Arcs in Planes of Even Order

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In a finite projective plane $\pi$ of order $n$ a $k$-arc is a set $K$ of $k$ points of $\pi$ with no three collinear. $K$ is complete if $K$ is not properly contained in a $k'$-arc with $k' > k$. In this note we study properties of a complete $n$-arc $K$ in $\pi$. By the results of B. Segre and G. Tallini (see [7, 9]), this forces $\pi$ to be non-Desarguesian. Throughout this note "tangents" and "secants" mean tangents and secants to $K$, which are defined in the usual way. Now, since $K$ is complete, each point $X$ off $K$ lies on at least one secant to $K$. We say that $X$ is $K$-special if $X$ lies on exactly one secant to $K$. Denniston [4] intimates that there are no $K$-special points for any complete 9-arc in the known non-Desarguesian planes of order 9.

**Theorem 1.** Let $K$ be a complete $n$-arc in a projective plane $\pi$ of order $n$ with $n$ even. Then each point $X$ of $\pi$ not on $K$ lies on at least $\frac{1}{4}n$ secants to $K$.

**Proof.** Let $X$ lie on exactly $\alpha$ tangents to $K$. If $\alpha = 0$, $X$ lies on $\frac{1}{2}n > \frac{1}{4}n$ secants and we are done. So assume $\alpha > 0$. Put

\[ A = \{\text{tangents not on } X\}; \]
\[ B = \{\text{all non-tangent lines of } \pi \text{ through } X\}; \]
\[ S = A \cup B. \]

Since $n$ is even each point $P$ of $\pi$ lies on an even number of tangents. Thus if a point $P \neq X$ of $\pi$ lies on a tangent from $X$, then $P$ lies on at least one tangent not passing through $X$. Thus

(i) each point $P$ of $\pi$ lies on at least one line of $S$.

Since $K$ is complete, no point of $\pi$ can lie on as many as $n$ tangents. Also, $\alpha > 0$. Thus we get

(ii) no point of $\pi$ has all the lines through it contained in $S$.

Properties (i) and (ii) guarantee that $S$ is a dual blocking set with $|S| = 3n + 1 - 2\alpha$. But from [2, 3], $|S| > n + \sqrt{n} + 1$. Thus $\alpha < n - \frac{1}{2}\sqrt{n}$. Now if $U$ denotes the secants on $X$ we have $|U| = \frac{1}{2}(n - \alpha) > \frac{1}{4}n$.

**Discussion.** I first obtained Theorem 1 for planes of order $n = 2 (4)$ by using methods of coding theory. For an interesting account of such methods we refer to Assmus and Mattson [1], Hering [6] and MacWilliams, Sloane and Thompson [8]. To describe the original proof of Theorem 1 for $n = 2 (4)$, we require another result which may be of interest in its own right. We adopt the terminology of [8]. Let $\pi$ denote a projective plane of order $n = 2 (4)$, let $C$ denote the code of the dual plane and let $\tilde{C}$ denote the extended code. It is known (see [1, 6, 8]) that $\tilde{C} = \tilde{C}^\perp$. Now let $K$ be any $k$-arc in $\pi$, complete or not. If $k$ is odd, then every point of $\pi$ lies on an odd number of tangents to $K$. If $k$ is even, then every point of $\pi$ lies on an even number of tangents to $K$. Using the fact that $\tilde{C} = \tilde{C}^\perp$, we get the following theorem.

* Research supported in part by the N.S.E.R.C. of Canada.
THEOREM 2. Let $K$ be a $k$-arc in a plane $\pi$ of order $n = 2 (4)$. Then
(a) if $k$ is even, the tangents to $K$ yield a codeword in $\tilde{C}$ of weight $k(n + 2 - k)$;
(b) if $k$ is odd, the tangents to $K$ form the "finite part" of a codeword in $\tilde{C}$.

REMARK. It is worth noting that if $k = n = 10$ and if $K$ is complete then the tangents to $K$ yield a primitive codeword $M_{20}$ of weight 20 in $\tilde{C}$. Primitive codewords of weight 20 are extensively studied in Hall [5].

We return to Theorem 1 for the case $n = 2 (4)$. From Theorem 2 the $2n$ tangents to $K$ there form a codeword $u$ in $C$ of even weight. The $n + 1$ lines of $\pi$ on $X$ from another codeword $v$ in $C$ of odd weight. Then $u + v$ is a codeword in $C$ of odd weight. Now $u + v$ corresponds to the set $S$. From [8, 2.4] $S$ yields a dual blocking set in $\pi$. As before, an appeal to the blocking set result [2, 3] completes the proof.

REFERENCES

Received 16 November 1981

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