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Note

Directed triangles in directed graphs

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Abstract

We show that each directed graph (with no parallel arcs) on n vertices, each with indegree and outdegree at least n/t where $t = 2.888997\ldots$ contains a directed circuit of length at most 3.

Keywords: Girth; Digraph; Degree condition; Directed triangle

In this paper, directed graphs have no loops or parallel arcs. It is an intriguing conjecture of Cacetta and Häggkvist [2] that any directed graph on n vertices, each with outdegree at least $\lceil n/k \rceil$ contains a directed circuit of length at most k. Surprisingly, the special case for $k = 3$ is still open.

Instead of proving the conjecture, one may look for values of s so that any directed graph on n vertices with minimum outdegree at least n/s, contains a directed triangle. The highest value of s is due to Shen [6], who obtained the value

$$s = \frac{1}{3 - \sqrt{7}} = 2.8228757\ldots \qquad (1)$$

Shen's result improved approximations by Cacetta and Häggkvist [2] and Bondy [1].

It is not even known whether any directed graph on n vertices, each with both indegree and outdegree at least $n/3$, contains a directed triangle. Again, one may look for values of t so that any directed graph on n vertices, each with both indegree and outdegree at least n/t contains a directed triangle. The best result on this problem is in

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[3], where using (1) it is shown that $t=\left(22-2\sqrt{7}+\left(1+\sqrt{7}\right)\sqrt{45-17\sqrt{7}}\right)/6\approx 2.875$. This improved the results obtained by Cacetta and Häggkvist [2] and Li and Brualdi [4].

In this note we use Shen's approximation (1) to show the following:

Theorem 1. *Any directed graph on n vertices, each with both indegree and outdegree at least n/t_0 where*

$$t_0=\frac{1}{72}\left(241-17\sqrt{7}+2\sqrt{4064-1522\sqrt{7}}\right)\cos\alpha,$$

where $\alpha=\frac{1}{3}\arctan\left(18\sqrt{1262428404\sqrt{7}-1131169991}/1367549\right)$ *contains a directed triangle.*

Note that $t_0\approx 2.8889971$.

The theorem is proved by extending the approach of [3]. Before doing so, we introduce some notation. For each $v\in V$ let E_v^+ and E_v^- denote the sets of outneighbours and inneighbours of v, respectively. For u, v, $w\in V$ let

$$E_{uv}^+=E_u^+\cap E_v^+, E_{uv}^-=E_u^-\cap E_v^-.$$

Moreover let

$$\varepsilon_v^+=|E_v^+|,\quad \varepsilon_v^-=|E_v^-|,\quad \varepsilon_{uv}^+=|E_{uv}^+|,\quad \varepsilon_{uv}^-=|E_{uv}^-|.$$

We recapitulate a number of earlier results in the following proposition.

Proposition 2 (de Graaf et al. [3]). *Let $D=(V,A)$ be a directed graph on n vertices with no directed triangle, where for each vertex $v\in V$ $\varepsilon_v^+\geqslant k$ and $\varepsilon_v^+\geqslant k$, such that deletion of any arc would violate this assumption. Then*

(1) *there exists a vertex v' with both indegree and outdegree equal to k,*
(2) *if (u,v), (v,w), $(u,w)\in A$ then $\varepsilon_{uv}^-+\varepsilon_{vw}^+\geqslant 4k-n$,*
(3) *for each arc (u,v) of D: $\varepsilon_{uv}^-\geqslant(3k-n)s$ and $\varepsilon_{uv}^+\geqslant(3k-n)s$.*

Now we are in a position to prove Theorem 1. With respect to [3], the stronger inequality in this paper is obtained because instead of showing that the total number of arcs in one of the graphs induced by $E_{v'}^+$, $E_{v'}^-$ exceeds $k^2/2$, we use the lower bound on the number of triangles in an undirected graph established by Moon and Moser [5].

Theorem 3 (Moon and Moser [5]). *Let $G=(V,E)$ be an (undirected) graph with $|V|=n$, $|E|=m$. Then G contains at least $m(4m-n^2)/3n$ (undirected)-triangles.*

Proof of Theorem 1. Suppose $D=(V,A)$ is a directed graph with $|V|=n$, each with both indegree and outdegree at least $k=\lceil n/t_0\rceil$, and without any directed triangle. We may assume that deleting any arc would give a vertex of indegree or outdegree less than k. Let $t = n/k$. For future reference we note that

$$3 - \frac{2}{10} < s < t < t_0 < 3 - \frac{1}{10}, \tag{2}$$

where the lower bound for t follows from (1).

According to Proposition 2 there is a vertex with both indegree and outdegree equal to k. Let v' be such a vertex. Let u' be a vertex of minimum indegree in the subgraph induced by $E^-_{v'}$ and let w' be a vertex of minimum outdegree in the subgraph induced by $E^+_{v'}$. So $\varepsilon^-_{u'v'} \leqslant \varepsilon^-_{uv'}$ for all $u \in E^-_{v'}$ and $\varepsilon^+_{v'w'} \leqslant \varepsilon^+_{v'w}$ for all $w \in E^+_{v'}$. By Shen's result we have

$$\varepsilon^-_{u'v'} < k/s \quad \text{and} \quad \varepsilon^+_{v'w'} < k/s. \tag{3}$$

Without loss of generality we may assume that $\varepsilon := \min\{\varepsilon^-_{u'v'}, \varepsilon^+_{v'w'}\} = \varepsilon^-_{u'v'}$. Next, we consider the subgraph induced by $E^+_{v'}$.

By Proposition 2 we know that for all $w \in E^+_{u'v'}$ we have

$$\varepsilon^+_{v'w} \geqslant 4k - n - \varepsilon^-_{u'v'} = 4k - n - \varepsilon. \tag{4}$$

For all other $k - \varepsilon^+_{u'v'}$ vertices in $E^+_{v'}$ we have

$$\varepsilon^+_{v'w} \geqslant \varepsilon^+_{v'w'} \geqslant \varepsilon. \tag{5}$$

As $n < 3k$ and $\varepsilon < k/s$ it follows that $4k - n - \varepsilon \geqslant k - \varepsilon \geqslant \varepsilon$. By removing arcs if necessary, we may assume that in (4) and (5) equality holds.

For the number of arcs in $E^+_{v'}$ we find, using $n = tk$,

$$\begin{aligned} m &= \varepsilon^+_{u'v'}(4k - n - \varepsilon) + (k - \varepsilon^+_{u'v'})\varepsilon \\ &= \varepsilon^+_{u'v'}((4 - t)k - 2\varepsilon) + \varepsilon k. \end{aligned}$$

Using Theorem 3 it follows that the number of transitive triangles T in the graph induced by $E^+_{v'}$ is bounded from below according to

$$\begin{aligned} T \geqslant & -(1/3k)(\varepsilon(k - 2\varepsilon^+_{u'v'}) - k(t - 4)\varepsilon^+_{u'v'}) \\ & \times(-4\varepsilon(k - 2\varepsilon^+_{u'v'}) + k(k + 4(-4 + t)\varepsilon^+_{u'v'})). \end{aligned} \tag{6}$$

Let $T_{\text{low}}(\varepsilon, \varepsilon^+_{u'v'}, t)$ denotes the lower bound for the number of transitive triangles given by the right-hand side of (6).

The number of transitive triangles is bounded from above by

$$\begin{aligned} T &\leqslant \sum_{w \in E^+_{v'}} \binom{\varepsilon^+_{v'w}}{2} \\ &= \varepsilon^+_{u'v'} \binom{4k - n - \varepsilon}{2} + (k - \varepsilon^+_{u'v'}) \binom{\varepsilon}{2} \\ &\leqslant \tfrac{1}{2}\varepsilon^+_{u'v'}((4 - t)k - \varepsilon)^2 + \tfrac{1}{2}(k - \varepsilon^+_{u'v'})\varepsilon^2. \end{aligned} \tag{7}$$

Let $T_{\text{up}}(\varepsilon, \varepsilon^+_{u'v'}, t)$ denote the upper bound for the number of transitive triangles given by (7). Let $U(\varepsilon, \varepsilon^+_{u'v'}, t) = T_{\text{low}}(\varepsilon, \varepsilon^+_{u'v'}, t) - T_{\text{up}}(\varepsilon, \varepsilon^+_{u'v'}, t)$. We obtain

$$U(\varepsilon, \varepsilon^+_{u'v'}, t) = c_0 + c_1\varepsilon^+_{u'v'} + c_2(\varepsilon^+_{u'v'})^2 \tag{8}$$

with $c_0 = \varepsilon(5\varepsilon - 2k)k/6$, $c_1 = (2\varepsilon + k(t-4))(k(14-3t) - 16\varepsilon)/6$ and $c_2 = 4(2\varepsilon + k(t-4))^2/3k$.

To conclude the proof, we will show that $U(\varepsilon, \varepsilon^+_{u'v'}, t) > 0$ for all $(3-t)ks \leqslant \varepsilon < k/s$ and $(3-t)ks \leqslant \varepsilon^+_{u'v'} < k/s$ and for t in the interval defined by (2). This is simplified by the following lemma.

Lemma 4. *For* $(3-t)ks \leqslant \varepsilon < k/s$ *and* $(3-t)ks \leqslant \varepsilon^+_{u'v'} < k/s$, *and with* t *in the interval defined by* (2), *it holds that* $U(\varepsilon, \varepsilon^+_{u'v'}, t) \geqslant U((3-t)ks, (3-t)ks, t)$.

This lemma will be proved at the end of this article. Using Lemma 4 we obtain the following inequality:

$$\begin{aligned} U(\varepsilon, \varepsilon^+_{u'v'}, t) \geqslant U((3-t)ks, (3-t)ks, t) = {} & \tfrac{1}{6} k^3 s(3-t) \\ & \times((-58+675s-1440s^2+864s^3)+(26-483s+1248s^2-864s^3)t \\ & +(-3+110s-352s^2+288s^3)t^2+(-8s+32s^2-32s^3)t^3). \end{aligned}$$

Multiplying by $3/s^4k^3(3-t)$ and substituting $s = 1/(3-\sqrt{7})$ leads to

$$\begin{aligned} & \frac{3}{s^4k^3(3-t)}(U((3-t)ks, (3-t)ks, t)) \\ & \quad = (8\sqrt{7} - 32)t^3 + (361 - 103\sqrt{7})t^2 + (383\sqrt{7} - 1254)t + 1062 - 319\sqrt{7}. \end{aligned} \tag{9}$$

As t_0 is a zero of the polynomial defined by (9), and, moreover, this polynomial is strictly positive on the interval for t defined by (2), it follows that $T_{\text{low}}(\varepsilon, \varepsilon^+_{u'v'}, t) > T_{\text{up}}(\varepsilon, \varepsilon^+_{u'v'}, t)$. This contradiction finishes the proof of Theorem 1. □

Proof of Lemma 4. We first show that $U(\varepsilon, \varepsilon^+_{u'v'}, t)$ (for fixed ε and t) is an increasing function of $\varepsilon^+_{u'v'}$, by showing that the derivative with respect to $\varepsilon^+_{u'v'}$ is strictly positive on the interval mentioned in Lemma 4.

$$\frac{\mathrm{d}U(\varepsilon, \varepsilon^+_{u'v'}, t)}{\mathrm{d}\varepsilon^+_{u'v'}} = \frac{2\varepsilon + k(t-4)}{6k} p(\varepsilon, \varepsilon^+_{u'v'}, t), \tag{10}$$

where

$$p(\varepsilon, \varepsilon^+_{u'v'}, t) = -16\varepsilon(k - 2\varepsilon^+_{u'v'}) + k(k(14-3t) + 16(t-4)\varepsilon^+_{u'v'}).$$

As $\varepsilon < k/s$ and $t < 3$ the first term in (10) is negative. We proceed by showing that also $p(\varepsilon, \varepsilon^+_{u'v'}, t) < 0$. As $\varepsilon^+_{u'v'} < k/s$ the coefficient of ε in $p(\varepsilon, \varepsilon^+_{u'v'}, t)$ is negative. So $p(\varepsilon, \varepsilon^+_{u'v'}, t) \leqslant p((3-t)ks, \varepsilon^+_{u'v'}, t)$ which is equal to

$$k^2(14-3t) - 16k^2s(3-t) + 16(2ks(3-t) + k(t-4))\varepsilon^+_{u'v'}. \tag{11}$$

As the coefficient of $\varepsilon^+_{u'v'}$ in (11) is negative, we obtain: $p((3-t)ks, \varepsilon^+_{u'v'}, t) \leqslant p((3-t)ks, (3-t)ks, t)$ where the latter equals

$$k^2\left(-3+\frac{169}{32-64s}+\frac{9}{64s}+16s(2s-1)\left(t-\left(3-\frac{32s-3}{64s^2-32s}\right)\right)^2\right).$$

As $3-(32s-3)/(64s^2-32s) < 3-2/10 < t < 3-1/10$, we obtain

$$\frac{1}{k^2}p((3-t)ks,(3-t)ks,t) \leqslant \frac{1}{50}(265+8s(2s-21)) = \frac{77}{50}-\frac{6\sqrt{7}}{5} < 0.$$

This shows that $U(\varepsilon, \varepsilon^+_{u'v'}, t) \geqslant U(\varepsilon, (3-t)ks, t)$. Next, taking the derivative with respect to ε yields

$$6k\,\frac{\mathrm{d}U(\varepsilon,(3-t)ks,t)}{\mathrm{d}\varepsilon} = \varepsilon k^2(10+64s(t-3)+64s^2(t-3)^2)+2k^3q(t), \qquad (12)$$

where $q(t)$ only depends on t. As the coefficient of ε is negative on the considered interval for t, we find that the right-hand side of (12) is minimized when $\varepsilon = k/2$, which is a relaxation of $\varepsilon < k/s$. This leads to

$$6k\frac{\mathrm{d}U(\varepsilon,(3-t)ks,t)}{\mathrm{d}\varepsilon} \geqslant k^3(3+2s(t-3)(-30+16s(-3+t)^2+11t)) > 0,$$

where the latter inequality follows by straightforward numerical evaluation using (2). This shows that

$$U(\varepsilon,(3-t)ks,t) \geqslant U((3-t)ks,(3-t)ks,t)$$

which finishes the proof of Lemma 4. □

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