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J. Math. Anal. Appl. 321 (2006) 675–690

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

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# Generalized convexities and generalized gradients based on algebraic operations ☆

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Received 18 April 2005

Available online 23 September 2005

Submitted by J.P. Dauer

**Abstract**

In this paper, we investigate properties of generalized convexities based on algebraic operations introduced by Ben Tal [A. Ben Tal, On generalized means and generalized convex functions, J. Optim. Theory Appl. 21 (1977) 1–13] and relations between these generalized convexities and generalized monotonicities. We also discuss the $(h,\varphi)$-generalized directional derivative and gradient, and explore the relation between this gradient and the Clarke generalized gradient. Definitions of some generalized averages of the values of a generalized convex function at $n$ equally spaced points based on the algebraic operations are also presented and corresponding results are obtained. Finally, the $(\varphi,\gamma)$-convexity is defined and some properties of $(\varphi,\gamma)$-convex functions are derived.



*Keywords:* Generalized convexity; Generalized monotonicity; $(\varphi,\gamma)$-Convexity; $(\varphi,\gamma)$-Subdifferential

## 1. Introduction

Convexity plays a central role in many aspects of mathematics including sufficient conditions and duality theory of mathematical programming [1,6,15], inequality, and monotonicity of various averages of the values of a function at $n$ equally spaced points [3]. Convex functions

☆ This research is supported in part by NSF and Air Force grants.

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0022-247X/$ – see front matter 

doi:10.1016/j.jmaa.2005.08.093

have many nice properties. However, these convexity assumptions must be weakened in order to handle practical problems. In recent years, many authors have been interested in generalized convexities and their properties. Several generalized convexities have been obtained and corresponding results in optimization have been derived, for instance, in [5,7–14,16–18]. Several authors [4,19,20] have introduced generalized convexities based on some algebraic operations introduced by Ben Tal [4]. In this paper, we consider these generalized convexities and investigate their further properties.

We first present the algebraic operations introduced by Ben Tal [1,4].

**Definition 1.** Let $E \subseteq \mathbb{R}^N$, $h : E \to \mathbb{R}^N$ be a continuous vector function. Suppose that the inverse function $h^{-1}$ of $h$ exists. Then the *h-vector addition* of $x \in E$ and $y \in E$ is defined by

$$x \oplus y = h^{-1}\big(h(x) + h(y)\big)$$

and the *h-scalar multiplication* of $x$ and $\lambda \in \mathbb{R}$ is defined by

$$\lambda \odot x = h^{-1}\big(\lambda h(x)\big).$$

Similarly, algebraic operations for scalar valued functions can be defined as follows.

**Definition 2.** Let $A \subseteq \mathbb{R}$ and $\varphi : A \to \mathbb{R}$ be a continuous and scalar valued function. Suppose that the inverse function $\varphi^{-1}$ of $\varphi$ exists. Then the *$\varphi$-addition* of $\alpha \in A$ and $\beta \in A$ is defined by

$$\alpha[+]\beta = \varphi^{-1}\big(\varphi(\alpha) + \varphi(\beta)\big)$$

and the *$\varphi$-scalar multiplication* of $\alpha \in A$ and $\lambda \in \mathbb{R}$ is defined by

$$\lambda[\cdot]\alpha = \varphi^{-1}\big(\lambda\varphi(\alpha)\big).$$

Finally, we have left one more operation to define.

**Definition 3.** The $(h, \varphi)$-inner product of $x \in E$ and $y \in E$ is defined by

$$\big(x^T y\big)_{(h,\varphi)} = \varphi^{-1}\big(h(x)^T h(y)\big).$$

From the above definitions, the corresponding subtractions can be defined easily as follows:

- The $h$-vector subtraction of $x \in E$ and $y \in E$:
$$x \ominus y = x \oplus \big((-1) \odot y\big) = h^{-1}\big(h(x) - h(y)\big).$$
- The $\varphi$-subtraction of $\alpha \in A$ and $\beta \in A$:
$$\alpha[-]\beta = \alpha[+]\big((-1)[\cdot]\beta\big) = \varphi^{-1}\big(\varphi(\alpha) - \varphi(\beta)\big).$$

Ben Tal [4] obtained some properties of $(h, \varphi)$-convex functions [1,4] based on the above generalized algebraic operations. These results were also applied to some problems in statistical decision theory. Xu and Liu [19] later introduced $(h, \varphi)$-generalized invex functions and established efficiency conditions and duality theorems for semi-infinite multiobjective programming under the generalized invexity assumptions. Zhang [20] have also generalized the concept of $(h, \varphi)$-convexity and have obtained some optimality conditions and several duality results for a class of nonsmooth programming under the generalized convexity assumptions.

Let $f$ be a Lipschitz and real-valued function defined on $\mathbb{R}^N$. For all $x, d \in \mathbb{R}^N$, the $(h,\varphi)$-generalized directional derivative of $f$ with respect to direction $d$ and the $(h,\varphi)$-generalized gradient of $f$ at $x$, denoted by $f^*(x;d)$ and $\partial^* f(x)$, respectively, are defined as follows:

$$f^*(x;d) = \lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \frac{1}{\mu}[\cdot]\big(f(y \oplus \mu \odot d)[-]f(y)\big),$$

$$\partial^* f(x) = \big\{\xi^* \bigm| f^*(x,d) \geqslant \big(\xi^{*T} d\big)_{(h,\varphi)},\ \forall d \in R^N\big\}.$$

We note that the above definitions can be seen as generalizations of the definitions introduced by Zhang [20].

Following Ben Tal [4], we use the notations:

$$\left[\sum_{i=1}^{m}\right] a_i = a_1[+]a_2[+]\cdots[+]a_m \quad \text{and} \quad \bigoplus_{i=1}^{m} \xi_i = \xi_1 \oplus \cdots \oplus \xi_m,$$

where $a_i \in A$, $\xi_i \in E$, $i = 1, 2, \ldots, m$.

We also denote Clarke generalized gradient by $\partial f(x)$ [6] and Clarke directional derivative by $f^\circ(x;d)$ [6].

This paper is organized as follows. In Section 2, we give some definitions which will be used in the rest part of the paper. Then we explore the relation between $(h,\varphi)$-generalized directional derivative and Clarke directional derivative in Section 3. We show that many properties of $(h,\varphi)$-generalized gradient can be conveniently derived based on this result in the section. We obtain relations between $(h,\varphi)$-generalized monotonicity and $(h,\varphi)$-generalized convexity in Section 4. In Section 5, we consider monotonicity of some averages based on the algebraic operations under the $\varphi$-convexity assumption. In the final section, $(\varphi,\gamma)$-convex functions are introduced and their properties are discussed.

## 2. Preliminaries

In this section, we consider a generalized convexity, namely $(h,\varphi)$-convexity [1,4], and some generalizations of the generalized convexity.

**Definition 4.** A function $f: E \to \mathbb{R}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi)$-convex, if for all $x_1, x_2 \in E$, the relation

$$f\big(\lambda \odot x_1 \oplus (1-\lambda) \odot x_2\big) \leqslant \lambda[\cdot]f(x_1)[+](1-\lambda)[\cdot]f(x_2), \quad \forall \lambda \in [0,1],$$

holds. If, for all $x_1, x_2 \in E$ and $x_1 \neq x_2$, the relation

$$f\big(\lambda \odot x_1 \oplus (1-\lambda) \odot x_2\big) < \lambda[\cdot]f(x_1)[+](1-\lambda)[\cdot]f(x_2), \quad \forall \lambda \in (0,1),$$

holds, then $f$ is said to be $(h,\varphi)$-strictly convex on $E$.

Next, we present the following extension of $(h,\varphi)$-convexity involving a scalar.

**Definition 5.** A function $f: E \to \mathbb{R}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi,\alpha)$-strongly convex, if there exists $\alpha > 0$ such that, for all $x_1, x_2 \in E$, $x_1 \neq x_2$,

$$f(x_2)[-]f(x_1) \geqslant \big((x_2 \ominus x_1)^T \xi^*\big)_{(h,\varphi)}[+]\alpha[\cdot]\big(\|x_2 \ominus x_1\|^2\big)_{(h,\varphi)}, \quad \forall \xi^* \in \partial^* f(x_1).$$

Since $(h,\varphi)$-convex and $(h,\varphi,\alpha)$-strongly convex functions have been defined, the concepts of $(h,\varphi)$-pseudoconvex [20] and $(h,\varphi,\alpha)$-strongly pseudoconvex functions can also be defined.

**Definition 6.** A function $f: E \to \mathbb{R}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi)$-(strictly) pseudoconvex if, for any $x_1, x_2 \in E$ $(x_1 \neq x_2)$,

$$f(x_2)[-]f(x_1) < (\leqslant) 0 \quad \Rightarrow \quad \big((x_2 \ominus x_1)^T \xi^*\big)_{(h,\varphi)} < 0, \quad \forall \xi^* \in \partial^* f(x_1),$$

holds.

**Definition 7.** A function $f: E \to \mathbb{R}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi,\alpha)$-strongly pseudoconvex, if there exists $\alpha > 0$ such that, for all $x_1, x_2 \in E$, $x_1 \neq x_2$,

$$\forall \xi^* \in \partial^* f(x_1),\ \big((x_2 \ominus x_1)^T \xi^*\big)_{(h,\varphi)} \geqslant 0 \quad \Rightarrow \quad f(x_2)[-]f(x_1) \geqslant \alpha[\cdot]\big(\|x_2 \ominus x_1\|^2\big)_{(h,\varphi)}$$

holds.

Therefore, the concept of generalized monotonicity is easily defined as follows.

**Definition 8.** A set-valued function $F: E \to 2^{\mathbb{R}^N}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi)$-(strictly) monotonic if, for all $x_1, x_2 \in E$ $(x_1 \neq x_2)$, the relation

$$\big((x_2 \ominus x_1)^T (\eta \ominus \xi)\big)_{(h,\varphi)} \geqslant (>) 0, \quad \forall \xi \in F(x_1),\ \eta \in F(x_2),$$

holds.

**Definition 9.** A set-valued function $F: E \to 2^{\mathbb{R}^N}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi,\alpha)$-strongly monotonic if there exists $\alpha > 0$ such that, for all $x_1, x_2 \in E$,

$$\big((x_2 \ominus x_1)^T (\eta \ominus \xi)\big)_{(h,\varphi)} \geqslant \alpha[\cdot]\big(\|x_2 \ominus x_1\|^2\big)_{(h,\varphi)}, \quad \forall \xi \in F(x_1),\ \eta \in F(x_2),$$

holds.

**Definition 10.** A set-valued function $F: E \to 2^{\mathbb{R}^N}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi)$-(strictly) pseudomonotonic if, for any $x_1, x_2 \in E$ $(x_1 \neq x_2)$,

$$\big((x_2 \ominus x_1)^T \eta\big)_{(h,\varphi)} < (\leqslant) 0, \quad \forall \eta \in F(x_2) \quad \Rightarrow \quad \big((x_2 \ominus x_1)^T \xi\big)_{(h,\varphi)} < 0, \quad \forall \xi \in F(x_1),$$

holds.

**Definition 11.** A set-valued function $F: E \to 2^{\mathbb{R}^N}$ defined on $E \subseteq \mathbb{R}^N$ is said to be $(h,\varphi)$-strongly pseudomonotonic if, for any $x_1, x_2 \in E$,

$$\begin{aligned} &\big((x_2 \ominus x_1)^T \xi\big)_{(h,\varphi)} \geqslant 0, \quad \forall \xi \in F(x_1) \\ &\quad \Rightarrow \quad \big((x_2 \ominus x_1)^T \eta\big)_{(h,\varphi)} \geqslant \alpha[\cdot]\big(\|x_2 \ominus x_1\|^2\big)_{(h,\varphi)}, \quad \forall \eta \in F(x_2), \end{aligned}$$

holds.

When $\varphi(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$ and $h(x) = x$ for all $x \in \mathbb{R}^N$, real valued functions defined by Definitions 4–7 are (strictly) convex, strongly convex, pseudoconvex, strongly pseudoconvex, respectively; set valued functions defined by Definitions 8 to 11 are (strictly) monotone, strongly monotone, pseudomonotone and strongly pseudomonotone [6], respectively. Hence $(h,\varphi)$-generalized convex and monotone functions are generalizations of generalized convex and monotone functions, respectively. In particularly, we have the following definition when $N=1$ and $h=\varphi$.

**Definition 12.** A real valued function $f : A \to R$ defined on $A \subseteq \mathbb{R}$ is said to be $\varphi$-convex function, if for all $\alpha, \beta \in A \subseteq \mathbb{R}$, and $\forall \lambda \in [0,1]$, the relation

$$f\big(\lambda[\cdot]\alpha[+](1-\lambda)[\cdot]\beta\big) \leqslant \lambda[\cdot]f(\alpha)[+](1-\lambda)[\cdot]f(\beta)$$

holds.

The following proposition can be derived easily from the definition motivated by Jensen's inequality [4].

**Proposition 1.** *Let $f$ be a $\varphi$-convex function on $A \subseteq \mathbb{R}$, $\alpha_i \in A$, $\lambda_i \in [0,1]$, $i = 1,\dots,m$, and $\sum_{i=1}^m \lambda_i = 1$. Then*

$$f\left(\left[\sum_{i=1}^m\right]\lambda_i[\cdot]\alpha_i\right) \leqslant \left[\sum_{i=1}^m\right]\lambda_i[\cdot]f(\alpha_i).$$

## 3. Generalized gradients and directional derivatives

In this section, some relations between $(h,\varphi)$-directional derivative and Clarke directional derivative as well as relations between $(h,\varphi)$-generalized gradient and Clarke generalized gradient will be discussed.

In order to present a theorem about the relation between $(h,\varphi)$-generalized directional derivative and Clarke directional derivative we consider the following lemma.

**Lemma 1.** *Let $\varphi(t)$ be strictly increasing and continuous on $\mathbb{R}$, and $h$ be a continuous vector function on $\mathbb{R}^N$. Then*

$$\lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \varphi^{-1}\left(\frac{1}{\mu}\big(\hat f(y+\mu d)-\hat f(y)\big)\right) = \varphi^{-1}\left(\lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \frac{1}{\mu}\big(\hat f(y+\mu d)-\hat f(y)\big)\right).$$

**Proof.** Let us denote

$$A = \lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \varphi^{-1}\left(\frac{1}{\mu}\big(\hat f(y+\mu d)-\hat f(y)\big)\right)$$

and

$$B = \lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \frac{1}{\mu}\big(\hat f(y+\mu d)-\hat f(y)\big).$$

It is enough to prove that $\varphi(A) = B$. Therefore, we prove only that $\varphi(A) \geqslant B$ since the part $\varphi(A) \leqslant B$ can be proved in the similar way. By the definition of upper limit, for any $i$, there exist $y_i$ and $\mu_i$ such that

$$\frac{1}{\mu_i}\big(\hat f(y_i+\mu_i d)-\hat f(y_i)\big) \geqslant B - \frac{1}{i}, \quad \forall i \in \mathbb{N},$$

and the sequence $\{y_i\}_{i=1}^{+\infty}$ converges to $x$ and $\{\mu\}_{i=1}^{+\infty} \downarrow 0$.

Since $\varphi$ is strictly increasing on $\mathbb{R}$, we can write that

$$\varphi^{-1}\left(\frac{1}{\mu_i}\big(\hat{f}(y_i+\mu_i d)-\hat{f}(y_i)\big)\right) \geqslant \varphi^{-1}\left(B-\frac{1}{i}\right), \quad \forall i \in \mathbb{N}.$$

Consequently,

$$\lim_{i\to+\infty} \varphi^{-1}\left(\frac{1}{\mu_i}\big(\hat{f}(y_i+\mu_i d)-\hat{f}(y_i)\big)\right) \geqslant \lim_{i\to+\infty} \varphi^{-1}\left(B-\frac{1}{i}\right) = \varphi^{-1}(B).$$

By definition of upper limit, we have

$$A \geqslant \lim_{i\to+\infty} \varphi^{-1}\left(\frac{1}{\mu_i}\big(\hat{f}(y_i+\mu_i d)-\hat{f}(y_i)\big)\right)$$

or $\varphi(A) \geqslant B$. □

The relation between $(h,\varphi)$-generalized directional derivative and Clarke directional derivative can be given by the following theorem.

**Theorem 1.** *Let $f$ be a real valued function, $\varphi(t)$ be strictly increasing and continuous on $\mathbb{R}$, and let $\hat{f}(t) \triangleq \varphi f h^{-1}(t)$. Then*

$$f^*(x;d) = \varphi^{-1}\big(\hat{f}^\circ\big(h(x),h(d)\big)\big).$$

**Proof.** Note that

$$\begin{aligned} f^*(x;d) &= \lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \frac{1}{\mu}[\cdot]\big(f(y\oplus\mu\odot d)[-]f(y)\big) \\ &= \lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \varphi^{-1}\left(\frac{\varphi f h^{-1}(h(y)+\mu h(d)) - \varphi f h^{-1}(h(y))}{\mu}\right) \\ &= \varphi^{-1}\left(\lim_{\substack{y\to x\\ \mu\downarrow 0}} \sup \frac{\hat{f}(h(y)+\mu h(d)) - \hat{f}(h(y))}{\mu}\right) \\ &= \varphi^{-1}\big(\hat{f}^\circ\big(h(x),h(d)\big)\big). \end{aligned}$$

Here, the third equality is due to Lemma 1. □

Therefore, we can give a similar theorem as Theorem 1 about the relation between the generalized gradients.

**Theorem 2.** *Let $f$ be a real valued function, $\varphi(t)$ be strictly increasing and continuous on $\mathbb{R}$, and let $\hat{f}(t) \triangleq \varphi f h^{-1}(t)$. Then*

$$\partial^* f(x) = h^{-1}\big(\partial \hat{f}\big(h(x)\big)\big) \triangleq \big\{h^{-1}(\xi) \;\big|\; \xi \in \partial\big(\hat{f}(t)|_{t=h(x)}\big)\big\}.$$

**Proof.** We prove only that $\partial^* f(x) \subseteq h^{-1}(\partial \hat{f}(h(x)))$ since $\partial^* f(x) \supseteq h^{-1}(\partial \hat{f}(h(x)))$ can be proved in the similar way. Let $\xi^* \in \partial^* f(x)$, then

$$f^*(x,d) \geqslant \big(\xi^{*T} d\big)_{(h,\varphi)}, \quad \forall d \in \mathbb{R}^N.$$

According to Theorem 1, it follows that

$$\varphi^{-1}\big(\hat{f}^{\circ}\big(h(x), h(d)\big)\big) \geqslant \varphi^{-1}\big(h(\xi^*)^T h(d)\big), \quad \forall d \in \mathbb{R}^N,$$

or

$$\hat{f}^{\circ}\big(h(x), h(d)\big) \geqslant \big(h(\xi^*)^T h(d)\big), \quad \forall d \in \mathbb{R}^N.$$

Taking $y = h(d)$ and noting that $h$ is one to one mapping, we can conclude that

$$\hat{f}^{\circ}\big(h(x), y\big) \geqslant \big(h(\xi^*)^T y\big), \quad \forall y \in \mathbb{R}^N.$$

The last inequality shows that $h(\xi^*) \in \partial \hat{f}(h(x))$ or $\xi^* \in h^{-1}(\partial \hat{f}(h(x)))$ which completes the proof. $\square$

Note that Theorems 1 and 2 are very important results. Based on these theorems, we can easily show that many important properties of Clarke generalized directional derivative and gradient in [6] are valid for $(h, \varphi)$-generalized directional derivative and gradient. For instance, we can give Propositions 2–5.

**Proposition 2.** *Let $f$ be a real valued function defined on $\mathbb{R}^N$. Suppose that $\hat{f}(t) \triangleq \varphi f h^{-1}(t)$ be Lipschitz near $x$. Then, we have*:

(1) *the function $x \to f^*(x; d)$ is well defined and, for $\lambda \in \mathbb{R}^+$, satisfies the following*:

$$f^*(x; \lambda \odot d) = \lambda[\cdot] f^*(x; d), \qquad f^*\big(x; (-1) \odot d\big) = \big((-1)[\cdot] f\big)^*(x; d);$$

(2) $f^*(x; d_1 \oplus d_2) \leqslant f^*(x; d_1)[+] f^*(x; d_2)$;

(3) $\partial^* f(x)$ *is nonempty and, for every $v \in \mathbb{R}^N$, the following holds*:

$$f^*(x; v) = \max\big\{\big(\xi^{*T} v\big)_{(h,\varphi)} \;\big|\; \xi^* \in \partial^* f(x)\big\}.$$

**Proof.** These properties can be easily obtained using Theorems 1, 2 and the properties of Clarke directional derivative [6]. $\square$

Let us consider some classical derivatives with respect to the function $\hat{f}(t) \triangleq \varphi f h^{-1}(t)$ and the following definition.

**Definition 13.** Let $f$ be a real valued function defined on $\mathbb{R}^N$ and let $\hat{f}(t) \triangleq \varphi f h^{-1}(t)$. $f$ is said to be $(h, \varphi)$-Gâteaux (Hadamard, strict or Fréchet) differentiable at $x$ if $\hat{f}$ admits a Gâteaux (Hadamard, strict or Fréchet) derivative $D\hat{f}(t)$ at $t = h(x)$. Here, $D^* f(x) \triangleq h^{-1}(D\hat{f}(t|_{t=h(x)}))$ denotes the corresponding $(h, \varphi)$-Gâteaux (Hadamard, strict or Fréchet) derivative at $x$.

**Proposition 3.** *Let $f$ be a real valued function defined on $\mathbb{R}^N$ and $\hat{f}(t) \triangleq \varphi f h^{-1}(t)$ be Lipschitz on $\mathbb{R}^N$. If $f$ admits a $(h, \varphi)$-Gâteaux (Hadamard, strict or Fréchet) derivative $D^* f(x)$ at $x$, then $D^* f(x) \in \partial^* f(x)$.*

**Proof.** According to Proposition 2.2.2 in [6], we have

$$D\hat{f}(t|_{t=h(x)}) \in \partial \hat{f}(t|_{t=h(x)}).$$

Therefore, by Theorem 2, we can write that

$$D^* f(x) = h^{-1}\big(D\hat{f}(t|_{t=h(x)})\big) \in h^{-1}\big(\partial \hat{f}(t|_{t=h(x)})\big) = \partial^* f(x). \qquad \square$$

**Definition 14.** Let $f$ be a real valued function defined on $\mathbb{R}^N$ and let $\hat{f}(t) \triangleq \varphi f h^{-1}(t)$. $f$ is said to be $(h,\varphi)$-regular at $x$ if $\hat{f}$ is regular at $t = h(x)$.

**Proposition 4.** *Let $f$ be a real valued function defined on $\mathbb{R}^N$ and let $\hat{f}_i(t) \triangleq \varphi f_i h^{-1}(t)$ $(i = 1,\dots,m)$ be Lipschitz on $\mathbb{R}^N$. Then*

(1) *for any $s_1 \in R$, $\partial^*(s_1[\cdot]f_1)(x) = s_1 \odot \partial^* f_1(x)$;*
(2) *for any $s_i \in R (i = 1,\dots,m)$, $\partial^*([\sum]_{i=1}^m s_i[\cdot]f_i)(x) \subset \bigoplus_{i=1}^m s_i \odot \partial^* f_i(x)$;*
(3) *let $f_i$ be $(h,\varphi)$-regular at $x$, and $s_i \in R^+$ $(i = 1,\dots,m)$, then*

$$\partial^*\left(\left[\sum_{i=1}^m\right] s_i[\cdot]f_i\right)(x) = \bigoplus_{i=1}^m s_i \odot \partial^* f_i(x).$$

**Proof.** By Propositions 2.3.1, 2.3.2 and Corollary 2 in [6] and Theorem 2 we can derive the results directly. □

**Proposition 5.** *Let $f$ be a real valued function defined on $\mathbb{R}^N$. Suppose that $\varphi f h^{-1}(t)$ is Lipschitz on an open set containing the line segment $[h(x), h(y)]$. Then there exists a point $\lambda \in (0,1)$ such that*

$$f(y)[-]f(x) \in \big(\partial^* f\big(\lambda \odot x \oplus (1-\lambda) \odot y\big), y \ominus x\big)_{(h,\varphi)}.$$

**Proof.** Using Theorem 2 and Theorem 2.3.7 in [6], we can derive the result. □

## 4. Generalized convexity and monotonicity

In this section, we discuss the relations between $(h,\varphi)$-generalized convexity and $(h,\varphi)$-generalized monotonicity. Throughout this section we assume that $\varphi(t)$ is strictly increasing and continuous on $\mathbb{R}$ and $\varphi(0) = 0$.

**Theorem 3.** *Let $f(x)$ and $\hat{f}(x) = \varphi f h^{-1}(x)$ be Lipschitz on $\mathbb{R}^N$. Then the following statements are equivalent*:

(1) *$f$ is $(h,\varphi)$-convex on $\mathbb{R}^N$;*
(2) *$\hat{f}$ is convex on $\mathbb{R}^N$;*
(3) *$\partial \hat{f}$ is monotonic on $\mathbb{R}^N$;*
(4) *$\partial^* f$ is $(h,\varphi)$-monotonic on $\mathbb{R}^N$.*

**Proof.** We will prove that (1) ⇔ (2) ⇔ (3) ⇔ (4).

(1) ⇔ (2). This part can be checked very easily using the definition of the algebraic operations.

(2) ⇔ (3). The proof is due to [6, Proposition 2.2.9].

(3) ⇔ (4). If $\partial \hat{f}$ is monotonic on $\mathbb{R}^N$, then

$$(y_2 - y_1)^T(\eta - \xi) \geqslant 0, \quad \forall \xi \in \partial \hat{f}(y_1),\ \eta \in \partial \hat{f}(y_2). \tag{1}$$

Let $x_1 = h^{-1}(y_1)$ and $x_2 = h^{-1}(y_2)$. Then according to Theorem 2, (1) implies that

$$\big(h(x_2) - h(x_1)\big)^T\big(h(\eta^*) - h(\xi^*)\big) \geqslant 0, \quad \forall \xi^* \in \partial^* f(x_1),\ \eta^* \in \partial^* f(x_2),$$

where $\xi^* = h^{-1}(\xi)$, $\eta^* = h^{-1}(\eta)$.

Therefore,

$$\big((x_2 \ominus x_1)^T(\eta^* \ominus \xi^*)\big)_{(h,\varphi)} \geqslant 0, \quad \forall \xi^* \in \partial^* f(x_1),\ \eta^* \in \partial^* f(x_2).$$

Moreover, the above steps are invertible, so the result follows. □

**Theorem 4.** *Let $f(x)$ and $\hat f(x) = \varphi f h^{-1}(x)$ be Lipschitz on $\mathbb{R}^N$. Then the following statements are equivalent*:

(1) *$f$ is $(h,\varphi,\alpha)$-strongly convex on $\mathbb{R}^N$*;
(2) *$\hat f$ is $\varphi(\alpha)$-strongly convex on $\mathbb{R}^N$*;
(3) *$\partial \hat f$ is $2\varphi(\alpha)$-strongly monotonic on $\mathbb{R}^N$*;
(4) *$\partial^* f$ is $(h,\varphi,\alpha)$-monotonic on $\mathbb{R}^N$.*

**Proof.** (1) ⇔ (2) and (3) ⇔ (4) can be proved on simple lines as the corresponding parts of Theorem 3. We now prove that (2) ⇔ (3).

If $\hat f$ is $\varphi(\alpha)$-strongly convex on $\mathbb{R}^N$, then, for all $y_1, y_2 \in \mathbb{R}^N$, we have

$$\hat f(y_2) - \hat f(y_1) \geqslant \xi^T(y_2 - y_1) + \varphi(\alpha)\|y_2 - y_1\|^2, \quad \forall \xi \in \partial\big(\hat f(y_1)\big),$$
$$\hat f(y_1) - \hat f(y_2) \geqslant \mu^T(y_1 - y_2) + \varphi(\alpha)\|y_2 - y_1\|^2, \quad \forall \mu \in \partial\big(\hat f(h_2)\big).$$

Adding the above inequalities together, we obtain

$$(y_1 - y_2)^T(\xi - \eta) \geqslant 2\varphi(\alpha)(y_1 - y_2)^T(y_1 - y_2), \quad \forall \xi \in \partial\big(\hat f(y_1)\big),\ \mu \in \partial\big(\hat f(y_2)\big),$$

which shows $\partial \hat f$ is $2\varphi(\alpha)$-strongly monotonic on $\mathbb{R}^N$.

Conversely, for any pair of points $y_1, y_2 \in \mathbb{R}^N$, let

$$\psi(\lambda) = \hat f\big((1-\lambda)y_1 + \lambda y_2\big), \quad \lambda \in [0,1].$$

Then $\psi(\lambda)$ is Lipschitz and differentiable almost everywhere on $\mathbb{R}$ [2]. Let us denote the set of nondifferentiable points of $\psi(\lambda)$ by $\Delta$, and define

$$\psi'(\lambda) = \begin{cases} \psi'(\lambda), & \lambda \in \Delta \setminus [0,1], \\ \lim_{\lambda' \to \lambda,\, \lambda' \in \Delta\setminus[0,1]} \psi'(\lambda'), & \lambda \in \Delta. \end{cases}$$

According to [6, Theorem 2.5.1], we have

$$\psi'(\lambda) \in \partial\psi(\lambda) \subset (y_2 - y_1)^T \partial \hat f\big((1-\lambda)y_1 + \lambda y_2\big).$$

Since $\partial \hat f$ is $2\varphi(\alpha)$-strongly monotonic on $\mathbb{R}^N$, from the above, we can conclude that

$$\big((1-\lambda)y_1 + \lambda y_2 - y_1\big)^T(\xi_\lambda - \xi) \geqslant 2\varphi(\alpha)\big\|\big((1-\lambda)y_1 + \lambda y_2\big) - y_1\big\|^2,$$

for all $\xi \in \partial \hat f(y_1)$, $\xi_\lambda \in \partial \hat f((1-\lambda)y_1 + \lambda y_2)$.

We can rewrite the above as

$$(y_2 - y_1)^T \xi_\lambda \geqslant (y_2 - y_1)^T \xi + 2\varphi(\alpha)\|y_2 - y_1\|^2.$$

Therefore,

$$\hat{f}(y_2) - \hat{f}(y_1) = \psi(1) - \psi(0) = \int_0^1 \psi'(\lambda)\, d\lambda \geqslant \int_0^1 \big((y_2 - y_1)^T \xi + 2\varphi(\alpha)\|y_2 - y_1\|^2\big)\, d\lambda$$
$$\geqslant (y_2 - y_1)^T \xi + \frac{\varphi(\alpha)}{2}\|y_2 - y_1\|^2.$$

This shows that $\hat{f}$ is $\varphi(\alpha)$-strongly convex on $\mathbb{R}^N$. $\square$

The properties in the above theorem can be kept for generalized pseudoconvex functions. Next, we state the following theorems without proofs since they can be obtained similarly as that in the previous theorem.

**Theorem 5.** *Let $f(x)$ and $\hat{f}(x) = \varphi f h^{-1}(x)$ be Lipschitz on $\mathbb{R}^N$. Then the following statements are equivalent*:

(1) *$f$ is $(h, \varphi)$-pseudoconvex on $\mathbb{R}^N$*;
(2) *$\hat{f}$ is pseudoconvex on $\mathbb{R}^N$*;
(3) *$\partial(\hat{f})$ is pseudomonotonic on $\mathbb{R}^N$*;
(4) *$\partial^* f$ is $(h, \varphi)$-pseudomonotonic on $\mathbb{R}^N$.*

**Theorem 6.** *Let $f(x)$ and $\hat{f}(x) = \varphi f h^{-1}(x)$ be Lipschitz on $\mathbb{R}^N$. Then the following statements are equivalent*:

(1) *$f$ is $(h, \varphi)$-strictly convex on $\mathbb{R}^N$*;
(2) *$\hat{f}$ is strictly convex on $\mathbb{R}^N$*;
(3) *$\partial(\hat{f})$ is strictly monotonic on $\mathbb{R}^N$*;
(4) *$\partial^* f$ is $(h, \varphi)$-strictly monotonic on $\mathbb{R}^N$.*

**Theorem 7.** *Let $f(x)$ and $\hat{f}(x) = \varphi f h^{-1}(x)$ be Lipschitz on $\mathbb{R}^N$. Then the following statements are equivalent*:

(1) *$f$ is $(h, \varphi)$-strictly pseudoconvex on $\mathbb{R}^N$*;
(2) *$\hat{f}$ is strictly pseudoconvex on $\mathbb{R}^N$*;
(3) *$\partial(\hat{f})$ is strictly pseudomonotonic on $\mathbb{R}^N$*;
(4) *$\partial^* f$ is $(h, \varphi)$-strictly pseudomonotonic on $\mathbb{R}^N$.*

**Lemma 2.** *Let $f(x)$ and $\hat{f}(x) = \varphi f h^{-1}(x)$ be Lipschitz on $\mathbb{R}^N$. Then*

(1) *$f$ is $(h, \varphi, \alpha)$-strongly pseudoconvex on $\mathbb{R}^N$ if and only if $\hat{f}$ is $\varphi(\alpha)$-strongly pseudoconvex on $\mathbb{R}^N$*;
(2) *$\partial^* f$ is $(h, \varphi)$-strongly pseudomonotonic on $\mathbb{R}^N$ if and only if $\partial(\hat{f})$ is $\varphi(\alpha)$-strongly pseudomonotonic on $\mathbb{R}^N$.*

**Theorem 8.** *Let $f(x)$ and $\hat{f}(x) = \varphi f h^{-1}(x)$ be Lipschitz on $\mathbb{R}^N$. If $\partial^* f$ is $(h, \varphi)$-strongly pseudomonotonic on $\mathbb{R}^N$, then $f$ is $(h, \varphi, \alpha)$-strongly pseudoconvex on $\mathbb{R}^N$. Furthermore, suppose that $\hat{f}$ is differentiable on $\mathbb{R}^N$. Then $\partial^* f$ is $(h, \varphi)$-strongly pseudomonotonic on $\mathbb{R}^N$ if and only if $f$ is $(h, \varphi, \alpha)$-strongly pseudoconvex on $\mathbb{R}^N$.*

Theorems 6 and 7 can be proved on simple lines as Theorems 3 and 5, respectively. The proof of Lemma 2 is analogous to the corresponding parts of Theorem 5. The first part of Theorem 8 can be proved similarly as (3) $\Rightarrow$ (2) part of Theorem 4. Then applying Lemma 2 and the differentiability assumption of $\hat{f}$, we can derive the second part.

## 5. Monotonic averages of $\varphi$-convex functions

The monotonicity of some averages of the values of a convex function at $n$ equally spaced points was studied by Bennett and Jameson in [3]. They obtained several interesting properties of these averages and relations between them. In this section we consider these averages from the viewpoint of generalized convex functions. In particularly, $\varphi$-convex functions are discussed. Following Bennett and Jameson, we define four averages of the values of a function at $n$ equally spaced points based on the algebraic operations by Ben Tal [4] as follows.

**Definition 15.** Define

$$s_n(f) = \frac{1}{n}[\cdot]\left[\sum_{r=0}^{n-1}\right] f\left(\frac{r}{n}\right), \qquad S_n(f) = \frac{1}{n}[\cdot]\left[\sum_{r=1}^{n}\right] f\left(\frac{r}{n}\right),$$

$$A_n(f) = \frac{1}{n-1}[\cdot]\left[\sum_{r=1}^{n-1}\right] f\left(\frac{r}{n}\right) \quad (n \geqslant 2),$$

$$B_n(f) = \frac{1}{n+1}[\cdot]\left[\sum_{r=0}^{n}\right] f\left(\frac{r}{n}\right) \quad (n \geqslant 0).$$

Obviously, if $\varphi(\alpha) = \alpha$, for all $\alpha \in \mathbb{R}$, then the above averages are the averages in [3] which have many good properties under the convexity assumption of $f$. These properties include the following.

**Lemma 3.** *Let $f$ be convex on* $[0, 1]$*, then $A_n(f)$ increases with $n$, and $B_n(f)$ decreases with $n$.*

**Theorem 9.** *Let $f$ be convex on* $[0, 1]$*, then $s_n(f)$ increases with $n$, and $S_n(f)$ decreases with $n$.*

Bennett and Jameson [3] presented a detailed proof for Theorem 9. However, their proof was quite lengthy. Now we give below a very simple proof for Theorem 9.

**Proof of Theorem 9.** When $f$ is a monotonic and convex function, Theorem 3A in [3] gives us the proof. So, we assume that $f$ is nonmonotonic and convex. Then there exists $x_0 \in (0, 1)$, such that

$$f(x) \geqslant f(x_0)$$

and $f$ is decreasing on interval $[0, x_0]$ and increasing on interval $[x_0, 1]$. Furthermore, without loss of generality, we can assume that $f(x_0) = 0$ (if not we consider the function $g(x) = f(x) - f(x_0)$). Now we consider the following two functions:

$$f_1(x) = \begin{cases} f(x), & x \in [0, x_0], \\ 0 & (x_0, 1], \end{cases} \qquad f_2(x) = \begin{cases} 0, & x \in [0, x_0], \\ f(x), & x \in (x_0, 1]. \end{cases}$$

It is easy to verify that $f_1$ and $f_2$ are monotonic and convex. Furthermore, $f_1(x)+f_2(x)=f(x)$ when $x\in[0,1]$. Hence

$$s_n(f)=s_n(f_1)+s_n(f_2),\qquad S_n(f)=S_n(f_1)+S_n(f_2).$$

Now applying [3, Theorem 3A], we obtain the result. □

Next, we show that the monotonicity of averages defined by Definition 15 will be kept under the $\varphi$-convexity assumption.

**Theorem 10.** *Let $\varphi$ be increasing and convex on $[0,1]$, and $f$ be increasing and $\varphi$-convex on $[0,1]$. Then $B_n(f)\geqslant A_n(f)$.*

**Proof.** Note that $\varphi^{-1}$ is increasing. Since $\varphi$ is convex on $[0,1]$, we can write the following:

$$\varphi\left(\frac{r}{n}\right)=\varphi\left(\left(1-\frac{r}{n}\right)\cdot 0+\frac{r}{n}\cdot 1\right)\leqslant\left(1-\frac{r}{n}\right)\varphi(0)+\frac{r}{n}\varphi(1)$$

or

$$\frac{r}{n}\leqslant\varphi^{-1}\left(\left(1-\frac{r}{n}\right)\varphi(0)+\frac{r}{n}\varphi(1)\right)=\left(1-\frac{r}{n}\right)[\cdot]0[+]\frac{r}{n}[\cdot]1.$$

Using this statement, we have

$$f\left(\frac{r}{n}\right)\leqslant f\left[\left(1-\frac{r}{n}\right)[\cdot]0[+]\frac{r}{n}[\cdot]1\right]\leqslant\left(1-\frac{r}{n}\right)[\cdot]f(0)[+]\frac{r}{n}[\cdot]f(1),\quad r=0,1,\dots,n.$$

Therefore,

$$f\left(\frac{r}{n}\right)[+]f\left(1-\frac{r}{n}\right)\leqslant f(0)[+]f(1)$$

or

$$A_n(f)\leqslant\frac{1}{2}[\cdot]\big(f(0)[+]f(1)\big)=B_1(f).$$

Hence,

$$B_n(f)=\frac{n-1}{n+1}[\cdot]A_n(f)[+]\frac{2}{n+1}[\cdot]B_1(f)\geqslant A_n(f).\qquad\Box$$

**Theorem 11.** *Let $\varphi$ be increasing and convex on $[0,1]$, and $f$ be increasing and $\varphi$-convex on $[0,1]$. Then, $A_n(f)$ increases with n, and $B_n(f)$ decreases with n.*

**Proof.** Obviously,

$$\frac{r}{n}=\frac{n-r}{n}\frac{r}{n+1}+\frac{r}{n}\frac{r+1}{n+1},\quad r=1,\dots,n.\tag{2}$$

Based on (2), we write that

$$f\left(\frac{r}{n}\right)\leqslant f\left(\left(1-\frac{r}{n}\right)[\cdot]\frac{r}{n+1}[+]\frac{r}{n}[\cdot]\frac{r+1}{n+1}\right)\leqslant\left(1-\frac{r}{n}\right)[\cdot]a_r[+]\frac{r}{n}[\cdot]a_{r+1},$$

where $a_r=f\big(\frac{r}{n+1}\big)$, $r=1,\dots,n$.

Therefore, it follows that

$$\begin{aligned}A_n(f) &= \frac{1}{n-1}[\cdot]\left[\sum_{r=1}^{n-1}\right] f\left(\frac{r}{n}\right) = \varphi^{-1}\left(\frac{1}{n-1}\varphi\left(\left[\sum_{r=1}^{n-1}\right] f\left(\frac{r}{n}\right)\right)\right)\\ &\leqslant \varphi^{-1}\left(\frac{1}{n-1}\varphi\left(\left[\sum_{r=1}^{n-1}\right]\left(\left(1-\frac{r}{n}\right)[\cdot]a_r[+]\frac{r}{n}[\cdot]a_{r+1}\right)\right)\right)\\ &= \varphi^{-1}\left(\frac{r}{n}\varphi\big(a_1[+]a_2[+]\cdots[+]a_n\big)\right) = A_{n+1}(f).\end{aligned}$$

Thus $A_n(f)$ increases with $n$. We also note that

$$\frac{r}{n} = \frac{r}{n}\frac{r-1}{n-1} + \left(1-\frac{r}{n}\right)\frac{r}{n-1}.$$

Then the rest part of the theorem can be proved similarly as that we have just done above. □

**Theorem 12.** *Let $\varphi$ be increasing and convex on* $[0, 1]$ *and* $\varphi(0) = 0$. *Moreover, let $f$ be increasing and $\varphi$-convex on* $[0, 1]$. *Then $s_n(f)$ increases with $n$, and $S_n(f)$ decreases with $n$.*

**Proof.** Without loss of generality we can assume that $f(0) = 0$. If not, we simply consider the function $g(x) = f(x)[-]f(0)$, and it is easy to verify that $g(x)$ is increasing and $\varphi$-convex on $[0, 1]$, and $g(0) = 0$. Therefore, it follows that

$$s_n(f) = \left(1-\frac{1}{n}\right)[\cdot]A_n(f), \qquad S_n(f) = \left(1+\frac{1}{n}\right)[\cdot]B_n(f).$$

By Theorem 11, the result is obtained. □

We note that the similar results as above can be obtained for $(h, \varphi)$-convex functions under some additional assumptions.

## 6. $(\varphi, \gamma)$-Convexity and its properties

In this section, we introduce $(\varphi, \gamma)$-convexity, generalization of $\gamma$-convexity [9–11], on real line and establish some properties of $(\varphi, \gamma)$-convex functions. Throughout this section we use the following assumptions.

Let $\gamma$ be a positive and fixed real number, $D$ be an interval in $\mathbb{R}$, $\varphi$ be increasing on $D$, $\varphi(0) \geqslant 0$ and $\sup_{t_1,t_2\in D}(\varphi(t_2) - \varphi(t_1)) \geqslant \varphi(\gamma)$.

**Definition 16.** Let $f$ be a real valued function defined on $D$. If, for all $t_1, t_2 \in D$, the relation

$$t_2[-]t_1 \geqslant \gamma \quad\Rightarrow\quad f\big(t_1[+]\gamma\big)[+]f\big(t_2[-]\gamma\big) \leqslant f(t_1)[+]f(t_2)$$

holds, then $f$ is said to be $(\varphi, \gamma)$-convex on $D$.

The relation between $\varphi$-convexity and $(\varphi, \gamma)$-convexity is given by the following proposition.

**Proposition 6.** *If $f$ is $\varphi$-convex on $D$, then $f$ is $(\varphi, \gamma)$-convex on $D$ for any real number $\gamma > 0$.*

**Proof.** Since $f$ is $\varphi$-convex on $D$, we have

$$t_2[-]t_1 \geqslant \gamma \quad\Leftrightarrow\quad \varphi(t_2) - \varphi(t_1) \geqslant \varphi(\gamma).$$

Let $\lambda \triangleq \frac{\varphi(\gamma)}{\varphi(t_2)-\varphi(t_1)}$. Then $\lambda \in (0,1)$, and it is easy to verify that

$$t_1[+]\gamma = \lambda[\cdot]t_2[+](1-\lambda)[\cdot]t_1,\ t_2[-]\gamma = (1-\lambda)[\cdot]t_2[+]\lambda[\cdot]t_1.$$

Hence

$$f\big(t_1[+]\gamma\big) \leqslant \lambda[\cdot]f(t_2)[+](1-\lambda)[\cdot]f(t_1) = \varphi^{-1}\big(\varphi\big((1-\lambda)\varphi(t_1)+\lambda\varphi(t_2)\big)\big), \tag{3}$$

$$f\big(t_2[-]\gamma\big) \leqslant (1-\lambda)[\cdot]f(t_2)[+]\lambda[\cdot]f(t_1) = \varphi^{-1}\big(\varphi\big(\lambda\varphi(t_1)+(1-\lambda)\varphi(t_2)\big)\big). \tag{4}$$

Combining (3), (4) with the monotonicity of $\varphi$ and according to the algebraic operation, we have

$$f\big(t_1[+]\gamma\big)[+]f\big(t_2[-]\gamma\big) \leqslant f(t_1)[+]f(t_2). \qquad \square$$

**Theorem 13.** *Let $\gamma$ be a positive real number and $f$ be a real valued function defined on $D$. Then the following statements are equivalent*:

(1) *$f$ is $(\varphi,\gamma)$-convex on $D$*;
(2) *$\varphi f \varphi^{-1}$ is $\varphi(\gamma)$-convex* [10] *on $\varphi(D)$, where $\varphi(D) = \{\varphi(x) \mid x \in D\}$*;
(3) *let $h^{\varphi}_{(f,\gamma)}(t) = \frac{1}{\gamma}[\cdot](f(t[+]\gamma)[-]f(t))$. Then $h^{\varphi}_{(f,\gamma)}(t)$ is nondecreasing, i.e.,*

$$h^{\varphi}_{(f,\gamma)}(t') \leqslant h^{\varphi}_{(f,\gamma)}(t''), \quad \text{if } \forall t' \leqslant t'' \text{ and } \big\{t', t''[+]\gamma\big\} \subset D.$$

**Proof.** (1) $\Leftrightarrow$ (2). By the definition, $f$ is $(\varphi,\gamma)$-convex on $D$ if and only if

$$\begin{aligned}&\varphi^{-1}\big(\varphi f\big(\varphi^{-1}\big(\varphi(t_1)+\varphi(\gamma)\big)\big)+\varphi f\big(\varphi^{-1}\big(\varphi(t_2)-\varphi(\gamma)\big)\big)\big)\\ &\quad\leqslant \varphi^{-1}\big(\varphi f\big(\varphi^{-1}\big(\varphi(t_1)\big)\big)+\varphi f\big(\varphi^{-1}\big(\varphi(t_2)\big)\big)\big).\end{aligned}$$

Letting $s_1 = \varphi(t_1)$ and $s_2 = \varphi(t_2)$, we obtain the desired result.

(1) $\Leftrightarrow$ (3). We can easily verify that

$$h_{(\varphi f\varphi^{-1},\varphi(\gamma))}\big(\varphi(t)\big)\cdot\frac{\varphi(\gamma)}{\gamma} = \varphi\big(h^{\varphi}_{(f,\gamma)}(t)\big). \tag{5}$$

By [9, Theorem 2.1], $\varphi f\varphi^{-1}$ is $\varphi(\gamma)$-convex on $\varphi(D)$ iff $h_{(\varphi f\varphi^{-1},\varphi(\gamma))}$ is nondecreasing on $\varphi(D)$. Therefore, due to (5), $\varphi f\varphi^{-1}$ is $\varphi(\gamma)$-convex on $\varphi(D)$ iff $h^{\varphi}_{(f,\gamma)}(t)$ is nondecreasing. $\square$

**Definition 17.** The set defined by

$$\begin{aligned}\partial^{\varphi}_{\gamma} f(t) \triangleq \big\{\xi \in \mathbb{R} \bigm| \exists t',\ t'' \in \big[t[-]\gamma, t\big] \cap D:&\\ f\big(t'[+]\gamma\big)[-]f(t') \leqslant \gamma[\cdot]\xi \leqslant f\big(t''[+]\gamma\big)[-]f(t'')\big\}&\end{aligned}$$

is called $(\varphi,\gamma)$-subdifferential of $f$ at $t$.

Next, we explore very important characterization of $(\varphi,\gamma)$-subdifferential by the following theorem.

**Theorem 14.**

$$\begin{aligned}\partial^{\varphi}_{\gamma} f(t) &= \mathrm{co}\big\{h^{\varphi}_{(f,\gamma)}(t') \bigm| t \in \big[t', t'[+]\gamma\big]\big\}\\ &\triangleq \big\{\lambda[\cdot]h^{\varphi}_{(f,\gamma)}(t_1)[+](1-\lambda)[\cdot]h^{\varphi}_{f,\gamma}(t_2) \bigm| t \in \big[t_1, t_1[+]\gamma\big],\\ &\qquad\qquad t \in \big[t_2, t_2[+]\gamma\big],\ \forall\lambda \in (0,1)\big\}.\end{aligned}$$

**Proof.** We first prove that if $\forall \xi_1,\ \xi_2 \in \partial_\gamma^\varphi f(t)$, then $\Rightarrow \lambda[\cdot]\xi_1[+](1-\lambda)[\cdot]\xi_2 \in \partial_\gamma^\varphi f(t)$. When $\xi_1, \xi_2 \in \partial_\gamma^\varphi f(t)$, there exist $t_1, t_1', t_2, t_2' \in [t[-]\gamma, t] \cap D$ such that

$$f\big(t_1[+]\gamma\big)[-]f(t_1) \leqslant \gamma[\cdot]\xi_1 \leqslant f\big(t_1'[+]\gamma\big)[-]f\big(t_1'\big),$$
$$f\big(t_2[+]\gamma\big)[-]f(t_2) \leqslant \gamma[\cdot]\xi_2 \leqslant f\big(t_2'[+]\gamma\big)[-]f\big(t_2'\big).$$

Hence,

$$\frac{\varphi f(t_1[+]\gamma) - \varphi f(t_1)}{\gamma} \leqslant \varphi(\xi_1) \leqslant \frac{\varphi f(t_1'[+]\gamma) - \varphi f(t_1')}{\gamma},$$
$$\frac{\varphi f(t_2[+]\gamma) - \varphi f(t_2)}{\gamma} \leqslant \varphi(\xi_2) \leqslant \frac{\varphi f(t_2'[+]\gamma) - \varphi f(t_2')}{\gamma}. \tag{6}$$

Let

$$a = \min\left\{\frac{\varphi f(t_1[+]\gamma) - \varphi f(t_1)}{\gamma}, \frac{\varphi f(t_2[+]\gamma) - \varphi f(t_2)}{\gamma}\right\}$$

and

$$b = \max\left\{\frac{\varphi f(t_1'[+]\gamma) - \varphi f(t_1')}{\gamma}, \frac{\varphi f(t_2'[+]\gamma) - \varphi f(t_2')}{\gamma}\right\}.$$

Then, by the above inequalities, we can write that

$$a \leqslant \lambda\varphi(\xi_1) + (1-\lambda)\varphi(\xi_2) \leqslant b \quad \text{or} \quad a \leqslant \varphi\big(\lambda[\cdot]\xi_1[+](1-\lambda)[\cdot]\xi_2\big) \leqslant b.$$

Therefore,

$$\lambda[\cdot]\xi_1[+](1-\lambda)[\cdot]\xi_2 \in \partial_\gamma^\varphi f(t).$$

Next, we prove our that $\partial_\gamma^\varphi f(t) = \mathrm{co}\{h_{(f,\gamma)}^\varphi(t') \mid t \in [t', t'[+]\gamma]\}$ based on the above property of $\partial_\gamma^\varphi f(t)$. Consider $\xi = h_{f,\gamma}^\varphi(t_1)$ such that $t \in [t_1, t_1[+]\gamma]$ and $t_1 \in D$. Then, taking $t' = t'' = t_1$, we can say that $\xi \in \partial_\gamma^\varphi f(t)$ by Definition 16. Hence, due to the above property we just proved, $\mathrm{co}\{h_{(f,\gamma)}^\varphi(t') \mid t \in [t', t'[+]\gamma]\} \subseteq \partial_\gamma^\varphi f(t)$.

On the other hand, let $\xi_1 \in \partial_\gamma^\varphi f(t)$. Then there exist $t_1, t_1' \in [t[-]\gamma, t] \cap D$ such that (6) holds. Therefore,

$$c_1 \triangleq h_{(f,\gamma)}^\varphi(t_1) \leqslant \xi_1 \leqslant h_{(f,\gamma)}^\varphi\big(t_1'\big) \triangleq c_2.$$

Denoting $\lambda \triangleq \frac{\varphi(\xi)-\varphi(c_1)}{\varphi(c_2)-\varphi(c_1)}$, we have

$$\xi = \lambda[\cdot]c_2[+](1-\lambda)[\cdot]c_1 \in \mathrm{co}\big\{h_{f,\gamma}^\varphi(t') \,\big|\, t \in \big[t', t'[+]\gamma\big]\big\}. \qquad \square$$

**Theorem 15.** *Let $\gamma$ be a positive real number and $f$ be a real valued function on $D$. Then, the following statements are equivalent*:

(1) *$f$ is $(\varphi, \gamma)$-convex on $D$*;
(2) *$\xi \leqslant \eta$ for all $\xi \in \partial_\gamma^\varphi f(t)$, $\eta \in \partial_\gamma^\varphi f(t[+]\gamma)$ and $\{t, t[+]\gamma\} \subset D$.*

**Proof.** Since $\{t, t[+]\gamma\} \subset D$, by Theorem 14, we have

$$\partial_\gamma^\varphi f(t) = \mathrm{co}\big\{h_{f,\gamma}^\varphi(t') \,\big|\, t' \in \big[t[-]\gamma, t\big]\big\},$$
$$\partial_\gamma^\varphi f\big(t[+]\gamma\big) = \mathrm{co}\big\{h_{f,\gamma}^\varphi(t'') \,\big|\, t'' \in \big[t, t[+]\gamma\big]\big\}.$$

These together with (5) and [9, Theorem 2.3] give the result. $\square$

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