Uniqueness of limit cycles in predator–prey system:
the role of weight functions

Karel Hasík

Institute of Mathematics, Silesian University, Bezručovo nam. 13, Opava 746 01, Czech Republic
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Abstract
We consider a Gause type model of interactions between predator and prey populations. Using the
ideas of Cheng and Liou we give a sufficient condition for uniqueness of the limit cycle, which is
more general than their condition. That is, we include a kind of weight function in the condition. It
was motivated by a result due to Hwang, where the prey isocline plays a role of weight function.
Moreover, we show that the interval where the condition from Hwang’s result is to be fulfilled can
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1. Introduction
The main purpose of this paper is to establish conditions that ensure uniqueness of the
limit cycle of the predator–prey system
\[
\begin{align*}
x' &= xg(x) - yp(x), \\
y' &= y[q(x) - y]
\end{align*}
\] (1)
\[(x(0) \geq 0, \ y(0) \geq 0)\] which was introduced by Gause [2].

The function \(g(x)\) represents the relative increase of the prey in terms of its density. For
low densities the number of offspring is greater than the number who have died, and so
$g(x)$ is positive. As the density increases, living conditions deteriorate and the death-rate is greater than the birth-rate, and hence $g(x)$ becomes negative. The function $q(x) - \gamma$ gives the total increase of the predator population. This is negative for low values of prey densities, when the prey population is insufficient to sustain the predator. The function $p(x)$, called the trophic function of the predator (or the functional response), expresses the number of prey consumed by a predator in a unit of time as a function of the density of the prey population.

System (1) has been studied in several papers in various modifications. Cheng [1] was the first one to prove uniqueness of the limit cycle for a special form of system (1), by using the symmetry of the prey isocline (the function $h(x) = xg(x)/p(x)$) with respect to its maximum. Liou and Cheng [6] generalized the method from [1] to a class of predator prey models with asymmetric prey isocline. Kuang and Freedman [5] and Huang and Merrill [4] transformed a class of the predator–prey models of the Gause type to a generalized Lienard system, where results concerning uniqueness of the limit cycle are known. System (1) with symmetric prey isocline was studied by the author [3] under the assumption $q(x) = cp(x)$.

In each of the mentioned papers, except [1], there is given a condition ensuring uniqueness of the limit cycle, which, rewritten for system (1), has the form

$$\frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - \gamma} \right) \leq 0,$$

where $h(x) = xg(x)/p(x)$. The interval where this condition is to be fulfilled is different in different papers. Recently, a similar condition, ensuring that the number of limit cycles does not exceed one, was found by Hwang [7] in the form

$$\frac{d}{dx} \left( \frac{p(x)h'(x)}{(q(x) - \gamma)(\alpha + \beta h(x))} \right) \leq 0 \text{ for } x \in (0, k) \setminus \{x^*\},$$

where $k$ is the carrying capacity of system (1) and $\alpha, \beta \geq 0$. In the present paper we study the question when the function $\alpha + \beta h(x)$ can be replaced by a more general function. We also show that the interval where condition (3) must be satisfied can be narrowed.

System (1) is studied under the following assumptions:

(i) There exists a number $k > 0$ such that

$$g(x) > 0 \text{ for } 0 \leq x < k; \quad g(k) = 0; \quad g(x) < 0 \text{ for } x > k.$$  

(ii)

$$p(0) = 0; \quad p'(x) > 0 \text{ for } x > 0; \quad p'_+(0) > 0. \quad q(0) = 0; \quad q'(x) > 0 \text{ for } x > 0; \quad q'_+(0) > 0.$$  

(iii) There exists a unique point $(x^*, y^*)$ with $0 < x^* < k, y^* > 0$ such that

$$q(x^*) - \gamma = 0, \quad x^* g(x^*) - y^* p(x^*) = 0.$$  

(iv) The prey isocline $h(x) := xg(x)/p(x)$ is a unimodal function and there exists $m, 0 < m < k$, such that $h'(x) > 0$ for $x \in (0, m)$, $h'(x) = 0$ for $x = m$ and $h'(x) < 0$ for $m < x$.  


(v) The functions $g(x)$, $p(x)$, $q(x)$ are as smooth as it is required.

The conditions (i)–(iii) are natural in the biological context described above. The last two conditions are needed for proofs.

Before stating our main result we recall the definition of transformation $T$ used by Liou and Cheng [6]. It is the mapping from $(0, m) \times (0, \infty)$ into $(m, k) \times (0, \infty)$ defined by

$$T(x, y) = \left( h_2^{-1} \circ h_1(x), y \right) = (\bar{x}, y),$$

where $h_1 = h|_{(0, m)}$, $h_2 = h|_{(m, k)}$.

2. Main result

Our main result is the following.

**Theorem 2.1.** Let for the system (1) the following assumptions be satisfied:

(i) $x^* < m$,

(ii) $\frac{d}{dx} \left( \frac{p(x)h'(x)}{(q(x) - y)(W(x))} \right) \leq 0$ for $x \in (0, x^*) \cup (\bar{x}^*, k)$,

where $W(x)$ is a smooth positive function such that $W(x) = W(\bar{x})$ for $x \in (0, x^*) \cup (\bar{x}^*, k)$, $W'(x)$ is negative for $x \in [0, x^*)$ and positive for $x \in (\bar{x}^*, k]$ and equality

$$W(x) = -\frac{(q(x) - y)}{p(x)h'(x)}$$

holds in no subinterval of the intervals $(0, x^*)$, $(\bar{x}^*, k)$.

Then system (1) possesses a unique limit cycle which is globally asymptotically stable in the positive quadrant.

This theorem is an extension of results due to Liou and Cheng [6]. Since we modify their proofs, we mainly devote attention to the places where our considerations are more general. When appropriate, we also adopt their notation. We need the following modifications of Lemmas 1 and 2 from [1] to be able to prove Theorem 2.1.

**Lemma 2.2.** Let $\Gamma$ be a nontrivial closed orbit of system (2). Then $\Gamma \subset \{(x, y), \ 0 < x < k, \ 0 < y\}$. Let $L$, $R$, $H$ and $J$ be the leftmost, rightmost, highest and lowest points of $\Gamma$, respectively. Then

$$L \in \{(x, y), \ 0 < x < x^*, \ y = h(x)\}, \quad H \in \{(x, y), \ x = x^*, \ y^* < y\},$$

$$R \in \{(x, y), \ x^* < x < k, \ y = h(x)\}, \quad J \in \{(x, y), \ x = x^*, \ 0 < y < y^*\},$$

where $h(x) = xg(x)/p(x)$.

The proof is clear and can be omitted here.
Lemma 2.3. Let $\Gamma$ be a nontrivial closed orbit of (1) and let

\begin{enumerate}[(i)]
  \item $x^* < m$,
  \item \[ \frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - \gamma(W(x))} \right) \leq 0 \quad \text{for } x \in (0, x^*) \cup (x^*, k), \]
\end{enumerate}

where $W(x)$ is the same function as in Theorem 2.1.

Then the image of the arc H\overline{L}J under the transformation $T$, i.e., the arc $\overline{H\overline{L}J}$, intersects the arc $\overline{BRA}$ exactly at two points (see Fig. 1) $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, with $y_2 < h(x_2)$ and $y_1 > h(x_1)$. Moreover, if $\overline{P} = (\bar{x}_1, \bar{y}_1)$ and $\overline{Q} = (\bar{x}_2, \bar{y}_2)$ are the images of the points $P, Q$ under the transformation $T$, then

\begin{equation}
0 > \frac{p(\bar{x}_1)h'_2(\bar{x}_1)}{q(\bar{x}_1) - \gamma} \geq \frac{p(x_1)h'_1(x_1)}{q(x_1) - \gamma} \tag{5}
\end{equation}

and

\begin{equation}
0 > \frac{p(\bar{x}_2)h'_2(\bar{x}_2)}{q(\bar{x}_2) - \gamma} \geq \frac{p(x_2)h'_1(x_2)}{q(x_2) - \gamma} \tag{6}
\end{equation}

Proof. Consider the function

\[ V(x, y) = \int_{x^*}^{x} \frac{q(\xi) - \gamma}{p(\xi)} \, d\xi + \int_{y^*}^{y} \frac{\eta - y^*}{\eta} \, d\eta \]

and obtain

\[ \frac{dV}{dt} = (q(x) - \gamma)(h(x) - y^*). \]
Moreover, if $\Gamma'$ is a periodic orbit of (1) then
\[
0 = \int_0^\tau dV = \int_{\Gamma'} (h(x) - y^*) \frac{dy}{y},
\]
where $\tau$ is the period of $\Gamma'$. Similarly as in [6], the assumption that the arc $\bar{H}\bar{L}\bar{J}$ does not intersect the arc $\text{BRA}$ leads to a contradiction. Now let $P = (x_1, y_1)$ be the highest intersection of arcs $\text{BRA}$ and $\bar{H}\bar{L}\bar{J}$. First assume that $y_1 > h(x_1)$. Similarly as in [6] inequality
\[
0 > (\frac{dy}{dx})_P \geq (\frac{dy}{dx})_{\bar{P}\bar{L}}
\]
implies inequality (5), where $(dy/dx)_P$ and $(dy/dx)_{\bar{P}\bar{L}}$ denote slopes of the arcs $\text{BRA}$ and $\bar{H}\bar{L}\bar{J}$ at point $P$, respectively.

Now we prove that arcs $\text{BRA}$ and $\bar{H}\bar{L}\bar{J}$ have exactly two intersections. The arc $\text{PR}$ satisfies the following differential equation
\[
(\frac{dy}{dx})_{\text{PR}} = \frac{y(q(x) - \gamma)}{p(x)(h_2(x) - y)} = \frac{y h_2'(x)}{h_2(x) - y} - \frac{yh_2'(x)}{h_2(x) - y}
\]
and the arc $\bar{P}\bar{L}$ satisfies (see [6])
\[
(\frac{dy}{dx})_{\bar{P}\bar{L}} = \frac{q(x_1) - \gamma}{x h_1'(x_1)} \cdot \frac{yh_2'(x)}{h_2(x) - y}.
\]
From the assumption (ii), properties of function $W(x)$ and (5) we have
\[
\frac{q(x_1) - \gamma}{x h_1'(x_1)} \cdot W(x_1) < \frac{q(x_1) - \gamma}{p(x_1)h_1'(x_1)} \cdot W(x_1)
\]
for all $x > x_1$.

Hence it follows
\[
0 > \frac{q(x) - \gamma}{x h_2'(x)} \cdot \frac{yh_2'(x)}{h_2(x) - y} \geq \frac{q(x) - \gamma}{x h_1'(x)} \cdot \frac{yh_2'(x)}{h_2(x) - y}
\]
for all $x > x_1$.

This proves (by the comparison theorem) that under assumption $y_1 > h(x_1)$ the arcs $\bar{H}\bar{L}$ and $\text{BR}$ have at most one point in common. Similar conclusion holds for the arcs $\bar{J}\bar{L}$ and $\text{AR}$ under the assumption that $y_2 < h(x_2)$. This means that $P$ cannot be one of the intersections of arcs $\bar{J}\bar{L}$ and $\text{AR}$ and $Q$ cannot be one of the intersections of arcs $\bar{H}\bar{L}$ and $\text{BR}$. Hence we have $y_1 > h(x_1)$ and $y_2 < h(x_2)$. This completes our proof. \hfill \Box

**Proof of Theorem 2.1.** Let $\Gamma$ be any nontrivial closed orbit of (1). Define
\[
e(x, y) = xg(x) - yp(x), \quad f(x, y) = y(q(x) - \gamma).
\]
Then
\[ \text{Div}(e,f) = g(x) + xp'(x) - yp'(x) + q(x) - \gamma, \]
\[ \oint_{\Gamma} (q(x) - \gamma) \, dt = \oint_{\Gamma} \frac{xg(x)}{p(x)} \, p'(x) \, dt, \]
\[ p(x)h'(x) = g(x) + xp'(x) - \frac{xg(x)}{p(x)} p'(x). \]

The last three conditions imply that
\[ \oint_{\Gamma} \text{Div}(e,f) \, dt = \left( \int_{A} \bar{Q} + \int_{\bar{Q}} \bar{P} + \int_{\bar{P}} \bar{Q} + \int_{\bar{Q}} A \right) p(x)h'(x) \, dt. \]

First consider the integral along the arcs $AQ$ and $\bar{Q}A$. Parametrizing the arc $\bar{Q}A$ by $(x, y_1(x))$, where $x_2 \leq x \leq m$, we obtain

\[ \int_{\bar{Q}A} p(x)h'(x) \, dt = \int_{x_2}^{m} \frac{h'(x)}{h(x) - y_1(x)} \, dx = \int_{x_2}^{m} \frac{h'_1(x)}{h_1(x) - y_1(x)} \, dx. \]

Now let $x = h^{-1}_1 \circ h_2(\bar{x})$, where $\bar{x} \in [m, x_2]$. Then

\[ \int_{\bar{x}_2}^{x} \frac{h'_1(x)}{h_1(x) - y_1(x)} \, dx = \int_{\bar{x}_2}^{x} \frac{h'_1(h^{-1}_1 \circ h_2(\bar{x})) \cdot (h^{-1}_1 \circ h_2(\bar{x}))' \cdot h'_2(\bar{x})}{h_1(h^{-1}_1 \circ h_2(\bar{x})) - y_1(h^{-1}_1 \circ h_2(\bar{x}))} \, d\bar{x} \]
\[ = \int_{x_2}^{m} \frac{h'_2(\bar{x})}{h_2(\bar{x}) - y_1(h^{-1}_1 \circ h_2(\bar{x}))} \, d\bar{x} \]
\[ = - \int_{m}^{x_2} \frac{h'_2(x)}{h_2(x) - y_1(h^{-1}_1 \circ h_2(x))} \, dx. \]

The arc $AQ$ can be parametrized by $(x, y_2(x))$, where $x \in [m, x_2]$. Then

\[ \int_{AQ} p(x)h'(x) \, dt = \int_{x}^{x_2} \frac{h'(x)}{h(x) - y_1(x)} \, dx = \int_{x}^{x_2} \frac{h'_2(x)}{h_2(x) - y_2(x)} \, dx. \]

Combining the last two equations, we obtain

\[ \left( \int_{\bar{Q}A} + \int_{AQ} \right) p(x)h'(x) \, dt = \int_{x}^{x_2} \frac{h'_2(x)[y_2(x) - y_1(h^{-1}_1 \circ h_2(x))]}{[h_2(x) - y_2(x)][h_2(x) - y_1(h^{-1}_1 \circ h_2(x))]} \, dx < 0. \]
Similarly we obtain
\[
\left( \int_{PB} + \int_{BP} \right) p(x)h'(x) \, dt < 0.
\]

Now we can write
\[
\left( \int_{PLQ} + \int_{RP} \right) p(x)h'(x) \, dt = \left( \int_{PLQ} + \int_{RP} \right) p(x)h'(x) \frac{y(q(x) - \gamma)}{y(q(x) - \gamma)} \, dy.
\]

Then we paste
\[
\left( \int_{PLQ} + \int_{LP} + \int_{RP} + \int_{QRP} \right) p(x)h'(x) \frac{y(q(x) - \gamma)}{y(q(x) - \gamma)} \, dy.
\]

Let the functions \( s_1(y) \) and \( \tilde{s}_1(y) \) describe the arcs \( PLQ \) and \( QLP \), respectively, and \( \tilde{x}_1 < \tilde{x}_2 \). (The case \( \tilde{x}_1 \geq \tilde{x}_2 \) can be treated by similarly.) Since the arcs \( PLQ \) and \( QLP \) lie above the intervals \((0, \tilde{x}_2)\) and \((\tilde{x}_2, 0)\), respectively, by condition (ii) in Theorem 2.1 and inequality (5) we have
\[
0 > \frac{p(s_1(y))h'(s_1(y))}{q(s_1(y)) - \gamma} W(s_1(y)) > \frac{p(\tilde{s}_1(y))h'(\tilde{s}_1(y))}{q(\tilde{s}_1(y)) - \gamma} W(\tilde{s}_1(y)).
\]

Then
\[
0 > \frac{p(s_1(y))h'(s_1(y))}{q(s_1(y)) - \gamma} > \frac{p(\tilde{s}_1(y))h'(\tilde{s}_1(y))}{q(\tilde{s}_1(y)) - \gamma}
\]

since \( W(s_1(y)) = W(\tilde{s}_1(y)) > 0 \). The arcs \( PLQ \) and \( QLP \) are traversed in counterclockwise sense so that the parametrization \( s_1(y) \) gives the arc \( PLQ \), but with the opposite orientation. Thus, because of the last inequality we obtain
\[
\left( \int_{PLQ} + \int_{QLP} \right) p(x)h'(x) \, dt
\]
\[
= - \int_{y_2}^{y_1} \frac{p(s_1(y))h'(s_1(y))}{y[q(s_1(y)) - \gamma]} \, dy + \int_{y_2}^{y_1} \frac{p(\tilde{s}_1(y))h'(\tilde{s}_1(y))}{y[q(\tilde{s}_1(y)) - \gamma]} \, dy < 0.
\]

Next since \( W(x) \) is increasing in \([\tilde{x}^2, k]\) and
\[
\frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - \gamma} \cdot \frac{1}{W(x)} \right) = \left( \frac{p(x)h'(x)}{q(x) - \gamma} \right)' \cdot \frac{1}{W(x)} + \frac{p(x)h'(x)}{q(x) - \gamma} \cdot \left( \frac{1}{W(x)} \right)' \leq 0,
\]
we have
\[
\frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - \gamma} \right) \leq 0 \quad \text{for} \ x \in [\tilde{x}^2, k].
\]

Hence by the Green theorem we get
\[
\left( \int_{\bar{P}LQ} + \int_{QRP} \right) p(x)h'(x) \, dt = \left( \int_{\bar{P}LQ} + \int_{QRP} \right) \frac{p(x)h'(x)}{y(q(x) - \gamma)} \, dy \\
= \iint_{\Omega} \frac{1}{y} \frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - \gamma} \right) \, dx \, dy \leq 0,
\]
where \( \Omega \) is the region bounded by arcs \( \bar{P}LQ \) and \( QRP \). Consequently
\[
\oint_{\Gamma} \text{Div}(e, f) \, dt < 0,
\]
i.e., \( \Gamma \) is orbitally stable and hence, unique. \( \square \)

**Remark 2.4.** Consider the case when \( W(x) = \alpha + \beta h(x) \). Then all steps from proofs of Theorem 2.1 and Lemma 2.3 can be repeated except for the last step in proof of Theorem 2.1, where the Green theorem was used. The reason is that if the condition
\[
\frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - \gamma} \right) \leq 0
\]
is satisfied for \( x \in [0, x^*] \cup [x^*, k] \) then the condition
\[
\frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - \gamma} \right) < 0
\]
is satisfied for \( x \in [0, x^*] \) but it is violated for \( x \in [x^*, k] \). Therefore the Green theorem cannot be used. Nevertheless, we have
\[
(q(x) - \gamma)(\alpha + \beta h(x)) = \frac{\alpha + \beta y}{y} \frac{dy}{dt} + \beta \frac{q(x) - \gamma}{p(x)} \frac{dx}{dt}.
\]
Therefore the Green theorem yields
\[
\left( \int_{\bar{P}LQ} + \int_{QRP} \right) p(x)h'(x) \, dt \\
= \left( \int_{\bar{P}LQ} + \int_{QRP} \right) \frac{p(x)h'(x)}{(q(x) - \gamma)(\alpha + \beta h(x))} \left( \frac{\alpha + \beta y}{y} \frac{dy}{dt} + \beta \frac{q(x) - \gamma}{p(x)} \frac{dx}{dt} \right) \\
= \iint_{\Omega} \frac{\alpha + \beta y}{y} \frac{d}{dx} \left( \frac{p(x)h'(x)}{(q(x) - \gamma)(\alpha + \beta h(x))} \right) \, dx \, dy \leq 0.
\]

From Lemma 2.3, Theorem 2.1 and Remark 2.4 follows that the interval, where condition (3) is to be fulfilled can be narrowed. So that we can formulate stronger theorem then Theorem 2.2 in [7].

**Theorem 2.5.** Let for system (1) the following assumptions be satisfied:
3. Examples

Example 1. Consider the system

\[
\begin{align*}
    x' &= xg(x) - yp(x), \\
    y' &= y(q(x) - \gamma),
\end{align*}
\]

where \( h(x) = xg(x)/p(x) \) is unimodal, strictly concave down, and symmetric with respect to its maximum. Then we can define

\[ W(x) = h'(x) \quad \text{for} \quad x \in (0, x^*) \quad \text{and} \quad W(x) = -h'(x) \quad \text{for} \quad x \in (x^*, k). \]

Hence, the uniqueness of a limit cycle is ensured when

\[ \frac{d}{dx}\left( \frac{p(x)}{q(x) - \gamma} \right) \leq 0 \quad \text{for} \quad x \in (0, x^*) \quad \text{and} \quad \geq 0 \quad \text{for} \quad x \in (x^*, k). \]

To be more specific consider system (7) where \( p(x) = x, q(x) - \gamma = (ax - b)/(cx + d) \), where \( a, b, c, d \) are positive constants. Then

\[ \frac{d}{dx}\left( \frac{p(x)}{q(x) - \gamma} \right) = \frac{d}{dx}\left( \frac{x(cx + d)}{ax - b} \right) = \frac{acx^2 - 2bxc - bd}{(ax - b)^2}. \]

Since the roots of the function \( acx^2 - 2bxc - bd \) are

\[ x_{1,2} = \frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} + \frac{bd}{ac}}, \]

it follows that system (7) has exactly one limit cycle when

\[ \left( \frac{b}{a}, \sqrt{\frac{b^2}{a^2} + \frac{bd}{ac}} \right) \subset \left( \frac{b}{a}, \frac{b}{a} \right). \]

Example 2. Consider the system

\[
\begin{align*}
    x' &= xg(x) - y \left( \frac{bx}{a + x} \right), \\
    y' &= y \left( \frac{bx}{a + x} - \gamma \right),
\end{align*}
\]

where \( a, b, c, \gamma \) are positive constants. The coefficient \( \gamma \) is the relative death-rate of the predator, \( b \) is the maximal relative biomass growth rate of the predator and \( a \) is the
Michaelis–Menten constant. It represents the amount of prey necessary for the relative biomass growth rate of the predator to be half its maximum. We have $0 < c < 1$, since the whole biomass of the prey is not transformed to the biomass of the predator and the constant $k$ (such that $g(k) = 0$) is the carrying capacity of the prey population.

Let us consider the situation when $h(x)$ is a unimodal polynomial of degree 4. If we put $a = 1, b = 1, g(x) = (x - 7)(-x^2 + 6x - 26), x^* = 2.8$, we can see from Fig. 2 that condition (2) and therefore also condition (3) are violated in $(0, x^*)$.

If the function

$$\frac{p(x)h'(x)}{q(x) - \gamma}$$

is increasing on some subinterval of $(0, x^*)$ then its product with the function $1/W(x)$ can be decreasing in $(0, x^*)$ and vice versa in $(x^*, k)$. If there exists a function $W(x)$ such that the function

$$\frac{p(x)h'(x)}{q(x) - \gamma} \frac{1}{W(x)}$$

is nonincreasing on $(0, x^*) \cup (x^*, k)$ then the considered system has the unique limit cycle.

The problem of finding the proper weight function $W(x)$ can be divided into two cases:

(a) $h(x)$ is symmetric with respect to its maximum,
(b) $h(x)$ is not symmetric with respect to its maximum.

In both cases functions in the class $[1/(\alpha + \beta h(x)), \alpha > 0, \beta > 0]$ (or the functions $\phi(h(x))$, where $\phi$ is a proper function) can be considered as candidates for the weight function $W(x)$. Moreover, in case (a), every function, which is symmetric with respect to its maximum and satisfies given conditions, is a possible candidate.
First consider case (a), i.e., $h(x)$ is a polynomial of degree 4 symmetric with respect to its maximum. Such a polynomial can be written in the form
\[
h(x) = (x + a)(x - k)(-x^2 + (k - a)x - \alpha),
\]
where $\alpha$ is a positive constant. Now we obtain
\[
\frac{p(x)h'(x)}{q(x) - \gamma} = 2 \frac{x}{x - x^*} \left( x - \left( \frac{k - a}{2} \right) \right) \left( -2x^2 + 2(k - a)x - (\alpha - ka) \right),
\]
where $x^* = ay/(bc - \gamma)$. Conditions (2), (3) can be violated as Fig. 2 illustrates. Thus we cannot use conditions (2), (3) for establishing the uniqueness of limit cycle in system (8) generally.

Now define $W(x) = (2x^2 - 2(k - a)x + (\alpha - ka))$. Then
\[
\frac{d}{dx} \left( \frac{p(x)h'(x)}{(q(x) - \gamma)W(x)} \right) = -2 \frac{x^2 - 2x^*x + \frac{k - a}{2} x^*}{(x - x^*)^2} < 0,
\]
when $x^* < (k - a)/2$ since in this case the polynomial in the numerator has no real roots. Therefore system (8) has a unique stable limit cycle if $x^* < (k - a)/2$ and $h(x)$ is a polynomial of degree 4 symmetric with respect to its maximum.

In the case (b), unfortunately, we do not know the functions $h_1^{-1}, h_2^{-1}$ explicitly, which is necessary since $W(h_1^{-1} \circ h_2(\bar{x})) = W(\bar{x}) = W(h_2^{-1} \circ h_1(x))$. Therefore we can take as a candidate for the weight function only functions of the form $1/(\alpha + \beta h(x))$, $\alpha > 0, \beta > 0$. Thus we have to find the number $\delta = \alpha/\beta$ such that the condition
\[
\frac{\beta}{x-x^*} \frac{d}{dx} h'(x) \left( \delta + h(x) \right) \leq 0 \quad \text{for} \quad x \in (0, x^* \cup (\bar{x}^*, 0)
\]
holds under the assumption
\[
\frac{d}{dx} \left( \frac{x}{x - x^*} h'(x) \right) \leq 0 \quad \text{for} \quad x \in (\bar{x}^*, 0).
\]
These last two conditions are equivalent to the conditions (differentiation)
\[
\left[ \delta + h(x) \right] \left[ x(x - x^*)h''(x) - x^* h'(x) \right] + x(x - x^*)h'^2(x) \leq 0, \quad (9)
\]
\[
x(x - x^*)h''(x) - x^* h'(x) \leq 0. \quad (10)
\]
The expression $x(x - x^*)h'^2(x)$ is always positive on $(\bar{x}^*, k)$. Then, the existence of the number $\delta_1$ such that inequality (9) holds in $(\bar{x}^*, 0)$ for $\delta \in [\delta_1, \infty)$ follows from (10). And if inequality (9) holds for $\delta_1$ on $(0, x^*)$, we have found the weight function $\delta_1 + h(x)$.

Consider the situation when
\[
h(x) = (x + 0.5)(x - 7)(-x^2 + 6x - 26), \quad x^* = 3.9.
\]
In this case $m = 4$ and $\tilde{0} \approx 6.57$. The functions in inequalities (9) and (10) are the polynomials of degree 8 and 4, respectively. Let us denote them by $P_8(x)$, $P_4(x)$. We use the Sturm sequence method for establishing the number of real roots of the polynomials $P_8(x)$, $P_4(x)$ in the intervals $(0, x^*), (\bar{x}^*, 0)$. The following table gives the results for $\delta = 200:

| $P_8(x)$ | $0$ | $0$ |
| $P_4(x)$ | $2$ | $0$ |
Since the polynomial $P_8(x)$ does not have roots in $(0, x^*) \cup (\bar{x}^*, 0)$ and the polynomial $P_4(x)$ has roots only in the interval $(0, x^*)$ it follows from this table that the function $200 + h(x)$ is a proper weight function.

Our numerical experience indicates a wide range of applicability of these weight functions. However, we have not analytic method for finding the proper value of $\delta$, although this is no so difficult to be established. It is a nontrivial question whether there exists a class of predator–prey systems with applications in biology for which the condition

$$
\beta \frac{d}{dx} \left( \frac{p(x)h'(x)}{q(x) - y} (\delta + h(x)) \right) \leq 0 \quad \text{for} \quad x \in (0, x^*) \cup (\bar{x}^*, 0)
$$

can be analytically solved.

We only note that it is not necessary to assume that condition (2) is violated. Using of a weight function can lead to simplification of the condition (2), as case (a) illustrates, where condition (2) is satisfied but where its verification is very complicated.

References