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Semiclassical Asymptotics, Gauge Fields, and Quantum Chaos

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A gauge field is given by a connection on a principal bundle $P \to M$. We consider the semiclassical behavior of a family of Schrödinger operators associated with a gauge field, in the limit as $\hbar \to 0$. We relate the spectral theory of such operators to behavior of the Hamiltonian flow on the natural phase space associated to a gauge field, examining in particular situations where this flow exhibits chaotic behavior.

Contents.

0. INTRODUCTION

In this paper we refine our study [24] of the spectral behavior in the limit as $\hbar \to 0$ of (nonrelativistic) Schrödinger operators associated with external gauge fields. We then exploit our asymptotic analysis in several ways, obtaining in particular information on the spectral theory of

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Schrödinger operators in cases where the corresponding classical dynamics exhibit chaotic behavior.

We introduce here the concepts we will be working with. A gauge field on a manifold M is given by a principal G-bundle $P \to M$ together with a connection on P. We will suppose G and M are compact, and that M is endowed with a Riemannian metric, and G with a bi-invariant Riemannian metric. Then a connection on P corresponds to a G-invariant metric on P; the horizontal space at $p \in P$ coincides with the orthogonal complement with respect to such a metric of the tangent space to the fiber through p, which is a G-orbit.

If G has a representation π_λ on a vector space V_λ, there is an associated vector bundle $E_\lambda \to M$, and we have a natural isomorphism

$$C^\infty(M, E_\lambda) \approx \{u \in C^\infty(P, V_\lambda): u(p \cdot g) = \pi_\lambda(g)^{-1}\, u(p)\} \tag{0.1}$$

of the space of smooth sections of E_λ with a certain subspace of smooth functions on P with values in V_λ; we write the action of G on P as a right action. There is an associated covariant derivative operator

$$\nabla_\lambda: C^\infty(M, E_\lambda) \to C^\infty(M, T^* \otimes E_\lambda), \tag{0.2}$$

where $T^* = T^*M$. There is also associated a natural connection on $T^* \otimes E_\lambda$, and we have

$$\nabla_\lambda: C^\infty(M, T^* \otimes E_\lambda) \to C^\infty(M, T^* \otimes T^* \otimes E_\lambda). \tag{0.3}$$

The Riemannian metric produces a bundle map of $T^* \otimes T^*$ to the trivial bundle of scalars; call this map γ. Then we have a composed operator

$$H^0_\lambda = -\gamma \circ \nabla_\lambda \circ \nabla_\lambda: C^\infty(M, E_\lambda) \to C^\infty(M, E_\lambda), \tag{0.4}$$

a nonnegative second order elliptic differential operator on sections of E_λ.

On the space $C^\infty(P)$ there is the ordinary Laplace–Beltrami operator Δ, which also acts componentwise on $C^\infty(P, V_\lambda)$. If we regard $C^\infty(M, E_\lambda)$ as a linear subspace of $C^\infty(P, V_\lambda)$, via (0.1), then as in [4, 39] we will use the identity

$$\Delta = -H^0_\lambda + \Delta^P_G \qquad \text{on } C^\infty(M, E_\lambda), \tag{0.5}$$

where Δ^P_G is the operator on $C^\infty(P)$ produced from the Laplace operator Δ_G on G via the G-action on P, also acting components on $C^\infty(P, V_\lambda)$.

If we consider a particle whose mass is normalized to be $\frac{1}{2}$, the operator H^0_λ gives the quantum mechanical Hamiltonian for motion in the gauge field considered above, when Planck's constant $\hbar = 1$. If there is in addition a scalar potential V defined on M, the Schrödinger operator is $H^0_\lambda + V$. In local coordinates, and given a local frame field for E_λ, this operator has the form

$$g^{jk}(i\partial_j + A_j)(i\partial_k + A_k) + b^j\partial_j + V. \tag{0.6}$$

We now incorporate Planck's constant as a parameter. For the case of a scalar field, one has

$$H_\hbar = -\hbar^2\Delta + V. \tag{0.7}$$

In the case of an abelian gauge field on flat Euclidean space, associated, for example, to a magnetic field, one considers the Schrödinger operators

$$\delta^{jk}(i\hbar\partial_j + A_j)(i\hbar\partial_k + A_k) + V. \tag{0.8}$$

However, such a prescription, written in this form, would not make sense in the general case, due to the transformation properties of the connection coefficients.

The following prescription makes invariant sense, and coincides with (0.8) in the case of abelian fields on flat Euclidean space. Suppose that, for $\hbar = 1$, we have the quantum Hamiltonian

$$H_1 = H^0_{\lambda_1} + V, \tag{0.9}$$

with λ_1 a given highest weight for an irreducible representation of G. Then we will vary the representation with $\hbar$, replacing λ_1 by $n\lambda_1$, subject to the relation $\hbar \sim 1/n$, and take

$$H_\hbar = \hbar^2 H^0_{n\lambda_1} + V. \tag{0.10}$$

For (0.10) to coincide with (0.9) when $n = 1$, we would take $\hbar = 1/n$, but the formulas which arise take a neater form if we relate $\hbar$ and n by

$$\hbar = |n\lambda_1 + \delta|^{-1}, \tag{0.11}$$

where δ is half the sum of the positive roots of the Lie algebra of G. What is behind this is the fact that, for an irreducible representation π_λ of G with highest weight λ, which belongs to a lattice in $\mathfrak{t}^*$, the dual space of the Lie algebra of a maximal torus of G, there is the formula

$$-\pi_\lambda(\Delta_G) = |\lambda + \delta|^2 - |\delta|^2. \tag{0.12}$$

Then (0.11) is equivalent to

$$\hbar^{-2} = -\pi_{n\lambda_1}(\Delta_G) + |\delta|^2. \tag{0.13}$$

Our first major goal is the following. Let $\sigma \in \mathscr{S}(R)$, i.e., σ is a smooth function on R which is rapidly decreasing, together with all its derivatives. We want to analyze the asymptotic behavior of

$$\operatorname{tr} \sigma(H_\hbar) \qquad \text{as } \hbar \to 0. \tag{0.14}$$

More generally, we want to analyze the behavior of

$$\operatorname{tr}[A(\hbar)\, \sigma(H_\hbar)], \qquad \hbar \to 0, \tag{0.15}$$

for a sufficiently rich class of families of observables $A(\hbar)$, described in more detail in the body of the paper.

Our attack on (0.14)–(0.15) will use a functional calculus based on the fundamental solution to the wave equation. We sketch here the basic approach to $\sigma(-\hbar^2 \Delta)$, which does not possess complications inherent in the analyses of $\sigma(-\hbar^2\Delta + V)$, or of $\sigma(H_\hbar)$ with $H_\hbar$ given by (0.10). Set $\rho(\tau) = \sigma(\tau^2)$, so ρ is an even function in $\mathscr{S}(R)$. Set $\rho_\hbar(\tau) = \rho(\hbar\tau)$. We use the formula

$$\rho_\hbar(\sqrt{-\Delta}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\rho}_\hbar(t) \cos t\sqrt{-\Delta}\, dt. \tag{0.16}$$

Note that

$$\hat{\rho}_\hbar(t) = \hbar^{-1} \hat{\rho}(t/\hbar), \tag{0.17}$$

which decreases rapidly outside any fixed neighborhood of the origin as $\hbar \to 0$, if $\rho \in \mathscr{S}(R)$. It follows that we need only analyze $\cos t\sqrt{-\Delta}$ for $|t|$ small. In view of the finite propagation speed for the solution operator $\cos t\sqrt{-\Delta}$ to the hyperbolic equation $(\partial^2/\partial t^2 - \Delta)u = 0$, we can easily localize the analysis of (0.16) to coordinate neighborhoods. In any given coordinate neighborhood on M, we can write $\cos t\sqrt{-\Delta}$ as a sum of Fourier integral operators

$$\cos t\sqrt{-\Delta}\, u = \sum_{\pm} \int a_\pm(t, x, \xi)\, e^{i\varphi_\pm(t,x,\xi)} \hat{u}(\xi)\, d\xi. \tag{0.18}$$

Here the phase functions $\varphi_\pm$ satisfy certain eikonal equations,

$$\partial\varphi_\pm/\partial t = \pm A_1(x, d\varphi_\pm), \tag{0.19}$$

where $A_1(x, \xi)$ is the principal symbol of $\sqrt{-\Delta}$, and we can take

$$\varphi_\pm(0, x, \xi) = x \cdot \xi. \tag{0.20}$$

The amplitudes $a_\pm(t, x, \xi)$ satisfy certain transport equations, and we can take

$$a_\pm(0, x, \xi) = \tfrac{1}{2}. \tag{0.21}$$

Substituting (0.18) into (0.16), we can write

$$\rho_\hbar(\sqrt{-\Delta})u = \sum_{\pm} \int \rho_\hbar(D_t)(a_\pm e^{i\varphi_\pm})|_{t=0}\, \hat{u}(\xi)\, d\xi. \tag{0.22}$$

Now the stationary phase method allows for a precise analysis of

$$\rho_\hbar(D_t)(a_\pm e^{i\varphi_\pm})|_{t=0} = b_\pm(\hbar, x, \xi)\, e^{ix\cdot\xi}, \tag{0.23}$$

and this leads to a fine analysis of

$$\rho_\hbar(\sqrt{-\Delta})u = \int b(\hbar, x, \xi)\, e^{ix\cdot\xi}\hat{u}(\xi)\, d\xi. \tag{0.24}$$

This approach to spectral theory was used in [30, Chap. XII]. There a study was also made of $q(\sqrt{-\Delta})$ when $q(\lambda)$ is a fixed function, not of rapid decrease but rather with symbolic behavior. Parallel to (0.22)–(0.24), one has, for $q(\lambda)$ even,

$$\begin{aligned} q(\sqrt{-\Delta})u &= \sum_{\pm} \int q(D_t)(a_\pm e^{i\varphi_\pm})|_{t=0}\, \hat{u}(\xi)\, d\xi \\ &= \int p(x, \xi)\, e^{ix\cdot\xi}\hat{u}(\xi)\, d\xi, \end{aligned} \tag{0.25}$$

where $p(x, \xi)$ is a symbol whose asymptotic expansion is obtained by the stationary phase method. In particular its leading term is

$$p_0(x, \xi) = q(A_1(x, \xi)). \tag{0.26}$$

To treat $\rho(\sqrt{-h^2\Delta + V})$, one could write

$$\rho(\sqrt{-\hbar^2\Delta + V}) = \rho_\hbar(\sqrt{-\Delta + \lambda^2 V}), \qquad \lambda = \hbar^{-1}. \tag{0.27}$$

This could be analyzed in a fashion parallel to (0.16)–(0.24), given a construction of a parametrix for $\cos t\sqrt{-\Delta + \lambda^2 V}$ which is increasingly accurate as $\lambda \to \infty$. We do this in Section 1. This works easily once V is adjusted to be positive. Then a construction of such a parametrix is essentially equivalent to the construction of a parametrix for

$$\cos t\sqrt{-\Delta - V\partial_\theta^2} \qquad \text{on } M \times S^1. \tag{0.28}$$

In fact, the analysis of the family of operators $\rho(\sqrt{-\hbar^2\Delta + V})$ can be obtained by the analysis of the single operator on functions on $M \times S^1$, with $D_\theta = (1/i)\,\partial_\theta$,

$$\rho(D_\theta^{-1}\sqrt{-\Delta - V\partial_\theta^2}) = \rho_1(\sqrt{-\Delta - V\partial_\theta^2}, D_\theta), \tag{0.29}$$

where $\rho_1(\mu, \nu) = \rho(\mu/\nu)$ is a symbol of order 0 in the two variables (μ, ν); since the operators D_θ and $\sqrt{-\Delta - V\partial_\theta^2}$ *commute*, the analysis (0.25)–(0.26) generalizes in a straightforward fashion to provide an analysis of (0.29) as a pseudodifferential operator. The idea of analyzing the spectrum of $-\hbar^2\Delta + V$ in terms of the joint spectrum of $-\Delta - V\partial_\theta^2$ and D_θ was used by Colin de Verdiere [9].

A similar technique, in a more elaborate form, is effective in the study of $\sigma(H_\hbar)$ when $H_\hbar$ is given by (0.10), in terms of a gauge field. We consider the following operators on $C^\infty(P)$, where P is the principal G-bundle over M discussed above:

$$L = \Delta + V_1(x)\,\Delta_G^P - |\delta|^2\, V(x), \tag{0.30}$$

$$A = -\Delta_G^P + |\delta|^2. \tag{0.31}$$

Here, Δ is the Laplace operator on P and Δ_G^P is the operator derived from the Laplace operator Δ_G on G by the G-action on P, as in (0.5). We suppose $V(x) \geqslant 1$, which can be arranged by a trivial shift, and set $V_1(x) = V(x) - 1$. Then $\sigma(-A^{-1}L)$ can be analyzed as a pseudodifferential operator on P:

$$\sigma(-A^{-1}L) \in \mathrm{OPS}^0(P). \tag{0.32}$$

We note that $\sigma(-A^{-1}L)$ commutes with the natural G-action on $C^\infty(P)$. The relation of (0.32) with $\sigma(H_\hbar)$, acting on $C^\infty(M, E_\lambda)$, $\lambda = n\lambda_1$, is described as follows. Denote by $\mathscr{D}_\lambda$ the subspace of $C^\infty(P)$ on which G acts as a sum of copies of π_λ. Then, as in [39], we use the isomorphism

$$\mathscr{D}_\lambda \approx \text{sum of } d_\lambda \text{ copies of } C^\infty(M, E_\lambda), \tag{0.33}$$

where d_λ is the dimension of the representation space V_λ of π_λ, and

$$\sigma(-A^{-1}L)\,|_{\mathscr{D}_\lambda} \approx \text{sum of } d_\lambda \text{ copies of } \sigma(H_\hbar). \tag{0.34}$$

In order to analyze the left side of (0.34), we make use of the following construction. If B is an operator on $C^\infty(P)$ with smooth integral kernel,

$$Bu(p) = \int_P b(p, q)\, u(q)\, d\,\mathrm{vol}(q), \tag{0.35}$$

we define $\mathrm{Tr}_G\, B$ to be an operator on $C^\infty(G)$, namely convolution by

$$\kappa(g) = \int_P b(p \cdot g, p)\, d\,\mathrm{vol}(p). \tag{0.36}$$

If B commutes with the G-action, then κ is a central function, so $\mathrm{Tr}_G\, B$ is a bi-invariant operator on $C^\infty(G)$. The transformation Tr_G can be defined on a certain class of pseudodifferential operators, and, as we will show in Section 4,

$$\mathrm{Tr}_G\, \sigma(-A^{-1}L) \in \mathrm{OPS}^m(G), \tag{0.37}$$

where $m = \dim M$. Given the bi-invariance of (0.37), it can be shown that

$$\mathrm{Tr}_G\, \sigma(-A^{-1}L)\pi_\lambda^{ij} = \beta(\lambda+\delta)\, \pi_\lambda^{ij}, \tag{0.38}$$

where $\beta(\lambda)$ is a symbol of order m in $\lambda \in \mathfrak{t}^*$. The principal term in the asymptotic expansion of $\beta(\lambda)$ is simply the restriction of the principal symbol of $\mathrm{Tr}_G\, \sigma(-A^{-1}L)$ to $\mathfrak{t}^*$, regarded as a subspace of $T_e^*G = \mathfrak{g}^*$. From (0.34) we have

$$d_\lambda^{-1} \operatorname{tr} \sigma(H_\hbar) = \beta(\lambda+\delta), \tag{0.39}$$

and hence as $\hbar = |\lambda+\delta|^{-1} \to 0$ this has a complete asymptotic expansion, with a simple explicit formula for its leading term. Further elaboration of this attack allows for an asymptotic analysis of (0.15), for certain families of pseudodifferential operators $A(\hbar)$, described in Section 2 and 3.

As described above, the study of the family of Schrödinger operators $H_\hbar$ associated with a gauge field is closely tied to the study of operators on $C^\infty(P)$ commuting with the G-action. The symbol of a pseudodifferential operator on P is a function on T^*P, the cotangent bundle. If one restricts attention to operators commuting with the G-action, the symbols are naturally defined on a vector bundle over T^*M, whose fibers are linearly isomorphic to $\mathfrak{g}^*$. Such a bundle, which we will call the WGS bundle, has a natural Poisson structure; it is foliated by symplectic manifolds, each of which is a fiber bundle over T^*M with fibers which are diffeomorphic to co-adjoint orbits of G. The "classical dynamics" defined by the gauge fields under consideration are defined by flows on such spaces. We will give a more detailed description of thse bundles and flows, introduced in [34, 27, 11], in Section 5.

We utilize the asymptotic expansions of (0.14)–(0.15) to obtain two different types of qualitative information on semiclassical limits. First we show how families of "smoothed out Chern forms" arise from a family of Schrödinger operators (0.10), with a smoothly parametrized family of gauge fields and potentials. Phenomena arising here are related to studies of Berry's phase [5, 7, 26], and Hannay's angle [13].

We also use the asymptotic behavior of (0.15) to obtain results on eigenfunctions of $H_\hbar$ in cases when the classical dynamics exhibit aspects of chaotic behavior, e.g., ergodicity. Such results were established for the Laplace operator on a compact manifold, with ergodic geodesic flow, in [10], and for Schrödinger operators with scalar potentials, in [14].

Ergodic flows on constant energy surfaces are not so easy to come by, and we give a result which is valid in nonergodic cases, as we will illustrate with some examples, in Section 8.

The complete asymptotic expansions of (0.14)–(0.15), given by (6.18)–(6.19) and (9.7)–(9.10), generalize the complete asymptotic expansions of $\operatorname{tr} e^{-\beta H_\hbar}$ as $\hbar \to 0$, for fixed $\beta > 0$, given in [24], which in turn refined the analysis of the leading behavior in [42]. We also mention related work of Guillemin and Uribe [12, 39] on the spectral behavior of Schrödinger operators in gauge fields. The paper [39] considers $H_\lambda + V$ as $\lambda = n\lambda_1 \to \infty$ rather than (0.10), a family of operators whose behavior is different in detail from the families considered here; nonetheless the WGS bundle arises in [39] to describe spectral behavior. In the paper [39] particular attention is paid to the case when all the geodesics on the principal bundle P are closed, with the same period, and delicate results on clustering of eigenvalues are derived. The paper [12] looks at the family (0.10), in the case $V \equiv 1$, with $\lambda \to \infty$ along a ray, and examines spectral asymptotics of a somewhat different nature than (0.14), using an averaging process that also sums over $\lambda = n\lambda_1$. An important phenomenon considered in [12] is the influence of closed trajectories in the WGS bundle on spectral behavior.

We now outline the contents of the main body of the paper. In Section 1 we develop an analysis of the operators (0.27), using a family of Fourier integral operators depending on a parameter λ, and derive results on the asymptotic behavior of (0.14) for $H_\hbar$ of the form (0.7), arising from a scalar field. The results of Section 1 of course have points in common with other approaches to asymptotic analysis of Schrödinger operators, such as those described in [9, 15, 19, 22, 40, 44, 46]. We discuss in more detail relations with some of this work at the end of Section 1. While the material of Section 1 partly serves a pedagogical purpose to orient the reader toward the attack we make in the gauge field case, it also contains some novel points, even when the analysis is specialized to functional calculus for the Laplace operator, as we also explain in further detail at the end of Section 1. Having obtained an analysis of $\sigma(H_\hbar)$ in Section 1, as a family of pseudodifferential operators, we introduce two classes of families of pseudodifferential operators, $\mathrm{OP}\,\Sigma^\mu$ and OPR^μ, in Sections 2 and 3, respectively, and derive some of their basic properties. In Section 3 we make an asymptotic expansion for (0.15), for $A(\hbar)$ belonging to one of these families, in the scalar field case. Connections between $\mathrm{OP}\,\Sigma^\mu$ and OPR^μ and classes of pseudodifferential operators on $M \times S^1$ are emphasized, and they serve as a guide for an appropriate generalization of these expansions to the gauge field case.

In Section 4 we analyze the transformation Tr_G and prove mapping properties which enable us to establish (0.37) once $\sigma(-A^{-1}L)$ is analyzed as a pseudodifferential operator (which is done in Section 6).

In Section 5 we give a description of the WGS bundle and its Poisson structure, close to that of [11]. We approach it here as a natural space on which to define the principal symbol of a G-invariant pseudodifferential operator on the bundle P, and emphasize the connection between the Poisson structure on the WGS bundle and Egorov's theorem, for conjugation by a G-invariant Fourier integral operator.

In Section 6 we obtain an asymptotic expansion of (0.14) in the gauge field case. The analysis of (0.15) is similarly obtained (as briefly described in Section 9). In Section 6 we also consider a more general family of Schrödinger operators,

$$H_\hbar = \hbar^2 H_\lambda^0 + i\hbar\pi_\lambda(X) + V, \tag{0.40}$$

where X is a section of the vector bundle $\mathfrak{g}_{\mathrm{ad}} = P \times_{\mathrm{ad}} \mathfrak{g}$ over M. The formula for the leading term of the expansion of (0.14) in this case lacks some of the gratuitous symmetry of that in the more restricted case (0.10), and the influence of the bundle P on this formula is more fully apparent.

Finally, in Sections 7–9, we apply the asymptotic expansions for (0.14)–(0.15) to the studies of smoothed out Chern forms and to reflection of chaotic behavior of the flow on the WGS bundle, on the spectral behavior of $H_\hbar$, as mentioned above. Section 8 also has a partly pedagogical purpose. The material of Section 8 is closely related to the recent paper [14]; we have presented our variant of this work in such a manner as to introduce the further analysis of Section 9.

We will make extensive use of pseudodifferential operators in this paper, and will follow notation used in [17, 30], some of which we briefly recall here. To a "symbol" $p(x, \xi)$, a smooth function of $(x, \xi) \in R^{2n}$ satisfying certain growth restrictions, we associate an operator $p(x, D)$, acting on functions on R^n, by

$$p(x, D)u = (2\pi)^{-n/2} \int p(x, \xi)\, e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \tag{0.41}$$

where $\hat{u}(\xi)$ is the Fourier transform of u. One frequently used class of symbols is denoted $S^m_{1,0}$. We say $p(x, \xi) \in S^m_{1,0}$ provided

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leqslant C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}, \tag{0.42}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. If $p(x, \xi) \in S^m_{1,0}$, the operator (0.42) is said to belong to $\mathrm{OPS}^m_{1,0}(R^n)$. We say $p(x, \xi) \in S^m$ if it belongs to $S^m_{1,0}$ and has an asymptotic expansion

$$p(x, \xi) \sim \sum_{j \geqslant 0} p_j(x, \xi), \tag{0.43}$$

where $p_j(x, \xi)$ is homogeneous of degree $m - j$ in ξ for $|\xi| \geqslant 1$. There are other classes of symbols of importance, including classes of families $p(\hbar, x, \xi)$, with $\hbar$ as a parameter, tending to 0, such as the symbol classes Σ^μ and R^μ mentioned above. For any class of symbols $\mathscr{C}$, we denote by $\mathrm{OP}\,\mathscr{C}$ the corresponding class of operators. In certain cases, one has invariance under diffeomorphisms, and $\mathrm{OP}\,\mathscr{C}(M)$ can be defined for a compact manifold M.

The symbol spaces Σ^μ and R^μ are two of a number of symbol spaces which arise in this paper, to describe various important specific properties that operators and families of operators may possess. To help the reader keep track of these symbols, we include at the end of this paper a list of symbol spaces, accompanied by formula numbers, near or at which definitions can be found, and we also include a list of other special symbols used in the paper.

1. Functional Calculus for $-\hbar^2 L + V$

Let L be a negative semi-definite second order elliptic differential operator on a compact Riemannian manifold M, acting on scalars (or more generally with scalar principal symbol), V a smooth real valued function on M. We want to understand the behavior of

$$\sigma(-\hbar^2 L + V) \tag{1.1}$$

as $\hbar \to 0$, given $\sigma \in \mathscr{S}(R)$. Replacing $\sigma(\tau)$ by $\sigma(\tau - C)$ with C a sufficiently large constant, we can suppose without loss of generality that V is *positive*, and this assumption will be in effect throughout this section.

Let $\rho(\tau) = \sigma(\tau^2)$, so ρ is an even function in $\mathscr{S}(R)$. Let $\rho_\hbar(\tau) = \rho(\hbar\tau)$. Then (1.1) is equal to

$$\rho_\hbar(\sqrt{-L + \lambda^2 V}), \tag{1.2}$$

with $\lambda = \hbar^{-1}$. We will use the following identity to study (1.2);

$$\rho_\hbar(\sqrt{-L + \lambda^2 V})u = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\rho}_\hbar(t) \cos t\sqrt{-L + \lambda^2 V}\, u\, dt, \tag{1.3}$$

where $\hat{\rho}_\hbar(t)$ is the Fourier transform of $\rho_\hbar(\tau)$. Note that

$$\hat{\rho}_\hbar(t) = \hbar^{-1}\hat{\rho}(t/\hbar). \tag{1.4}$$

For each $\hbar > 0$, $\lambda \in R$ (not necessarily equal to $\hbar^{-1}$), (1.2) is a smoothing operator, with a smooth integral kernel $r(\hbar, \lambda, x, y)$, $x, y \in M$. We first note the local nature of the behavior of this kernel as $\hbar \to 0$; this result has been given in [8] and in [30, Chap. 12] in case $V=0$. The following holds for $V \geqslant 0$.

PROPOSITION 1.1. *The kernel function $r(\hbar, \lambda, x, y)$ is rapidly decreasing off the diagonal $x=y$ as $\hbar \to 0$. For any neighborhod U of the diagonal $\{(x, x) \in M \times M\}$, we have*

$$|r(\hbar, \lambda, x, y)| \leqslant C(N, U)\, \hbar^N \qquad \text{if } (x, y) \in M \times M \setminus U, \tag{1.5}$$

as $\hbar \to 0$, where $C(N, U)$ is independent of λ.

Proof. We use the fact that $\cos t\sqrt{-L+\lambda^2 V}$ satisfies the finite propagation speed condition; its distribution kernel *vanishes* for $\operatorname{dist}(x, y) \leqslant C\,|t|$, for some constant C, so to evaluate $r(\hbar, \lambda, x, y)$ we need only integrate (1.3) over $|t| \geqslant C^{-1} \operatorname{dist}(x, y)$, on which set $\hat{\rho}_\hbar(t) = \hbar^{-1}\hat{\rho}(t/\hbar)$ is rapidly decreasing as $\hbar \to 0$ if $(x, y) \in M \times M \setminus U$.

Let $\beta \in C_0^\infty(R)$ be supported on $|t| \leqslant C_0$ and equal to 1 on $|t| \leqslant C_0/2$, where C_0 is chosen small enough that $\operatorname{dist}(x, y) \leqslant CC_0$ implies $(x, y) \in U$, and let $\gamma(t) = 1 - \beta(t)$. Then write

$$R_{1,\hbar,\lambda} = (2\pi)^{-1/2} \int \beta(t)\, \hat{\rho}_\hbar(t) \cos t\sqrt{-L+\lambda^2 V}\, dt \tag{1.6}$$

and

$$R_{2,\hbar,\lambda} = (2\pi)^{-1/2} \int \gamma(t)\, \hat{\rho}_\hbar(t) \cos t\sqrt{-L+\lambda^2 V}\, dt. \tag{1.7}$$

It follows that the integral kernel of $R_{1,\hbar,\lambda}$ is supported in $(x, y) \in U$. It remains to estimate (1.7). Note that $\cos t\sqrt{-L+\lambda^2 V}$ has operator norm $\leqslant 1$ on $L^2(M)$, for all $\lambda \in R$, provided $V \geqslant 0$. Integration by parts produces

$$\begin{aligned} R_{2,\hbar,\lambda} = (2\pi)^{-1/2} &(-L+\lambda^2 V+1)^{-k} \int (D_t^2+1)^k\, (\gamma(t)\, \hat{\rho}_\hbar(t)) \\ &\cdot \cos t\sqrt{-L+\lambda^2 V}\, dt. \end{aligned} \tag{1.8}$$

Since $(-L+\lambda^2 V+1)^{-k} = (-L+1)^{-k}\,(-L+1)^k\,(-L+\lambda^2 V+1)^{-k}$ maps $L^2(M)$ to $H^{2k}(M)$ with norm independent of $\lambda \in R$, desired sup norm estimates on the integral kernel of $R_{2,\hbar,\lambda}$, and also of all x- and y-derivatives, follow.

Consequently, we can get a very accurate picture of the behavior of (1.2) by working in local coordinates on M, and we need only analyze the operator $\cos t\sqrt{-L+\lambda^2 V}$ for $|t| \leqslant C_0$, with C_0 as small a fixed positive number as we like. It is convenient to make such an analysis by applying the method of geometrical optics to (1.3). Such an approach has been pursued in [33] and in [30, Chap. 12] following the work [16], in the case $V=0$.

For $|t|$ small, we write

$$\cos t\sqrt{-L+\lambda^2 V}\,u(x)=\sum_{\pm}\int a^{\pm}(t,x,\xi,\lambda)\,e^{i\varphi^{\pm}(t,x,\xi,\lambda)}\hat{u}(\xi)\,d\xi, \tag{1.9}$$

given u supported on a coordinate patch in M. The phase functions $\varphi^{\pm}$ are determined by the eikonal equations

$$\begin{aligned}(\partial\varphi^{\pm}/\partial t)^2&=L_2(x,\nabla_x\varphi^{\pm})+\lambda^2 V(x),\\ \varphi^{\pm}(0,x,\xi,\lambda)&=x\cdot\xi.\end{aligned} \tag{1.10}$$

Here $L_2(x,\xi)$ is the principal symbol of $-L$, a polynomial homogeneous of degree 2 in ξ, which is strictly positive for $\xi\neq 0$. We take the positive square root to specify φ^{+} and the negative square root to specify φ^{-}. This first order nonlinear PDE has a solution for $|t|$ small. Next we set

$$a^{\pm}\sim\sum_{j\geqslant 0}a_j^{\pm}(t,x,\xi,\lambda) \tag{1.11}$$

with $a_j^{\pm}$ homogeneous of degree $-j$ in (ξ,λ), defined by the following transport equations (with the convention $a_{-1}^{\pm}=0$):

$$\begin{aligned}&i(2\varphi_t^{\pm}\partial_t-2\langle\nabla_x\varphi^{\pm},\nabla_x\rangle+(L^b\varphi^{\pm}))\,a_j^{\pm}(t,x,\xi,\lambda)\\ &\quad=-(\partial_t^2-L)\,a_{j-1}^{\pm}(t,x,\xi,\lambda),\end{aligned} \tag{1.12}$$

$$a_0^{\pm}(0,x,\xi,\lambda)=\tfrac{1}{2},\ a_j^{\pm}(0,x,\xi,\lambda)=0\qquad\text{for } j\geqslant 1. \tag{1.13}$$

In (1.12), $\langle\ ,\ \rangle$ stands for the bilinear form obtained by polarizing $L_2(x,\xi)$, a quadratic form in ξ, and L^b stands for L with the zero order term omitted. We clearly have the following symmetries:

$$\begin{aligned}\varphi^{-}(t,x,\xi,\lambda)&=\varphi^{+}(-t,x,\xi,\lambda),\\ a_j^{-}(t,x,\xi,\lambda)&=a_j^{+}(-t,x,\xi,\lambda).\end{aligned} \tag{1.14}$$

We note that the construction of the phase functions $\varphi^{\pm}$ and amplitudes $a^{\pm}$ in (1.10)–(1.13) is equivalent to the geometrical optics construction for the equation (which is hyperbolic if $V>0$)

$$(\partial^2/\partial t^2-L-V\partial_\theta^2)u=0 \tag{1.15}$$

for $u=u(t,x,\theta)$ defined on $R\times M\times S^1$. Thus the standard estimates for this construction show without further work that, if the sum in (1.11) is restricted to $0 \leqslant j \leqslant K$, and K is sufficiently large, then the difference between $\cos t\sqrt{-L+\lambda^2 V}$, for $|t| \leqslant C_0$, and the resulting Fourier integral operator is a family of operators mapping $L^2(M)$ to $H^N(M)$, with norm which is $O(\langle\lambda\rangle^{-N})$, and N can be taken arbitrarily large provided K is sufficiently large.

Continuing our analysis of (1.3), we have, modulo a remainder which is a smoothing operator, $O(\hbar^N)+O(\langle\lambda\rangle^{-N})$ for N arbitrarily large,

$$\rho_\hbar(\sqrt{-L+\lambda^2 V})u=\sum_{\pm}\int \rho_\hbar(D_t)(\beta(t)\,a^{\pm}e^{i\varphi^{\pm}})|_{t=0}\,\hat{u}(\xi)\,d\xi. \qquad (1.16)$$

Here, we let $\hbar$ and λ be independent, though later we will identify λ with $\hbar^{-1}$. Since $\varphi_t^{\pm}$ is nonzero, we can apply the stationary phase method, in the form of the fundamental asymptotic expansion formula for pseudodifferential operators (see, e.g., [30, p. 186]), to write

$$\rho_\hbar(D_t)(\beta(t)\,a^{\pm}e^{i\varphi^{\pm}})|_{t=0}=b^{\pm}(\hbar,x,\xi,\lambda)\,e^{ix\cdot\xi}. \qquad (1.17)$$

Here

$$\begin{aligned}\sum_{\pm} b^{\pm} &= b(\hbar,x,\xi,\lambda)\\ &\sim \sum_{j\geqslant 0}\hbar^j\rho^{(j)}(\sqrt{L_2(x,\hbar\xi)+(\hbar\lambda)^2\,V(x)})\,p_j(x,\xi,\lambda),\end{aligned} \qquad (1.18)$$

where $p_j(x,\xi,\lambda)$ is given explicitly by the formula

$$p_j(x,\xi,\lambda)=\sum_{\pm}(\partial/\partial t)^j\,(a^{\pm}e^{i\rho^{\pm}})|_{t=0} \qquad (1.19)$$

with

$$\rho^{\pm}(t,x,\xi,\lambda)=\varphi^{\pm}(t,x,\xi,\lambda)-x\cdot\xi-t\varphi_t^{\pm}(t,x,\xi,\lambda). \qquad (1.20)$$

Therefore, outside $(\xi,\lambda)=(0,0)$, $p_j(x,\xi,\lambda)$ is a classical symbol, satisfying

$$p_j(x,\xi,\lambda)\in S^{[1/2j]}(M,R^{n+1}_{\xi,\lambda}). \qquad (1.21)$$

We note that, in addition to (1.14), we have

$$\rho^{-}(t,x,\xi,\lambda)=\rho^{+}(-t,x,\xi,\lambda),$$

so $p_j(x,\xi,\lambda)=0$ for j odd. Hence we can restrict attention to j *even* in (1.18). Since $\rho^{(j)}(\tau)$ is even if j is even, we have, for $\hbar\in(0,1]$, and for $0\leqslant\ell\leqslant j$, if j is even,

$$\hbar^j\rho^{(j)}(\sqrt{L_2(x,\hbar\xi)+(\hbar\lambda)^2\,V(x)})=O(\hbar^{j-\ell}) \quad \text{in } S^{-\ell}_{1,0}(M,R^{n+1}_{\xi,\lambda}). \qquad (1.22)$$

Furthermore, if we sum over $0 \leqslant j \leqslant 2J$ in (1.18), the remainder term is $O(\hbar^{2J-\ell})$ in $S_{1,0}^{J-\ell}(M, R_{\xi,\lambda}^{n+1})$, for $0 \leqslant \ell \leqslant 2J$.

If we write

$$p_{2j}(x, \xi, \lambda) \sim \sum_{\ell \geqslant 0} p_{j\ell}(x, \xi, \lambda) \tag{1.23}$$

with $p_{j\ell}(x, \xi, \lambda)$ homogeneous of degree $j-\ell$ in (ξ, λ), then

$$b(\hbar, x, \xi, \hbar^{-1}) \sim \sum_{j,\ell \geqslant 0} \hbar^{j+\ell} \rho^{(2j)}(\sqrt{L_2(x, \hbar\xi) + V(x)})\, q_{j\ell}(x, \hbar\xi), \tag{1.24}$$

where

$$q_{j\ell}(x, \xi) = p_{j\ell}(x, \xi, 1) \in S^{j-\ell}(M, R_\xi^n). \tag{1.25}$$

Rearranging the asymptotic sum (1.24), we have the following analysis of (1.1).

PROPOSITION 1.2. *For $\sigma(-\hbar^2 L + V)$, $\sigma \in \mathscr{S}(R)$, there is the following Fourier integral representation in local coordinates, modulo a smoothing operator which is rapidly decreasing as $\hbar \to 0$:*

$$\sigma(-\hbar^2 L + V) = (2\pi)^{-n/2} \int c(\hbar, x, \xi)\, e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \tag{1.26}$$

where

$$c(\hbar, x, \xi) \sim \sum_{k \geqslant 0} \hbar^k c_k(x, \hbar\xi). \tag{1.27}$$

Here, each $c_k(x, \xi)$ belongs to $S^{-\infty}$, i.e., is smooth in all arguments, even at $\xi = 0$, and rapidly decreasing with all derivatives as $|\xi| \to \infty$. In particular,

$$\begin{aligned} c_0(x, \xi) &= \rho(\sqrt{L_2(x, \xi) + V(x)}) \\ &= \sigma(L_2(x, \xi) + V(x)). \end{aligned} \tag{1.28}$$

Note that, since the left side of (1.26) is an even operator valued function of $\hbar$, the amplitude $c(\hbar, x, \xi)$ in (1.26) is an even function of $\hbar$. It follows that, for the expansion (1.27), $c_k(x, \xi)$ is an *even* function of ξ when k is even and an *odd* function of ξ when k is odd.

The trace of an operator of the form

$$Bu(x) = (2\pi)^{-n/2} \int b(x, \xi)\, e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi \tag{1.29}$$

is equal to

$$(2\pi)^{-n} \int b(x, \xi)\, d\xi\, dx \tag{1.30}$$

if $|b(x, \xi)| \leqslant C\langle \xi \rangle^{-n-1}$ and $b(x, \xi)$ is compactly supported in x. We apply this to (1.26)–(1.27). Since $\int c_k(x, \hbar\xi)\, d\xi = 0$ for k odd, we have the following conclusion.

PROPOSITION 1.3. *For $\sigma \in \mathcal{S}(R)$, we have*

$$\operatorname{tr} \sigma(-\hbar^2 L + V) \sim \hbar^{-n}(a_0 + a_2\hbar^2 + a_4\hbar^4 + \cdots), \tag{1.31}$$

where $n = \dim M$, with

$$a_0 = (2\pi)^{-n} \int_{T^*M} \sigma(L_2(x, \xi) + V(x))\, d\xi\, dx. \tag{1.32}$$

We point out some notable features of the construction given above. The form (1.26)–(1.27) for $\sigma(-\hbar^2 L + V)$ exhibits explicitly the phenomenon of rapid decrease off the diagonal demonstrated in Proposition 1.1, a property that relies very strongly on the fact that L is a differential operator, so that finite propagation speed can be exploited, and would fail for a general elliptic (negative self adjoint) second order pseudodifferential operator. Now the geometrical optics construction used above depends on L being a differential operator; for the eikonal equation (1.10) to have a solution smooth for $(\xi, \lambda) \neq (0, 0)$, we require $L_2(x, \xi)$ to be smooth at $\xi = 0$, so it must be a polynomial in ξ; this is also required for the symbol estimates (1.22); similarly the rest of the symbol of L must be a polynomial in ξ for the transport Eqs. (1.12)–(1.13) to have smooth solutions. The fact that L is forced to be a differential operator for this geometrical optics construction to work gives the formulas enough extra structure to exhibit this special phenomenon, and other properties special to the differential operator case. In particular, the form (1.26)–(1.27) leads easily to the special form of the asymptotic expansion (1.31), involving only even powers of $\hbar$ in the parentheses. In the case of $V = 0$, the expansion of $\operatorname{tr} \sigma(-\hbar^2 L)$ for L a pseudodifferential operator would generally involve also odd powers of $\hbar$ and, as pointed out by Duistermaat and Guilleman [38] (for $\sigma(\tau^2) = \exp(-\tau^2)$), also terms including a factor of $\log \hbar$. Therefore, even when we are interested in functional calculus for L alone (with $V = 0$), in the differential operator case it is convenient to throw in a term $V = 1$, trivially shifting the spectrum, and use the construction above, to obtain results which are more precise than one would get using a geometrical optics construction valid for pseudodifferential operators L.

The approach to functional calculus used above, via (1.3) and the Fourier integral representation for $\cos t\sqrt{-L+\lambda^2 V}$, follows the development of functional calculus in [33], which was further expounded in [30, Chap. 12]. This work and work of Colin de Verdiere [9], which was published about the same time, have a number of points in common; both considered more generally functions of commuting pseudodifferential operators $L_1, ..., L_k$, such that $-L=L_1^2+\cdots+L_k^2$ is elliptic. One difference is that, while [9] relied on the analysis of Strichartz [29] to obtain a basic identification of various operators $f(L_1, ..., L_k)$ as pseudodifferential operators, [33] emphasized that this property was a particularly simple consequence of the geometrical optics construction. This direct approach has advantages; for example, the method of [29] requires L to have compact resolvent, while use of (1.3) adapts readily to cases of operators on noncompact manifolds where L may have continuous spectrum; some applications of this are made in [8, 37].

Another approach to functional calculus, based on the Mellin transform, has been developed by Helffer and Robert [41, 22, 45]. It starts from a formula equivalent to

$$\begin{aligned} f(-L) &= g(\log(-L)) \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \hat{g}(s)(-L)^{is}\,ds, \end{aligned} \tag{1.33}$$

where $g(\lambda)=f(e^\lambda)$. Construction of approximations to $(-L)^{is}$ can be done via purely pseudodifferential operator methods. An advantage of this approach is that quite general sorts of elliptic operators can be treated, in cases where it would not be easy to produce a Fourier integral approximation to e^{itL} or to $e^{it\sqrt{-L}}$. In return a price is paid. The integral in (1.33) cannot be effectively localized near $s=0$, as can (1.3), and as approximations to $(-L)^{is}$ do not improve as $|s|\to\infty$, the remainder estimates one gets for approximations to $f(-L)$ and to $f(-\hbar^2 L+V)$ by this method are not as sharp as via (1.3). For example, the works [41, 22, 45] contain results which imply that $\sigma(-\hbar^2 L+V)$ in Proposition 1.2 is an *admissible* family of operators, while the analysis above shows that here it is a *strongly admissible* family, in the terminology of [22]. The analysis of $f(-L)$ via the Mellin transform, when $\operatorname{spec}(-L)\subset[0,\infty)$, and similarly of $f(-\hbar^2 L+V)$, also requires that $f(\lambda)$ vanish to infinite order at $\lambda=0$, which allows one to study the spectrum in intervals bounded away from 0 more easily that in intervals containing 0. A further price, noted at the end of [22], is that the approach via the Mellin transform is not convenient for studying functions of *commuting* pseudodifferential operators. The approach via (1.3) is easily modified to treat this case, and in subsequent sections of this paper we will make essential use of such a functional calculus for commuting operators.

Another attack made on the spectral asymptotics of $-\hbar^2 L + V$, operating on functions on R^n, is via a Fourier integral representation of $\exp it(-\hbar^2 L + V)$. This approach is used in [15, 22], among other places. In this case, of course, the eikonal equation is quite different from (1.10), and the Fourier integral operators produced are of a different nature from the ones used here. This construction works for Schrödinger operators on R^n, with a variety of potentials $V(x)$, ranging from short range potentials to those resembling that of the harmonic oscillator. Such an approach does not work on compact manifolds. As noted by Duistermaat and Guillemin [38], the operator $e^{it\Delta}$ on a compact M can be expected to squirt singularities all over the place instantly; one does not expect to construct a useful parametrix for such operators when M is compact.

We also mention another approach to semiclassical spectral asymptotics, taken by Shubin in [46, Appendix 2]. This involves taking the family of pseudodifferential operators with complete symbol $\sigma_\hbar(-\hbar^2 L_2(x, \xi) + V(x))$, for a certain class of functions $\sigma_\hbar$, approximating the characteristic function of an interval, as a first approximation to the operators $\sigma_\hbar(-\hbar^2 L + V)$. It is shown in [46] that this can be used to give the leading behavior of the spectral asymptotics. An approach that starts with an a priori guess of the symbol of an approximating operator is limited when it comes to producing further terms in an asymptotic expansion, though it is likely that the methods of Strichartz [29] could be exploited in the context of the analysis of [46] to produce such expansions.

2. Special Families of Smoothing Operators

Here we present some general results about families of operators with symbols of the form (1.27), which arose in the description of $\sigma(-\hbar^2 L + V)$. For starters, we will work on R^n, with coordinates denoted x. A family of operators that has been used before in [30], which we will denote OP $\Sigma^\mu_\#$, consists of operators $a(\hbar, x, D)$ with symbols $a(\hbar, x, \xi)$ belonging to $\Sigma^\mu_\#$, which means the following: $a(\hbar, x, \xi)$ is smooth for $\hbar \in (0, 1]$, $x, \xi \in R^n$, and

$$\hbar^k D^j_\hbar a(\hbar, \cdot, \cdot) \text{ is bounded in } S^{\mu+j-k}_{1,0}(R^n) \qquad \text{for } \hbar \in (0, 1], \tag{2.1}$$

i.e.,

$$|\hbar^k D^j_\hbar D^\beta_x D^\alpha_\xi a(\hbar, x, \xi)| \leqslant C_{\alpha\beta jk} \langle \xi \rangle^{\mu+j-k-|\alpha|}. \tag{2.2}$$

We also sometimes take the parameter $\hbar$ in $[-1, 1]$. The family of operators of particular interest to use here is denoted $\mathrm{OP}\,\Sigma^\mu$, where $a(\hbar, x, \xi) \in \Sigma^\mu$ if and only if $a(\hbar, x, \xi) \in \Sigma^\mu_\#$ and

$$a(\hbar, x, \xi) \sim |\hbar|^{-\mu} \sum_{j \geq 0} \hbar^j a_j(x, \hbar\xi), \qquad a_j(x, \xi) \in S^{-\infty}(R^n), \tag{2.3}$$

in the sense that the difference between $a(\hbar, x, \xi)$ and the sum over $j<k$ belongs to $\Sigma^{\mu-k}_\#$, and furthermore $\hbar^{-\ell}$ times this difference belongs to $\Sigma^{\mu-k+\ell}_\#$, for $0 \leq \ell \leq k$. In (2.3) we insist $a_j(x, \xi)$ be smooth, even at $\xi = 0$. From Proposition 1.2 it follows that, in any local coordinate system,

$$\sigma(-\hbar^2 L + V) \in \mathrm{OP}\,\Sigma^0. \tag{2.4}$$

We recall that $\sigma(-\hbar^2 L + V)$ has more structure, which is captured as follows. Set

$$\Sigma^\mu_e = \{a(\hbar, x, \xi) \in \Sigma^\mu\text{: (2.3) holds with } a_j(x, -\xi) = (-1)^j a_j(x, \xi)\}. \tag{2.5}$$

Then, in any local coordinate system,

$$\sigma(-\hbar^2 L + V) \in \mathrm{OP}\,\Sigma^0_e. \tag{2.6}$$

Note that

$$\Sigma^\mu_e = (|\hbar|^{-\mu}\,\Sigma^0_e) \cap \Sigma^\mu, \tag{2.7}$$

and Σ^0_e can be characterized as consisting of functions $a(\hbar, x, \xi)$, defined and smooth for $(\hbar, x, \xi) \in [-1, 1] \times R^n \times R^n$, belonging to Σ^0, and *even* in $\hbar$.

One important property of $\mathrm{OP}\,\Sigma^\mu$ is that it captures the phenomenon described in Proposition 1.1. Indeed, if $a(\hbar, x, D)$ has the symbol of the form (2.3), then

$$a(\hbar, x, D)\, u(x) = \int A(\hbar, x, y)\, u(y)\, dy \tag{2.8}$$

with

$$\begin{aligned}(2\pi)^n A(\hbar, x, y) &\sim |\hbar|^{-\mu} \sum_{j \geq 0} \hbar^j \int a_j(x, \hbar\xi)\, e^{i(x-y)\cdot\xi}\, d\xi \\ &\sim |\hbar|^{-\mu-n} \sum_{j \geq 0} \hbar^j A_j(x, (x-y)/\hbar),\end{aligned} \tag{2.9}$$

where $A_j(x, z)$ is C^∞ in x and z rapidly decreasing as $|z| \to \infty$. In fact, (2.9) with such $A_j(x, z)$ precisely specifies the class of integral kernels of elements of $\mathrm{OP}\,\Sigma^\mu$, when $\mu = 0$.

We note the following algebraic properties of OP Σ^μ and OP Σ_e^μ, whose proofs are routine.

PROPOSITION 2.1. *Suppose $b(x, D)$ is a differential operator of order m, and $a(\hbar, x, D) \in \mathrm{OP}\,\Sigma^\mu$. Then*

$$b(x, D)\, a(\hbar, x, D) \text{ and } a(\hbar, x, D)\, b(x, D) \text{ belong to } \mathrm{OP}\,\Sigma^{\mu+m}, \tag{2.10}$$

if the parameter set for $\hbar$ is $(0, 1]$. *Also*

$$a(\hbar, x, D)^* \in \mathrm{OP}\,\Sigma^\mu. \tag{2.11}$$

Furthermore, if $c(\hbar, x, D) \in \mathrm{OP}\,\Sigma^\nu$, then

$$a(\hbar, x, D)\, c(\hbar, x, D) \in \mathrm{OP}\,\Sigma^{\mu+\nu}. \tag{2.12}$$

The results (2.11)–(2.12) *holds with Σ^* replaced by Σ_e^*.*

We remark that (2.10) fails in general when $b(x, D)$ is a pseudodifferential operator. A more substantial result is the following.

PROPOSITION 2.2. *The classes OP Σ^μ and OP Σ_e^μ are invariant under diffeomorphisms of R^n which are linear outside some compact set.*

Proof. Suppose $a(\hbar, x, D) \in \mathrm{OP}\,\Sigma^\mu$. Under such a diffeomorphism χ, the standard transformation law, given, e.g., in [30, Chap. 2], implies that the family of conjugated operators is of the form $\tilde{a}(\hbar, x, D)$, with symbol

$$\tilde{a}(\hbar, \chi(x), \xi) \sim \sum_{\alpha \geqslant 0} \varphi_\alpha(x, \xi)\, D_\xi^\alpha a(\hbar, x, D\chi(x)^t\, \xi), \tag{2.13}$$

where $\varphi_\alpha(x, \xi)$ is a polynomial in ξ, of degree $\leqslant \frac{1}{2}|\alpha|$, $\varphi_0(x, \xi) = 1$. Given $a(\hbar, x, \xi)$ of the form (2.3), it is easy to express (2.13) in a similar form, giving $\tilde{a}(\hbar, x, D) \in \mathrm{OP}\,\Sigma^\mu$. If $a(\hbar, x, D) \in \mathrm{OP}\,\Sigma_e^0$, the evenness in $\hbar$ perists for $\tilde{a}(\hbar, x, D)$, and the invariance of OP Σ_e^μ follows from this.

Thus there is a natural notion of OP $\Sigma^\mu(M)$ and of OP $\Sigma_e^\mu(M)$, for any compact manifold M, such that $a(\hbar, x, D) \in \mathrm{OP}\,\Sigma^\mu(M)$ has integral kernel which is rapidly decreasing as $\hbar \to 0$, outside the diagonal in $M \times M$. In particular, (2.6) gives

$$\sigma(-\hbar^2 L + V) \in \mathrm{OP}\,\Sigma_e^0(M), \tag{2.14}$$

with L and V as in Section 1.

Now we discuss an isomorphism between OP $\Sigma^\mu(M)$ and a certain algebra of pseudodifferential operators on $M\times S^1$. Given $a(\hbar, x, D)\in$ OP $\Sigma^\mu(M)$, define an operator A on $C^\infty(M\times S^1)$ by

$$A(u(x)\,e^{ik\theta})=a(k^{-1}, x, D)\,u(x)\,e^{ik\theta}\qquad (0 \text{ if } k=0)\tag{2.15}$$

Then A is a pseudodifferential operator, with symbol

$$A(x, \theta, \xi, \lambda)=a(\lambda^{-1}, x, \xi)\sim|\lambda|^\mu\sum_{j\geqslant 0}\lambda^{-j}a_j(x, \xi/\lambda),\tag{2.16}$$

if $a(\hbar, x, \xi)$ satisfies (2.3). For a general symbol $a_j(x, \xi)$, (2.16) would be singular at $\lambda=0$, but the hypothesis $a_j\in S^{-\infty}(M)$ implies that this singularity is removable. $A(x, \theta, \xi, \lambda)$ is C^∞ in all its arguments, away from $(\xi, \lambda)=(0, 0)$, and belongs to $S^\mu(M\times S^1)$. Furthermore, the complete symbol of $A(x, \theta, D_{x,\theta})$ vanishes to infinite order at $\lambda=0$, i.e., on the subbundle $\mathfrak{h}^*(M\times S^1)$ of $T^*(M\times S^1)$ defined by

$$\mathfrak{h}^*_{(x,\theta)}(M\times S^1)=\{\xi\in T^*_{(x,\theta)}(M\times S^1)\colon \xi \text{ is orthogonal to the tangent space to the fibers of } M\times S^1\to M\}.\tag{2.17}$$

We will denote the class of symbols in $S^\mu(M\times S^1)$ vanishing to infinite order on this subbundle by

$$S^\mu_{\mathfrak{h}}(M\times S^1),\tag{2.18}$$

and the associated operator class by $\mathrm{OPS}^\mu_{\mathfrak{h}}(M\times S^1)$. Denote by $\mathrm{OPS}^\mu_{\mathfrak{h}S}(M\times S^1)$ the subset consisting of operators commuting with the S^1-action. The correspondence $a(\hbar, x, D)\mapsto A$ of (2.15) defines a transformation

$$\mathscr{J}\colon \mathrm{OP}\,\Sigma^\mu(M)\to\mathrm{OPS}^\mu_{\mathfrak{h}S}(M\times S^1).\tag{2.19}$$

Note that, if $A(x,\theta, \xi, \lambda)\in S^\mu_{\mathfrak{h}S}(M\times S^1)$, then

$$A(x, \theta, \xi, \lambda)\sim\sum_{j\geqslant 0}|\lambda|^\mu A_j(x, \xi, \lambda)\tag{2.20}$$

with $A_j(x, \xi, \lambda)$ homogeneous of degree $-j$ in (ξ, λ), vanishing to infinite order at $\lambda=0$. This implies $A_j(x, \xi, 1)$ vanishes to infinite order as $|\xi|\to\infty$, since $A_j(x, \xi, 1)=|\xi|^j A_j(x, \xi/|\xi|, |\xi|^{-1})$. Thus $A(x, \theta, \xi, \lambda)$ corresponds under (2.16) to $\sum_{j\geqslant 0}|\hbar|^{-\mu+j}A_j(x, \hbar\xi, 1)$. This proves the following:

Proposition 2.3. *Modulo* $\mathrm{OP}\,\Sigma^{-\infty}$ *and* $\mathrm{OPS}^{-\infty}_{\mathfrak{h}S}(M\times S^1)$ *the map* (2.19) *is invertible.*

The following result, whose proof is obvious, provides a convenient characterization of the class $\mathrm{OPS}^\mu_{\mathfrak{h}}(M\times S^1)$.

PROPOSITION 2.4. *Given $P \in \mathrm{OPS}^{\mu}(M \times S^1)$, P belongs to $\mathrm{OPS}^{\mu}_{\mathfrak{b}}(M \times S^1)$ if and only if, for each $j \geq 1$, there exists $P_j \in \mathrm{OPS}^{\mu-j}(M \times S^1)$ such that*

$$P = D_\theta^j P_j \qquad \mathrm{mod}\ \mathrm{OPS}^{-\infty}(M \times S^1). \tag{2.21}$$

Also, for each $\nu \in R$, $|D_\theta|^\nu P \in \mathrm{OPS}^{\mu+\nu}_{\mathfrak{b}}(M \times S^1)$ provided $P \in \mathrm{OPS}^{\mu}_{\mathfrak{b}}(M \times S^1)$.

Note that

$$\mathscr{J}(\sigma(-\hbar^2 L + V)) = \sigma(-D_\theta^{-2} L + V), \tag{2.22}$$

and (2.19) implies this operator belongs to $\mathrm{OPS}^0_{\mathfrak{b}S}(M \times S^1)$. We now give an alternative proof of this, touching on a point mentioned in the Introduction. Namely, we have

$$\sigma(-D_\theta^{-2} L + V) = \rho_1(\sqrt{-L - V\partial_\theta^2}, D_\theta) \tag{2.23}$$

with

$$\rho_1(\mu, \nu) = \sigma(\mu^2/\nu^2). \tag{2.24}$$

Since $\sigma \in \mathscr{S}(R)$, the apparent singularity at $\nu = 0$ is removable; ρ_1 is C^∞ on $R^2 \setminus (0, 0)$, and homogeneous of degree 0. The operator $L + V\partial_\theta^2$ is elliptic on $M \times S^1$ and commutes with D_θ. Hence the results of [30, Chap. 12], or of [29], apply to show that the operator (2.23) belongs to $\mathrm{OPS}^0(M \times S^1)$; obviously it commutes with the S^1-action. Furthermore, mod $\mathrm{OPS}^{-\infty}$,

$$\rho_1(\sqrt{-L - V\partial_\theta^2}, D_\theta) = D_\theta^{2j} \gamma_j(\sqrt{-L - V\partial_\theta^2}, D_\theta) \tag{2.25}$$

with

$$\gamma_j(\mu, \nu) = \nu^{-2j} \sigma(\mu^2/\nu^2), \tag{2.26}$$

also smooth on $R^2 \setminus (0, 0)$ and in $S^{-2j}(R^2 \setminus (0, 0))$, so $\gamma_j(\sqrt{-L - V\partial_\theta^2}, D_\theta)$ belongs to $\mathrm{OPS}^{-2j}(M \times S^1)$. This reproves

$$\sigma(-D_\theta^{-2} L + V) \in \mathrm{OPS}^0_{\mathfrak{b}S}(M \times S^1). \tag{2.27}$$

In view of Proposition 2.3, this provides a second proof that $\sigma(-\hbar^2 L + V)$ belongs to $\mathrm{OP}\,\Sigma^0(M)$, which is slightly weaker than (2.14).

Of course, this just reiterates the equivalence between the geometrical optics construction of Section 1 and the geometrical optics construction for ordinary hyperbolic equations in one more variable.

3. REGULAR FAMILIES OF OPERATORS AND FURTHER ASYMPTOTICS

Here we introduce a class of symbols somewhat larger than the class Σ^μ discussed in Section 2, defined by (2.3). We take as our motivation for the more general class the correspondence (2.15) taking $\mathrm{OP}\,\Sigma^{\mu}(M)$ to $\mathrm{OPS}^{\mu}(M \times S^1)$. We consider $a(\hbar, x, \xi)$, defined for

$$\hbar \in (0, \infty) \cup (-\infty, 0) \cup \{\infty\}, \tag{3.1}$$

and we set

$$A(x, \xi, \lambda) = a(\lambda^{-1}, x, \xi). \tag{3.2}$$

We say

$$a(\hbar, x, \xi) \in R^{\mu}(M) \Leftrightarrow A(x, \xi, \lambda) \in S^{\mu}(M \times S^1), \qquad (3.3))$$

and we say $a(\hbar, x, D) \in \mathrm{OPR}^{\mu}(M)$ is a *regular* family of operators. As in Section 2, the correspondence on the operator level is given by

$$A(u(x)\, e^{ik\theta}) = a(k^{-1}, x, D)\, u(x)\, e^{ik\theta}. \tag{3.4}$$

We continue to denote the correspondence $a(\hbar, x, D) \mapsto A$ by $\mathscr{J}$, so

$$\mathscr{J}: \mathrm{OPR}^{\mu}(M) \to \mathrm{OPS}^{\mu}_{S}(M \times S^1), \tag{3.5}$$

where $\mathrm{OPS}^{\mu}_{S}(M \times S^1)$ denotes the set of operators in $\mathrm{OPS}^{\mu}(M \times S^1)$ which commute with the S^1-action. Clearly

$$\mathrm{OP}\,\Sigma^{\mu}(M) \subset \mathrm{OPR}^{\mu}(M). \tag{3.6}$$

The map (3.5) is invertible from $\mathrm{OPR}^{\mu}(M)/\mathrm{OP}\,\Sigma^{-\infty}(M)$ to $\mathrm{OPS}^{\mu}_{S}(M \times S^1)/\mathrm{OPS}^{-\infty}_{S}(M \times S^1)$, the inverse of (3.2) being

$$a(\hbar, x, \xi) = A(x, \xi, \hbar^{-1}). \tag{3.7}$$

Note that, if

$$A(x, \xi, \lambda) \sim \sum_{j \geq 0} A_j(x, \xi, \lambda) \qquad \text{as } |\xi| + |\lambda| \to \infty, \tag{3.8}$$

with $A_j(x, \xi, \lambda)$ homogeneous of degree $\mu - j$ in (ξ, λ), we have

$$a(\pm\hbar, x, \xi) \sim \sum_{j \geq 0} \hbar^{-\mu + j} a_j^{\pm}(x, \hbar\xi) \qquad \text{for } \hbar > 0, \tag{3.9}$$

where

$$a_j^{\pm}(x, \xi) = A_j(x, \xi, \pm 1) \in S^{\mu - j}(M). \tag{3.10}$$

Our class $\mathrm{OPR}^{\mu}(M)$ is a sub-class of the class of "admissible" operators considered in [14, 15]. We develop it there to advertise its basic simplicity and rather desirable properties.

To give some examples of elements of OPR^{μ}, we note that if L and V are as in Sections 1–2, then

$$-L+\hbar^{-2}V\in \mathrm{OPR}^{2}(M), \tag{3.11}$$

and

$$(-\hbar^{2}L+V)^{-1}\in \mathrm{OPR}^{0}(M), \tag{3.12}$$

but of course

$$-\hbar^{2}L+V\notin \mathrm{OPR}^{0}(M). \tag{3.13}$$

Also, any *differential operator* of order k on M (independent of $\hbar$) belongs to $\mathrm{OPR}^{k}(M)$, though of course a pseudodifferential operator of order μ on M independent of $\hbar$ typically does not belong to $\mathrm{OPR}^{\mu}(M)$.

We also note that the operator calculus on $M\times S^{1}$ immediately implies that $\mathrm{OPR}^{\mu}(M)$ is invariant under coordinate changes on M and for products of regular families we have

$$\mathrm{OPR}^{\mu_1}(M)\cdot \mathrm{OPR}^{\mu_2}(M)\subset \mathrm{OPR}^{\mu_1+\mu_2}(M). \tag{3.14}$$

If $A_j(\hbar)\in \mathrm{OPR}^{\mu_j}(M)$, we have from (3.4)

$$\mathscr{J}(A_1(\hbar)\cdot A_2(\hbar))=(\mathscr{J}A_1(\hbar))\cdot(\mathscr{J}A_2(\hbar)). \tag{3.15}$$

Since the product of two pseudodifferential operators $P_j\in \mathrm{OPS}^{\mu_j}(M\times S^{1})$ belongs to $\mathrm{OPS}_{\mathfrak{H}}^{\mu_1+\mu_2}(M\times S^{1})$ if either factor belongs to $\mathrm{OPS}_{\mathfrak{H}}^{\mu_j}(M\times S^{1})$, we also have

$$A_1(\hbar)\,A_2(\hbar)\in \mathrm{OP}\,\Sigma^{\mu_1+\mu_2}(M) \tag{3.16}$$

if $A_j(\hbar)\in \mathrm{OPR}^{\mu_j}(M)$ and either factor belongs to $\mathrm{OP}\,\Sigma^{\mu_j}(M)$. In particular, if L and V are as above, given $\sigma\in\mathscr{S}(R)$, $A(\hbar)\in \mathrm{OPR}^{\mu}(M)$, we have

$$A(\hbar)\,\sigma(-\hbar^{2}L+V)\in \mathrm{OP}\,\Sigma^{\mu}(M). \tag{3.17}$$

Next we prove an Egorov-type theorem. A related result is given in [14, 22].

PROPOSITION 3.1. *Let L be a second order elliptic operator on M, negative semidefinite, as above, let $V\in C^{\infty}(M)$, $V>0$, and let $A(\hbar)=a(\hbar, x, D)\in \mathrm{OPR}^{\mu}(M)$. Consider*

$$B(\hbar)=\exp[i\hbar^{-1}t\sqrt{-\hbar^{2}L+V}]\,A(\hbar)\exp[-i\hbar^{-1}t\sqrt{-\hbar^{2}L+V}], \tag{3.18}$$

for any fixed t. Then $B(\hbar) \in \mathrm{OPR}^{\mu}(M)$. Furthermore,

$$B(\hbar) = b_0(\hbar, x, D) \qquad \mathrm{mod}\ \mathrm{OPR}^{\mu-1}(M), \tag{3.19}$$

where, if

$$a(\pm\hbar, x, \xi) \sim \sum_{j \geq 0} \hbar^{-\mu+j} a_j^{\pm}(x, \hbar\xi), \tag{3.20}$$

then

$$b_0(\pm\hbar, x, \xi) = \hbar^{-\mu} \beta^{\pm}(x, \hbar\xi), \tag{3.21}$$

where

$$\beta^{\pm}(x, \xi) = a_0^{\pm}((\exp tH_W)(x, \xi)), \tag{3.22}$$

H_W denoting the Hamiltonian vector field associated with the function

$$W(x, \xi) = [L_2(x, \xi) + V(x)]^{1/2}, \tag{3.23}$$

and $L_2(x, \xi)$ denoting the principal symbol of $-L$.

Proof. The operator $\mathscr{J}$ defined by (3.4) transforms the right side of (3.18) to

$$B^{\#} = \exp[it\sqrt{-(L + V\partial_\theta^2)}] \cdot \mathscr{J}A(\hbar) \cdot \exp[-it\sqrt{-(L + V\partial_\theta^2)}] \tag{3.24}$$

acting on $C^\infty(M \times S^1)$. Since $L + V\partial_\theta^2$ is an elliptic second order negative semidefinite operator on $M \times S^1$, Egorov's theorem gives $B^{\#} = B_0(x, D_{x,\theta})$ mod $\mathrm{OPS}^{\mu-1}(M \times S^1)$, with

$$B_0(x, \xi, \lambda) = A_0((\exp tH_U)(x, \xi, \lambda)), \tag{3.25}$$

where $A_0(x, \xi, \pm\lambda) = \lambda^\mu a_0^{\pm}(x, \xi/\lambda)$ and $U = [L_2(x, \xi) + V(x)\lambda^2]^{1/2}$. Since $\partial U/\partial\theta = 0$ and A_0 is independent of θ, we can write (3.25) as

$$B_0(x, \xi, \lambda) = A_0((\exp tY)(x, \xi, \lambda)), \tag{3.26}$$

where

$$Y = (\partial U/\partial\xi)\, \partial/\partial x - (\partial U/\partial x)\, \partial/\partial\xi. \tag{3.27}$$

From this (3.21)–(3.22) is an immediate consequence, and the proposition is proved.

Note that if $A(\hbar) \in \mathrm{OP}\,\Sigma^{\mu}(M)$, so $\mathscr{J}A(\hbar) \in \mathrm{OPS}^{\mu}(M \times S^1)$, then $B^{\#}$ also belongs to $\mathrm{OPS}_0^{\mu}(M \times S^1)$, and hence $B(\hbar) \in \mathrm{OP}\,\Sigma^{\mu}(M)$. In this case, we also have (3.19) holding modulo $\mathrm{OP}\,\Sigma^{\mu-1}(M)$.

We make brief mention of a few other families of operators. In analogy with the definition of $\mathrm{OPR}^{\mu}(M)$, we say $a(\hbar, x, D)$ belongs to

$$\mathrm{OPR}^{\mu}_{1,0}(M) \tag{3.28}$$

provided the operator on $C^{\infty}(M \times S^1)$ given by (3.4), i.e., $\mathscr{J}a(\hbar, x, D)$, belongs to $\mathrm{OPS}^{\mu}_{1,0}(M \times S^1)$. As in the previous cases, any element of $\mathrm{OPS}^{\mu}_{1,0}(M \times S^1)$ which commutes with the S^1-action comes from an element of $\mathrm{OPR}^{\mu}_{1,0}(M)$. Similarly, we say $a(\hbar, x, D)$ belongs to $\mathrm{OP}\,\Sigma^{\mu}_{1,0}(M)$ provided its image under $\mathscr{J}$ is an element of $\mathrm{OPS}^{\mu}_{1,0}(M \times S^1)$ whose complete symbol vanishes to infinite order on $\mathfrak{h}^*(M \times S^1)$. It is clear that Proposition 3.1 extends to these classes of operators.

We next derive a result on the asymptotic behavior of the trace of operators of the form (3.17). In the proof we find it convenient to anticipate a result which will be proved in the next section, on the G-trace, in the case $G = S^1$. The map Tr_G has been defined in the Introduction, in (0.35)–(0.36).

PROPOSITION 3.2. *With L and V as above, $\sigma \in \mathscr{S}(R)$, and $A(\hbar) \in \mathrm{OPR}^{\mu}(M)$, having symbol expansion of the form* (3.9), *we have as $\hbar \searrow 0$,*

$$\operatorname{tr} A(\hbar)\, \sigma(-\hbar^2 L + V) \sim \hbar^{-n-\mu}(\alpha_0 + \alpha_1 \hbar + \alpha_2 \hbar^2 + \cdots), \tag{3.29}$$

where $n = \dim M$, and

$$\alpha_0 = \int_{T^*M} a_0^+(x, \xi)\, \sigma(L_2(x, \xi) + V(x))\, d\xi\, dx. \tag{3.30}$$

Proof. Set $A^{\#} = \mathscr{J}A(\hbar) \in \mathrm{OPS}^{\mu}(M \times S^1)$. We have

$$\mathscr{J}\sigma(-\hbar^2 L + V) = \sigma(-D_\theta^{-2} L + V) \in \mathrm{OPS}^0_{\mathfrak{h}}(M \times S^1), \tag{3.31}$$

and

$$\mathscr{J}(A(\hbar)\, \sigma(-\hbar^2 L + V)) = A^{\#} \sigma(-D_\theta^{-2} L + V) \in \mathrm{OP}^{\mu}_{\mathfrak{h}}(M \times S^1). \tag{3.32}$$

This operator clearly commutes with the S^1-action, and we have

$$\operatorname{tr} A(\hbar)\, \sigma(-\hbar^2 L + V) = e^{-ik\theta} \Phi(e^{ik\theta}) \qquad \text{for } \hbar = k^{-1}, \tag{3.33}$$

with

$$\Phi = \mathrm{Tr}_{S^1}\, \mathscr{J}(A(\hbar)\, \sigma(-\hbar^2 L + V)). \tag{3.34}$$

As shown in Proposition 4.2, this implies that

$$\Phi \in \mathrm{OPS}^{\mu + n}(S^1), \tag{3.35}$$

which gives (3.29), and the principal symbol calculation for Φ given by (4.12) yields (3.30). (In this case, Lemma 4.3 and (4.13) suffice.)

If one considers a sequence $\sigma_j \in C_0^\infty(R)$ converging to the characteristic function of an interval $I = [a, b] \subset R$, a standard limiting argument from (3.29) produces the following well known result on the asymptotic behavior of the spectrum of $-\hbar^2 L + V$ (cf. [30,15]).

PROPOSITION 3.3. *For a given compact interval I, let*

$$N(\hbar, I) = \text{number of eigenvalues of } -\hbar^2 L + V \text{ belonging to } I, \tag{3.36}$$

counting multiplicities. Then

$$\lim_{\hbar \to 0} \hbar^n N(\hbar, I) = \operatorname{meas} \mathcal{O}_{L,V}(I), \tag{3.37}$$

where

$$\mathcal{O}_{L,V}(I) = p_{L,V}^{-1}(I), \qquad p_{L,V}(x, \xi) = L_2(x, \xi) + V(x). \tag{3.38}$$

In fact, for (3.37) we need (3.29) only with $A(\hbar) = 1$, i.e., Proposition 1.3 suffices. Choosing more general $A(\hbar)$ produces a more detailed picture of the spectral behavior of $-\hbar^2 L + V$, as we will see.

If the eigenvalues of $-\hbar^2 L + V$ are denoted

$$e_1(\hbar) \leqslant e_2(\hbar) \leqslant e_3(\hbar) \leqslant \cdots \tag{3.39}$$

we let

$$\Lambda(\hbar, I) = \{ j : e_j(\hbar) \in I \}. \tag{3.40}$$

Thus the quantity $N(\hbar, I)$ above is the cardinality of $\Lambda(\hbar, I)$. We denote the associated normalized eigenfunctions of $-\hbar^2 L + V$ by $\varphi_j^\hbar$:

$$(-\hbar^2 L + V)\, \varphi_j^\hbar = e_j(\hbar)\, \varphi_j^\hbar. \tag{3.41}$$

Following [35, 10, 14], we introduce a set $\mu_j^\hbar$ of probability measures associated to the eigenfunctions $\varphi_j^\hbar$ by the following device. Let $a(\hbar, x, \xi) \in R_{1,0}^0(M)$, with $B(x, \xi, \lambda) = a(\lambda^{-1}, x, \xi) \in S_{1,0}^0(M \times S^1)$. An operator $B \in \mathrm{OPS}_{1,0}^0(M \times S^1)$ with principal symbol $B(x, \xi, \lambda)$ is well defined modulo $\mathrm{OPS}_{1,0}^{-1}(M \times S^1)$, and there exists an association $B(x, \xi, \lambda) \mapsto B^{\mathrm{F}}$ (defined by Friedrichs symmetrization) such that $B^{\mathrm{F}} \in \mathrm{OPS}_{1,0}^0(M \times S^1)$ has principal symbol $B(x, \xi, \lambda)$, and B^{F} is a positive semidefinite operator provided $B(x, \xi, \lambda) \geqslant 0$. We can also suppose B^{F} commutes with the S^1-action. Then we can define $A^{\mathrm{F}}(\hbar) \in \mathrm{OPR}_{1,0}^0(M)$ satisfying

$$A^{\mathrm{F}}(\hbar)\, u(x) = e^{-ik\theta} B^{\mathrm{F}}(u(x)\, e^{ik\theta}), \qquad \hbar = k^{-1}, \tag{3.42}$$

and we have $A^{\mathrm{F}}(\hbar)$ positive semidefinite when $a(\hbar, x, \xi) \geqslant 0$. Furthermore,

$$A^{\mathrm{F}}(\hbar) = a(\hbar, x, D) \qquad \text{mod } \mathrm{OPR}_{1,0}^{-1}(M). \tag{3.43}$$

The choice of $A^{\mathrm{F}}(\hbar)$ is not unique, but we can arrange the map $a(\hbar, x, D) \mapsto A^{\mathrm{F}}(\hbar)$ to be linear, and we can also arrange that, if $a(\hbar, x, D)u = \varphi u$, $\varphi \in C^{\infty}(M)$, is a multiplication operator, independent of $\hbar$, then $A^{\mathrm{F}}(\hbar)u = \varphi u$ also. Given a choice of the correspondence $a(\hbar, x, D) \mapsto A^{\mathrm{F}}(\hbar)$, there is a unique probability measure $\mu_j^\hbar$ on $\widehat{T^*M}$ such that

$$\int a(x, \xi)\, d\mu_j^\hbar(x, \xi) = (A^{\mathrm{F}}(\hbar)\, \varphi_j^\hbar, \varphi_j^\hbar), \qquad \text{with} \quad a(\hbar, x, \xi) = a(x, \hbar\xi), \tag{3.44}$$

for all $a(x, \hbar\xi) \in R_{1,0}^0(M)$. Here $\widehat{T^*M}$ is a compactification of T^*M, namely the maximal ideal space of the uniform closure of the set of all bounded functions $a(x, \xi)$ on T^*M with the property that $a(x, \hbar\xi)$ belongs to $R_{1,0}^0(M)$. Note that (3.43) implies

$$(A^{\mathrm{F}}(\hbar)\, \varphi_j^\hbar, \varphi_j^\hbar) = (a(\hbar, x, D)\, \varphi_j^\hbar, \varphi_j^\hbar) \qquad \text{mod } O(\hbar). \tag{3.45}$$

Now, as in [14], we define a set of probability measures $m_I^\hbar$ (for any compact interval I) by

$$m_I^\hbar = N(\hbar, I)^{-1} \cdot \sum_{j \in A(\hbar, I)} \mu_j^\hbar. \tag{3.46}$$

Now standard limiting arguments applied to (3.29) yield

$$\int a(x, \xi)\, dm_I^\hbar(x, \xi) \to [\text{meas } \mathcal{O}_{L,V}(I)]^{-1} \int_{\mathcal{O}_{L,V}(I)} a(x, \xi)\, d\xi\, dx, \tag{3.47}$$

as $\hbar \to 0$, for each $a(x, \xi)$ such that $a(x, \hbar\xi) \in R_{1,0}^0(M)$. Therefore

$$m_I^\hbar \to [\text{meas } \mathcal{O}_{L,V}(I)]^{-1}\, \chi_{\mathcal{O}_{L,V}(I)}(d\xi\, dx) \qquad \text{as } \hbar \to 0, \tag{3.48}$$

the convergence being in the weak* topology on the space of finite measures on $\widehat{T^*M}$. In other words, in a certain mean sense, $\mu_j^\hbar$ converge weakly to the (normalized) Liouville measure as $\hbar \to 0$. In Section 8 we will establish some results which in particular strengthen this limiting behavior in cases where the classical dynamical system is ergodic on constant energy surfaces.

4. The G-Trace

Suppose P is a compact Riemannian manifold on which a compact Lie group G acts as a group of isometries. We write the action as a right action, so G has a unitary action on $L^2(P)$:

$$\alpha(g)\, u(p) = u(p \cdot g). \tag{4.1}$$

Let B be an operator on $C^\infty(P)$. For now suppose B has a smooth integral kernel:

$$Bu(p) = \int_P b(p, q)\, u(q)\, dV(q). \tag{4.2}$$

We define

$$\mathrm{Tr}_G\, B \colon C^\infty(G) \to C^\infty(G) \tag{4.3}$$

to be convolution (on the left) by

$$\kappa(g) = \int_P b(p \cdot g, p)\, dV(p). \tag{4.4}$$

Lemma 4.1. *If B commutes with the G-action* (4.1), *then* $\mathrm{Tr}_G\, B$ *is a bi-invariant operator on $C^\infty(G)$.*

Proof. The relation $\alpha(g)B = B\alpha(g)$ is equivalent to

$$b(p \cdot g, q) = b(p, q \cdot g^{-1}). \tag{4.5}$$

If this holds, then, for all $g, g' \in G$,

$$\begin{aligned} \kappa(gg') &= \int_P b(p \cdot g, p \cdot (g')^{-1})\, dV(p) = \int_P b((q \cdot g') \cdot g, q)\, dV(q) \\ &= \kappa(g'g). \end{aligned} \tag{4.6}$$

Hence $\kappa(g)$ is a central element of the convolution algebra $C^\infty(G)$.

We next consider the extension of Tr_G to a class of pseudodifferential operators. We make the hypothesis that G acts on P without fixed points, indeed, that each $p \in P$ has a neighborhood U diffeomorphic to $\mathcal{O} \times G$, with G acting on the second factor, by right translation. This hypothesis, which holds when P is a principal G-bundle over a compact manifold M, will be in effect throughout the rest of this section. We define a subbundle $\mathfrak{h}^*P$ of T^*P by

$$\mathfrak{h}_p^*P = \{\xi \in T_p^*P \colon \xi \text{ annihilates vectors tangent to the } G\text{-orbit through } p\}. \tag{4.7}$$

Then set

$$\mathrm{OPS}^{\mu}_{\mathfrak{h}}(P) = \{A \in \mathrm{OPS}^{\mu}(P)\text{: the complete symbol of } A \bmod S^{-\infty}(P) \text{ vanishes to infinite order on } \mathfrak{h}^*P\}. \tag{4.8}$$

Similarly define the subspace $\mathrm{OPS}^{\mu}_{1,0\mathfrak{h}}(P)$ of $\mathrm{OPS}^{\mu}_{1,0}(P)$. Note that (2.17)–(2.18) is a special case of (4.7)–(4.8). The following result, for that special case, was invoked in the proof of Proposition 3.2. In its general form, it will play an important role in subsequent sections. This generalizes [24, Proposition 4.7].

PROPOSITION 4.2. *The transformation* Tr_G *has a unique continuous extension to*

$$\mathrm{Tr}_G\colon \mathrm{OPS}^{\mu}_{1,0\mathfrak{h}}(P) \to \mathrm{OPS}^{\mu+n}_{1,0}(G), \tag{4.9}$$

where

$$n = \dim P - \dim G. \tag{4.10}$$

We have

$$\mathrm{Tr}_G\colon \mathrm{OPS}^{\mu}_{\mathfrak{h}}(P) \to \mathrm{OPS}^{\mu+n}(G). \tag{4.11}$$

If $A \in \mathrm{OPS}^{\mu}_{\mathfrak{h}}(P)$ *has principal* $a_0(x, \xi)$, *then* $B = \mathrm{Tr}_G A$ *has a principal symbol which satisfies*

$$b_0(e, \lambda) = \int_{\mathfrak{h}^*P} a_0(p, \xi + \lambda)\, dV(\xi, p), \tag{4.12}$$

for $\lambda \in T_e^*G \approx \mathfrak{g}^*$.

Here the G-invariant metric on P gives rise to a natural injection $\mathfrak{g}^* \hookrightarrow T_p^*P$ for each p, and it also gives a natural volume element on $\mathfrak{h}^*P$. In (4.9) and (4.11), if A commutes with $\alpha(g)$, then $\mathrm{Tr}_G A$ is bi-invariant.

Writing A as a sum of a smoothing operator and an operator with distributional kernel supported very near the diagonal in $P \times P$, we can obtain the proof of Proposition 4.2 as a consequence of the following two lemmas.

LEMMA 4.3. *Consider the case* $G = \mathbf{T}^m$, $P = X \times \mathbf{T}^m$. *Then* (4.9)–(4.12) *hold in this case.*

Proof. In this case, if $A \in \mathrm{OPS}^{\mu}_{1,0\mathfrak{h}}(X \times \mathbf{T}^m)$ has symbol, in local coordinates $x \in X$, $y \in \mathbf{T}^m$, $a(x, y, \xi, \lambda)$, then $\mathrm{Tr}_G A$ has symbol

$$b(\lambda) = \int a(x, y, \xi, \lambda)\, d\xi\, dx\, dy. \tag{4.13}$$

The hypothesis $a \in S^{\mu}_{1,0\hbar}(X \times \mathbf{T}^m)$ implies $a(x, y, \xi, \lambda)$ is rapidly decreasing as $|\xi| \to \infty$, for fixed λ, which guarantees the integral above is convergent, and it is easy to see that $b(\lambda)$ has the properties stated in Proposition 4.2 in this case.

In the context of Proposition 4.2, we can deduce that, for $A \in \mathrm{OPS}^{\mu}_{1,0\hbar}(P)$, with distribution kernel supported near the diagonal, $\mathrm{Tr}_G A$ is given by convolution by $B^{\#}\delta_e$, with $B^{\#} \in \mathrm{OPS}^{\mu+n}_{1,0}(V)$ having principal symbol satisfying (4.12), where V is a (small) neighborhood of 0 in R^m $(m = \dim G)$, identified with a neighborhood of e in G via the exponential map. The following result finishes the proof of Proposition 4.2.

LEMMA 4.4. *Let $B^{\#} \in \mathrm{OPS}^{\nu}_{1,0}(G)$, and let B be convolution (on the left) by*

$$\kappa = B^{\#}\delta_e. \tag{4.14}$$

Then $B \in \mathrm{OPS}^{\nu}_{1,0}(G)$ and the symbols $b_0^{\#}(g, \lambda)$ and $b_0(g, \lambda)$ are related by

$$b_0(e, \lambda) = b_0^{\#}(e, \lambda) \qquad \mathrm{mod}\ S^{\nu-1}_{1,0}. \tag{4.15}$$

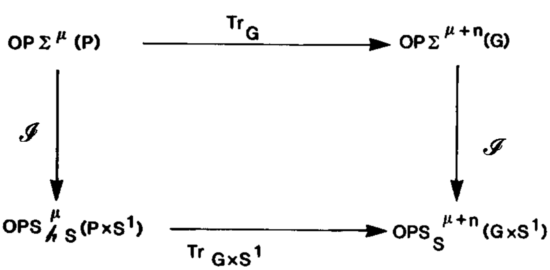
If $B^{\#} \in \mathrm{OPS}^{\nu}(G)$, then $B \in \mathrm{OPS}^{\nu}(G)$.

A proof of Lemma 4.4 can be found in [32, Chap. I] or in [28].

Though we will not make direct use of it, we note the following simple consequence of the reasoning given above, namely we have

$$\mathrm{Tr}_G\colon \mathrm{OP}\,\Sigma^{\mu}(P) \to \mathrm{OP}\,\Sigma^{\mu+n}(G), \tag{4.16}$$

and in fact the following diagram is commutative:



5. THE WGS BUNDLE

Here we discuss the space which is the natural phase space for the classical version of motion in a gauge field. This space and the associated flow were defined by Wong [34] for the special case of an $SU(2)$-bundle over flat Euclidean space, and by Guillemin and Sternberg [11, 27] in general (see also Weinstein [47]). A detailed discussion of the relation between these studies is given by Montgomery [43]. From our point of view, this space arises as a natural space on which to define symbols for G-invariant pseudodifferential operators on a principal G-bundle $P \to M$.

As before, let $P \to M$ be a principal G-bundle over a compact Riemannian manifold M, with a connection, associated to a G-invariant metric on P. The G-action will be written

$$R_g p = p \cdot g. \tag{5.1}$$

Let $A \in \mathrm{OPS}^\mu(P)$ commute with the G-action. Then the principal symbol $A_0(p, \zeta)$, defined on T^*P, satisfies

$$A_0(R_g p, \zeta) = A_0(p, DR_g(p)' \zeta). \tag{5.2}$$

Thus A_0 can be specified by a "symbol" defined on the quotient space of T^*P under the equivalence relation

$$(R_g p, \zeta) \sim (p, DR_g(p)' \zeta). \tag{5.3}$$

This quotient space is a vector bundle over T^*M in a natural fashion, which we proceed to describe.

In fact, the G-invariant metric on P (corresponding to the connection) produces isomorphisms

$$\begin{aligned} T_p^*P &\approx \mathfrak{h}_p^*P \oplus \mathfrak{g}^* \\ &\approx T_x^*M \oplus \mathfrak{g}^*, \end{aligned} \tag{5.4}$$

where x is the image point of p in M. The G-action on T^*P can then be described as

$$(p, \zeta) \cdot g = (R_g p, DR_g(p \cdot g)^{-1\,\prime} \zeta), \tag{5.5}$$

or as

$$(p, \xi + \lambda) \cdot g = (p \cdot g, \xi + (\mathrm{Ad}^* g)^{-1} \lambda) \tag{5.6}$$

with

$$\xi \in T_x^*M, \qquad \lambda \in \mathfrak{g}^*. \tag{5.7}$$

By (5.4), we see that

$$\mathfrak{h}^*P \to T^*M \tag{5.8}$$

is a principal G-bundle, namely the lift of $P \to M$ to T^*M, and in view of (5.3), the equivalence relation on $T^*P \approx \mathfrak{h}^*P \oplus \mathfrak{g}^*$ produces the vector bundle

$$\mathfrak{g}^{\#}_{\mathrm{ad}} = \mathfrak{h}^*P \times_{\mathrm{Ad}^*} \mathfrak{g}^* \tag{5.9}$$

over T^*M, associated with the principal bundle (5.8) via the coadjoint representation of G on $\mathfrak{g}^*$, The bundle (5.9) is the WGS-bundle, and, by (5.2), we have

$$A_0 = A_0^{\#} \circ q, \tag{5.10}$$

where

$$q\colon T^*P \to \mathfrak{g}^{\#}_{\mathrm{ad}} \tag{5.11}$$

is the projection defined by the equivalence relation (5.3), and

$$A_0^{\#} \in S^{\mu}(\mathfrak{g}^{\#}_{\mathrm{ad}}), \tag{5.12}$$

where to define the space of symbols on $\mathfrak{g}^{\#}_{\mathrm{ad}}$, we regard $\mathfrak{g}^{\#}_{\mathrm{ad}}$ as a vector bundle over M, and use the vector space structures on the fibers over points of M to define a dilation on $\mathfrak{g}^{\#}_{\mathrm{ad}}$, giving meaning to the notion of a function on the space $\mathfrak{g}^{\#}_{\mathrm{ad}}$ being homogeneous of a given order.

A symbol defined on $\mathfrak{g}^{\#}_{\mathrm{ad}}$ gives rise to a natural flow on this space. We produce this flow in the following fashion. Let $A \in \mathrm{OPS}^{\mu}(P)$ be G-invariant, and let $M \in \mathrm{OPS}^1(P)$ be G-invariant and self adjoint. Then, by Egorov's theorem, for each $t \in R$,

$$A(t) = e^{itM} A e^{-itM} \in \mathrm{OPS}^{\mu}(P), \tag{5.13}$$

and $A(t)$ is G-invariant. Thus the principal symbol $A_0(t)$ of $A(t)$ is of the form

$$A_0(t) = A_0^{\#}(t) \circ q, \tag{5.14}$$

with q as in (5.11) and $A_0^{\#} \in S^{\mu}(\mathfrak{g}^{\#}_{\mathrm{ad}})$. The flow on T^*P generated by the Hamiltonian vector field H_{M_1} (M_1 being the principal symbol of M) is a flow which preserves the class of symbols of G-invariant operators, and it induces a flow on the WGS-bundle $\mathfrak{g}^{\#}_{\mathrm{ad}}$, whose generator we denote $W_{M_1^{\#}}$ (where $M_1 = M_1^{\#} \circ q$). We call the corresponding flow the WGS-flow. The pairing

$$(A_0^{\#}, M_1^{\#}) \mapsto W_{M_1^{\#}} A_0^{\#} \tag{5.15}$$

defines a *Poisson structure* on $\mathfrak{g}^{\#}_{\mathrm{ad}}$; this bundle is foliated by symplectic manifolds, which are in fact bundles over T^*M with fibers which are diffeomorphic to coadjoint orbits for G. (For more details on this, see [11, 43].) The formula for principal symbols given by Egorov's theorem, when applied to these G-invariant operators on $C^\infty(P)$, can be stated as

$$A_0^\#(t) = A_0^\# \circ (\exp tW_{M_1^\#}). \tag{5.16}$$

Since the flow on $\mathfrak{g}_{\mathrm{ad}}^\#$ generated by $W_{M_1^\#}$ preserves homogeneity, with respect to the dilations of $\mathfrak{g}_{\mathrm{ad}}^\#$ described above, it induces a flow on the unit sphere bundle $S(\mathfrak{g}_{\mathrm{ad}}^\#)$ (regarding $\mathfrak{g}_{\mathrm{ad}}^\#$ as a vector bundle over M). We denote the generator of this last flow by S_{M_1}.

6. Functional Calculus and Trace Asymptotics in the Gauge Field Case

We turn now to an analysis of the family of operators $H_\hbar$ associated to a gauge field as in (0.10), i.e.,

$$H_\hbar = \hbar^2 H_\lambda^0 + V, \tag{6.1}$$

where

$$\lambda = n\lambda_1, \qquad \hbar = |\lambda + \delta|^{-1}. \tag{6.2}$$

As indicated in the Introduction, our approach is via analysis on the principal bundle P. As in [4, 39], we make use of the identity

$$\Delta = -H_\lambda^0 + \Delta_G^P \qquad \text{on } C^\infty(M, E_\lambda), \tag{6.3}$$

where $C^\infty(M, E_\lambda)$ is identified with a linear subspace of $C^\infty(P, V_\lambda)$, Δ is the Laplace operator on P, and Δ_G^P the operator derived from the Laplace operator Δ_G on G by the G-action on P. The representation theory of G implies that, if $\mathscr{D}_\lambda$ is the subspace of $C^\infty(P)$ on which G acts like copies of π_λ, then in fact $\mathscr{D}_\lambda$ is isomorphic to a direct sum of d_λ copies of $C^\infty(M, E_\lambda)$ (cf. [39, Lemmas 5.3 and 5.4]. Furthermore, if we set

$$L = \Delta + V_1 \Delta_G^P - |\delta|^2 V \tag{6.4}$$

with $V_1(x) = V(x) - 1$, and

$$A = -\Delta_G^P + |\delta|^2, \tag{6.5}$$

since $\pi(\Delta_G) = -(|\lambda + \delta|^2 - |\delta|^2)$, it follows that

$$-A^{-1} L|_{\mathscr{D}_\lambda} \approx \text{sum of } d_\lambda \text{ copies of } H_\hbar. \tag{6.6}$$

We suppose $V(x) > 1$, so $V_1 > 0$. Thus L is a negative semidefinite elliptic differential operator on P. We have the following result.

PROPOSITION 6.1. *For* $\sigma \in \mathscr{S}(R)$,

$$\sigma(-A^{-1}L) \in \mathrm{OPS}^0_{\mathfrak{h}}(P). \tag{6.7}$$

Proof. We have

$$\sigma(-A^{-1}L) = \rho_1(\sqrt{-L}, \sqrt{-L+\tfrac{1}{2}A}), \tag{6.8}$$

where

$$\rho_1(\mu, \nu) = \sigma(\mu^2/2(\nu^2 - \mu^2)). \tag{6.9}$$

The argument of σ is singular at $\mu = \pm\nu$, but if $\sigma \in \mathscr{S}(R)$, then this singularity is removable; $\rho_1 \in C^\infty(R^2 \setminus (0, 0))$. Clearly ρ_1 is homogeneous of degree 0 in (μ, ν). Since $-L$ and $-L + \frac{1}{2}A$ are both elliptic, and commute, the functional calculus described in [30, Chap. 12], or that described in [29], applies to show $\sigma(-A^{-1}L) \in \mathrm{OPS}^0(P)$. Furthermore,

$$\sigma(-A^{-1}L) = (\tfrac{1}{2}A)^j \gamma_j(\sqrt{-L}, \sqrt{-L+\tfrac{1}{2}A}) \tag{6.10}$$

with

$$\gamma_j(\mu, \nu) = (\nu^2 - \mu^2)^{-j} \sigma(\mu^2/2(\nu^2 - \mu^2)), \tag{6.11}$$

smooth on $R^2 \setminus (0, 0)$ and homogeneous of degree $-2j$ in (μ, ν), so $\gamma_j(\sqrt{-L}, \sqrt{-L+\frac{1}{2}A})$ belongs to $\mathrm{OPS}^{-2j}(P)$. This implies that the complete symbol of $\sigma(-A^{-1}L)$ vanishes to infinite order on the bundle $\mathfrak{h}^*P$, defined by (4.7), so the proposition is proved.

From Proposition 4.2 it follows that

$$\mathrm{Tr}_G\, \sigma(-A^{-1}L) = B \in \mathrm{OPS}^n(G), \tag{6.12}$$

where $n = \dim M$. The operator B is bi-invariant, so it is a scalar on each linear span

$$\mathscr{V}_\lambda = \mathrm{span}\{\pi_\lambda^{ij} : 1 \leqslant i, j \leqslant d_\lambda\}. \tag{6.13}$$

By (6.6), we have

$$d_\lambda^{-1}\, \mathrm{tr}\, \sigma(H_\hbar) = B|_{\mathscr{V}_\lambda}. \tag{6.14}$$

The analytical tool which provides an asymptotic analysis of (6.14) is given by the following result, which was proved in [30, Chap. 12] and used in

[24]. It is a microlocalization of a theorem of Zelobenko [36]. For its statement, we recall that the set of highest weights λ for irreducible representations of G is characterized as being the intersection of a Weyl chamber and a lattice in $\mathfrak{t}^*$, the dual space of the Lie algebra $\mathfrak{t}$ of a maximal torus in G. Using the bi-invariant metric on G, we can identify $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$.

PROPOSITION 6.2. *If $B \in \mathrm{OPS}^m(G)$ is bi-invariant, then*

$$B\pi_\lambda^{ij} = \beta(\lambda + \delta)\, \pi_\lambda^{ij} \tag{6.15}$$

with

$$\beta(\lambda) \in S^m(\mathfrak{t}^*), \tag{6.16}$$

invariant under the Weyl group. The principal symbol $B_m(g, \lambda)$ of B and the principal term $\beta_m(\lambda)$ in the expansion of $\beta(\lambda)$ are related by the identity

$$B_m(e, \lambda) = \beta_m(\lambda), \qquad \lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*, \tag{6.17}$$

which uniquely determines the correspondence between $B_m(g, \lambda)$ and $\beta_m(\lambda)$.

This allows us to obtain the main result of this section.

PROPOSITION 6.3. *For $\sigma \in \mathcal{S}(R)$, we have, as $\hbar \to 0$,*

$$d_\lambda^{-1} \operatorname{tr} \sigma(H_\hbar) \sim \hbar^{-n}(a_0 + a_1\hbar + a_2\hbar^2 + \cdots), \tag{6.18}$$

with

$$a_0 = \int_{T^*M} \sigma(|\xi|_x^2 + V(x))\, d\xi\, dx. \tag{6.19}$$

Proof. By (6.14)–(6.15), we have

$$d_\lambda^{-1} \operatorname{tr} \sigma(H_\hbar) = \beta(\lambda + \delta) \tag{6.20}$$

with

$$\beta(\lambda) \sim \beta_0(\lambda) + \beta_1(\lambda) + \cdots, \tag{6.21}$$

where $\beta_j(\lambda)$ is homogeneous of degree $n - j$ in λ. This proves (6.18). The leading term $\beta_0(\lambda)$ is equal to the principal symbol of $\mathrm{Tr}_G\, \sigma(-A^{-1}L)$ at (e, λ). By Proposition 4.2, this is given by

$$B_0(e, \lambda) = \int_{\mathfrak{h}^* P} \sigma(|\lambda|^{-2}\, L_2(p, \xi + \lambda))\, dV(\xi, p), \tag{6.22}$$

where L_2 is the principal symbol of $-L$. Since

$$\begin{aligned} L_2(p, \xi+\lambda) &= |\xi|_x^2 + |\lambda|^2 + (V(x)-1)\,|\lambda|^2 \\ &= |\xi|_x^2 + V(x)\,|\lambda|^2, \end{aligned} \tag{6.23}$$

we have

$$B_0(e, \lambda) = |\lambda|^n \int_{T^*M} \sigma(|\xi|_x^2 + V(x))\, d\xi\, dx, \tag{6.24}$$

which proves (6.19).

It is also of interest to consider the more general class of Schrödinger operators

$$H_\hbar = \hbar^2 H_\lambda^0 + i\hbar\pi_\lambda(X) + V, \tag{6.25}$$

where X is a section of the vector bundle $\mathfrak{g}_{\text{ad}}$ over M, defined by

$$\mathfrak{g}_{\text{ad}} = P \times_{\text{Ad}} \mathfrak{g}. \tag{6.26}$$

In analogy with (6.6), we have a vector field Y on P, tangent to the fibers of $P \to M$, such that

$$Y|_{\mathscr{D}_\lambda} \approx \text{sum of } d_\lambda \text{ copies of } \pi_\lambda(X). \tag{6.27}$$

The flow generated by Y commutes with the G-action on P. Thus, we might attack the analysis of $\sigma(H_\hbar)$ by using

$$A^{-1}(-L + A^{1/2} Y)|_{\mathscr{D}_\lambda} \approx \text{sum of } d_\lambda \text{ copies of } H_\hbar \tag{6.28}$$

in this case. A technical problem arises because $A^{1/2}$ is not a pseudodifferential operator on P; its "symbol" is singular on $\mathfrak{h}^*P$. We "resolve" this problem of a singular symbol by adding one more variable.

Thus we work on $P \times S^1$, with $\partial_\theta = \partial/\partial\theta$ on $C^\infty(S^1)$, $D_\theta = (1/i)\,\partial_\theta$. Also let α be a parameter, taking values in a small neighborhood of $|\lambda_1|$. We will make a partial replacement of $A^{1/2}$ by αD_θ. We set

$$\mathscr{L} = \Delta + (\tilde{V} - 1)\,\Delta_G^P - |\delta|^2\,\tilde{V} + \alpha\partial_\theta Y + K\alpha^2\partial_\theta^2, \tag{6.29}$$

where

$$\tilde{V} = V - K. \tag{6.30}$$

K is a positive constant, and we assume $V > K + 1$, so $\tilde{V} > 1$; in fact, we will suppose $\tilde{V}$ is bounded below by a sufficiently large constant; as noted before, this can be done without loss of generality, since one can easily adjust (6.25) by adding a suitable constant. Then we replace the left side of (6.28) by

$$-A^{-1/2}(\alpha D_\theta)^{-1}\,\mathscr{L}. \tag{6.31}$$

If we set

$$\mathscr{D}_{\lambda,k}=\{u\in C^\infty(P\times S^1)\colon G \text{ acts like } \pi_\lambda,\ D_\theta=k\}, \tag{6.32}$$

then

$$-A^{-1/2}(\alpha D_\theta)^{-1}\,\mathscr{L}\,|_{\mathscr{D}_{\lambda,k}}\approx d_\lambda \text{ copies of } H_\hbar \tag{6.33}$$

provided

$$\hbar=|\lambda+\delta|^{-1}=(\alpha k)^{-1}. \tag{6.34}$$

It is clear that the operator $\mathscr{L}$ is strongly elliptic provided $\tilde{V}>1$ and K is taken to be sufficiently large. Furthermore, if $\tilde{V}$ is greater than a sufficiently large constant, we can guarantee that $-\mathscr{L}$ is positive definite.

The operators $\mathscr{L}$, A, and D_θ all commute, and we have

$$\sigma(-A^{-1/2}(\alpha D_\theta)^{-1}\mathscr{L})=\gamma(\sqrt{-\mathscr{L}},\sqrt{-\mathscr{L}+A},\alpha D_\theta) \tag{6.35}$$

with

$$\gamma(\mu,\nu,\eta)=\sigma(\mu^2/|\nu^2-\mu^2|^{1/2}\,\eta). \tag{6.36}$$

Given $\sigma\in\mathscr{S}(R)$, γ is C^∞ on R^3 except for the union ℓ of the lines $(0,0,\eta)$ and $(0,\nu,0)$ $\eta,\nu\in R$, on which γ is generally singular. Since we are interested in functions on $P\times S^1$ on which $A^{1/2}\sim\alpha D_\theta$, we can make the following construction. The closed conic subset $\mathscr{C}$ of R^3 defined by

$$\eta=|\nu^2-\mu^2|^{1/2}$$

intersects the set ℓ only at the origin $(0,0,0)$; thus we can pick a function $\mathfrak{d}\in C^\infty(R^3\backslash(0,0,0))$, homogeneous of degree 0, equal to 1 on a neighborhood of $\mathscr{C}$, and equal to 0 on a neighborhood of ℓ. If we set

$$\tilde{\gamma}(\mu,\nu,\eta)=\mathfrak{d}(\mu,\nu,\eta)\,\gamma(\mu,\nu,\eta), \tag{6.37}$$

then $\tilde{\gamma}\in C^\infty(R^3\backslash(0,0,0))$, homogeneous of degree 0, and it follows from (6.33) that

$$\tilde{\gamma}(\sqrt{-\mathscr{L}},\sqrt{-\mathscr{L}+A},\alpha D_\theta)\,|_{\mathscr{D}_{\lambda,k}}\approx d_\lambda \text{ copies of } \sigma(H_\hbar) \tag{6.38}$$

provided (6.34) holds. Now we have

$$\tilde{\gamma}\,(\sqrt{-\mathscr{L}},\sqrt{-\mathscr{L}+A},\alpha D_\theta)\in \mathrm{OPS}^0(P\times S^1). \tag{6.39}$$

Also it is easily verified that its complete symbol vanishes to infinite order on the orthogonal complement to the fibers of $P \times S^1 \to M$. Thus we can apply $\mathrm{Tr}_{G \times S^1}$ to (6.39), obtaining a bi-invariant operator in $\mathrm{OPS}^n(G \times S^1)$. Here α enters as a parameter, and all quantities are smooth in α. Taking $\alpha = |\lambda + \delta|/k$ with $\lambda = k\lambda_1$ so that (6.34) holds, we have the following extension of Proposition 6.3.

PROPOSITION 6.4. *If $H_\hbar$ is given by* (6.25), $\sigma \in \mathcal{S}(R)$, *then, as* $\hbar \to 0$, d_λ^{-1} tr $\sigma(H_\hbar)$ *has an expansion of the form* (6.18), *with*

$$a_0 = \int_{\mathfrak{h}^* P} \sigma(|\xi|_x^2 + \langle X(p), \hat{\lambda} \rangle + V(x))\, dV(\xi, p), \tag{6.40}$$

where $X(p)$ is the section X in (6.25)–(6.26), *regarded as a function on P with values in* $\mathfrak{g}$, *and* $\hat{\lambda} = \lambda_1/|\lambda_1|$.

7. Quantum Chern Forms

Let Y be a parameter space for a family of Hamiltonians $H_\hbar = H_\hbar(y)$. If $H_\hbar$ is of the form $-\hbar^2 \Delta + V(x)$, then the Riemannian metric on M, hence the Laplace operator Δ, and also the potential function $V(x)$, can depend on $y \in Y$. In the case $H_\hbar = H_\lambda$ arises from a gauge field, the gauge field can also depend on $y \in Y$. We might write $H_\hbar(y) = -\hbar^2 \Delta(y) + V(x, y)$, for example, in the case of scalar fields, but we will usually suppress the y-dependence in our notation. We will associate a family of differential forms on Y, depending on the parameter $\hbar$, and consider the semiclassical limit $\hbar \to 0$.

We start with a description of curvature and some characteristic classes for a certain class of vector bundles on a smooth manifold Y. Let F be a Hilbert space, and let $P(y)$ be a smooth family of orthogonal projections on F, with range E_y. This gives rise to a smooth vector bundle $E \to Y$, with fibers E_y. E is a subbundle of the trivial bundle $Y \times F$. A section of E is a special case of a function on Y with values in F. If X is a vector field on Y, u a smooth F-valued function on Y, we denote by $D_X u$ the componentwise X-derivative of u. Then a connection on E is defined by

$$\nabla_X u(y) = P(y)\, D_X u(y), \qquad u \text{ a smooth section of } E. \tag{7.1}$$

The curvature of this connection has the intrinsic definition

$$\Omega(X_1, X_2) u = [\nabla_{X_1}, \nabla_{X_2}] u - \nabla_{[X_1, X_2]} u, \tag{7.2}$$

where X_j are smooth vector fields, u a section of E. Thus, Ω is an

End(E)-valued 2-form on Y. Gauss' formula for curvature (in Cartan's notation) reads

$$\Omega = P\, dP \wedge dP\, P. \tag{7.3}$$

A set of characteristic classes is defined as follows. Suppose the fiber dimension of E is K; say E_y is a complex vector space of dimension K. Let $p(A)$ be a polynomial defined on the space of $K \times K$ complex matrices, homogeneous of degree ℓ, which is invariant under conjugation

$$p(BAB^{-1}) = p(A)$$

for any invertible matrix B. Then there is naturally defined

$$p(\Omega), \qquad \text{a } 2\ell\text{-form on } Y. \tag{7.4}$$

The form $p(\Omega)$ is closed. It will be worthwhile to recall a proof of this. The End(F)-valued 1-form

$$\tau = dP\, P = (I - P)\, dP \tag{7.5}$$

is related to the curvature form by the identities

$$\Omega \wedge \tau = 0 \tag{7.6}$$

and

$$d\Omega = \tau \wedge \Omega, \tag{7.7}$$

hence

$$d\Omega = \tau \wedge \Omega - \Omega \wedge \tau. \tag{7.8}$$

Assuming p is homogeneous of degree ℓ, we denote by q the ℓ-linear function polarizing p, and the conjugation invariance of p when differentiated implies

$$\sum q(A, ..., [A, B], ..., A) = 0 \tag{7.9}$$

and hence

$$\sum q(\Omega, ..., \Omega \wedge \tau - \tau \wedge \Omega, ..., \Omega) = 0. \tag{7.10}$$

By (7.8), this gives

$$\sum q(\Omega, ..., d\Omega, ..., \Omega) = 0, \tag{7.11}$$

and hence

$$dp(\Omega) = 0. \tag{7.12}$$

Hence $p(\Omega)$ defines a deRham cohomology class on Y. Furthermore, it can be shown that such a cohomology class is independent of the choice of connection on E (see [21]). The particular examples which we will take are

$$p_\ell(A) = \text{trace } A^\ell. \tag{7.13}$$

The associated classes are $[p_\ell(\Omega)] \in H^{2\ell}(Y, \mathbf{C})$. The case $\ell = 1$ corresponds to $2\pi i$ times the first Chern class. Other characteristic classes are obtained as polynomials in the classes $[p_\ell(\Omega)]$.

A particular case which gives rise to a *line bundle* is the following: Suppose $H(y)$ is a smooth family of self adjoint operators and suppose $e_0(y)$ is a smooth function on Y such that $e_0(y)$ is an eigenvalue of $H(y)$, with one dimensional eigenspace E_y. For the resulting vector bundle with connection, the first Chern class provides an analysis of Berry's phase, as discussed in [26].

We now proceed to set up a certain generalization of the notion of vector subbundles of F, of a connection, and of curvature. Let $P(y)$ be a smooth family of self adjoint operators on F, not necessarily projections. We will suppose each $P(y)$ has finite rank, though this hypothesis could be generalized; the hypothesis that each $P(y)$ is Hilbert–Schmidt would be adequate. We continue to associate a "smoothed out curvature form" Ω to P, by the formula (7.3), and "smoothed out characteristic forms"

$$p_\ell(\Omega) \in \Lambda^\ell(Y), \tag{7.14}$$

with p_ℓ given by (7.13), i.e.,

$$p_\ell(\Omega) = \text{trace } \Omega \wedge \cdots \wedge \Omega \qquad (\ell \text{ factors}), \tag{7.15}$$

where the product $\Omega \wedge \cdots \wedge \Omega$ is an End(F)-valued 2ℓ-form, and the trace is applied to the End(F)-coefficients.

We apply this construction to the following situation. Let

$$P(\hbar, y) = \sigma(H_\hbar(y)), \tag{7.16}$$

where $\sigma \in C_0^\infty(R)$ and $H_\hbar(y)$ is a family of quantum Hamiltonians. Thus we consider the family of End(F)-valued 2-forms

$$\Omega_\hbar = \sigma(H_\hbar(y))\, d_y\sigma(H_\hbar(y)) \wedge d_y\sigma(H_\hbar(y))\, \sigma(H_\hbar(y)), \tag{7.17}$$

where $F = L^2(M)$ (or $L^2(M, E_\lambda)$), and we consider the family of $2k$-forms on Y:

$$F_\hbar^k = \text{trace}[\Omega_\hbar \wedge \cdots \wedge \Omega_\hbar] \qquad (k \text{ factors}). \tag{7.18}$$

The following describes the behavior of $F^k_\hbar$ as $\hbar \to 0$. We concentrate on the gauge field case, though of course the case of scalar fields is included, with $d_\lambda = 1$. The form (7.20) in the scalar case was first observed in [2].

PROPOSITION 7.1. *As* $\hbar \to 0$, $\hbar = |\lambda + \delta|^{-1}$, *we have*

$$d_\lambda^{-1} F^k_\hbar \sim \hbar^{-m+k}(\mathfrak{h}_{k0} + \mathfrak{h}_{k1}\hbar + \mathfrak{h}_{k2}\hbar^2 + \cdots), \tag{7.19}$$

where $\mathfrak{h}_{kj}$ *are* $2k$*-forms on* Y. *In particular,*

$$\mathfrak{h}_{k0} = i^k \int_{T^*M} (\Phi^2 \{d_y\Phi, d_y\Phi\})^k \, d\xi \, dx, \tag{7.20}$$

where, if $H_\hbar$ *is given by* (6.1), *with* $y \in Y$ *as a parameter,*

$$\Phi(x, \xi, y) = \sigma(|\xi|^2_{x,y} + V(x, y)). \tag{7.21}$$

In local corrdinates on Y,

$$\{d_y\Phi, d_y\Phi\} = \sum_{j,k} \{\partial\Phi/\partial y_j, \partial\Phi/\partial y_k\} \, dy_j \wedge dy_k,$$

$\{\ ,\ \}$ *denoting the usual bracket of functions on* T^*M.

The proof of Proposition 7.1 is very much like that given in Section 6. Recall the operators on the principal bundle $P \to M$,

$$L = \Delta + V_1(x)\,\Delta^P_G - |\delta|^2\, V(x), \qquad A = -\Delta^P_G + |\delta|^2.$$

The operator L depends on $y \in Y$ as a parameter. Then, if $\mathscr{D}_\lambda$ is the subspace of $C^\infty(P)$ on which G acts as copies of π_λ, we have the restriction to $\mathscr{D}_\lambda$ of

$$\Gamma = \sigma(-A^{-1}L)\, d_y\sigma(-A^{-1}L) \wedge d_y\sigma(-A^{-1}L)\, \sigma(-A^{-1}L) \tag{7.22}$$

acting as an orthogonal sum of d_λ copies of $\Omega_\hbar$, given by (7.17). Here, Γ is a 2-form on Y, with values in $\mathrm{OPS}^{-1}_{\mathfrak{h}}(P)$. Note that

$$\Gamma = \sum_{j,k} \sigma(-A^{-1}L)[\partial_{y_j}\sigma(-A^{-1}L), \partial_{y_k}\sigma(-A^{-1}L)]\, \sigma(-A^{-1}L)\, dy_j \wedge dy_k.$$

Moreover, $\Gamma^k|_{\mathscr{D}_\lambda}$ is an orthogonal sum of d_λ copies of $\Omega^k_\hbar$. Therefore, with $F^k_\hbar$ given by (7.10),

$$F^k_\hbar = d_\lambda \rho_k(\lambda + \delta), \tag{7.23}$$

where

$$(\mathrm{Tr}_G\, \Gamma^k)\, \pi_\lambda^{ij} = \rho_k(\lambda + \delta)\, \pi_\lambda^{ij}. \tag{7.24}$$

Here, $\mathrm{Tr}_G\, \Gamma^k$ is a $2k$-form on Y with values in the space of bi-invariant operators in $\mathrm{OPS}^{m-k}(G)$, $m = \dim M$, and hence $\rho_k(\lambda)$ is a $2k$-form on Y with values in the symbol space $S^{m-k}(\mathfrak{t})$. There is an asymptotic expansion

$$\rho_k(\lambda) \sim \rho_{k0}(\lambda) + \rho_{k1}(\lambda) + \cdots \tag{7.25}$$

with $\rho_{kj}(\lambda)$ homogeneous in λ of degree $m-k-j$, and hence (7.19) holds, with

$$\mathfrak{h}_{k0} = \rho_{k0}(\lambda/|\lambda|). \tag{7.26}$$

Since $\rho_{k0}(\lambda)$ is obtained from the principal symbol of Γ^k, which is

$$i^k[\varphi^2\{d_y\varphi, d_y\varphi\}]^k, \tag{7.27}$$

where

$$\varphi(x, \xi, \lambda, y) = \sigma(|\lambda|^{-2}\,(|\xi|^2_{x,y} + V(x, y)\,|\lambda|^2)), \tag{7.28}$$

by the process described in Section 6, the formula (7.20) follows, and Proposition 7.1 is proved.

Having characterized this semiclassical limit, we note an important contrast with the characteristic classes described in the beginning of this section. Namely, the forms $F_\hbar^k$ defined by (7.18) are not generally closed. In fact, the proof that the characteristic forms associated with a connection on a vector bundle are closed described in (7.5)–(7.12) uses the identity $P = PP$ in an essential way, an identity which does not generally hold for $\sigma(H_\hbar(y))$. Similarly, the forms $\mathfrak{h}_{k0}$ in (7.20) cannot be expected to be closed. In the case $k = 1$, we have

$$d_y\mathfrak{h}_{10} = 2i \int_{T^*M} \Phi\, d_y\Phi \wedge \{d_y\Phi, d_y\Phi\}\, d\xi\, dx. \tag{7.29}$$

Note that $d_y\{\Phi, d_y\Phi\} = \{d_y\Phi, d_y\Phi\}$, but there is no reason to conclude that (7.29) vanishes. Of course, if $\dim M = 2k$, then $F_\hbar^k$ is closed, and so are all the coefficients in the expansion 7.19.

We can also treat the more general case where $H_\hbar$ is given by (6.25), with $y \in Y$ as a parameter. In that case, using Proposition 6.4 in place of Proposition 6.3, we see that the expansion (7.19) continues to hold, with

$$\mathfrak{h}_{k0} = i^k \int_{\mathfrak{h}^*P} (\Phi^2\{d_y\Phi, d_y\Phi\})^k\, dV(\xi, p), \tag{7.30}$$

where

$$\Phi(p, \xi, \lambda, y) = \sigma(|\xi|^2_{x,y} + \langle X(p, y), \hat{\lambda}\rangle + V(x, y)). \tag{7.31}$$

As before, $\hat{\lambda} = \lambda_1/|\lambda_1|$. In (7.30) the Poisson bracket on $\mathfrak{h}^*P$ is used; the formula could also be written in terms of symbols defined on the WGS bundle.

As remarked to us by A. Connes, one could replace the four σ's in (7.17) by four different functions in $\mathcal{S}(R)$, $\sigma_1, ..., \sigma_4$, and obtain a similar asymptotic expansion in this more general case, by the same arguments. More generally, we can replace the $4k$ σ's in (7.18) by $\sigma_1, ..., \sigma_{4k}$; the resulting object defines a current.

8. Ergodic Classical Motion and Quantum Chaos: Scalar Fields

In this section, we prove the following result. Let M be a compact Riemannian manifold, with Laplace operator Δ. Let $e_j(\hbar)$, $N(\hbar, I)$, $A(\hbar, I)$, $\mu_j^\hbar$, $m_I^\hbar$, etc., be as in Section 3, with $L = \Delta$. As before, we take $V \in C^\infty(M)$, $V > 0$. Given a compact interval I, we set

$$\mathcal{O}(I) = \{(x, \xi) : p_V(x, \xi) \in I\}, \tag{8.1}$$

where

$$p_V(x, \xi) = |\xi|^2_x + V(x). \tag{8.2}$$

We will assume that

$$p_V(x, \xi) \text{ has no critical points in } \mathcal{O}(I). \tag{8.3}$$

Theorem 8.1. *Let $a(\hbar, x, \xi) \in R^0(M)$ be given, of the form $a(\hbar, x, \xi) = a(x, \hbar\xi)$. Assume that there is an interval $\tilde{I} \supset I$ such that, for almost all $(x, \xi) \in \mathcal{O}(\tilde{I})$,*

$$\lim_{T\to\infty} T^{-1} \int_0^T a((\exp tX)(x, \xi))\, dt = \Psi_a(p_V(x, \xi)) = \bar{a}(x, \xi), \tag{8.4}$$

where $\Psi_a(p)$ is the average value of $a(x, \xi)$ on the surface $p_V(x, \xi) = p$, with *respect to the natural Liouville measure, and X is the Hamiltonian vector field*

$$X = H_{\sqrt{p_V}}. \tag{8.5}$$

Then we have (for any $\varepsilon > 0$*)*

$$\lim_{\hbar \to 0} N(\hbar, I)^{-1} \operatorname{card} \left\{ j \in \Lambda(\hbar, I) : \left| \int a \, d\mu_j^\hbar - \Psi_a(e_j(\hbar)) \right| < \varepsilon \right\} = 1. \tag{8.6}$$

The property (8.4) holds for any $a(x, \hbar\xi) \in R^0(M)$ if the flow generated by X is ergodic on $\{p_V(x, \xi) = p\}$ for almost all $p \in \tilde{I}$. On the other hand, even if this ergodicity fails, (8.4) may hold for certain interesting $a(x, \xi)$, if not for all such functions, and it seems worthwhile to phrase Theorem 8.1 in this general fashion.

Theorem 8.1 is similar to the main result of [14], but is differs in that we consider a fixed interval of energies rather than a family of intervals shrinking to a single point. Our proof of Theorem 8.1 follows along the lines of [10, 14]; we give the proof principally as a warm-up for the arguments of the following section.

Adapting notation from [14], we set

$$a_j^\hbar = \int a \, d\mu_j^\hbar - \Psi_a(e_j(\hbar)), \tag{8.7}$$

$$b_{j,T}^\hbar = \int \left(a - T^{-1} \int_0^T (a \circ \exp tX) \, dt \right) d\mu_j^\hbar, \tag{8.8}$$

and

$$c_{j,T}^\hbar = \int \left| T^{-1} \int_0^T (a \cdot \exp tX) \, dt - \Psi_a(p_V(x, \xi)) \right| d\mu_j^\hbar$$

$$= \int |E_T(x, \xi)| \, d\mu_j^\hbar. \tag{8.9}$$

We also get

$$d_j^\hbar = \int \Psi_a(p_V(x, \xi)) \, d\mu_j^\hbar - \Psi_a(e_j(\hbar)). \tag{8.10}$$

The aim of the proof of (8.6) is to show that $|a_j^\hbar| < \varepsilon$ for most $j \in \Lambda(\hbar, I)$, if $\hbar$ is small, and we approach that via the estimate

$$|a_j^\hbar| \leqslant |b_{j,T}^\hbar| + c_{j,T}^\hbar + |d_j^\hbar|. \tag{8.11}$$

To estimate $b^{\hbar}_{j,T}$, we write

$$b^{\hbar}_{j,T}=T^{-1}\int_0^T ([e^{it\hbar^{-1}\sqrt{H_\hbar}}A^F(\hbar)\,e^{-it\hbar^{-1}\sqrt{H_\hbar}}-A^F_t(\hbar)]\,\varphi^{\hbar}_j,\varphi^{\hbar}_j)\,dt, \quad (8.12)$$

where $A^F(\hbar)\in \mathrm{OPR}^0_{1,0}(M)$ is the family associated with $A(\hbar)=a(\hbar, x, D)\in \mathrm{OPR}^0(M)$ via the Friedrichs symmetrization process, discussed in Section 3, and $A^F_t(\hbar)\in \mathrm{OPR}^0_{1,0}(M)$ is similarly associated with $a_t(x, \hbar\xi)$, where $a_t(x,\xi)=a((\exp tX)(x,\xi))$. As before, $H_\hbar = -\hbar^2\Delta + V$. By Proposition 3.1, we have for any given T the estimate

$$|b^{\hbar}_{j,T}|\leqslant C_T\hbar, \quad (8.13)$$

and hence there exists $h_1(T,\xi)>0$ such that $|b^{\hbar}_{j,T}|<\varepsilon/3$ for all $j\in\Lambda(\hbar, I)$ provided $\hbar<h_1(T,\xi)$.

It is in the estimate of (8.9) that the hypothesis (8.4) comes in. We have $E_T(x,\xi)$ uniformly bounded on $\mathcal{O}(I)$, and as $T\to\infty$, $E_T(x,\xi)\to 0$ a.e., with respect to the smooth measure $d\,\mathrm{vol}(x,\xi)$. Thus, for any $\delta>0$, we can pick $T_0=T_0(\delta)$ such that

$$\int_{\mathcal{O}(I)} |E_{T_0}(x,\xi)|\; d\,\mathrm{vol}(x,\xi)<\delta/3. \quad (8.14)$$

Now the weak convergence result (7.48) implies that there is an $h_2(T_0,\delta)>0$ such that

$$\int_{\mathcal{O}(I)} |E_{T_0}(x,\xi)|\; dm^{\hbar}_I<\delta/2 \qquad \text{for } \hbar<h_2(T_0,\delta). \quad (8.15)$$

Since $m^{\hbar}_I$ is the mean of the measures $\mu^{\hbar}_j$ for $j\in\Lambda(\hbar, I)$, this says

$$N(\hbar, I)^{-1}\left\{\sum_{j\in\Lambda(\hbar,I)}\int_{\mathcal{O}(I)} |E_{T_0}(x,\xi)|\; d\mu^{\hbar}_j\right\}<\delta \quad (8.16)$$

for $\hbar<h_2(T_0,\delta)$, given T_0 large enough that (8.14) holds. Thus, by Tchebicheff's inequality, the proportion of $j\in\Lambda(\hbar, I)$ such that

$$|c^{\hbar}_{j,T_0}|<\varepsilon/3 \quad (8.17)$$

is greater than $1-\sqrt{\delta}$, provided $\delta<\varepsilon^2/9$ and $\hbar<h_2(T_0,\delta)$.

To complete the proof of Theorem 8.1, we estimate (8.10). Note that hypothesis (8.3) implies $\Psi_a(p_V(x,\xi))$ is C^∞ on $\mathcal{O}(I)$. In fact, it is C^∞ on

$\mathcal{O}(\tilde{I}) \supset \mathcal{O}(I)$ for a slightly larger interval $\tilde{I}$. Let us temporarily suppose $a(x, \xi)$ is actually supported in $\mathcal{O}(\tilde{I})$. In that case, we also have $\bar{a}(x, \xi) = \Psi_a(p_V(x, \xi))$ in $C_0^\infty(\mathcal{O}(\tilde{I}))$, i.e., $\Psi_a \in C_0^\infty(\tilde{I})$. In this case, recall that

$$\begin{aligned} \int \bar{a}(x, \xi)\, d\mu_j^\hbar &= (\bar{a}^{\mathrm{F}}(\hbar, x, D)\, \varphi_j^\hbar, \varphi_j^\hbar) \\ &= (\bar{a}(\hbar, x, D)\, \varphi_j^\hbar, \varphi_j^\hbar) + O(\hbar), \end{aligned} \tag{8.18}$$

where $\bar{a}(\hbar, x, \xi) = \bar{a}(x, \hbar\xi) = \Psi_a(\hbar^2\, |\xi|_x^2 + V(x))$, which belongs to $\mathrm{OP}\, \Sigma^0(M)$. Now the functional calculus gives

$$\Psi_a(H_\hbar) = \bar{a}(\hbar, x, D) \qquad \mathrm{mod}\ \mathrm{OP}\, \Sigma^{-1}(M). \tag{8.19}$$

Thus (8.18) yields

$$\int \bar{a}(x, \xi)\, d\mu_j^\hbar = (\Psi_a(H_\hbar)\, \varphi_j^\hbar, \varphi_j^\hbar) + O(\hbar) \tag{8.20}$$

and the inner product on the right is of course just $\Psi_a(e_j(\hbar))$.

Putting together our estimates of (8.8)–(8.10), we see that the desired conclusion (8.6) holds whenever $a(x, \xi)$ belongs to $C_0^\infty(\mathcal{O}(\tilde{I}))$. But the general case of $a(x, \hbar\xi) \in R^0(M)$ can be reduced to this, by the following argument. Fix $\rho \in C_0^\infty(\tilde{I})$ such that $\rho = 1$ on a neighborhood of I, and consider $\rho(H_\hbar) \in \mathrm{OP}\, \Sigma^0$. Now let

$$a_1(x, \xi) = a(x, \xi)\, \rho(p_V(x, \xi)) \in C_0^\infty(\mathcal{O}(\tilde{I})). \tag{8.21}$$

Set $a_1(\hbar, x, \xi) = a_1(x, \hbar\xi) \in \Sigma^0(M)$. Then (8.6) holds with a replaced by a_1, and

$$a_1(\hbar, x, D) = a(\hbar, x, D)\, \rho(H_\hbar) \qquad \mathrm{mod}\ \mathrm{OP}\, \Sigma^{-1}(M). \tag{8.22}$$

Thus

$$\begin{aligned} \int a\, d\mu_j^\hbar &= (a(\hbar, x, D)\, \varphi_j^\hbar, \varphi_j^\hbar) + O(\hbar) \\ &= (a(\hbar, x, D)\, \rho(H_\hbar)\, \varphi_j^\hbar, \varphi_j^\hbar) \qquad \text{if } j \in \Lambda(\hbar, I) \\ &= \int a_1\, d\mu_j^\hbar + O(\hbar) \end{aligned} \tag{8.23}$$

and

$$\Psi_a(e_j(\hbar)) = \Psi_{a_1}(e_j(\hbar)) \qquad \text{if } j \in \Lambda(\hbar, I), \tag{8.24}$$

so Theorem 8.1 is proved in general.

We make a few complementary remarks on Theorem 8.1. When $V=0$, the flow generated by X is the geodesic flow on T^*M, identified with TM via the Riemannian metric, and thus (8.4) holds for all $a \in R^0(M)$ if and only if the geodesic flow is ergodic on each nonzero constant energy surface in T^*M, i.e., on the unit sphere bundle. It is the content of the paper [10] that (8.6) holds in such a case. We remark that if $M = \mathbf{T}^m$ is a flat torus, such ergodicity obviously fails, but nevertheless the hypothesis (8.4) does hold for a certain class of $a(x, \xi)$, namely when

$$a(x, \xi) = a(x). \tag{8.25}$$

Indeed, in such a case, the limit on the left in (8.4) is equal to the mean value of $a(x)$ over $\mathbf{T}^m$ for all $\xi = (\xi_1, ..., \xi_m)$ for which the ξ_j are pairwise incommensurable, which is almost everywhere on $T^*\mathbf{T}^m$. Thus Theorem 8.1 applies to all multiplication operators by $a \in C^\infty(\mathbf{T}^m)$. This says that, except for a sequence of density 0, the eigenfunctions of Δ on $\mathbf{T}^m$ are equidistributed over $\mathbf{T}^m$, consistent with the obvious fact that

$$\left\{ e^{i\ell \cdot x} : \sum_1^m \ell_j^2 = \lambda \right\}$$

spans the eigenspace of $-\Delta$ with eigenvalue λ, and all these eigenfunctions have constant modulus. However, Theorem 8.1 applies to an arbitrary choice of basis for such eigenspaces, and, especially for large m, there may be many such choices, so the conclusion of Theorem 8.1 may not be completely obvious in such a situation. On the other hand, when $M = S^n$, the standard sphere, if $a(x, \xi)$ is of the form (8.25), it need not follow that the left side of (8.4) is equal to $\bar{a}(x, \xi)$ for almost every (x, ξ), and indeed there are a good many spherical harmonics concentrated near the equator.

In any case, it is worth remarking that there are often special classes of interesting symbols $a(x, \xi)$ for which (8.4) holds, even in the absence of ergodicity on constant energy surfaces, particularly since examples of classical Hamiltonian flows for motion other than geodesic flow on certain types of Riemannian manifolds, exhibiting such ergodic behavior, are hard to come by. Some recent examples which do exhibit such ergodicity are given in [18].

As further preparation for the result of Section 9, we produce a variant of Theorem 8.1. Observe that, with $\hbar = k^{-1}$, the eigenfunctions $\varphi_j^\hbar$ of $H_\hbar = -\hbar^2 \Delta + V$ with eigenvalue $e_j(\hbar)$ correspond to the functions

$$\tilde{\varphi}_{j,k} = \varphi_j^\hbar(x)\, e^{ik \cdot \theta} \in C^\infty(M \times S^1) \tag{8.26}$$

which are simultaneously eigenfunctions of $L = \Delta + V(x)\, \partial_\theta^2$ with eigenvalue

$-k^2 e_j(k^{-1})$ and eigenfunctions of D_θ with eigenvalue k. Note that L and D_θ commute and L is elliptic on $M \times S^1$. The condition that $e_j(\hbar)$ belong to the interval $I = [a, b]$ is equivalent to the condition that $(k[e_j(k^{-1})]^{1/2}, k)$ belong to the wedge in R^2,

$$W_I = \{(\lambda, \mu) \in R^2 \colon \lambda > 0, (\lambda/\mu)^2 \in I\}. \tag{8.27}$$

Thus our problem can be translated into a problem of considering the asymptotic behavior of the joint spectrum of the pair $(\sqrt{-L}, D_\theta)$ within the wedge W_I.

If $a(x, \xi)$ belongs to $C_0^\infty(\mathcal{O}(\tilde{I}))$ as above, we have $a(\hbar, x, \xi) \in \Sigma^0(M)$, and also $b_a(x, \theta, \xi, \lambda) = a(x, \xi/\lambda) \in S^0(M \times S^1)$; let us denote the associated operator in $\mathrm{OPS}^0(M \times S^1)$ by B_a; if B_a^{F} is obtained by Friedrichs symmetrization, then the measure $\mu_j^\hbar$, restricted to $\mathcal{O}(\tilde{I})$, satisfies

$$\int a \, d\mu_j^\hbar = (B_a^{\mathrm{F}} \tilde{\varphi}_{j,k}, \tilde{\varphi}_{j,k})_{L^2(M \times S^1)}, \tag{8.28}$$

if $\hbar = k^{-1}$, by the construction of (3.42)–(3.44). Note that the hypothesis (8.3) is equivalent to the following hypothesis on the mapping

$$Q \colon T^*(M \times S^1) \to R^2 \tag{8.29}$$

given by

$$Q(x, \theta, \xi, \lambda) = (\sqrt{|\xi|_x^2 + V(x)\,\lambda^2}, \lambda), \tag{8.30}$$

namely

$$Q \text{ has no critical points in } \mathcal{W}(I), \tag{8.31}$$

i.e., Q has surjective differential everywhere in $\mathcal{W}(I)$, where

$$\begin{aligned} \mathcal{W}(I) &= Q^{-1}(W_I) \\ &= \{(x, \theta, \xi, \lambda) \colon \lambda > 0, |\xi/\lambda|_x^2 + V(x) \in I\}. \end{aligned} \tag{8.32}$$

Meanwhile, hypothesis (8.4) is equivalent to the hypothesis for $b = b_a$,

$$\begin{aligned} \lim_{T \to \infty} T^{-1} \int_0^T b((\exp tY)(x, \theta, \xi, \lambda))\, dt &= \Phi_b(\sqrt{L_2(x, \xi, \lambda)}, \lambda) \\ &= \bar{b}(x, \theta, \xi, \lambda), \end{aligned} \tag{8.33}$$

for almost every $(x, \theta, \xi, \lambda)$ in $\mathcal{W}(I)$, where $L_2(x, \xi, \lambda) = |\xi|_x^2 + V(x)\,\lambda^2$ is the principal symbol of $-L$, $\Phi_b(p, q)$ is the mean value of $b(x, \theta, \xi, \lambda)$ over the (codimension 2) surface $\{\sqrt{L_2(x, \xi, \lambda)} = p,\ \lambda = q\}$, with its natural measure, and Y is the Hamiltonian vector field $Y = H_{\sqrt{L_2}}$.

The following is a restatement of Theorem 8.1. A generalization of the case of gauge fields will be proved in the next section.

PROPOSITION 8.2. *Given a, b_a as above, if hypotheses* (8.31) *and* (8.33) *are assumed to hold, then*

$$\lim_{\hbar \to 0} N(\hbar, I)^{-1} \operatorname{card} \left\{ j \in \Lambda(\hbar, I): \left| \int a\, d\mu_j^\hbar - \Phi_{b_a}(\sqrt{e_j(\hbar)}, 1) \right| < \varepsilon \right\} = 1. \quad (8.34)$$

Note that, by homogeneity,

$$\Phi_{b_a}(\sqrt{e_j(\hbar)}, 1) = \Phi_{b_a}(k\sqrt{e_j(\hbar)}, k).$$

9. QUANTUM CHAOS AND ERGODIC MOTION: GAUGE FIELDS

In this section our goal is to produce an analogue of Theorem 8.1 for gauge fields. As a first step, we will need the gauge field version of Propositions 3.2 and 3.3. Since we have gotten used to the notion of relating $\hbar$ and λ, we will alter our notation a bit, replacing $H_\hbar$ as in (2.12) by H_λ,

$$H_\lambda = \hbar^2 H_\lambda^0 + V \qquad \text{on } C^\infty(M, E_\lambda), \quad (9.1)$$

where λ belongs to the lattice parametrizing the irreducible representations of G, and, as before $\hbar = |\lambda + \delta|^{-1}$. Therefore, with L as in (6.4),

$$L = \Delta + V_1(x)\, \Delta_G^P - |\delta|^2\, V(x), \quad (9.2)$$

an elliptic operator on the principal bundle P, and with A as in (6.5),

$$A = -\Delta_G^P + |\delta|^2, \quad (9.3)$$

we have, as in (6.6),

$$-H_\lambda = A^{-1} L \quad (9.4)$$

on the subspace $\mathscr{D}_\lambda$ of $C^\infty(P)$ on which G acts as copies of π_λ, a space which is isomorphic to a direct sum of d_λ copies of $C^\infty(M, E_\lambda)$.

Our replacement for a family in $\mathrm{OPR}^0(M)$ which was used in the case of scalar fields is naturally obtained from an operator $B \in \mathrm{OPS}^0(P)$ which commutes with the G action. Clearly, if $\sigma \in \mathscr{S}(R)$, then

$$B\sigma(A^{-1}L) \in \mathrm{OPS}^0(P) \tag{9.5}$$

commutes with the G action. The restriction of B to the space $\mathscr{D}_\lambda \subset C^\infty(P)$ produces a family of operators

$$b(\lambda, x, D): C^\infty(M, E_\lambda) \to C^\infty(M, E_\lambda), \tag{9.6}$$

generalizing the notion of an element of OPR^0. Note that

$$\operatorname{tr} b(\lambda, x, D)\, \sigma(H_\lambda) = d_\lambda \beta^b(\lambda + \delta), \tag{9.7}$$

where

$$\mathrm{Tr}_G\, B\sigma(A^{-1}L)\, \pi_\lambda^{ij} = \beta^b(\lambda + \delta)\, \pi_\lambda^{ij}. \tag{9.8}$$

From (9.5) it follows that $\beta^b(\lambda) \in S^m(\mathfrak{t}^*)$, having an expansion

$$\beta^b(\lambda) \sim \beta_m^b(\lambda) + \beta_{m-1}^b(\lambda) + \cdots \tag{9.9}$$

in terms $\beta_{m-j}^b(\lambda)$ homogeneous of degree $m-j$ in λ, with

$$\beta_m^b(\lambda) = |\lambda|^m \int_{\mathfrak{h}^* P} B_0(p, \xi + \lambda/|\lambda|)\, \sigma(|\xi|_p^2 + V(x))\, d\, \mathrm{vol}(\xi, p). \tag{9.10}$$

This simultaneously generalizes Propositions 6.3 and 3.2. Recall from (5.2)–(5.6) that B_0, the principal symbol of B, satisfies the invariance property

$$B_0(p \cdot g, \xi + \lambda) = B_0(p, \xi + Ad^* g\lambda), \tag{9.11}$$

where we consider $\xi \in \mathfrak{h}_p^* P \approx T_x^* M$, $\lambda \in \mathfrak{g}^*$.

We denote the eigenvalues of H_λ on $C^\infty(M, E_\lambda)$ by

$$e_1(\lambda) \leqslant e_2(\lambda) \leqslant e_3(\lambda) \leqslant \cdots \tag{9.12}$$

and let $\Lambda(\lambda, I)$ denote the set of $e_j(\lambda)$ belonging to an interval $I = [a, b]$, $N(\lambda, I)$ the cardinality of $\Lambda(\lambda, I)$, and φ_j^λ the assocciated normalized eigenfunctions of H_λ:

$$H_\lambda \varphi_j^\lambda = e_j(\lambda)\, \varphi_j^\lambda. \tag{9.13}$$

The generalization of Proposition 3.3 is

$$\lim_{|\lambda| \to \infty} d_\lambda^{-1} |\lambda|^{-m} N(\lambda, I) = \text{meas } \mathcal{O}(I), \tag{9.14}$$

for any fixed compact interval I, where $\mathcal{O}(I)$ is given as in (8.1)–(8.2) by

$$\mathcal{O}(I) = \{(x, \xi) \in T^*M: |\xi|_x^2 + V(x) \in I\}. \tag{9.15}$$

The result (9.14) follows directly from (9.7)–(9.10), in the special case where B is the identity operator; in other words it follows from Proposition 6.3.

To each φ_j^λ satisfying (9.13), there is a d_λ-dimensional space of eigenfunctions of $A^{-1}L$ in $\mathscr{D}_\lambda$, which we denote by $E_j(\lambda)$:

$$-A^{-1}L\tilde{\varphi}_{j,\lambda} = e_j(\lambda)\, \tilde{\varphi}_{j,\lambda} \qquad \text{for } \tilde{\varphi}_{j,\lambda} \in E_j(\lambda). \tag{9.16}$$

Pick an orthonormal basis for $E_j(\lambda)$, say

$$\tilde{\varphi}_{j,\lambda,\ell} \in E_j(\lambda), \qquad 1 \leqslant \ell \leqslant d_\lambda. \tag{9.17}$$

We can now define a family of measures μ_j^λ on the unit sphere $S(\mathfrak{g}_{\mathrm{ad}}^\#)$ of the WGS bundle $\mathfrak{g}_{\mathrm{ad}}^\# \to M$ generalizing the notion of the measures $\mu_j^\hbar$, as defined by (8.28). Let $a \in C^\infty(S(\mathfrak{g}_{\mathrm{ad}}^\#))$; this gives rise to a function homogeneous of degree 0, $a_0 \in S^0(\mathfrak{g}_{\mathrm{ad}}^\#)$, coinciding with a on $S(\mathfrak{g}_{\mathrm{ad}}^\#)$. This in turn gives rise to an element $b_a \in S^0(P)$, satisfying the invariance condition (9.11):

$$b_a = a_0 \circ q, \qquad \text{where } q: T^*P \to \mathfrak{g}_{\mathrm{ad}}^\# \text{ is the natural projection.} \tag{9.18}$$

Using local coordinates, we have in the usual way an associated operator $B_a \in \mathrm{OPS}^0(P)$, and we let $B_a^{\mathrm{F}} \in \mathrm{OPS}_{1,0}^0(P)$ be associated with this by Friedrichs symmetrization. Then μ_j^λ is characterized by

$$\int a\, d\mu_j^\lambda = d_\lambda^{-1} \sum_\ell (B_a^{\mathrm{F}} \tilde{\varphi}_{j,\lambda,\ell}, \tilde{\varphi}_{j,\lambda,\ell}). \tag{9.19}$$

In analogy with (3.46), we define probability measures m_I^λ on $S(\mathfrak{g}_{\mathrm{ad}}^\#)$ by

$$m_I^\lambda = N(\lambda, I)^{-1} \sum_{j \in A(\lambda, I)} \mu_j^\lambda. \tag{9.20}$$

As a consequence of (9.10), we have

$$m_I^\lambda \to \tilde{m}_I^{\lambda_1} \qquad \text{for } \lambda = n\lambda_1,\ n \to \infty, \tag{9.21}$$

in the weak* topology of measures, where the probability measure $\tilde{m}_I^{\lambda_1}$ on $S(\mathfrak{g}_{\mathrm{ad}}^{\#})$ is defined by

$$\int a\, d\tilde{m}_I^{\lambda_1} = [\text{meas}\ \mathcal{O}(I)]^{-1} \int_{r^{-1}(\mathcal{O}(I))} b_a(p, \xi + \lambda_1/|\lambda_1|)\, d\,\text{vol}(\xi, p). \quad (9.22)$$

Here, $r\colon \mathfrak{h}^*P \to T^*M$ is the natural projection, so

$$\text{meas}\ \mathcal{O}(I) = \int_{r^{-1}(\mathcal{O}(I))} d\,\text{vol}(\xi, p), \quad (9.23)$$

and $\mathcal{O}(I)$ is given by (9.15). In (9.21), b_a is the function on T^*P, obtained from $a \in C^\infty(S(\mathfrak{g}_{\mathrm{ad}}^{\#}))$ by the prescription above (cf. (9.18)).

To parallel the use of the form of Egorov's theorem given in Proposition 3.1 in the analysis of the term $b_{j,T}^{\hbar}$ given by (8.8), we recall the effect of conjugating an operator $B \in \mathrm{OPS}^{\mu}(P)$ which commutes with the G action by $e^{it\sqrt{-L}}$, with L as in (9.2), or more generally the action of conjugating such B by e^{itM}, given $M \in \mathrm{OPS}^1(P)$ self adjoint and G-invariant. As we have mentioned in Section 5, by Egorov's theorem,

$$B(t) = e^{itM} B e^{-itM} \in \mathrm{OPS}^{\mu}(P) \quad (9.24)$$

for each t. Also, each $B(t)$ is G-invariant. Since B is G-invariant, its principal symbol B_0, defined on T^*P, can be written in the form

$$B_0 = B_0^{\#} \circ q, \quad (9.25)$$

where $q\colon T^*P \to \mathfrak{g}_{\mathrm{ad}}^{\#}$ is the projection mentioned in (9.18), and $B_0^{\#}$ is defined on the vector bundle $\mathfrak{g}_{\mathrm{ad}}^{\#} \to M$, with symbolic properties. Similarly, if M_1 is the principal symbol of M, we have

$$M_1 = M_1^{\#} \circ q \quad (9.26)$$

with $M_1^{\#}$ defined on $\mathfrak{g}_{\mathrm{ad}}^{\#}$. The flow on T^*P generated by the Hamiltonian vector H_{M_1} is a flow which preserves the class of symbols of G-invariant operators, and it induces a smooth flow on the WGS bundle $\mathfrak{g}_{\mathrm{ad}}^{\#}$, whose generator we denote $W_{M_1^{\#}}$. This flow preserves homogeneity, so it induces a flow on the unit sphere bundle $S(\mathfrak{g}_{\mathrm{ad}}^{\#})$, whose generator we denote $S_{M_1^{\#}}$, or, loosely, S_{M_1}. We note that differentiating $B_0^{\#}$ with respect to $W_{M_1^{\#}}$ produces the Poisson structure on the WGS bundle $\mathfrak{g}_{\mathrm{ad}}^{\#}$, described in Section 5, and, from a different perspective, in [11]. The significance of this flow for the description of (9.24) is that, by the result of Egorov on the principal symbol B_0^t of $B(t)$, we can write $B_0^t = B_0^{t\#} \circ q$ with

$$B_0^{t\#} = B_0^{\#} \circ (\exp tS_{M_1}). \quad (9.27)$$

Recall that $\exp tS_{M_1}$ is the WGS flow on $S(\mathfrak{g}^{\#}_{\mathrm{ad}})$ defined by the symbol M_1.

As a further preliminary remark for our main analysis in this section, we will find it convenient to use the following notation, in place of $\int a\, d\mu^{\lambda}_j$, sometimes. Namely, with $a \in C^{\infty}(S(\mathfrak{g}^{\#}_{\mathrm{ad}}))$, to which is associated $a_0 \in C^{\infty}(\mathfrak{g}^{\#}_{\mathrm{ad}} \backslash 0)$, homogeneous of degree 0, and also $b_a \in C^{\infty}(T^*P \backslash 0)$, homogeneous of degree 0, it is convenient to regard μ^{λ}_j as acting on each of these functions, so we make the identifications

$$\int a\, d\mu^{\lambda}_j = \int a_0\, d\mu^{\lambda}_j = \int b_a\, d\mu^{\lambda}_j. \tag{9.28}$$

Which notion is used should be clear in context. In the second integral in (9.28) we think of μ^{λ}_j as a measure on the σ-algebra of Borel subsets of $\mathfrak{g}^{\#}_{\mathrm{ad}} \backslash 0$ which are invariant under the radial action on the fibers of $\mathfrak{g}^{\#}_{\mathrm{ad}} \to M$, and the third integral in (9.28) we think of μ^{λ}_j as a measure on the σ-algebra of radially invariant Borel subsets of $T^*P \backslash 0$ which are also invariant under the G-action: $(p, \xi, \lambda) \cdot g = (p \cdot g, \xi, \mathrm{Ad}^* g^{-1} \lambda)$. We make the same conventions for m^{λ}_I and $\tilde{m}^{\lambda_1}_I$. Note that $\tilde{m}^{\lambda_1}_I$, pictured as defined on a σ-algebra of subsets of $T^*P \backslash 0$, is supported on a submanifold of $T^*P \approx \mathfrak{h}^*P \times \mathfrak{g}^*$. In fact, it is "supported" on $\mathfrak{h}^*P \times \mathcal{O}_{\lambda_1}$ where $\mathcal{O}_{\lambda_1} \subset \mathfrak{g}^*$ is the co-adjoint orbit containing $\lambda_1/|\lambda_1|$; more precisely it is supported on the union of the set of rays in $T^*P \backslash 0$ through points in $\mathfrak{h}^*P \times \mathcal{O}_{\lambda_1}$.

We now present the following generalization of Theorem 8.1.

THEOREM 9.1. *Let λ_1 be fixed and let $I = [a, b]$ be a compact interval in $(0, \infty)$. Let $B \in \mathrm{OPS}^0(P)$ be G-invariant, with principal symbol B_0. Let $\mathfrak{a}(I) \subset \mathfrak{h}^*(P)$ denote the inverse image of $\mathcal{O}(I)$ under $\mathfrak{h}^*(P) \to T^*M$. Assume that, for almost all $(p, \xi) \in \mathfrak{a}(I)$, we have*

$$\begin{aligned}\lim_{T \to \infty} T^{-1} \int_0^T B_0((\exp tX)(p, \xi, \hat{\lambda}_1))\, dt &= \bar{B}_0(p, \xi, \hat{\lambda}_1)\\ &= \Phi_{B_0}(\sqrt{L_2(p, \xi, \hat{\lambda}_1)}, \hat{\lambda}_1).\end{aligned} \tag{9.29}$$

Here, $\Phi_{B_0}(\bar{\mu}, \hat{\lambda})$ is assumed to be homogeneous of degree 0 in $(\bar{\mu}, \hat{\lambda}) \in \underline{R} \times \mathfrak{g}^* \backslash (0, 0)$, C^{∞} in a neighborhood of $I_1 \times \{\hat{\lambda}_1\}$, where $I_1 = [\sqrt{a}, \sqrt{b}]$, $\hat{\lambda}_1 = \lambda_1/|\lambda_1|$. It is also assumed to satisfy the identity

$$\Phi_{B_0}(\bar{\mu}, \mathrm{Ad}^* g \hat{\lambda}) = \Phi_{B_0}(\bar{\mu}, \hat{\lambda}) \qquad \textit{for all } g \in G. \tag{9.30}$$

In (9.29), X is the Hamiltonian vector field, $X = H_{\sqrt{L_2}}$. Then we have

$$\lim_{n\to\infty} N(\lambda, I)^{-1} \operatorname{card}\left\{ j \in \Lambda(\lambda, I) : \left| \int B_0 \, d\mu_j^\lambda - \Phi_{B_0}(|\lambda| \, e_j(\lambda), \lambda) \right| < \varepsilon \right\} = 1, \tag{9.31}$$

where $\lambda = n\lambda_1$ *in* (9.31).

Proof. Parallel to the proof of Theorem 8.1, we adopt the following notation:

$$a_j^\lambda = \int B_0 \, d\mu_j^\lambda - \Phi_{B_0}(|\lambda| \sqrt{e_j(\lambda)}, \lambda), \tag{9.32}$$

$$b_{j,T}^\lambda = \int \left(B_0 - T^{-1} \int_0^T (B_0 \circ \exp tX) \, dt \right) d\mu_j^\lambda, \tag{9.33}$$

$$c_{j,T}^\lambda = \int \left| T^{-1} \int_0^T (B_0 \circ \exp tX) \, dt - \Phi_{B_0}(\sqrt{L_2(p, \xi, \bar\lambda)}, \bar\lambda) \right| d\mu_j^\lambda(p, \xi, \bar\lambda)$$

$$= \int |E_T(p, \xi, \bar\lambda)| \, d\mu_j^\lambda(p, \xi, \bar\lambda), \tag{9.34}$$

and

$$d_j^\lambda = \int \Phi_{B_0}(\sqrt{L_2(p, \xi, \bar\lambda)}, \bar\lambda) \, d\mu_j^\lambda(p, \xi, \bar\lambda) - \Phi_{B_0}(|\lambda| \sqrt{e_j(\lambda)}, \lambda). \tag{9.35}$$

Note that

$$\sqrt{-L} = |\lambda + \delta| \sqrt{e_j(\lambda)} \qquad \text{on } E_j(\lambda). \tag{9.36}$$

Since $\Phi_{B_0}(\bar\mu, \bar\lambda)$ is homogeneous of degree 0, and smooth near $I_1 \times \{\hat\lambda_1\}$, it follows that

$$\Phi_{B_0}(|\lambda| \sqrt{e_j(\lambda)}, \lambda) - \Phi_{B_0}(|\lambda + \delta| \sqrt{e_j(\lambda)}, \lambda) \to 0 \tag{9.37}$$

as $n \to \infty$, with $\lambda = n\lambda_1$.

As before, the goal of the proof of (9.31) is to show that $|a_j^\lambda| < \varepsilon$ for most $j \in \Lambda(\lambda, I)$ if $\lambda = n\lambda_1$, with n large, and we tackle this using

$$|a_j^\lambda| \leqslant |b_{j,T}^\lambda| + c_{j,T}^\lambda + |d_j^\lambda|. \tag{9.38}$$

To estimate $b_{j,T}^\lambda$, write

$$b_{j,T}^\lambda = d_\lambda^{-1} \sum_\ell T^{-1} \int_0^T ([e^{it\sqrt{-L}} B^{\mathrm F} e^{-it\sqrt{-L}} - B_t^{\mathrm F}] \, \tilde\varphi_{j,\lambda,\ell}, \tilde\varphi_{j,\lambda,\ell}) \, dt. \tag{9.39}$$

Here B^{F} is a Friedrichs symmetrization of B, as in (9.19), and B_t^{F} is the symmetrization of B_t, which for each t is an element of $\mathrm{OPS}^0(P)$ with principal symbol $B_0 \circ \exp tX$. By Egorov's theorem, for each t,

$$D(t) = e^{it\sqrt{-L}} B^{\mathrm{F}} e^{-it\sqrt{-L}} - B_t^{\mathrm{F}} \in \mathrm{OPS}_{1,0}^{-1}(P). \tag{9.40}$$

Consequently $D(t)\sqrt{-L} \in \mathrm{OPS}_{1,0}^0(P)$. By (9.36) that implies that, for any T,

$$|b_{j,T}^{\lambda}| \leqslant C_T\, |\lambda + \delta|^{-1}, \tag{9.41}$$

for all $j \in \Lambda(\lambda, I)$. This is an adequate estimate for $b_{j,T}^{\lambda}$.

As before, the hypothesis (9.29) figures into the estimate of $c_{j,T}^{\lambda}$. We have $E_T(p, \xi, \bar{\lambda})$ uniformly bounded and, as $T \to \infty$, $E_T(p, \xi, \bar{\lambda}) \to 0$ almost everywhere with respect to the measure $\tilde{m}_I^{\lambda_1}$, since the invariance property $E_T(p \cdot g, \xi, \bar{\lambda}) = E_T(p, \xi, \mathrm{Ad}^*\, g\bar{\lambda})$ together with (9.30) shows that the conclusion of (9.29) holds with λ_1 replaced by $\mathrm{Ad}^*\, g\lambda_1$ for all $g \in G$. Thus, for any $\varepsilon_2 > 0$, we can pick $T_0 = T_0(\varepsilon_2)$ such that

$$\int |E_{T_0}(p, \xi, \bar{\lambda})|\; d\tilde{m}_I^{\lambda_1} < \varepsilon_2/3. \tag{9.42}$$

Now the weak convergence result (9.21) implies that there is an $h_2(T_0, \varepsilon_2) > 0$ such that

$$\int |E_{T_0}(p, \xi, \bar{\lambda})|\; dm_1^{\lambda} < \varepsilon_2/2 \qquad \text{for } |\lambda + \delta|^{-1} < h_2(T_0, \varepsilon_2). \tag{9.43}$$

In other words,

$$N(\lambda, I)^{-1} \left\{ \sum_{j \in \Lambda(\lambda, I)} \int |E_{T_0}(p, \xi, \bar{\lambda})|\; d\mu_j^{\lambda} \right\} < \varepsilon_2 \tag{9.44}$$

for $|\lambda + \delta|^{-1} < h_2(T_0, \varepsilon_2)$, given T_0 large enough that (9.42) holds. Thus the proportion of $j \in \Lambda(\lambda, I)$ such that

$$|c_{j,T_0}^{\lambda}| < \varepsilon/3 \tag{9.45}$$

is greater than $1 - \sqrt{\varepsilon_2}$, provided $\varepsilon_2 < \varepsilon^2/9$ and $|\lambda + \delta|^{-1} < h_2(T_0, \varepsilon_2)$.

We turn to the estimate of d_j^{λ}. We first suppose B_0 is supported on a small conic neighborhood of $\mathfrak{a}(I) \times \{\bigcup_g \mathrm{Ad}^*\, g\bar{\lambda}_1\}$, so that the function

$$\bar{B}_0(p, \xi, \bar{\lambda}) = \Phi_{B_0}(\sqrt{L_2(p, \xi, \bar{\lambda})}, \bar{\lambda}) \tag{9.46}$$

is C^∞ on the support of B_0 (outside the 0-section of T^*P). We have

$$\int \Phi_{B_0}(\sqrt{L_2(p,\xi,\bar{\lambda})},\bar{\lambda})\,d\mu_j^\lambda = d_\lambda^{-1}\sum_\ell (\bar{B}^{\mathrm{F}}\tilde{\varphi}_{j,\lambda,\ell},\tilde{\varphi}_{j,\lambda,\ell}), \tag{9.47}$$

where B^{F} is an element of $\mathrm{OPS}^0_{1,0}(P)$ with principal symbol (mod $S^{-1}_{1,0}(P)$) equal to B_0, given by (9.46). As in the proof of [30, Chap. 12, Theorem 6.3], by the invariance (9.30) for $\Phi_{B_0}(\bar{\mu},\bar{\lambda})$, together with a theorem of Matner [20], we can write

$$\Phi_{B_0}(\sqrt{L_2},\bar{\lambda}) = \tilde{\Phi}(\sqrt{L_2},\sigma_1(\bar{\lambda}),\ldots,\sigma_\nu(\bar{\lambda})), \tag{9.48}$$

where $\sigma_j(\bar{\lambda})$ are smooth on $\mathfrak{g}^*\setminus 0$, homogeneous of degree 1, and invariant under Ad^*g, $g\in G$. Thus the functional calculus, as developed in [30, Chap. 12], implies

$$\bar{B}^{\mathrm{F}}\tilde{\varphi}_{j,\lambda,\ell} = \Phi_{B_0}(|\lambda+\delta|\sqrt{e_j(\lambda)},\bar{\lambda})\,\tilde{\varphi}_{j,\lambda,\ell} + O(|\lambda+\delta|^{-1}). \tag{9.49}$$

This gives

$$|d_j^\lambda| \leqslant C\,|\lambda+\delta|^{-1}, \tag{9.50}$$

under the hypothesis above on the support of B_0. We can remove this hypothesis by the same sort of argument as that used in the proof of Theorem 8.1, to get (9.50) in general. The estimates on $b^\lambda_{j,T}$, $c^\lambda_{j,T}$, and d^λ_j established above complete the proof of Theorem 9.1.

Theorem 9.1 can of course be reinterpreted, in terms of symbols and flows on the WGS bundle $\mathfrak{g}^\#_{\mathrm{ad}}$, in a straightforward fashion. Also the results of this section extend readily to Schrödinger operators of the form (6.25).

APPENDIX A:
List of Symbol Classes for Pseudodifferential Operators

Symbol class	Reference
$S^m_{1,0}$	(0.42)
S^m	(0.43)
$\Sigma^\mu_\#$	(2.1)–(2.2)
Σ^μ	(2.3)
Σ^μ_e	(2.5)
$S^\mu_{\mathfrak{h}}$	(2.18), (4.8)
$S^\mu_{\mathfrak{h}S}$	(2.19)
R^μ	(3.3)
$S^\mu_{1,0\mathfrak{h}}$	(4.8)

APPENDIX B: OTHER SYMBOLS

Symbol	Reference
H_λ^0	(0.4)
Δ_G^P	(0.5), (6.3)
$\mathscr{D}_\lambda$	(0.33), (6.6)
L	(0.30), (6.4)
A	(0.31), (6.5)
$\mathfrak{h}^*(M\times S^1)$	(2.17)
$\mathscr{I}$	(2.19), (3.5)
$N(\hbar, I)$	(3.36)
$N(\lambda, I)$	(9.12)
$\mathcal{O}_{L,V}(I)$	(3.38)
$\mathcal{O}(I)$	(8.1), (9.15)
$e_j(\hbar)$	(3.39)
$e_j(\lambda)$	(9.12)
$\Lambda(\hbar, I)$	(3.40)
$\Lambda(\lambda, I)$	(9.12)
$\varphi_j^\hbar$	(3.41)
φ_j^λ	(9.13)
$A^F(\hbar)$	(3.42)
$\mu_j^\hbar$	(3.44)
μ_j^λ	(9.19)
$m_I^\hbar$	(3.46)
m_I^λ	(9.20)
Tr_G	(4.3)–(4.4)
$\mathfrak{h}^*P$	(4.7)
$\mathfrak{g}_{\mathrm{ad}}^{\#}$	(5.9)
$W_{M_1^{\#}}$	(5.15)
S_{M_1}	(5.16)
$\Omega_\hbar$	(7.17)
$F_\hbar^k$	(7.18)
$a_j^\hbar$	(8.7)
$b_{j,T}^\hbar$	(8.8)
$c_{j,T}^\hbar$	(8.9)
$d_j^\hbar$	(8.10)
a_j^λ	(9.32)
$b_{j,T}^\lambda$	(9.33)
$c_{j,T}^\lambda$	(9.34)
d_j^λ	(9.35)

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