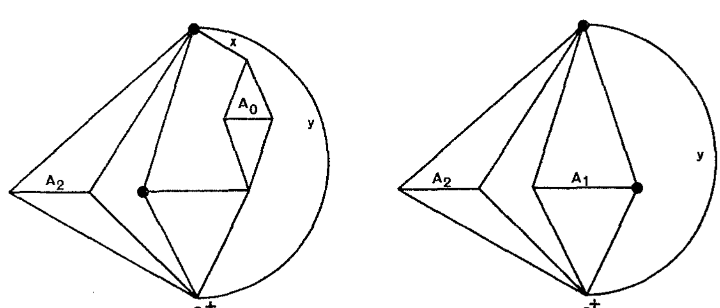

FIG. 2. $S^+ = A_0 \cup B_0 = A_0 \cup x \cup B_1$, where $i_1 = 1$; $S_1 = B_1$, and $S_1^+ = A_1 \cup B_0' = A_1 \cup y \cup B_1'$, where $B_1' = A_2$, and $i_2 = 1$.

By reversing the steps in the above procedure we reconstruct S^+ from the A_j's and the extra edges. Indeed, starting with arbitrary T_1 nets $\{A_j\}$ and as many extra edges as needed, we (theoretically) can construct all gnomons of a given order m. Then by solving the corresponding electrical networks we can determine all compound perfect squared squares of order $m - 1$.

With the aid of some simplifying observations we are able to list and analyze a set of gnomons sufficient for the proof of Theorem 5. We begin with three lemmas, whose proofs are easy and are omitted:

LEMMA 7. *If S^+ is decomposed according to the above scheme, at least two of the A_j's are subsets of S^+.*

LEMMA 8. *Let G be a gnomon of order n. Let C be a polar subnet of G. If the squared rectangle corresponding to C is imperfect, then so is any squared rectangle of order $n - 1$ derived by solving an electrical network obtained from G by placing a battery in some edge of $G - C$ and unit resistances in all other edges of G.*

COROLLARY. *Let G be a gnomon of order n. Let C_1 and C_2 be polar subnets of G which correspond to imperfect squared rectangles and which have no edges in common. Then no squared rectangle of order $n - 1$ derived from G is perfect.*

LEMMA 9. *If $n > 1$, a perfect squared rectangle with a width-to-length ratio of $1 : n$ is composed of at least $3n - 1$ squares.*

A polar net A may be inserted as a polar subnet of a polar net S in four ways. These come from rotating A so as to interchange the positions of its poles and from reflecting A in the line through its poles. Each of these orientations of A as a polar subnet of S corresponds to one of the four ways to orient the rectangle corresponding to A in the rectangle corresponding to S. We consider these orientations to be equivalent because the rectangles corresponding to S in each case are either all perfect or all imperfect, and each can be obtained from any other.

The 3-connected nets of six to ten edges are illustrated in Figure 3. They will be referred to frequently in the ensuing discussion. Only one (Fig. 3f) can be dissected to produce perfect squared rectangles. The two nets of nine edges (Fig. 3c and d) are *duals* of each other: there exists a one-to-one correspondence between the edges of one and the edges of the other such that vertices and meshes of one net are mapped, respectively, into meshes and vertices of the other. Each of the remaining nets pictured is *self-dual.* Dissections corresponding to dual nets yield the same squared

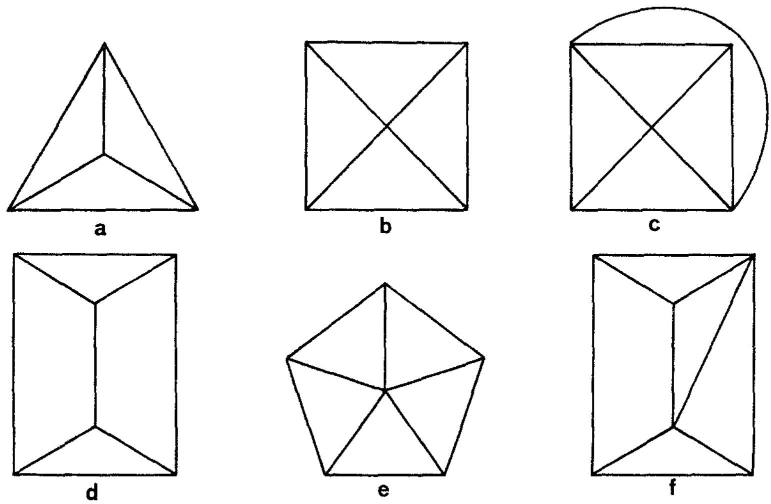

FIG. 3. The 3-connected nets of 6, 8, 9, and 10 edges.

rectangles [1, pp. 321–323], so that only one of a net and its dual need be dissected.

We next state two lemmas that give properties of the squared rectangles obtained by inserting a net corresponding to a perfect squared rectangle into one of the T_1 nets whose completions are pictured in Figure 3. These lemmas will be used in the proof of Theorem 1. They have the following hypothesis in common:

HYPOTHESIS H. *The graph A is a polar subnet of S^+, A is a T_1 net, and S is a T_4 net in $\mathscr{L}_n$. The graph B is $\overline{S^+ - A}$, and the rectangle corresponding to B has a width-to-length ratio of b.*

An example of such a net S is illustrated in Figure 4.

LEMMA 10. *Under Hypothesis* H, *let A^+ be the net shown in Figure* 3a. *Suppose the squared rectangle R corresponding to S has order n and a width-to-length ratio of c. If R is perfect, then c cannot equal* 1, 3/5, *or* 13/24. *If in addition n is* 29 *or less, then c cannot equal* 2/3, 5/8, *or* 8/13 *either.*

LEMMA 11. *Under Hypothesis* H, *let A^+ be the net shown in Figure* 3b (3c *or* 3d, 3e). *Then S cannot correspond to a perfect squared square of order n unless $b =$* 5/8 (8/13, 13/24).

Proof of Lemma 10. By Lemma 8, with A playing the role of C, the battery in the electrical analog of S^+ must be placed in an edge of A. To

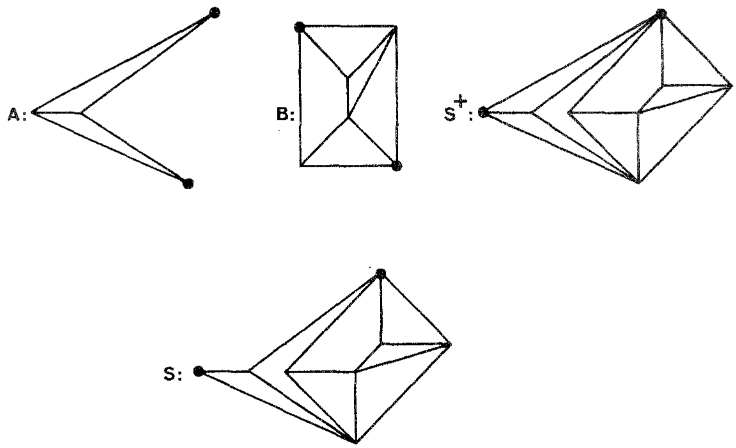

FIG. 4. Example of a net satisfying Hypothesis H.

solve the network corresponding to S, we can replace the subnetwork corresponding to B by a single wire of resistance b. The network may be solved for the currents, with c as a parameter. (See, for example, [10].) Solutions corresponding to $c = 1, 3/5$, and $13/24$ do not yield geometrically realizable perfect squared rectangles. Solutions with $c = 2/3, 5/8$, and $8/13$ yield solutions with $b = 9$, 25, and 41 (respectively), requiring R to have order 30 or more by Lemma 9. ∎

Lemma 11 is proved similarly. Symmetry arguments may be used to reduce the number of possible locations for b.

5. Nonexistence of a Compound Perfect Squared Square

THEOREM 5. *There exists no compound perfect squared square of order* 21 *or less.*

Remark. In dissecting a gnomon, one may ignore any dissections with zero currents because the corresponding squared rectangles generate nets of lower orders.

Proof of Theorem 5. By Theorem 4 and the above remark, it is sufficient to examine all T_4 nets with 21 or fewer edges to determine all compound perfect squared squares of corresponding order. Lemmas 7, 8, 10, and 11 remove some nets from consideration. The rest are examined with the aid of a computer.

Let S be a T_4 net of order 21 or less. We recall the decomposition of S induced by Theorem 2 and described early in Section 4. The number of

T_1 nets in this decomposition is $r + 1$, and the number of extra edges of types x and y is q. If each T_1 net A_j has m_j edges and $p = \sum_{j=0}^{r} m_j$, then S has $p + q - r$ edges; not $p + q$ edges, because each step in the decomposition procedure decomposes the *completion* of a T_4 net.

To prove Theorem 5 we must first find all T_4 nets S such that $p + q - r \leqslant 21$ or, equivalently, all gnomons of order 22 or less. Recall that $m_j \geqslant 5$ for each j. Since the squared rectangle corresponding to a T_4 net S is to be perfect and compound, it must contain at least one simple perfect subrectangle. This forces $m_k \geqslant 9$ for some k. Thus

$$\begin{aligned} 21 &\geqslant p + q - r \\ &\geqslant (9 + 5r) + 0 - r, \end{aligned}$$

so that $r \leqslant 3$. Consequently there are three main cases to consider—$r = 1$, $r = 2$, and $r = 3$:

(I) $r = 3$. In this case $9 + 4r = 21$. Therefore the only way that a T_4 net of 21 or fewer edges can be realized with $r = 3$ is if it has order 21, A_k is a net of nine edges for some k, and each A_j is the unique net, call it W, in $\mathscr{L}_5$ for $j \neq k$. By Lemma 7, at least two of the T_1 nets A_j ($j = 0,..., 3$) are subsets of S^+. By Lemma 8, exactly two of them are subsets of S^+ since no perfect squared rectangle of order 5 exists. Lemma 8 also implies that the battery must be placed in the 5-edge T_1 net W that is a subset of S^+, and Lemma 10 with $A = W$ implies that the dissection of S does not yield a perfect squared square of order 21.

(II) $r = 2$. In this instance S determines T_1 nets A_0, A_1, and $A_2 = S_2$, T_4 nets S_0 and S_1, and $q = i_1 + i_2$ extra edges.

Case 1. Each of A_0, A_1, A_2 is a subset of S^+. In this case, the corollary to Lemma 8 implies that at least two of them correspond to perfect squared rectangles. Thus their orders are at least 9, $p \geqslant 2 \cdot 9 + 5$, and

$$p + q - r \geqslant 23 + 0 - 2 = 21.$$

But we assume that the order of S is 21 or less. Thus S has order 21, $p = 23$, and one of the A_j's, say A_0, is the net W in $\mathscr{L}_5$. It follows that $A_i \subset S$ ($i = 1, 2$), $A_0 \subset S^+$, and the battery is in an edge of A_0. We conclude, using Lemma 10 with $A = A_0$, that S does not correspond to a perfect squared square of order 21.

Case 2. Exactly one of A_0, A_1, and A_2, say A_2, is not a polar subnet of S^+. Recall that at least two A_j's are subsets of S^+ by Lemma 7. Thus Cases 1 and 2 are all-inclusive. If the battery is placed in an edge of $\overline{(S^+ - A_0) - A_1}$, then for S to yield a perfect dissection the inequalities $m_0 \geqslant 9$, $m_1 \geqslant 9$, and (of course) $m_2 \geqslant 5$ must hold. Thus $p \geqslant 23$. Since r is 2 and the order $p + q - r$ of S is 21 or less, this forces $q = 0$, $m_0 = 9$, $m_1 = 9$, and $m_2 = 5$. For these values of the m_j and q there are twelve essentially different gnomons to consider. We drew these, coded them according to Duijvestijn's scheme, and directed the computer, an IBM 360/67, to dissect each gnomon in each possible way. No perfect squared squares resulted.

Otherwise the battery is placed in an edge of A_0 or A_1. At least one of A_0 and A_1 (say A_1) does not contain the edge $\overline{S^+ - S}$, so that A_1 is contained in S. (See the remark following Theorem 2.) If the battery is placed in an edge of A_0 (see Fig. 5 for an example), $m_1 \geqslant 9$, $m_2 \geqslant 5$, and $p + q - r \leqslant 21$, so that

$$m_0 + 9 + 5 - 2 \leqslant p + q - r \leqslant 21,$$

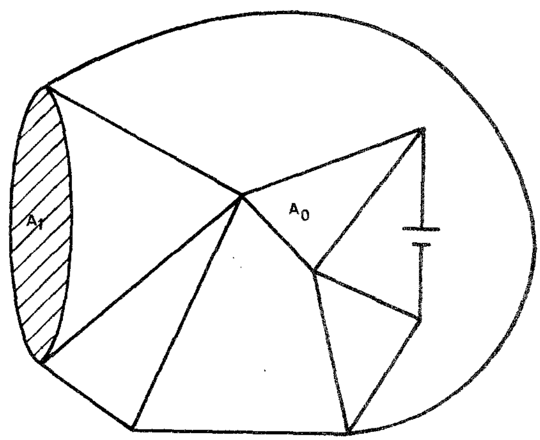

FIG. 5. Example for Case 2.

or $m_0 \leqslant 9$. If A_0^+ is the net of Figure 3a, Lemma 10 with $A = A_0$ implies that S does not correspond to a perfect squared square. If A_0^+ is the net of Figure 3f, then there are four gnomons to consider in addition to some of the twelve mentioned earlier in Case 2. Again we drew these nets and used the computer to show that their dissections yield no perfect squared squares.

The remaining possibilities are that A_0^+ is one of the three nets mentioned in Lemma 11.

Case 2a. The net A_0^+ is the wheel of ten edges (Fig. 3e). Here $m_0 = 9$, $m_1 \geqslant 9$, and $m_2 \geqslant 5$ so that

$$21 \geqslant p + q - r \geqslant 23 + q - 2 = 21 + q.$$

Thus $q = 0, S^+ = A_0 \cup S_1$, and $S_1^+ = A_1 \cup A_2$. Moreover, $m_2 = 5$ since $m_0 + m_1 + m_2 - 2 \leqslant 21$ and $m_0 + m_1 \geqslant 18$. By Lemma 10 with $A = A_2$, no dissection of S_1^+ can yield a 13 by 24 perfect rectangle of order 13. Then Lemma 11 with $A = A_0$ and $S^+ = A \cup S_1$ implies that S does not correspond to a perfect squared square of order 21.

Case 2b. The net A_0^+ is a net of nine edges. Then $m_2 = 5$ (there exists no 3-connected net of order 7), $9 \leqslant m_1 \leqslant 10$, and $q = 0$ or 1. If $m_1 = 10$, or if $m_1 = 9$ and $i_1 = 0$, we argue as in the preceding paragraph. Otherwise $i_1 = 1$ and $S_1^+ = A_1 \cup A_2$. By Lemma 10 with $A = A_2$, S_1 does not correspond to a 5 by 8 rectangle of order 13. But $S^+ = A_0 \cup t \cup S_1$, where t is an edge of type x or y in Theorem 2. Thus, since S_1 does not correspond to a 5 by 8 rectangle, neither $S_1 \cup x$ nor $S_1 \cup y$ corresponds to an 8 by 13 rectangle. Finally, by Lemma 11 with $A = A_0$, we conclude that S does not correspond to a perfect squared square of order 21.

Case 2c. The net A_0^+ is the wheel of eight edges (Fig. 3b). There are two subcases:

(1) The net A_2 has five edges ($m_2 = 5$) and $i_1 = 0$, 1, or 2. By applying Lemmas 10 and 11 and using techniques previously introduced, one shows that this case yields no perfect squared square of order $p + q - r$.

(2) The net A_2 has seven edges ($m_2 = 7$). Here, for S to have order 21 or less, m_1 must be 9. We are thus led to consider four more gnomons $S_1^+ = A_1 \cup A_2$. Using the computer, we showed that none of these yielded a 5 by 8 rectangle (corresponding to an S_1). We now use Lemma 11 with $A = A_0$, and we conclude that S does not correspond to a perfect squared square.

Case 2 is completed.

(III) $r = 1$. There are numerous gnomons falling in this case, about 17000 of which we generated and dissected by computer. We proceeded as follows. In this case, by Lemma 7, both A_0 and A_1 are polar subnets of S^+, so that S^+ is the union of T_1 nets A_0 and A_1 and $q = i_1$ edges of the types of x and y in Theorem 2. Using a program based upon Duijvestijn's [3], we generated and stored on the computer all simple perfect squarings of rectangles of orders up to and including 16. (All simple perfect squarings of rectangles up to and including order 18 were implicitly generated by Bouwkamp, Duijvestijn and Medema [2] and are available in implicit form.) None had width-to-length ratios of 1/2, 2/3, 3/5, 5/8, 8/13, 2/11, 11/13, or 13/24, so that there exist no perfect squared rectangles (compound or simple) with width-to-length ratios of 5/8, 8/13, or 13/24 that are derived from T_1, T_2, or T_3 nets of order 16 or less. As usual, arguments using Lemmas 8, 10, and 11 show that neither A_0^+ nor A_1^+ can be a net pictured in Figure 3a–e.

In the case considered the conditions

$$p + q \leqslant 22, \qquad m_0 \geqslant 9, \qquad \text{and} \qquad m_1 \geqslant 9$$

hold. At least one of each set of *equivalent* gnomons that satisfy these restrictions was constructed (except for those involving the 10-wheel Fig. 3e) and dissected on the computer, unless the corollary to Lemma 8 applied. Two gnomons are *equivalent* if one is the dual of the other, or if the dissections of one correspond to those of the other except for the orientation of the subrectangles, or if each is equivalent to a third in either of the above ways. A brief description of the method used for constructing all these gnomons follows.

We began with pairs of 3-connected nets. An edge was deleted from each member of a pair, forming two T_1 nets A_0 and A_1. These nets were joined at their poles (the end-points of the deleted edges) to produce a gnomon. If the orders of A_0^+ and A_1^+ are m and n, then all such constructions using these two nets will yield mn gnomons of order $m + n - 2$. This collection was sorted to eliminate most of the duplicates. (Occasionally the sorting program was unable to assign an identification number to a net. In these few instances, no sorting was done.) If neither of the original 3-connected nets was self-dual, we replaced one by its dual and repeated the process. We considered all admissible pairs of 3-connected nets. All resulting gnomons, after sorting within collections to eliminate duplicates, were dissected. None yielded a perfect squared square.

To add an edge of type y to a gnomon, we need only identify two non-adjacent vertices, each being a vertex of a common pair of meshes, and connect them with an edge. The edge may be drawn through either of the two meshes. The resulting gnomons are equivalent. An edge of type x in a gnomon corresponds to one of type y in the dual net, and conversely. These edges are thus also easy to add to a gnomon to get a gnomon. After each such edge is added the net is dissected. Lemma 6 of Section 3 implies that one may alternate adding edges of types x and y in the construction.

None of the gnomons formed in this way yielded a perfect squared square. Thus one of each set of equivalent gnomons of order 22 or less has been dissected or has otherwise been shown to produce no perfect squared square. Therefore, by Theorem 4, we finally conclude, having exhausted all possible cases, that no compound perfect squared squares of order 21 or less exist. ∎

6. Examples

By computer we found and printed all dissections of rectangles corresponding to 3-connected nets with 17 or fewer edges. We also generated all such nets of orders 18 and 19, dissected them, and printed perfect dissections yielding ratios p/q with $p + q < 300$. (Duijvestijn [3] counts eight 3-connected planar nets of 12 edges. We found nine. All other counts agree. [The referee has informed us that other versions of this thesis exist in which the correct Figure 9 appears.])

In Table I we give the Bouwkamp codes of *the* simple perfect squared rectangles of orders 16, 17, and 18 having sides with reduced ratios p/q such that $p + q < 30$. (The edge lengths of squares whose upper edges belong to the same p_i^h are grouped in parentheses; the groups are listed in order of decreasing levels of the p_i^h.)

TABLE I

Order	p/q	Bouwkamp code
16	14/15	(87, 95) (39, 48) (40, 55) (27, 12) (3, 60, 25) (15) (10, 45) (42) (35)
16	11/18	(70, 73) (67, 3) (76) (39, 9, 7, 12) (2, 5) (11) (8, 85) (19) (58)
17	13/14	(51, 30, 88) (13, 17) (8, 5) (1, 16) (6) (56, 3) (9) (25) (19, 94) (75)
17	5/7	(17, 19, 27, 21) (15, 2) (13, 8) (5, 16) (1, 4) (33, 3) (7) (28) (23)
17	3/5	(40, 29) (13, 16) (36, 4) (2, 8, 3) (6) (19) (14) (12, 21) (39, 9) (30)
17	14/15	(145, 93) (45, 48) (42, 3) (23, 28) (110, 35) (18, 5) (33) (75, 2) (20) (53)
17	11/15	(67, 52, 46)(6, 19, 21) (28, 30) (17, 2) (54, 13) (23) (41) (39, 8) (31)
18	9/10	(163, 98) (44, 54) (21, 23) (13, 41) (127, 57) (36) (8, 33) (19, 25) (70, 6) (64)
18	13/14	(123, 150) (91, 32) (6, 144) (23, 8, 1) (7) (15) (38) (80, 11) (9, 29) (20) (49)
18	14/15	(95, 61, 30, 54) (17, 13) (7, 6) (14, 3) (1, 5) (11) (59) (34, 52) (129) (111)
18	11/15	(76, 69, 119)(19, 50)(64, 12)(31)(21, 67, 112)(85)(39, 28)(11, 17)(135)(129)
18	11/15	(60, 29, 43) (15, 14) (1, 56) (16) (37, 39) (32, 5) (3, 11, 81) (8) (19) (51)
18	10/11	(129, 61, 70) (23, 29, 9) (20, 59) (17, 6) (11, 44) (28) (157) (5, 54) (49) (103)

References

1. R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares, *Duke Math. J.* 7 (1940), 312–340.
2. C. J. Bouwkamp, A. J. W. Duijvestijn, and P. Medema, Table of c-Nets of Orders 8 to 19, Inclusive, 1960 (available in UMT File of *Math. Comp. Abstract* in *Math. Comp.* **24** (1970), 995–997).
3. A. J. W. Duijvestijn, Electronic Computation of Squared Rectangles, Thesis, Technische Wetenschap aan de Tech. Hogeschool te Eindhoven, 1962.
4. P. J. Federico, Note on some low-order perfect squared squares, *Canad. J. Math.* **15** (1963), 350–362.

5. P. J. FEDERICO, Some simple perfect 2×1 rectangles, *J. Combinatorial Theory* **8** (1970), 244–246.
6. R. SPRAGUE, Beispiel einer Zerlegung des Quadrats in lauter verschiedene Quadrate, *Math. Z.* **45** (1939), 607–608.
7. W. T. TUTTE, Squaring the square, *Canad. J. Math.* **2** (1950), 197–209.
8. W. T. TUTTE, The dissection of equilateral triangles into equilateral triangles, *Proc. Cambridge Philos. Soc.* **44** (1948), 463–482.
9. W. T. TUTTE, The quest of the perfect square, *Amer. Math. Monthly* **72**, No. 2, Part II (1965), 29–35.
10. W. T. TUTTE, Squaring the square, "The Second Scientific American Book of Mathematical Puzzles and Diversions," by Martin Gardner, Simon & Schuster, New York, 1961, pp. 186–209 (reprinted with addendum and bibliography from *Scientific American*, November, 1958, pp. 136–142.)
11. T. H. WILLCOCKS, Problem 7795 and solution, *Fairy Chess Rev.* **7** (1948), 97, 106.
12. T. H. WILLCOCKS, A note on some perfect squared squares, *Canad. J. Math.* **3** (1951), 304–308.