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The Shape of Axisymmetric Rotating Fluid

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The equations for axisymmetric self-gravitating rotating fluid have been studied extensively since Poincaré. The model derives its primary interest from celestial mechanics, where it can be used to study the geometry of stars and planets. Existence of a solution for both the compressible and the incompressible cases is known. The smoothness of the boundary of the fluid is studied, and, in particular, it is proved that the rotating fluid has at most a finite number of rings.

INTRODUCTION

In this paper we consider the equilibrium figure of an axisymmetric self-gravitating fluid rotating about the z-axis. The fluid is either compressible or or incompressible; in both cases it is subject to either an angular velocity law or an angular momentum law. The existence of an equilibrium figure was established by Auchmuty and Beals [2] (for the compressible case) and by Auchmuty [1] (for the incompressible case). Here our purpose is to study the shape of the boundary of the fluid. This problem, of great interest in analyzing the structure of the stars and planets, has received considerable attention since the work of Poincaré [22, 23] (see [7, 16, 24]).

The main purpose of this paper is to prove that the fluid consists of at most a finite number of rings (i.e., torus-shaped regions) about the axis of rotation.

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Another problem of motion of incompressible fluid without gravity was studied by Fraenkel and Berger [10]. In this model the vorticity is prescribed; it is positive in the liquid, and it vanishes outside the liquid. Our methods apply also to this model and show that the number of vortex rings is finite.

1. The Model

We are interested in a steady fluid in R^3 rotating around the z-axis, which is axisymmetric with respect to the z-axis as well as with respect to the plane $z = 0$. Auchmuty and Beals [2] have considered the case when the fluid is compressible and self-gravitating with given total mass M and with given law of either

(a) angular velocity $\Omega(r)$, or

(b) angular momentum per unit mass, $j(m)$.

Here we use cylindrical coordinates (r, θ, z).

We introduce the functions

$$J(r) = \int_0^r s\Omega^2(s)\, ds, \tag{1.1}$$

$$L(m) = j^2(m), \qquad m_\rho(r) = \int_{r(x)<r} \rho(x)\, dx, \tag{1.2}$$

where x will henceforth denote a point in R^3 and $r(x)$ is the r-coordinate of x.

In case (a), the total energy of the fluid is

$$E_1(\rho) = \int_{\mathbb{R}^3} A(\rho(x))\, dx - \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\,\rho(y)}{|x-y|}\, dx\, dy - \int_{\mathbb{R}^3} J(r)\,\rho(x) \tag{1.3}$$

and in case (b) the total energy is

$$E_2(\rho) = \int_{\mathbb{R}^3} A(\rho(x))\, dx - \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\,\rho(y)}{|x-y|}\, dx\, dy + \frac{1}{2}\int_{\mathbb{R}^3} \frac{\rho(x)}{r^2(x)} L(m_\rho(x))\, dx, \tag{1.4}$$

where $A(r)$ is a given function. Recall that one is interested in a density $\rho(x)$ satisfying

$$\rho(x) = \rho(r, z), \qquad \rho(r, z) = \rho(r, -z), \qquad \rho \text{ measurable}, \qquad \rho \geqslant 0, \qquad \int_{\mathbb{R}^3} \rho\, dx = M. \tag{1.5}$$

The functions A, J, L are subject to the following conditions:

$$A(s) = s\int_0^s \frac{f(t)}{t^2}\,dt, \qquad \text{where } f \in C^1[0,\infty),\ f \geqslant 0,\ f' > 0,$$
$$\lim_{s\to 0}\frac{f(s)}{s^{4/3}} = 0, \qquad \lim_{s\to\infty}\frac{f(s)}{s^{4/3}} = \infty; \tag{1.6}$$

$$J(0) \geqslant 0, \qquad J(\infty) < \infty, \qquad J \in C^1[0,\infty), \qquad J'(r) \geqslant 0,$$
$$r(J(\infty) - J(r)) \to 0 \qquad \text{if} \quad r \to \infty; \tag{1.7}$$

$$L(0) = 0, \qquad L(r) \geqslant 0, \qquad L \in C^1[0,\infty). \tag{1.8}$$

By a general physical principle, the equilibrium figure of the fluid has density which minimizes the total energy. The following results are proved in [2].

THEOREM 1.1. *If* (1.6), (1.7) *hold then there exists a function* $\rho_1(x)$ *which minimizes* $E_1(\rho)$ *is the class of functions* $\rho(x)$ *given by* (1.5). *Further,* ρ_1 *is Hölder continuous and has compact support.*

THEOREM 1.2. *If* (1.6), (1.8) *hold then there exists a function* $\rho_2(x)$ *which minimizes* $E_2(\rho)$ *in the class of functions* $\rho(x)$ *given by* (1.5). *Further,* ρ_2 *is Hölder continuous and has compact support.*

In condition (1.6), the last restriction may be replaced by

$$\underline{\lim}_{s\to\infty}\frac{f(s)}{s^{4/3}} = K > 0$$

provided M (in (1.4)) is taken to be $<M_0$, for some M_0 depending on K (see [2]).

As shown in [2], the solution ρ_i of the minimization problem satisfies, for some real number λ_i,

$$E_i'(\rho_i) - \lambda_i = 0 \qquad \text{a.e., where } \rho_i > 0,$$

$$E_i'(\rho_i) - \lambda_i \geqslant 0 \qquad \text{a.e.;}$$

that is,

$$A'(\rho_i(x)) - J_i(r) - \int_{\mathbb{R}^3} \frac{\rho_i(y)}{|x-y|}\,dy = \lambda_i \qquad \text{a.e. if } \rho_i(x) > 0,$$
$$\geqslant \lambda_i \qquad \text{a.e. if } \rho_i(x) = 0, \tag{1.9}$$

where

$$J_1(r) = J(r),$$
$$J_2(r) = -\int_r^\infty s^{-3} L(m_\rho(s))\,ds. \tag{1.10}$$

In a recent paper Auchmuty [1] studied the analogous problem for incompressible fluid, that is, the fluid is rotating about the z-axis and is symmetric with respect to the z-axis and the plane $z = 0$. Here the total energy of the fluid is given by

$$\tilde{E}_1(\rho) = -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\,\rho(y)}{|x-y|}\,dx\,dy - \int_{\mathbb{R}^3} J(r)\,\rho(x)\,dx \tag{1.11}$$

in case (a), and by

$$\tilde{E}_2(\rho) = -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\,\rho(y)}{|x-y|}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^3} \frac{\rho(x)}{r^2(x)} L(m_\rho(r(x)))\,dx \tag{1.12}$$

in case (b). The function ρ varies in the class:

$$\begin{gathered} \rho(x) = \rho(r, z), \qquad \rho(r, z) = \rho(r, -z), \\ \rho \text{ measurable}, \qquad 0 \leqslant \rho \leqslant 1, \qquad \int_{\mathbb{R}^3} \rho\,dx = M. \end{gathered} \tag{1.13}$$

The assumption on J, L are somewhat different here than above, namely,

$$J(0) = 0, \qquad J(\infty) < \infty, \qquad J \in C^1[0, \infty), \qquad J'(r) \geqslant 0, \tag{1.14}$$

$$L(0) = 0, \qquad L \in C^1[0, \infty), \qquad L'(r) \geqslant 0. \tag{1.15}$$

The following results are proved in [1].

THEOREM 1.3. *If* (1.6), (1.14) *hold then there exists a function* $\tilde{\rho}_1$ *which minimizes* $\tilde{E}_1(\rho)$ *in the class* (1.13). *Further, there is a bounded set* $G_1 \subset \mathbb{R}^3$ *such that*

$$\tilde{\rho}_1 = 1 \text{ a.e. in } G_1, \qquad \tilde{\rho}_1 = 0 \text{ a.e. in } \mathbb{R}^3 \backslash G_1. \tag{1.16}$$

THEOREM 1.4. *If* (1.6), (1.15) *hold then there exists a function* $\tilde{\rho}_2$ *which minimizes* $\tilde{E}_2(\rho)$ *in the class* (1.13). *Further, there is a bounded set* $G_2 \subset \mathbb{R}^3$ *such that*

$$\tilde{\rho}_2 = 1 \text{ a.e. in } G_2, \qquad \tilde{\rho}_2 = 0 \text{ a.e. in } \mathbb{R}^3 \backslash G_2. \tag{1.17}$$

It is also shown in [1] that, for some real numbers $\tilde{\lambda}_i$,

$$\begin{aligned} \tilde{E}_i'(\tilde{\rho}_i) - \tilde{\lambda}_i &\geqslant 0 \qquad \text{a.e. in } G_i, \\ &\leqslant 0 \qquad \text{a.e. in } \mathbb{R}^3 \backslash G_i; \end{aligned}$$

that is,

$$-J_i(r) - \int_{\mathbb{R}^3} \frac{\tilde{\rho}_i(y)}{|x-y|}\, dy \geqslant \tilde{\lambda}_i \qquad \text{a.e. in } G_i\,, \qquad\qquad \leqslant \tilde{\lambda}_i \qquad \text{a.e. in } \mathbb{R}^3\backslash G_i\,. \tag{1.18}$$

We introduce the potential functions

$$u_i = J_i(r) + \int_{\mathbb{R}^3} \frac{\rho_i(y)}{|x-y|}\, dy + \lambda_i \tag{1.19}$$

and

$$\tilde{u}_i = J_i(r) + \int_{G_i} \frac{dy}{|x-y|} + \tilde{\lambda}_i \tag{1.20}$$

for the solutions ρ_i and $\tilde{\rho}_i$ asserted in Theorems 1.1, 1.2 and 1.3, 1.4, respectively. Then (1.9) gives

$$A'(\rho_i) = u_i \qquad \text{if} \quad \rho_i > 0, \qquad 0 \geqslant u_i \qquad \text{if} \quad \rho_i = 0.$$

It follows that

$$\rho_i = \gamma(u_i), \tag{1.21}$$

where $\gamma(u)$ is the monotone increasing function defined by

$$\gamma(u) = (A')^{-1}(u) \qquad \text{if} \quad u > 0, \qquad = 0 \qquad \text{if} \quad u \leqslant 0. \tag{1.22}$$

Introduce the elliptic operator in two independent variables

$$\mathscr{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\,. \tag{1.23}$$

Applying the Laplacian $\Delta = \sum \partial^2/\partial x_i^2$ (in $\mathbb{R}^3$) to both sides of (1.19) we get

$$\Delta u_i + \gamma(u_i) = \mathscr{L} J_i\,. \tag{1.24}$$

Similarly the function $\tilde{u}_i$ satisfies

$$\Delta \tilde{u}_i + I_{G_i} = \mathscr{L} J_i\,, \tag{1.25}$$

where I_G is the indicator function of the set G; (1.25) can also be written in the form

$$\Delta \tilde{u}_i + I_{\{\tilde{u}_i > 0\}} = \mathscr{L} J_i\,. \tag{1.26}$$

In this paper we shall study the boundary of the sets $\{\rho_i > 0\}$, $\{\tilde{\rho}_i = 1\}$. We find it useful to work with the potential functions u_i, $\tilde{u}_i$ rather than with the density. The boundary of the above sets (i.e., the sets occupied by the fluid) is called the *free boundary*.

The free boundary problem

$$\Delta u + \lambda u^+ = 0 \text{ in a bounded domain } \Omega \subset R^2 \qquad (\lambda > 0),\ u = \varphi \text{ on } \partial\Omega,$$

arises in plasma and was studied in [4, 13, 25, 26]. In this case one can show (see [15]) that the set $\{u > 0\}$ is connected. Another problem of type (1.24) or (1.26) arises in the theory of vortex rings; see [9] and the references given therein. In this case the solution u satisfies

$$\mathscr{L}u + r^2 f(u) = 0 \text{ in } \mathbb{R}^2, \tag{1.27}$$

where $f(u) = 0$ if $u \leqslant 0$, $f'(u) \geqslant 0$ if $u > 0$, and generally $f(0+) > 0$ (i.e., $f(u)$ is discontinuous at $u = 0$). We shall study this problem in Section 7.

We finally mention that a free boundary problem of the type

$$\Delta u + \gamma(u) = f \text{ in } \mathbb{R}^3 \tag{1.28}$$

arises in the Thomas–Fermi atomic model [3, 6, 17]; here, however, $\gamma(u)$ is a monotone decreasing function (instead of monotone increasing). This difference is important, for in this (decreasing) case one can apply the maximum principle to (1.28).

2. Description of the Results

In Sections 3 and 4 we study the solution u_i (given by (1.19)) and the corresponding free boundary. In Section 3 we study some general properties of u_i. It is proved that $u_i = u_i(r, z)$, that is, u_i is independent of θ, and further

$$u_i(r, -z) = u(r, z), \qquad \frac{\partial}{\partial z} u_i(r, z) < 0 \qquad \text{if } z > 0. \tag{2.1}$$

We also establish some properties of the free boundary. In Section 4 it is proved that the region occupied by the fluid ρ_1 consists of a finite number of rings provided $J(r)$ is analytic. The same result for ρ_2 is established without any analyticity assumptions.

The incompressible case is studied in Sections 5 and 6. In Section 5 we study general properties of $\tilde{u}_i$ and establish the analog of (2.1). Unlike the solution u_i which has continuous second derivatives, the solution $\tilde{u}_i$ is only known to have a bounded Laplacian.

In Section 6 we prove that the number of rings of the rotating fluid corresponding to $\tilde{u}_1$ is finite in any r-interval where $\mathscr{L}J \geqslant 0$.

In Section 7 we deal with the problem of vortex rings studied by Fraenkel and Berger [10] and show that the number of vortex rings is finite.

In Section 8 we briefly discuss the regularity of the boundary of the rings in the incompressible case near the line $z = 0$.

3. The Compressible Case: General Properties

We shall assume that

$$\mathscr{L}J(r) \text{ is Hölder continuous (exponent } \beta) \text{ in any interval} \quad 0 \leqslant r \leqslant r_0, \qquad r_0 < \infty, \tag{3.1}$$

and that

$$\begin{gathered} L \in C^2[0, \infty) \cap C^3[0, \delta_0) \qquad \text{for some } \delta_0 > 0, \\ L(0) = L'(0) = L''(0) = 0. \end{gathered} \tag{3.2}$$

It is easily computed that

$$J_2'(r) = L(m_\rho(r))/r^3,$$

$$\frac{d}{dr} m_\rho(r) = 2\pi \int \rho(r, z)\, dz,$$

so that

$$\mathscr{L}J_2(r) = \frac{2\pi}{r^3} L'(m_\rho(r)) \int \rho(r, z)\, dz - 2\frac{L(m_\rho(r))}{r^4}. \tag{3.3}$$

Since ρ is Hölder continuous and since

$$\begin{gathered} | m_\rho(r) - m_\rho(s)| \leqslant C\, | r - s |, \\ m_\rho(r) \leqslant Cr^2 \qquad (C \text{ constant}), \end{gathered} \tag{3.4}$$

the conditions in (3.2) imply that, for some $\beta > 0$,

$$\mathscr{L}J_2 \text{ is Hölder continuous (exponent } \beta) \text{ in } [0, \infty). \tag{3.5}$$

If

$$f(s)/s^p \sim \text{constant as } s \to 0$$

then $\gamma(u) \sim \text{constant} \cdot u^q$ as $u \downarrow 0$, where $q = 1/(p-1)$. (Note that (1.6) implies that $p > \frac{4}{3}$, so that $q < 3$.) (In the physical problem usually $p < \frac{5}{3}$, so that $q > \frac{3}{2}$.)

In this section we shall assume:

$$\begin{aligned}
&\gamma(u) \text{ is a monotone increasing function of } u,\\
&\gamma(u) = 0 \text{ if } u \leqslant 0, \qquad\qquad (3.6)\\
&\gamma(u) \text{ is Hölder continuous (exponent } \beta) \text{ in any bounded interval in } \mathbb{R}^1.
\end{aligned}$$

THEOREM 3.1. *The solution u_i ($i = 1, 2$) satisfies the following properties:*

(i) $u_i \in C^{2+\beta}(\mathbb{R}^3)$;

(ii) *u_i does not depend on θ, that is, $u_i(x) = u_i(r, z)$ if $x = (r, \theta, z)$;*

(iii) $u_i(r, -z) = u_i(r, z)$;

(iv) $u_{iz}(r, 0) = 0$, $u_{ir}(0, z) = 0$, $u_{irz}(r, 0) = 0$ ($r \geqslant 0$, $z \geqslant 0$);

(v) $u_{iz}(r, z) < 0$ *if* $z > 0$;

(vi) $u_{izz}(r, 0) < 0$ *if* $r \geqslant 0$.

Proof. It suffices to prove the theorem for u_1. Set $u = u_1$, $\rho = \rho_1$ (ρ_1 is defined in Theorem 1.1). Assertion (i) follows from (1.24) and the Schauder estimates.

Denote by τ_φ the orthogonal mapping $(r, \theta, z) \to (r, \theta + \varphi, z)$ and set

$$(B\rho)(x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{|x - y|}\, dy. \tag{3.7}$$

Then, by substituting $y \to \tau_\varphi y$,

$$(B\rho)(\tau_\varphi x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{|\tau_\varphi x - y|} = \int_{\mathbb{R}^3} \frac{\rho(\tau_\varphi y)}{|\tau_\varphi x - \tau_\varphi y|}\, dy.$$

Since $|\tau_\varphi x - \tau_\varphi y| = |x - y|$, $\rho(\tau_\varphi y) = \rho(y)$, we conclude that $(B\rho)(\tau_\varphi x) = (B\rho)(x)$. Recalling (1.19), assertion (ii) follows.

The proof of (iii) is similar to the proof of (ii); here we replace τ_φ by the mapping $\tau\colon (r, \theta, z) \to (r, \theta, -z)$.

From (iii) it follows that $u_z(r, 0) = 0$, $r \geqslant 0$; hence also $u_{rz}(r, 0) = 0$. From (ii) and the fact that $u \in C^{2+\beta}$ it follows that u_r is continuous up to $r = 0$ and $u_r(0, z) = 0$. Thus (iv) is established.

We next have (see [1, 2])

$$\rho(r, z)\downarrow \text{ if } z\uparrow, \qquad z \geqslant 0. \tag{3.8}$$

Indeed, a symmetric rearrangement of $\rho(r, z)$ with respect to z (see [11]) does not change the integrals

$$\int_{\mathbb{R}^3} A(\rho(x))\, dx, \qquad \int_{\mathbb{R}^3} J(r)\, \rho(x)\, dx$$

but it strictly decreases the convolution

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\,\rho(y)}{|\,x - y\,|}\,dx\,dy,$$

unless ρ already coincides a.e. with its symmetric rearrangement ρ^*. Thus

$$E_1(\rho^*) \leqslant E_1(\rho)$$

and equality holds if and only if $\rho^* = \rho$ a.e. Since ρ minimizes the functional E_1, it follows that indeed $\rho^* = \rho$ a.e. Since, finally, the minimal density is continuous (by [2]), $\rho^* \equiv \rho$. This gives (3.8).

From relation (1.21) and (3.8) we deduce that $u_z \leqslant 0$ in the open set

$$\Omega^+ = \{x;\, u(x) > 0,\, z > 0\}. \tag{3.9}$$

Hence also

$$u_z \leqslant 0 \text{ in } \overline{\Omega^+}. \tag{3.10}$$

In the open set

$$\Omega^- = \{x;\, u(x) < 0,\, z > 0\}$$

$\gamma(u) = 0$ and, by differentiating (1.24) with respect to z,

$$\Delta u_z = 0 \text{ in } \Omega^-.$$

Since also $u_z \leqslant 0$ on $\partial\Omega^-$ (by (3.10) and the fact that $u_z(r, 0) = 0$), the maximum principle gives: $u_z \leqslant 0$ in Ω^-.

We have thus proved that $u_z \leqslant 0$ if $z \geqslant 0$. Hence

$$\Delta u_z = -\gamma'(u)\,u_z \geqslant 0 \text{ in } z \geqslant 0,$$

where the derivatives are taken in the distribution sense. Applying the strong maximum principle in the set $z > 0$ we obtain assertion (v). Applying the boundary version of the strong maximum principle we obtain assertion (vi).

From Theorem 3.1 it follows that the set occupied by mass ρ_i has the form

$$\Omega_i = \{(r, z);\, -\psi_i(r) < z < \psi_i(r)\} \qquad (i = 1, 2),$$

where $\psi_i(r) \geqslant 0$; if $\psi_i(r) \equiv 0$ on a set I, then there is no mass in the strip $\{r \in I, z \in \mathbb{R}^1\}$.

The set where $\psi_i(r) > 0$ consists of a countable number of open disjoint intervals λ_{ij}.

From Theorem 3.1(v) it readily follows that there are no vertical segments on the cross sections of the boundary of the fluid, i.e., $\psi_i(r)$ is continuous at the points where it vanishes.

DEFINITION. The set

$$\Omega_{\lambda_{ij}} = \{(r, z); r \in \lambda_{ij}, -\psi_i(r) < z < \psi_i(r)\} \tag{3.11}$$

is called a *ring* with *base* λ_{ij}.

Note that $\psi_i(r)$ vanishes at the endpoints of λ_{ij} except possibly at an endpoint which coincides with $r = 0$.

THEOREM 3.2. *Let Ω_λ be a ring (for u_i) with base $\lambda = (a, b)$. Then $\psi_i(r)$ is analytic in $a < r < b$.*

Proof. By Theorem 3.1,

$$u_{iz}(r, \psi_i(r)) \neq 0,$$

and $u \in C^{2+\beta}$. Now apply the implicit function theorem to deduce that $\psi_i(r)$ belongs to $C^{2+\beta}(a, b)$. To prove analyticity we use the hodograph mapping (as in [13, 14])

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = u_i(r, z).$$

Since $u_{iz} \neq 0$ we can solve $z = w(y)$. Set $y' = (y_1, y_2)$ and define

$$\psi = w(y) \text{ if } y_3 \geqslant 0,$$
$$\eta(y', y_3) = w(y', -y_3).$$

Then

$$\psi = \eta, \qquad \psi_{y_3} = -\eta_{y_3} \text{ on } y_3 = 0.$$

Further, ψ and η satisfy, for $y_3 > 0$, the nonlinear elliptic system

$$F(\psi) = \mathscr{L} J_i(r) - \gamma(y_3) \qquad (r = (y_1^2 + y_2^2)^{1/2}),$$
$$F(\eta) = -\mathscr{L} J_i(r) + \gamma(-y_3),$$

where

$$F(z) = -\frac{z_{33}}{(z_3)^3} + \sum_{i=1}^{2} \left[-\left(\frac{z_i}{z_3}\right)_i + \frac{z_i}{z_3}\left(\frac{z_i}{z_3}\right)_3 \right] \qquad \left(z_i = \frac{\partial z}{\partial y_i}\right).$$

We can now use standard elliptic theory to show that the derivatives

$$D_{y'}^m \psi, \qquad D_{y'}^m \eta$$

exist, are continuous, and are bounded by

$$C_0 C^m (m-2)! \qquad \text{for all} \quad m \geqslant 1.$$

Thus ψ, η are analytic in y'. Since the free boundary $z = \psi_i(r)$ is given by

$$z = w(y_1, y_2, 0),$$

the analyticity of $\psi_i(r)$ follows.

DEFINITION. A point $R \geqslant 0$ is called a *point of accumulation* of rings (of u_i) if there exists a sequence of rings $\Omega_{\lambda_{ij'}}$ such that $\lambda_{ij'} \to R$ if $j' \to 0$, that is,

$$\operatorname{dist}(R, \lambda_{ij'}) \to 0 \qquad \text{if} \quad j' \to 0.$$

THEOREM 3.3. *If $\mathscr{L} J_i(R) \geqslant 0$ then R is not a point of accumulation of rings of u_i.*

Proof. Indeed, suppose R is a point of accumulation of rings for, say, $u = u_1$. Then $u(r, 0)$ oscillates an infinite number of times as r tends to R from one side. It follows that

$$u_r(R, 0) = 0, \qquad u_{rr}(R, 0) = 0. \tag{3.12}$$

Since also $\gamma(u(R, 0)) = \gamma(0) = 0$, $u_{zz}(R, 0) < 0$, (1.24) gives $\mathscr{L} J(R) < 0$, a contradiction.

We shall now discuss briefly the behavior of rings near the r-axis. A more definitive description will be given in Theorems 4.2 and 4.4.

Let Ω_λ be a ring for u ($u = u_1$ or $u = u_2$) with $\lambda = (a, b)$. If

$$u_r(b, 0) \neq 0 \tag{3.13}$$

then, by the implicit function theorem, we can represent $\partial\Omega_\lambda$ in a neighborhood of $(b, 0)$ in the form

$$r = \varphi(z) \qquad (-\delta < z < \delta;\ \delta \text{ small}), \tag{3.14}$$

where $\varphi \in C^{2+\beta}$ and $\varphi'(0) = 0$ (since $u_z(b, 0) = 0$). Clearly $u_r(b, 0) < 0$ so that there is no mass in some interval $b < r < b + \epsilon_0$ ($\epsilon_0 > 0$). By the proof of Theorem 3.2, $\varphi(z)$ is analytic in z near $z = 0$.

Suppose next that

$$u_r(b, 0) = 0, \qquad u_{rr}(b, 0) \neq 0. \tag{3.15}$$

Then

$$\nu \equiv u_{rr}(b, 0) > 0.$$

Setting $\mu = -u_{zz}(b, 0)$ ($\mu > 0$) we can write

$$2u(r, z) = \nu(r - b)^2 - \mu z^2 + O(((r - b)^2 + z^2)^{1+\beta/2}).$$

Hence $\partial\Omega_\lambda$ in a neighborhood of $(b, 0)$ is given by

$$z = \pm \frac{\nu}{\mu} | r - b | + O(| r - b |^{1+\beta/2}) \tag{3.16}$$

and there is a ring $\Omega_{\lambda'}$ with $\lambda' = (b, b_1)$ adjacent to Ω_λ on the right, whose boundary is also given by (3.16).

Suppose next that

$$u_r(b, 0) = 0, \qquad u_{rr}(b, 0) = 0. \tag{3.17}$$

Since also $u_{rz}(r, 0) = 0$, we have

$$2u(r, z) = -\mu z^2 + O(((r - b)^2 + z^2)^{1+\beta/2}).$$

Thus Ω_λ is contained in a cusp-like region near $(b, 0)$:

$$| z | \leqslant C | r - b |^{1+\beta/2} \qquad (C > 0). \tag{3.18}$$

4. Finite Number of Rings for u_i

In this section we impose additional restrictions on J and $\gamma(u)$:

$$J(r) \text{ is analytic in } r,\ r \geqslant 0, \tag{4.1}$$

$$\gamma(u) \leqslant Cu^q \text{ if } 0 < u < 1, \text{ where } C > 0,\ q > 1. \tag{4.2}$$

THEOREM 4.1. *Under the additional conditions* (4.1), (4.2), *the number of rings for u_1 is finite.*

Proof. If the assertion is not true then there is a number $R \geqslant 0$ which is a point of accumulation of rings. Setting

$$d = d(x) = ((r - R)^2 + z^2)^{1/2} \qquad \text{if} \quad x = (r, \theta, z),$$

we shall prove by induction that $u = u_1$ satisfies

$$\gamma(u(x)) = C_{\beta_m} d^{\beta_m} \qquad (d \leqslant d_0,\ C_{\beta_m} \geqslant N^{\beta_m}); \tag{4.3}$$

the constants β_m, C_{β_m} will be determined in the inductive proof, and d_0, N are positive constants independent of m.

We proceed to describe the passage from m to $m+1$. We have

$$\Delta(u - J) = -\gamma(u).$$

Using (4.3), we can apply Lemma 3.1 of [6] to conclude that

$$u - J = P_m + Q_m, \tag{4.4}$$

where P_m is a polynomial of degree $[\beta_m] + 2$ and Q_m satisfies

$$|Q_m(x)| \leqslant CC_{\beta_m}\beta_m d^{\beta_m+2}, \qquad C \text{ independent of } m. \tag{4.5}$$

Here β_m is required to be any positive noninteger such that

$$\beta_m - [\beta_m] \geqslant c, \qquad c > 0 \text{ independent of } m. \tag{4.6}$$

Since u and J depend only on r, z, the same must be true of P_m, i.e., $P_m = P_m(r, z)$. Hence, by (4.4), also $Q_m = Q_m(r, z)$.

The function $u(r, 0)$ oscillates an infinite number of times as $r \to R$ from one side. Since also

$$u(r, 0) = J(r) + P_m(r, 0) + Q_m(r, 0),$$

it follows (by (4.5)) that each Taylor coefficient of $(r - R)^k$ (with $k \leqslant [\beta_m] + 2$) must vanish. Hence

$$u(r, 0) = Q_m(r, 0) + \frac{(r - R)^k}{k!} J^{(k)}(\tilde{R}),$$

where $k = [\beta_m] + 3$ and $\tilde{R}$ lies in the interval with endpoints R, r.

The analyticity of J implies that for all $k \geqslant 1$,

$$\left| \frac{J^{(k)}(r)}{k!} \right| \leqslant C_0^k \qquad \text{if } \; |r - R| < d_0,$$

where C_0 is a positive constant. Taking $N > C_0$ we obtain

$$u(r, 0) \leqslant CC_{\beta_m}\beta_m |r - R|^{\beta_m+2}. \tag{4.7}$$

Recalling that $u_z(r, z) \leqslant 0$ if $z \geqslant 0$ and using (4.2), we obtain

$$\gamma(u(r, z)) \leqslant (CC_{\beta_m}\beta_m |r - R|^{\beta_m+2})^q. \tag{4.8}$$

Thus the inductive estimate for $m + 1$ follows with

$$\begin{aligned} C_{\beta_{m+1}} &= (CC_{\beta_m}\beta_m)^q, \\ \beta_{m+1} &= q(\beta_m + 2) - \theta_m, \end{aligned} \tag{4.9}$$

where $\theta_m \in (0, 1)$ is chosen so that

$$\beta_{m+1} - [\beta_m] \geqslant c.$$

Having established (4.3) for all m with C_{β_m}, β_m satisfying (4.9), we can now apply a unique continuation argument used in [6, Lemma 4.1] in order to deduce that

$$\gamma(u(r, z)) = 0 \qquad \text{if} \quad (r - R)^2 + z^2 < d_1,$$

where d_1 is sufficiently small; this contradicts the assumption that R is a point of accumulation of rings.

THEOREM 4.2. *Assume that* (4.1), (4.2) *hold and let* Ω_λ *be a ring with base* (a, b) *for* u_1. *Then* $\partial\Omega_\lambda$ *is given in a neighborhood of* $(b, 0)$ *by*

$$z = \pm\nu(b - r)^{k/2}(1 + O(| r - b |^\delta)) \qquad (\delta > 0), \tag{4.10}$$

where ν *is a positive number and* k *is a positive integer.*

Proof. The proof of Theorem 4.1 (with $R = b$) shows that there exists an m for which the Taylor expansion of $J(r) - P_m(r, 0)$ about $r = b$ must have a nonvanishing first term of order $<[\beta_m] + 2$. But then

$$u(r, 0) = c(b - r)^k + O(| b - r |^{k+\gamma}) \tag{4.11}$$

for some $0 < \gamma < 1$, $c \neq 0$, where k is a positive integer. Writing

$$u(r, z) = u(r, 0) + z^2(u_{zz}(r, 0) + O(| z |^\beta))$$

and noting that

$$u_{zz}(r, 0) = u_{zz}(b, 0) + O(| r - b |^\beta),$$

we obtain the expansion

$$u(r, z) = c(b - r)^k + O(| b - r |^{k+\gamma}) - \mu z^2(1 + O(| z |^\beta) + O(| r - b |^\beta)), \tag{4.12}$$

where $\mu = -u_{zz}(b, 0) > 0$. It follows that $c > 0$ and the set $\{u = 0; r < b\}$ in a neighborhood of $(b, 0)$ is given by (4.10).

Remark. The expansion (4.12) is valid in an entire neighborhood of $(b, 0)$. If k is odd then $u(r, 0) < 0$ if $b < r < b + \epsilon$ for some ϵ sufficiently small, so that there is no mass in the strip $b < r < b + \epsilon$. If k is even then $u(r, 0) > 0$ if $b < r < b + \epsilon$, so that $(b, 0)$ lies also on the boundary of another ring $\Omega_{\lambda'}$ with some base $\lambda' = (b, b_1)$.

We shall now generalize Theorems 4.1, 4.2 to the case of u_2.

THEOREM 4.3. *Assume that* (4.2) *holds. Then the number of rings for u_2 is finite.*

Proof. We proceed as in the proof of Theorem 4.1. Assuming (4.3) (for $u = u_2$), we consider

$$\Delta(u - J_2) = -\gamma(u)$$

and deduce (by Lemma 3.1 of [6]) the expansion

$$u - J_2 = P_m + Q_m \tag{4.13}$$

with P_m, Q_m as in (4.4).

Using (4.3) and relation (1.2), we have

$$\begin{aligned} m_\rho(R) - m_\rho(r) &= \int_r^R \int \gamma(u(s, z))\, dz\, ds \\ &\leqslant C \int_r^R \gamma(u(s, 0))\, ds \leqslant CC_{\beta_m} d^{\beta_m+1}. \end{aligned}$$

Writing

$$\begin{aligned} J_2(r) = J_2(R) &+ \int_R^r \frac{L(m_\rho(R))}{s^3}\, ds + \int_R^r L'(m_\rho(R))(m_\rho(s) - m_\rho(R)) \frac{ds}{s^3} \\ &+ \frac{1}{2} \int_R^r L''(\tilde{m})(m_\rho(s) - m_\rho(R))^2 \frac{ds}{s^3} \end{aligned}$$

and expanding $1/s^3$ in powers of $r - R$, we find that

$$J_2(r) = \sum_{j=0}^{[\beta_m]+2} a_j (R - r)^j + O((R - r)^{\beta_m+2}).$$

Substituting this into (4.13) we obtain an expansion

$$u(r, 0) = \sum_{j=0}^{[\beta_m]+2} c_j (R - r)^j + O((R - r)^{\beta_m+2}). \tag{4.14}$$

If all the c_j vanish then we obtain estimate (4.8). If this holds for all m, then the proof of Theorem 4.1 shows that $\gamma(u) \equiv 0$ in a neighborhood of $(R, 0)$. But then R cannot be a point of accumulation of rings. Thus there must exist an expansion of form (4.14) with some $c_j \neq 0$. This again implies that R is not a point of accumulation of rings.

The above proof shows that expansion (4.14) with some $c_j \neq 0$ is valid for some n. Thus:

THEOREM 4.4. *Assume that* (4.2) *holds. Then the assertion of Theorem* 4.2 *is valid for any ring* Ω_λ *for* u_2.

Remark. Theorems 4.2, 4.4 extend to the left endpoint $r = a$ of Ω_λ, even in case $a = 0$.

We conclude this section with an example where the set $\{\mu > 0\}$ is not topologically a ball.

EXAMPLE. Suppose

$$\begin{aligned} J(r) = \text{const} = J(\infty) \quad & \text{if} \quad r > 2R, \\ 0 \leqslant J(r) < J(\infty) - \delta \quad & \text{if} \quad 0 < r < R, \\ J(r) - \delta \leqslant J(r) \leqslant J(\infty) \quad & \text{if} \quad R < r < 2R, \end{aligned} \tag{4.15}$$

where $R > 0$, $\delta > 0$.

THEOREM 4.5. *If M is sufficiently small then the solution $\rho = \rho_1$ satisfies*

$$\rho(r, z) = 0 \qquad \textit{if} \quad r < R. \tag{4.16}$$

Proof. Suppose

$$M_0 \equiv \int_{r(x)<R} \rho(x)\, dx > 0;$$

we shall derive a contradiction. Let R^* be such that $\rho = 0$ if $r > R^*$. Define a new density function $\tilde{\rho}$ such that

$$\tilde{\rho} = \rho \qquad \text{if} \quad R < r < R^*$$

and

$$\int_{R^*<r<R^*+1} \tilde{\rho}\, dx = M_0 .$$

We can distribute $\tilde{\rho}$ in the set $R^* < r < R^* + 1$ so that

$$\int_{r>R^*} A(\tilde{\rho}) \leqslant \int_{r<R} A(\rho).$$

Then

$$E_1(\tilde{\rho}) - E_1(\rho) < -\delta M_0 + \int_{r(x)<R} \int_{\mathbb{R}^3} \frac{\rho(x)\, \rho(y)}{|x - y|}\, dy\, dx. \tag{4.17}$$

The last term on the right-hand side is bounded by

$$K = M_0 \sup_{r(x)<R} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x - y|}\, dy.$$

Now,

$$\int \frac{\rho(y)}{|x-y|}\,dy \leqslant \int_{|y-x|>1} \rho(y)\,dy + \int_{|y-x|<1} \frac{\rho(y)}{|x-y|}\,dy$$
$$\leqslant M + \left(\int_{|y-x|<1} \frac{1}{|x-y|^2}\,dy\right)^{1/2} \left(\int \rho^2(y)\,dy\right)^{1/2}.$$

Since $\rho(y) \leqslant C$, we obtain $K \leqslant CM_0 M^{1/2}$; here C is constant independent of M. It follows that

$$E_1(\tilde{\rho}) - E_1(\rho) < -\delta M_0 + CM_0 M^{1/2} < 0 \qquad \text{if} \quad M^{1/2} < \delta/C.$$

This contradicts the minimality of ρ.

5. The Incompressible Case: General Properties

We now consider the solution $\tilde{u}_i$ of (1.26); $\tilde{u}_i$ is defined by (1.20), where G_i is defined in Theorems 1.3, 1.4. We assume that conditions (3.1), (3.2) are satisfied. Let R_1 be a positive number such that $G_i \subset \{(r, t): 0 \leqslant r < R_1 - 1\}$.

Theorem 5.1. *The solution $\tilde{u}_i$ $(i = 1, 2)$ satisfies the following properties*:

(i) $\Delta\tilde{u}_i \in L^\infty(\mathbb{R}^3)$, *so that* $\tilde{u}_i \in W^{2,p}(\mathbb{R}^3)$ *for any* $p < \infty$;

(ii) $\tilde{u}_i$ *does not depend on* θ, *i.e.*, $\tilde{u}_i(x) = \tilde{u}_i(r, z)$;

(iii) $\tilde{u}_i(r, -z) = \tilde{u}_i(r, z)$;

(iv) $\tilde{u}_{iz}(r, 0) = 0$, $\tilde{u}_{ir}(0, z) = 0$ $(r \geqslant 0, z \geqslant 0)$;

(v) *for any* $\delta_0 > 0$, $\tilde{u}_{iz}(r, z) \leqslant -Cz$ *if* $0 \leqslant r \leqslant R_1$, $0 \leqslant z \leqslant \delta_0$, *where* $C = C(\delta_0) > 0$.

Proof. The proof of (i)–(iv) is similar to the proof of (i)–(iv) in Theorem 3.1. Next, with $u = \tilde{u}_i$,

$$\Delta u_z = -\frac{\partial}{\partial z} I_{(u>0)} \geqslant 0$$

in the distribution sense. Hence, by the strong maximum principle (see [18]), $u_z < 0$ if $z < 0$. Since also $u_z(r, 0) = 0$, u_z takes its maximum in the set where $z \geqslant 0$ on $z = 0$.

Comparing u_z with a suitable barrier-type function which vanishes on $z = 0$ we find that, for some $\delta_0 > 0$, $u_z \leqslant -Cz$ if $0 \leqslant r \leqslant R_1$, $0 \leqslant z \leqslant \delta_0$, where C is a sufficiently small positive constant.

From Theorem 5.1 it follows that the set $\tilde{\Omega}_i$, where $\tilde{u}_i > 0$, is given by

$$-\tilde{\psi}_i(r) < z < \tilde{\psi}_i(r);$$

the function $\tilde{\psi}_i$ is $\geqslant 0$. We next define the concept of a ring as in Section 3. Note that in the open sets $\{u > 0\}$, $\{u < 0\}$, u belongs to $C^{2+\beta}$.

THEOREM 5.2. *Let Ω_λ be a ring (for $\tilde{u}_i$) with base $\lambda = (a, b)$. Then $\tilde{\psi}_i(r)$ is analytic in $a < r < b$.*

Proof. The proof is similar to the proof of Theorem 3.2. The only difference is that now we begin with u in $C^{1+\alpha}$ (for any $0 < \alpha < 1$) instead of u in $C^{2+\beta}$. Hence we write for ψ, η a system of elliptic equations in divergence weak form, namely,

$$G(\psi) = (\mathscr{L} J_i(r) - \gamma(y_3))\,\psi_3\,,$$
$$G(\eta) = (\mathscr{L} J_i(r) - \gamma(-y_3))\,\eta_3\,,$$

where

$$G(z) = \sum_{i=1}^{2} \left[-z_{ii} + \left(\frac{1 + z_i^2}{z_3}\right)_3\right].$$

By the Schauder estimates for such equations ($C^{1+\alpha}$ estimates) we can deduce (working with finite differences) that ψ, η belong to $C^{2+\beta}$ up to $y_3 = 0$. We can now proceed as in the proof of Theorem 3.2.

The function $\tilde{\psi}_i(r)$ is readily seen to be continuous at the points $r = a$, $r = b$.

THEOREM 5.3. *If $\mathscr{L} J_i(r) \geqslant 1$ for all r in some neighborhood of R, then R is not a point of accumulation of rings of $\tilde{u}_i$.*

Proof. Otherwise there is a sequence of points $r_n \to R$ such that $\tilde{u}_i(r_n, 0) > 0$, $\tilde{u}_{ir}(r_n, 0$, $\tilde{u}_{irr}(r_n, 0) < 0$. Since also $\tilde{u}_{izz}(r_n, 0) < 0$, we conclude from (1.25) that $\mathscr{L} J_i(r_n) < 1$. Taking $n \to \infty$ we get a contradiction.

6. Finite Number of Rings for $\tilde{u}_1$

In this section we shall prove the following result.

THEOREM 6.1. *If $\mathscr{L} J(R) \geqslant 0$ for some $R \geqslant 0$, then R is not a point of accumulation of rings for $\tilde{u}_1$.*

Denote by $B_\rho(r_0, z_0)$ the disk with center (r_0, z_0) and radius ρ, and set $B_\rho(r_0) = B_\rho(r_0, 0)$.

LEMMA 6.2. *For any $\epsilon_0 > 0$ there exist positive constants γ_1, γ_2, C such that for any disk $B_\rho(r_0, z_0)$ with $\rho < \epsilon_0$,*

$$u(r_0, z_0) = \frac{\gamma_\rho(r_0, z_0)}{\rho^2} \iint_{B_\rho(r_0, z_0)} u(z, r)\, dr\, dz - \iint_{B_\rho(r_0, z_0)} G_\rho(r, z, r_0, z_0)\, \mathscr{L} u(r, z)\, dr\, dz, \qquad (6.1)$$

where $\gamma_\rho(r_0, z_0)$, $G_\rho(r, z, r_0, z_0)$ *are functions satisfying*

$$\begin{aligned} \gamma_1 \leqslant \gamma_\rho(r_0, z_0) \leqslant \gamma_2, \\ 0 \leqslant G_\rho(r, z, r_0, a_0) \leqslant C \left| \log \frac{(r-r_0)^2 + (z-z_0)^2}{\rho^2} \right|. \end{aligned} \tag{6.2}$$

Proof. Introduce coordinates

$$\xi = \frac{r - r_0}{a}, \qquad \eta = \frac{z - z_0}{a} \qquad (0 < a < r_0 - \epsilon_0)$$

and set $\tilde{u}(\xi, \eta) = u(r, z)$. Then

$$\tilde{\mathscr{L}}\tilde{u} \equiv \frac{\partial^2 u}{\partial \xi^2} + \frac{a}{r}\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial^2 \tilde{u}}{\partial \eta^2} = a^2 \mathscr{L} u.$$

Denote by $\tilde{G}_a(\xi, \eta)$ the Green's function for $\tilde{\mathscr{L}}$ in the unit disk with pole at (0, 0). Then

$$\tilde{u}(0, 0) = -\iint_{\xi^2+\eta^2=1} \tilde{u} \frac{\partial \tilde{G}_a}{\partial \nu_{\xi\eta}} dS_{\xi\eta} - \iint_{\xi^2+\eta^2<1} \tilde{G}_a(\xi, \eta)\, \tilde{\mathscr{L}}\tilde{u}\, d\xi\, d\eta. \tag{6.3}$$

By the construction of $\tilde{G}_a$,

$$\tilde{G}_a(\xi, \eta) = -\frac{1}{2\pi}\log(\xi^2 + \eta^2)^{1/2} + \Gamma_a(\xi, \eta), \tag{6.4}$$

where Γ_a is bounded (independently of a) and varies continuously with a. We also have (for instance, by the maximum principle)

$$\hat{c}_1 \leqslant -\frac{\partial \tilde{G}_a}{\partial \nu_{\xi\eta}} \leqslant \hat{c}_2 \qquad (\hat{c}_1 > 0, \hat{c}_2 > 0), \tag{6.5}$$

where $\hat{c}_1$, $\hat{c}_2$ are independent of a.

We can write (6.3) in the form

$$\begin{aligned} u(r_0, z_0) = &-\int_{(r-r_0)^2+(z-z_0)^2=a^2} u \frac{\partial}{\partial \nu_{rz}} \tilde{G}_a\left(\frac{r-r_0}{a}, \frac{z-z_0}{a}\right) dS_{rz} \\ &- \iint_{(r-r_0)^2+(z-z_0)^2<a^2} \tilde{G}_a\left(\frac{r-r_0}{a}, \frac{z-z_0}{a}\right) \mathscr{L}u(r, z)\, dr\, dz. \end{aligned} \tag{6.6}$$

Using (6.5) we find that

$$\frac{\partial}{\partial \nu_{rz}} \tilde{G}_a\left(\frac{r-r_0}{a}, \frac{z-z_0}{a}\right) = \frac{\Phi_a(r, z)}{a},$$

where

$$c_1 \leqslant \Phi_a(r, z) \leqslant c_2 \qquad (c_1 > 0, c_2 > 0). \tag{6.7}$$

Integrating both sides of (6.6) with respect to a, $0 < a < \rho$, and using (6.7) and (6.4), the assertion of the lemma follows.

We set

$$u = \tilde{u}_1, \qquad \psi(r) = \tilde{\psi}_1(r)$$

and denote by Ω_λ a ring for $\tilde{u}_1$ with base λ, that is, if $\lambda = (a, b)$,

$$\Omega_\lambda = \{(r, z); -\psi(r) < z < \psi(r), a < r < b\}$$

and $\psi(r) > 0$ if $a < r < b$, $\psi(a) = \psi(b) = 0$. The number

$$h_\lambda = \max_{r \in \bar{\lambda}} \psi(r)$$

is called the *height* of Ω_λ. We denote by $|\lambda|$ the length of λ and by $|\Omega_\lambda|$ the area of Ω_λ.

We shall denote by C, c generic positive constants independent of λ.

LEMMA 6.3. *For any ring* Ω_λ

$$h_\lambda \leqslant C|\lambda|, \tag{6.8}$$

$$|\Omega_\lambda| \leqslant C|\lambda|^2. \tag{6.9}$$

Proof. Suppose $h_\lambda = \psi(\bar{r})$, $\bar{r} \in \lambda$. Then

$$-u(\bar{r}, 0) = u(\bar{r}, h_\lambda) - u(\bar{r}, 0) = \int_0^{h_\lambda} u_z(\bar{r}, z)\, dz$$

$$\leqslant -C \int_0^{h_\lambda} z\, dz = -\frac{C}{2} h_\lambda^2$$

since $u_z \leqslant -Cz$. Thus

$$h_\lambda^2 \leqslant \frac{2}{C} u(\bar{r}, 0). \tag{6.10}$$

On λ we have

$$\frac{1}{r}(ru_r)_r = -u_{zz} - 1 + \mathscr{L}J \geqslant -c \qquad (c > 0)$$

since $u_{zz} < 0$. Also $u(a, 0) = u(b, 0) = 0$. The function

$$w(r) = -\frac{c}{2}(r - a)^2 + \frac{c}{2}(b - a)^2$$

satisfies

$$\frac{1}{r}(rw_r)_r = -c - \frac{c(r-a)}{r} \leqslant -c \qquad \text{if} \quad a < r < b,$$

and $w(b) = 0$, $w(a) > 0$. Comparing it with $u(r, 0)$, $r \in \lambda$, we conclude that $w(r) \geqslant u(r, 0)$; in particular,

$$u(\bar{r}, 0) \leqslant w(\bar{r}, 0) \leqslant \frac{c}{2}(b-a)^2 = \frac{c}{2} | \lambda |^2.$$

Substituting this into (6.10), we obtain inequality (6.8). Assertion (6.9) follows immediately from (6.8).

LEMMA 6.4. *For any ring* Ω_λ,

$$u(r, z) \leqslant C\lambda^2 \qquad \text{if} \quad (r, z) \in \Omega_\lambda, \qquad z \geqslant 0. \tag{6.11}$$

Proof. Indeed, if $r \in \lambda$,

$$\begin{aligned} -u(r, 0) &= \int_0^{\psi(r)} u_z(r, z)\, dz \leqslant -C \int_0^{\psi(r)} z\, dz. \\ &\leqslant -\frac{C}{2}\psi^2(r) \end{aligned}$$

and $\psi(r) \leqslant C | \lambda |$, by (6.8).

In Lemmas 6.3, 6.4 we have not made use of the condition $\mathscr{L}J(R) \geqslant 0$. In the next lemma we shall use this condition.

LEMMA 6.5. *Suppose* $\mathscr{L}J(R) \geqslant 0$ *and let* Ω_λ *be a ring with* $\operatorname{dist}(R, \lambda) < \delta$. *If* δ *is sufficiently small then*

$$| \Omega_\lambda | \geqslant c | \lambda |^2 \qquad (c > 0). \tag{6.12}$$

Proof. We suppose that

$$| \Omega_\lambda | < \epsilon | \lambda |^2, \tag{6.13}$$

where ϵ is sufficiently small, and proceed to derive a contradiction. Let r_0 be the midpoint of λ and let

$$S_1 = \left\{ r \in \lambda; \psi(r) > \frac{| \lambda |}{8} \right\},$$

$$S_2 = \lambda \backslash S_1 .$$

Then

$$| S_1 | \frac{| \lambda |}{8} \leqslant \int_\lambda \psi(r)\, dr < \epsilon \, | \lambda |^2,$$

where $| S_i |$ denotes the measure of S_i, i.e.,

$$| S_1 | < 8\epsilon \, | \lambda |. \tag{6.14}$$

If $r \in S_2$ then $\psi(r) < | \lambda |/8$ and, therefore,

$$\begin{aligned} u(r, z) &= \int_{\psi(r)}^{z} u_\zeta(r, \zeta)\, d\zeta \leqslant -\int_{\psi(r)}^{z} C\zeta \, d\zeta = -\frac{C}{2}(z^2 - \psi^2(r)) \\ &\leqslant -C \, | \lambda |^2 \qquad \text{if} \quad \frac{| \lambda |}{2} < z < | \lambda |. \end{aligned} \tag{6.15}$$

Since $| S_2 | > | \lambda | (1 - 8\epsilon)$, if ϵ is sufficiently small then we conclude that

$$\iint_{B_\rho(r_0)} u^- \leqslant -c \, | \lambda |^4 \qquad \left(c > 0, \rho = \frac{| \lambda |}{2}\right). \tag{6.16}$$

On the other hand, by (6.11), (6.13) we have

$$\iint_{B_\rho(r_0)} u^+ \leqslant C\epsilon \, | \lambda |^4. \tag{6.17}$$

Applying Lemma 6.2 with $z_0 = 0$ and using (6.16), (6.17), we obtain

$$u(r_0, 0) < -c \, | \lambda |^2 - \iint_{B_\rho} G_\rho \mathscr{L} u \qquad (c > 0),$$

where $B_\rho = B_\rho(r_0)$, provided ϵ is sufficiently small. Since $u(r_0, 0) > 0$, it follows that

$$\iint_{B_\rho} G_\rho \mathscr{L} u \leqslant -c \, | \lambda |^2. \tag{6.18}$$

Now

$$\begin{aligned} \mathscr{L} u &= -1 + \mathscr{L} J \geqslant -C \text{ in } B_\rho \cap \Omega_\lambda, \\ &= \mathscr{L} J \geqslant -\eta \text{ in } B_\rho \backslash \Omega_\lambda, \end{aligned}$$

where $\eta \to 0$ if $\delta \to 0$. Using (6.2) we conclude that

$$\begin{aligned} -\iint_{B_\rho} G_\rho \mathscr{L} u &\leqslant C \iint_{B_\rho \cap \Omega_\lambda} \log \frac{| \lambda |^2}{(r - r_0)^2 + z^2}\, dr\, dz \\ &\quad + \eta \iint_{B_\rho} \log \frac{| \lambda |^2}{(r - r_0)^2 + z^2}\, dr\, dz \equiv J_1 + J_2 . \end{aligned}$$

J_1 is increased if we replace $B_\rho \cap \Omega_\lambda$ by a disk B' with center $(r_0, 0)$ and radius $\rho' = c\,|\,\lambda\,|\,\epsilon^{1/2}$ having the same area as $B_\rho \cap \Omega_\lambda$. Introducing in this disk polar coordinates (R', θ), where

$$r - r_0 = R' \cos\theta, \qquad z = R' \sin\theta$$

and then substituting $t = R'/|\,\lambda\,|$, we find that

$$J_1 \leqslant C\,|\,\lambda\,|^2 \int_0^{\lambda\epsilon^{1/2}} t\,|\log t\,|\,dt \leqslant C\,|\,\lambda\,|^2\,\epsilon^{1/4}.$$

Similarly we find that

$$J_2 \leqslant C\,|\,\lambda\,|^2 \eta.$$

Hence

$$\iint_{B_\rho} G_\rho \mathscr{L} u \geqslant -C(\epsilon^{1/4} + \eta)\,|\,\lambda\,|^2, \tag{6.19}$$

thus contradicting (6.18) if δ and ϵ are sufficiently small.

Set

$$\Omega = \bigcup_\lambda \Omega_\lambda$$

and suppose:

$$R > 0;\ R \text{ is a point of accumulation of rings.} \tag{6.20}$$

For definiteness we assume that a sequence of rings which accumulate at R lies to the right of the line $r = R$.

LEMMA 6.6. *Assume that* $\mathscr{L} J(R) \geqslant 0$ *and that* (6.20) *holds. Then the density of* $\Omega \cap \{r > R\}$ *at* $(R, 0)$ *cannot be equal to zero, that is,*

$$\limsup_{\rho \to 0} \frac{1}{\rho^2} \int_R^{R+\rho} \psi(r)\,dr > 0. \tag{6.21}$$

Proof. Suppose (6.21) is not true, that is,

$$\int_R^{R+\rho} \psi(r)\,dr = o(\rho^2) \qquad \text{as} \quad \rho \downarrow 0. \tag{6.22}$$

We shall derive a contradiction.

Choose ρ small, and such that

$$u(R + \rho, 0) \leqslant 0 \tag{6.23}$$

and consider the sets

$$S_1 = \{r \in (R, R+\rho);\, \psi(r) \geqslant \epsilon_0 \rho\}, \qquad S_2 = (R, R+\rho)\backslash S_1$$

for any small $\epsilon_0 > 0$. Then the measure $|S_1|$ of S_1 satisfies

$$\epsilon_0 \rho \mid S_1 \mid = o(\rho^2). \tag{6.24}$$

For any $r \in S_2$, $\psi(r) < \epsilon_0 \rho$ and consequently (since $u_z \leqslant -Cz$)

$$u(r, z) \leqslant -C\rho^2 \qquad \text{if} \quad c_1 \rho < z < c_2 \rho,$$

where c_1, c_2 are any positive numbers; ϵ_0 is taken so that $\epsilon_0 < c_1/2$. Since, by (6.24),

$$\mid S_2 \mid \geqslant \rho - o(\rho) \qquad (\rho \to 0),$$

we deduce that, for some $c > 0$,

$$\iint_{B_{\rho/8}(r)} u^- \leqslant -c\rho^4 \qquad \text{for any} \quad r \in \left(R + \frac{\rho}{3}, R + \frac{2\rho}{3}\right). \tag{6.25}$$

Condition (6.23) implies that if $R < r < R + \rho$ and $(r, 0) \in \Omega_\lambda$ then $\lambda \subset (R, R + \rho)$. Hence, by Lemma 6.4,

$$u(r, 0) \leqslant C\rho^2 \qquad \text{if} \quad R < r < R + \rho.$$

Using also (6.22) we obtain

$$\iint_{B_{\rho/8}(r)} u^+ \leqslant C\rho^2 \int_R^{R+\rho} \psi(r)\, dr = 0(\rho^4). \tag{6.26}$$

Next, arguing as in the proof of Lemma 6.5 (cf. the proof of (6.19)) we obtain the estimate

$$-\iint_{B_{\rho/8}(r)} G_{\rho/8} \mathscr{L} u \leqslant o(\rho^4). \tag{6.27}$$

Using Lemma 6.2 and (6.25)–(6.27) we conclude that

$$u(r, 0) < 0 \qquad \text{if} \quad R + \frac{\rho}{3} < r < R + \frac{2\rho}{3}. \tag{6.28}$$

Now, the function $u(r, 0)$ cannot be negative for all $R < r < R + \rho/3$. Hence there exists a number r_1 such that

$$R < r_1 < R + \frac{\rho}{3}, \quad u(r_1, 0) = 0, \quad u(r, 0) < 0 \quad \text{if } r_1 < r < R + \frac{\rho}{3}.$$

Set $\rho_1 = r_1 - R$ and apply result (6.28) with ρ replaced by $2\rho_1$ (noting that $u(R + 2\rho_1, 0) < 0$). It follows that $u(r_1, 0) < 0$, a contradiction. Thus assumption (6.22) is false, and the proof of the lemma is complete.

LEMMA 6.7. *Assume that* $\mathscr{L}J(R) \geqslant 0$ *and* (6.20) *holds. Then there exist a sequence of rings* Ω_{λ_n} *with base* $\lambda_n = (R + a_n, R + b_n)$ *and a positive constant* c *such that* $a_n \downarrow 0$ *if* $n \uparrow \infty$ *and*

$$|\lambda_n| > ca_n .$$

Proof. Otherwise, for any $\epsilon > 0$ and for any ring Ω_λ with $\lambda = (R + a, R + b)$, $b < \delta$ ($\delta = \delta(\epsilon)$) we have

$$|\lambda| < \epsilon a.$$

By Lemma 6.3,

$$\psi(r) < C|\lambda| \leqslant C\epsilon a \leqslant C\epsilon r \qquad (r \in \lambda)$$

and thus

$$\int_R^{R+\rho} \psi(r)\,dr \leqslant \frac{C}{2}\epsilon\rho^2 \qquad \text{if} \quad \rho \leqslant \rho_0 \qquad (\rho_0 = \bar{\rho}_0(\epsilon)).$$

This contradicts assertion (6.21) of Lemma 6.6.

LEMMA 6.8. *Let the assumptions of Lemma* 6.7 *hold. Set* $\rho_n = |\lambda_n|/4$ *and denote by* r_n *the midpoint of* λ_n. *Then*

$$u \geqslant -C|\lambda_n|^2 \text{ in } B_{\rho_n}(r_n). \tag{6.29}$$

Proof. Clearly, for r', r near R,

$$\mathscr{L}(u \pm C(r' - r)^2) \gtrless 0 \qquad \text{(as a function of } r\text{)}$$

if C is large. Hence, by Lemma 6.2 (recalling that $G_\rho \geqslant 0$),

$$-C\rho^2 + \frac{c_1}{\rho^2}\iint_{B_\rho(r',z')} u \leqslant u(r', z') \leqslant \frac{c_2}{\rho^2}\iint_{B_\rho(r',z')} u + C\rho^2, \tag{6.30}$$

where c_1, c_2 are positive constants. We now suppose that

$$u(\bar{r}, \bar{z}) < -2A \mid \lambda_n \mid^2, \qquad \text{where} \quad ((\bar{r} - r_n)^2 + \bar{z}^2)^{1/2} < \rho_n , \tag{6.31}$$

and derive a contradiction if A is sufficiently large.

By (6.30)

$$\frac{c_1}{\rho_n{}^2} \iint_{B_{\rho_n}(\bar{r}, \bar{z})} u \leqslant -A \mid \lambda_n \mid^2 \text{ if } A \text{ is large.} \tag{6.32}$$

Also, again by (6.30),

$$0 \leqslant u(r_n , 0) \leqslant \frac{c_2}{(2\rho_n)^2} \iint_{B_{2\rho_n}(r_n, 0)} u + C(2\rho_n)^2. \tag{6.33}$$

Using the estimate $u \leqslant C \mid \lambda_n \mid^2$ in $B_{2\rho_n}(r_n , 0)$ and (6.32), we obtain

$$\frac{c_2}{(2\rho_n)^2} \iint_{B_{2\rho_n}(r_n, 0)} u \leqslant -\frac{A}{2} \mid \lambda_n \mid^2$$

if A is sufficiently large. Substituting this into (6.33), we conclude that $0 \leqslant u(r_n , 0) < 0$, a contradiction.

LEMMA 6.9. *If* $\mathscr{L}J(R) \geqslant 0$ *then* (6.20) *cannot hold.*

Proof. We suppose that (6.20) is satisfied and derive a contradiction. From Lemmas 6.4, 6.8 we obtain

$$\mid u \mid \leqslant C \mid \lambda_n \mid^2 \text{ in } B_{\rho_n}(r_n) \qquad (\rho_n = \mid \lambda_n \mid /4). \tag{6.34}$$

For any $\frac{1}{2}\rho_n < t < \rho_n$, set

$$B_t = B_t(r_n), \qquad M_t = \{(r, z); t^2 < (r - r_n)^2 + z^2 < \delta_0\}$$

for some small δ_0 (so that $r \geqslant \text{const} > 0$ if $(r, z) \in M_t$).

Denote by $\Gamma^n(r, z)$ the fundamental solution of the adjoint $\mathscr{L}^*$ of $\mathscr{L}$ with pole at $(r_n , 0)$. Then

$$\Gamma^n(r, z) = \frac{1}{2\pi} \log \frac{1}{(\rho^2 + z^2)^{1/2}} + N^n \qquad (\rho = \mid r - r_n \mid),$$

where N^n is a bounded function, and $D^\alpha N^n$ is bounded by

$$o\left(\left| D^\alpha \left(\log \frac{1}{(\rho^2 + z^2)^{1/2}}\right)\right|\right).$$

By standard potential theory, since $\mathscr{L}J$ is Hölder continuous,

$$\left| \iint_{M_t} D_z^2\Gamma^n \cdot \mathscr{L}J \right| \leqslant C, \tag{6.35}$$

where C is independent of n, t. Note also that

$$\mathscr{L}^*\Gamma_{zz}^n = (\mathscr{L}^*\Gamma^n)_{zz} = 0.$$

Integrating Green's identity

$$\begin{aligned}\mathscr{L}u \cdot \Gamma_{zz}^n - u\mathscr{L}^*\Gamma_{zz}^n &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\Gamma_{zz}^n - u\frac{\partial}{\partial r}\Gamma_{zz}^n + \frac{u}{r}\Gamma_{zz}^n\right)\\ &\quad + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\Gamma_{zz}^n - u\frac{\partial}{\partial z}\Gamma_{zz}^n\right)\end{aligned}$$

over M_t, we obtain

$$-\iint_{M_t} \mathscr{L}u \cdot \Gamma_{zz}^n = \int_{\partial B_t}\left[\frac{\partial u}{\partial \nu}\Gamma_{zz}^n - u\frac{\partial}{\partial \nu}\Gamma_{zz}^n + \frac{u}{r}\Gamma_{zz}^n\cos(r, \nu)\right] + O(1), \tag{6.36}$$

where ν is the outward normal to ∂B_t.

We shall now integrate with respect to t, $\frac{1}{2}\rho_n < t < \rho_n$. First

$$\int_{\rho_n/2}^{\rho_n}\int_{\partial B_t}\frac{\partial u}{\partial \nu}\Gamma_{zz}^n = \left[\int_{\partial B_t} u\Gamma_{zz}^n\right]_{t=\rho_n/2}^{t=\rho_n} - \int_{\rho_n/2}^{\rho_n}\int_{\partial B_t} u\Gamma_{zz\nu}^n. \tag{6.37}$$

Since

$$| D^j\Gamma^n | \leqslant C/t^j \text{ on } \partial B_t$$

and since, by (6.34), $u = O(t^2)$ on ∂B_t, we find that the right-hand side of (6.37) is bounded by $C\rho_n$. Since also

$$\left| \int_{\partial B_t}\left[-u\frac{\partial}{\partial \nu}\Gamma_{zz}^n + \frac{u}{r}\Gamma_{zz}^n\cos(r, \nu)\right]\right| \leqslant C,$$

we obtain from (6.36),

$$\frac{1}{\rho_n}\int_{\rho_n/2}^{\rho_n} dt \int_{M_t} \mathscr{L}u \cdot \Gamma_{zz}^n \text{ is bounded.}$$

Recalling (6.35), we conclude that

$$\left| \frac{1}{\rho_n}\int_{\rho_n/2}^{\rho_n} dt \iint_{M_t\cap(u>0)} \Gamma_{zz}^n \right| \leqslant C. \tag{6.38}$$

Since

$$\Gamma_z{}^n(r, z) = \frac{1}{2\pi} \frac{z}{\rho^2 + z^2} (1 + 0(1)), \qquad \rho = | r - r_n |,$$

we find that

$$\int_{R+a_{n+1}}^{R+\delta_0} \frac{\psi(r)}{(r - r_n)^2 + \psi^2(r)} \, dr \leqslant C$$

for any n; hence

$$\int_R^{R+\delta_0} \frac{\psi(r)}{(r - R)^2 + \psi^2(r)} \, dr < \infty. \tag{6.39}$$

We proceed to derive a contradiction. Let

$$S_n' = \{r \in \lambda_n , \psi(r) > A \mid \lambda_n \mid\}, \qquad | S_n' | = \text{meas } S_n' ,$$

where A is sufficiently large. Then

$$| S_n' | \, A \, | \lambda_n | \leqslant \int_{\lambda_n} \psi(r) \, dr = | \Omega_{\lambda_n} | \leqslant c \, | \lambda_n |^2$$

by Lemma 6.3. Hence

$$| S_n' | \leqslant \frac{c}{A} | \lambda_n |. \tag{6.40}$$

Let

$$S_n'' = \{r \in \lambda_n; \psi(r) > \epsilon_0 \mid \lambda_n \mid\}, \qquad | S_n'' | = \text{meas } S_n'' ,$$

where ϵ_0 is positive and sufficiently small. By Lemmas 6.5 and 6.3,

$$c \, | \lambda_n |^2 \leqslant \int_{\lambda_n} \psi(r) \, dr \leqslant | S_n'' | \, C \, | \lambda_n | + \epsilon_0 \, | \lambda_n |^2.$$

Consequently, if ϵ_0 is sufficiently small,

$$| S_n'' | \geqslant c \, | \lambda_n |, \qquad c > 0. \tag{6.41}$$

From (6.40), (6.41) it follows that the inequality

$$\epsilon_0 \, | \lambda_n | \leqslant \psi(r) \leqslant A \, | \lambda_n |$$

holds on a subset S_n of λ_n with measure $| S_n | > \theta \, | \lambda_n |$, $\theta > 0$. We also have, by Lemma 6.7:

$$\text{if } r \in S_n \text{ then } r - R < C \, | \lambda_n |.$$

On the set S_n,

$$\frac{\psi(r)}{(r-R)^2+\psi^2(r)} \geqslant \frac{c}{|\lambda_n|} \qquad (c>0).$$

Since meas $S_n \geqslant \theta\,|\lambda_n|$, it follows that the integral in (6.39) diverges to ∞; a contradiction.

Lemma 6.9 establishes Theorem 6.1 in case $R>0$. It remains to prove the corresponding result for $R=0$. Thus we suppose that

$$R=0 \text{ is a point of accumulation of rings,} \tag{6.42}$$

and proceed to derive a contradiction.

We shall need the analog of Lemma 6.2 for the Laplace operator in $\mathbb{R}^3$ (see [5]),

$$u(0)=\frac{1}{|B_R|}\iiint_{B_R} u - \iiint_{B_R} \Delta u \cdot G_R \qquad (\text{for any } R>0), \tag{6.43}$$

where B_R is the ball $r^2+z^2<R^2$,

$$G_R=\frac{1}{4\pi}\left(\frac{1}{\rho}-\frac{1}{R}\right)-\frac{\gamma}{R^3}(R^2-\rho^2) \qquad (\text{for some } \gamma>0)$$

and $\rho=(r^2+z^2)^{1/2}$. Take a point R such that $u(R,0)\leqslant 0$. Then, by Lemmas 6.3, 6.4,

$$\psi(r)\leqslant CR \qquad \text{if} \quad 0<r<R, \tag{6.44}$$

$$u(r,z)\leqslant CR^2 \qquad \text{if} \quad 0<r<R. \tag{6.45}$$

We claim that there exist positive constants c_0, θ independent of R such that

$$\text{the set of } r\in(0,R) \text{ for which } \psi(r)\geqslant c_0R \text{ has measure } \geqslant \theta R. \tag{6.46}$$

Indeed, otherwise we can use (6.44), (6.45) and the inequality $u_z<-Cz$ to deduce that

$$\frac{1}{|B_R|}\iiint_{B_R} u \leqslant -cR^2;$$

also, if $\mathscr{L}J(0)\geqslant 0$ and if (6.46) is not true,

$$-\iiint_{B_R} \Delta u\cdot G_R \leqslant \epsilon R^2$$

with $\epsilon\to 0$ if $R\to 0$. Using these inequalities in (6.43), we obtain $u(0)<0$, which contradicts (6.42).

With (6.44)–(6.46) at hand, we can use the argument of singular integrals with Γ_{zz}, where

$$\Gamma = \frac{1}{(r^2 + z^2)^{1/2}}$$

is the fundamental solution of the Laplace operator in R^3. We now integrate Green's identity for u, Γ_{zz} in a shell $t < \rho < R$ and then further integrate with respect to t,

$$R'/2 < t < R',$$

where R' is chosen so that $u(R', 0) \leqslant 0$. Using the inequality

$$u \leqslant CR'^2$$

on the inner boundary of the shell and proceeding as in the proof of Lemma 6.9, we arrive at the conclusion (after choosing $R' \downarrow 0$) that

$$I_R \equiv \int_0^R \frac{\psi(r)r}{(r^2 + \psi^2(r))^{3/2}}\, dr \leqslant C. \tag{6.47}$$

Now, assumption (6.42) implies that there exists a sequence of R_n such that $R_n \downarrow 0$ and $u(R_n, 0) \leqslant 0$. Since (6.44)–(6.46) hold for all R_n (with the same C, c_0, θ), we have

$$I_{R_n} = \int_0^{R_n} \frac{\psi(r)r}{(r^2 + \psi^2(r))^{3/2}}\, dr \geqslant \left(\frac{\theta}{2}\right)^2 R_n^2 \frac{c_0 R_n}{(R_n^2 + C^2 R_n^2)^{3/2}} = c > 0.$$

This, however, is impossible, since (6.47) implies that $I_R \to 0$ if $R \to 0$.

The function $\mathscr{L}J(r)$ cannot be nonnegative for all $r \geqslant 0$ since $J(\infty)$ must be finite. However, if one is looking for a local minimum of $\tilde{E}_1(\rho)$, the condition $J(\infty) < \infty$ is not a necessary condition for existence; then the condition $\mathscr{L}J(r) \geqslant 0$ for all $r > 0$ may be satisfied. An important special case is that where the angular velocity $\Omega(r)$ is a small positive constant ω. In this case $J(r) = r^2\omega/2$ and $\mathscr{L}\ J(r) = 3\omega > 0$.

THEOREM 6.10. *If $\rho_2(r, z)$ is a local minimum for $\tilde{E}_2(\rho)$ and $\rho_2(r, z)$ is monotone decreasing in z, and if $\mathscr{L}J(r) \geqslant 0$ for all $r \geqslant 0$, then the number of rings is finite.*

Indeed, the proof is the same as for Theorem 6.1.

Poincare [23; pp. 17–25] has constructed for the case $\Omega(r) = \omega$ examples of a local minimum with any given finite number of rings. His proof is somewhat formal.

Remark. Theorem 6.1 extends also to $\tilde{\rho}_2$, i.e., if one a priori knows that $\mathscr{L}J_2(R) \geqslant 0$ then R is not a point of accumulation of rings. Unfortunatly, one cannot give a simple sufficient condition which ensures that $\mathscr{L}J_2(R) \geqslant 0$ for a given R (unless one already has some information on $\tilde{\rho}_2$). If $r_0 = \inf\{s; \tilde{\rho}_2(s) \neq 0\}$, then one can easily compute that $\mathscr{L}J_2(r_0) = 0$. Thus, the inner core (i.e., $r = r_0$) is never a point of accumulation of rings.

7. Incompressible Fluid with Given Positive Vorticity

The methods of this paper apply to more general equations

$$Au + \gamma(u, r) = f(r),$$

where A is an elliptic operator in R^3 and $\gamma(u, z)$ is monotone increasing in u, with possible discontinuity at $u = 0$. An important example arises in the case of incompressible axisymmetric fluid without gravity, when the vorticity curl $\mathbf{q}$ ($\mathbf{q}$ the velocity) has positive magnitude in the fluid and vanishes outside the fluid.

Existence theorems have been established by Fraenkel and Berger [10], and construction of specific solutions was carried out by Hill [12], Fraenkel [8, 9], and Norbury [20, 21].

As shown by Fraenkel and Berger [10], one formulation of the minimization problem leads to the equation

$$\mathscr{L}u + \lambda r^2 f(u) = 0, \tag{7.1}$$

where $u = u(r, z)$, $\lambda > 0$,

$$f(t) = 0 \quad \text{if} \quad t \leqslant 0,$$
$$f'(t) \geqslant 0 \quad \text{if} \quad t > 0,$$

and $f(t)$ may be discontinuous at $t = 0$. In the liquid $u > 0$, and outside the liquid $u < 0$. The function $u(r, z)$ satisfies

$$u_z(r, z) < 0 \text{ if } z > 0, \; u(r, -z) = u(r, z).$$

In this problem rings are usually called *vortex rings*. It is proved in [10] that if $f(t)$ is Lipschitz continuous at $t = 0$ then the number of vortex rings is finite. We shall now eliminate this assumption on f:

THEOREM 7.1. *The number of vortex rings is finite.*

Proof. If $f(t)$ is continuous at $t = 0$ then the proof is the same as in Theorem 3.3. If $f(t)$ is discontinuous at $t = 0$ then the proof of Theorem 3.3 shows that $R = 0$ is not a point of accumulation of vortex rings (since $r^2 f \to 0$ if $r \to 0$).

Finally, if $f(t)$ is discontinuous then the proof of Theorem 6.1 shows that $R > 0$ is not a point of accumulation of vortex rings.

8. The Regularity of the Boundary Near $z = 0$

We return to the setting of Sections 5 and 6, and discuss the regularity of the boundary of a ring Ω_λ ($\lambda = (a, b)$) in the incompressible case, near $z = 0$.

Consider such a ring for $\tilde{u}_1$ and suppose $(b, 0)$ is not a boundary point of another ring. Then, for a disk $B_\rho(b)$ with center $(b, 0)$ and radius ρ sufficiently small we have

$$\mathscr{L}\tilde{u}_1 = \mathscr{L}J \text{ in } B_\rho(b)\backslash\Omega_\lambda,$$
$$u_1 < 0 \text{ in } B_\rho(b)\backslash\Omega_\lambda,$$

and $u_1 = 0$ in $\partial\Omega_\lambda$. Assume that

$$\mathscr{L}J \geqslant 0 \text{ in } B_\rho(b)\backslash\Omega_\lambda. \tag{8.1}$$

Then the maximum principle gives

$$u_{1r}(b, 0) \neq 0.$$

Hence, by the implicit function theorem, $\partial\Omega_\lambda$ can be represented in a neighborhood of $(b, 0)$ in the form

$$r = \varphi(z), \varphi \in C^{1,\alpha} \text{ for any } \alpha < 1. \tag{8.2}$$

The same considerations cannot be applied to a ring Ω_λ for $\tilde{u}_2$, since

$$\mathscr{L}J_2 \leqslant 0 \text{ in } B_\rho(b)\backslash\Omega_\lambda,$$

as seen immediately from (3.3).

Consider next the situation of two rings Ω_λ and Ω_{λ_0} with a common boundary point $(R, 0)$, i.e., $\lambda = (a_0, R)$ and $\lambda_0 = (R, b_0)$, where $a_0 < R < b_0$. We shall show that

$$\partial\Omega_\lambda \text{ and } \partial\Omega_{\lambda_0} \text{ cannot both be smooth (say } C^{2+\alpha}\text{) in a neighborhood of } (R, 0). \tag{8.3}$$

Proof. Any small disk $B_{r_0}(R)$ with center $(R, 0)$ and radius r_0 is divided by $\partial\Omega_\lambda$, $\partial\Omega_{\lambda_0}$ into four regions G_j $(1 \leqslant j \leqslant 4)$. In G_j,

$$\mathscr{L}\tilde{u}_i = f_{ij}, f_{ij} \text{ smooth,}$$

and $\tilde{u}_i = 0$ on the two arcs of ∂G_j which meet at $(R, 0)$. Since $\tilde{u}_i \in C^{1,\beta}$ for any $0 < \beta < 1$, the angle formed by these two arcs must be $\leqslant \pi/2$. This is true for each G_j; therefore each of these angles must actually be equal to $\pi/2$. But then $\tilde{u}_i$ is in $C^{1,1}$ in each G_j, and hence also in $C^{1,1}(B_{r_0}(R))$. Set $u = \tilde{u}_i$.

Suppose for simplicity that $\partial\Omega_{\lambda_0}$ forms an angle $\pi/4$ with the positive r-axis. Let Γ denote the fundamental solution of with singularity at $(R, 0)$. We apply Green's formula in $B_{r_0}(R)\backslash B_\epsilon(R)$ with the functions u and Γ_{zz}. Noting that $u = 0$, $\nabla u = 0$ at $(R, 0)$ we have that

$$u(r, z) = O((r - R)^2 + z^2).$$

Hence, taking $\epsilon \to 0$ we deduce (cf. proof of Lemma 6.9) that

$$\iint_{G_\rho(R)} \mathscr{L}u \cdot \Gamma_{zz} \text{ is bounded for } 0 < \rho < \frac{R}{2}, \tag{8.4}$$

where $G_\rho(R) = B_{r_0}(R)\backslash B_\rho(R)$.

Note next that

$$\Gamma_{zz}(r, z) = \frac{(r - R)^2 - z^2}{((r - R)^2 + z^2)^2} (1 + 0(1))$$

and that the right-hand side is positive if $|z| < |r - R|$ and negative if $|z| > |r - R|$. Also, $u = -1 + f$ ($f = \mathscr{L}J_i$) in "approximately" the region where $|z| < |r - R|$, and $u = f$ in "approximately" the region where $|z| > |r - R|$. Putting these facts together, one easily derives a contradiction to (8.4).

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