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Generating bricks

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Abstract

A brick is a 3-connected graph such that the graph obtained from it by deleting any two distinct vertices has a perfect matching. The importance of bricks stems from the fact that they are building blocks of the matching decomposition procedure of Kotzig, and Lovász and Plummer. We prove a "splitter theorem" for bricks. More precisely, we show that if a brick H is a "matching minor" of a brick G, then, except for a few well-described exceptions, a graph isomorphic to H can be obtained from G by repeatedly applying a certain operation in such a way that all the intermediate graphs are bricks and have no parallel edges. The operation is as follows: first delete an edge, and for every vertex of degree two that results contract both edges incident with it. This strengthens a recent result of de Carvalho, Lucchesi and Murty.

Keywords: Graph; Perfect matching; Brick; Generating theorem; Splitter theorem

1. Introduction

All graphs in this paper are finite and simple; that is, may not have loops or multiple edges. The following well-known theorem of Tutte [15] describes how to generate all 3-connected graphs, but first a definition. Let v be a vertex of a graph H, and let N_1, N_2 be a partition of the neighbors of v into two disjoint sets, each of size at least two. Let G be obtained from $H \setminus v$ (we use $\setminus$ for deletion and $-$ for set-theoretic difference) by adding two vertices v_1 and v_2, where v_i has neighbors $N_i \cup \{v_{3-i}\}$. We say that G was obtained from H by *splitting a vertex*. Thus for 3-connected graphs splitting a vertex is the inverse of contracting an edge that belongs to no

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triangle. A *wheel* is a graph obtained from a cycle by adding a vertex joined to every vertex of the cycle.

(1.1) *Every* 3*-connected graph can be obtained from a wheel by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. Seymour [14] extended (1.1) as follows.

(1.2) *Let H be a* 3*-connected minor of a* 3*-connected graph G such that H is not isomorphic to K_4 and G is not a wheel. Then a graph isomorphic to G can be obtained from H by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

Our objective is to prove an analogous theorem for bricks, where a *brick* is a 3-connected bicritical graph, and a graph G is *bicritical* if $G \setminus u \setminus v$ has a perfect matching for every two distinct vertices $u, v \in V(G)$. A related notion is that of a *brace*, by which we mean a connected bipartite graph such that every matching of size at most two is contained in a perfect matching. Bricks and braces are important, because they are the building blocks of the matching decomposition procedure of Kotzig, and Lovász and Plummer [8], which we now briefly review.

Let G be a graph, and let $X \subseteq V(G)$. We use $\delta(X)$ to denote the set of edges with one end in X and the other in $V(G) - X$. A *cut* in G is any set of the form $\delta(X)$ for some $X \subseteq V(G)$. A cut C is *tight* if $|C \cap M| = 1$ for every perfect matching M in G. Every cut of the form $\delta(\{v\})$ is tight; those are called *trivial*, and all other tight cuts are called *nontrivial*. Let $\delta(X)$ be a nontrivial tight cut in a graph G, let G_1 be obtained from G by identifying all vertices in X into a single vertex and deleting all resulting parallel edges, and let G_2 be defined analogously by identifying all vertices in $V(G) - X$. Then many matching-related problems can be solved for G if we are given the corresponding solutions for G_1 and G_2. As an example, consider $\mathrm{lat}(G)$, the *matching lattice* of a graph G, defined as the set of all integer linear combinations of characteristic vectors of perfect matchings of G. It is not hard to see that a description of $\mathrm{lat}(G)$ can be read off from descriptions of $\mathrm{lat}(G_1)$ and $\mathrm{lat}(G_2)$. We will return to the matching lattice shortly.

The above decomposition process can be iterated, until we arrive at graphs with no nontrivial tight cuts. Lovász [7] proved that the list of indecomposable graphs obtained at the end of the procedure does not depend on the choice of tight cuts made during the process. These indecomposable graphs were characterized by Edmonds, Lovász and Pulleyblank [2,3]:

(1.3) *Let G be a connected graph such that every edge of G belongs to a perfect matching. Then G has no nontrivial tight cut if and only if G is a brick or a brace.*

Coming back to the matching lattice, Lovász [6] proved that if G is a brace, then $\mathrm{lat}(G)$ consists of all integral vectors $\mathbf{w} \in \mathbf{Z}^{E(G)}$ such that $\mathbf{w}(\delta(v)) = \mathbf{w}(\delta(v'))$ for every two vertices $v, v' \in V(G)$. This is not true for bricks, because the Petersen graph is a counterexample. However, Lovász [7] proved the following deep result.

(1.4) *Let G be a brick other than the Petersen graph. Then* $\mathrm{lat}(G)$ *consists precisely of all vectors $\mathbf{w} \in \mathbf{Z}^{E(G)}$ such that $\mathbf{w}(\delta(v)) = \mathbf{w}(\delta(v'))$ for every two vertices $v, v' \in V(G)$.*

Our motivation for generating bricks came from Pfaffian orientations [4]. An orientation D of a graph G is *Pfaffian* if every even cycle C such that $G \setminus V(C)$ has a perfect matching has an odd number of edges directed in either direction of the cycle. A graph is *Pfaffian* if it has a Pfaffian orientation. This is an important concept, because the number of perfect matchings in a Pfaffian graph can be computed efficiently [4]. No polynomial-time algorithm to recognize Pfaffian graphs is known, even though there is one for bipartite graphs [11,13], using a structure theorem obtained in [10,13]. The above-mentioned tight cut decomposition procedure can be used to reduce the Pfaffian graph decision problem to bricks and braces [5,16]. Thus it remains to understand which bricks have a Pfaffian orientation, but that seems to be a much harder problem than the corresponding question for braces. Using the main theorem of this paper we managed to shed some light on this perplexing question, but the structure of Pfaffian graphs remains a mystery. We will report on these findings elsewhere. A characterization of Pfaffian graphs in terms of drawings in the plane (with crossings) has been recently obtained by the first author [12].

Let us now describe our theorem. We need a few definitions first. Let G be a graph, and let v_0 be a vertex of G of degree two incident with the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let H be obtained from G by contracting both e_1 and e_2 and deleting all resulting parallel edges. We say that H was obtained from G by *bicontracting* or *bicontracting the vertex* v_0, and write $H = G/v_0$. Let us say that a graph H is a *reduction* of a graph G if H can be obtained from G by deleting an edge and bicontracting all resulting vertices of degree two. By a *prism* we mean the unique 3-regular planar graph on six vertices. The following is a generation theorem of de Carvalho, Lucchesi and Murty [1].

(1.5) *If G is a brick other than K_4, the prism, and the Petersen graph, then some reduction of G is a brick other than the Petersen graph.*

Thus if a brick G is not the Petersen graph, then the reduction operation can be repeated until we reach K_4 or the prism. By reversing the process (1.5) can be viewed as a generation theorem. It is routine to verify that (1.5) implies (1.4), and that demonstrates the usefulness of (1.5). Our main theorem strengthens (1.5) in two respects. (We have obtained our result independently of [1], but later. We are indebted to the authors of [1] for bringing their work to our attention.) The first strengthening is that the generation procedure can start at graphs other than K_4 or the prism, as we explain next. Let a graph J be a subgraph of a graph G. We say that J is a *central subgraph* of G if $G \setminus V(J)$ has a perfect matching. We say that a graph H is a *matching minor* of G if H can be obtained from a central subgraph of G by repeatedly bicontracting vertices of degree two. Thus if H can be obtained from G by repeatedly taking reductions, then H is isomorphic to a matching minor of G. We will denote the fact that G has a matching minor isomorphic to H by writing $H \hookrightarrow G$. This is consistent with our notation for embeddings, to be introduced in Section 4. Since every brick has a matching minor isomorphic to K_4 or the prism by [8, Theorem 5.4.11], the following implies (1.5).

(1.6) *Let G be a brick other than the Petersen graph, and let H be a brick that is a matching minor of G. Then a graph isomorphic to H can be obtained from G by repeatedly taking reductions in such a way that all the intermediate graphs are bricks not isomorphic to the Petersen graph.*

We say that a graph H is a *proper reduction* of a graph G if it is a reduction in such a way that the bicontractions involved do not produce parallel edges. We would like to further strengthen (1.6) by replacing reductions by proper reductions; such an improvement is worthwhile, because

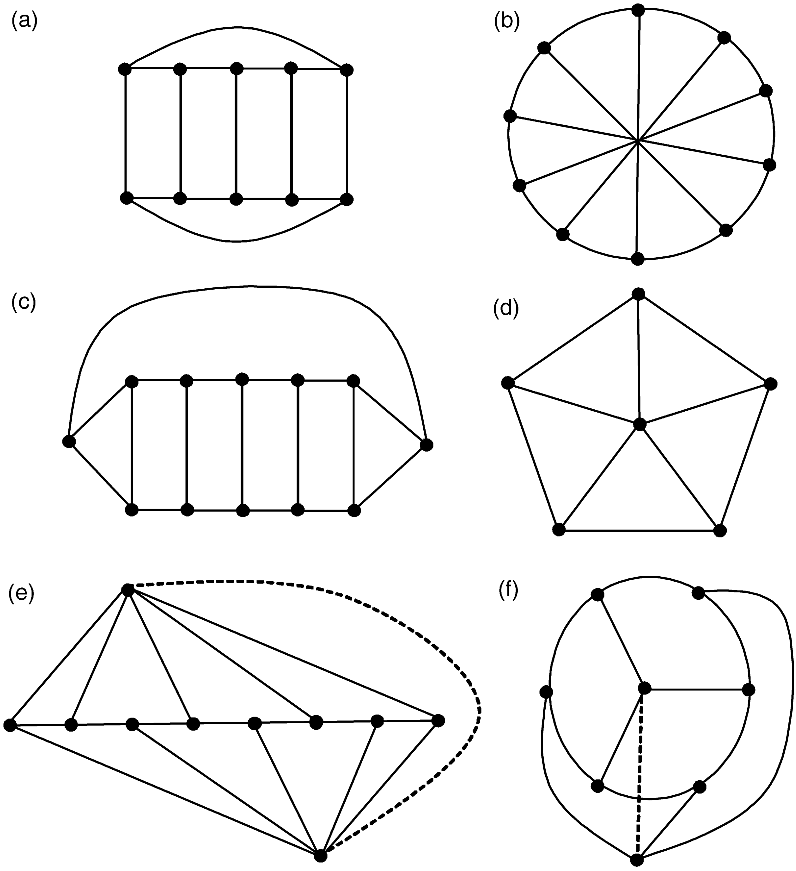

Fig. 1. Exceptional families.

in applications it reduces the number of cases that need to be examined. Unfortunately, (1.6) does not hold for proper reductions, but all the exceptions can be conveniently described. Let us do that now. We refer to Fig. 1(a)–(e).

Let C_1 and C_2 be two vertex-disjoint cycles of length $n \geqslant 3$ with vertex-sets $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ (in order), respectively, and let G_1 be the graph obtained from the union of C_1 and C_2 by adding an edge joining u_i and v_i for each $i = 1, 2, \ldots, n$. We say that G_1 is a *planar ladder*. Let G_2 be the graph consisting of a cycle C with vertex-set $\{u_1, u_2, \ldots, u_{2n}\}$ (in order), where $n \geqslant 2$ is an integer, and n edges with ends u_i and u_{n+i} for $i = 1, 2, \ldots, n$. We say that G_2 is a *Möbius ladder*. A *ladder* is a planar ladder or a Möbius ladder. Let G_1 be a planar ladder as above on at least six vertices, and let G_3 be obtained from G_1 by deleting the edge u_1u_2 and contracting the edges u_1v_1 and u_2v_2. We say that G_3 is a *staircase*. Let $t \geqslant 2$ be an integer, and let P be a path with vertices $v_1, v_2, \ldots, v_t$ in order. Let G_4 be obtained from P by adding two distinct vertices x, y and edges xv_i and yv_j for $i = 1, t$ and all even $i \in \{1, 2, \ldots, t\}$ and $j = 1, t$ and all odd $j \in \{1, 2, \ldots, t\}$. Let G_5 be obtained from G_4 by adding the edge xy. We say that G_5 is an *upper prismoid*, and if $t \geqslant 4$, then we say that G_4 is a *lower prismoid*. A *prismoid* is a lower prismoid or an upper prismoid. We are now ready to state our main theorem.

(1.7) *Let H, G be bricks, where H is isomorphic to a matching minor of G. Assume that H is not isomorphic to K_4 or the prism, and G is not a ladder, wheel, staircase or prismoid. Then a graph isomorphic to H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are bricks not isomorphic to the Petersen graph.*

If H is a brick isomorphic to a matching minor of a brick G and G is a ladder, wheel, staircase or prismoid, then H itself is a ladder, wheel, staircase or prismoid, and can be obtained from a graph isomorphic to G by taking (possibly improper) reductions in such a way that all intermediate graphs are bricks. Thus (1.7) implies (1.6). (Well, this is not immediately clear if the graph H from (1.6) is a K_4 or a prism, but in those cases the implication follows with the aid of the next theorem.)

As a counterpart to (1.7) we should describe the starting graphs for the generation process of (1.7). Notice that K_4 is a wheel, a Möbius ladder, a staircase and an upper prismoid, and that the prism is a planar ladder, a staircase and a lower prismoid. Later in this section we show

(1.8) *Let G be a brick not isomorphic to K_4, the prism or the Petersen graph. Then G has a matching minor isomorphic to one of the following seven graphs: the graph obtained from the prism by adding an edge, the lower prismoid on eight vertices, the staircase on eight vertices, the staircase on ten vertices, the planar ladder on ten vertices, the wheel on six vertices, and the Möbius ladder on eight vertices.*

McCuaig [9] proved an analogue of (1.7) for braces. To state his result we need another exceptional class of graphs, depicted in Fig. 1(f). Let C be an even cycle with vertex-set $v_1, v_2, \ldots, v_{2t}$ in order, where $t \geqslant 2$ is an integer and let G_6 be obtained from C by adding vertices v_{2t+1} and v_{2t+2} and edges joining v_{2t+1} to the vertices of C with odd indices and v_{2t+2} to the vertices of C with even indices. Let G_7 be obtained from G_6 by adding an edge $v_{2t+1}v_{2t+2}$. We say that G_7 is an *upper biwheel*, and if $t \geqslant 3$ we say that G_6 is a *lower biwheel*. A *biwheel* is a lower biwheel or an upper biwheel. McCuaig's result is as follows.

(1.9) *Let H, G be braces, where H is isomorphic to a matching minor of G. Assume that if H is a planar ladder, then it is the largest planar ladder matching minor of G, and similarly for Möbius ladders, lower biwheels and upper biwheels. Then a graph isomorphic to H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are braces.*

Actually, (1.9) follows from a version of our theorem stated in Section 11.

Let us now introduce terminology that we will be using in the rest of the paper. Let $H, G, v_0, v_1, v_2, e_1, e_2$ be as in the definition of bicontraction. Assume that both v_1 and v_2 have degree at least three and that they have no common neighbors except v_0; then no parallel edges are produced during the contraction of e_1 and e_2. Let v be the new vertex that resulted from the contraction. If both v_1 and v_2 have degree at least three, then we say that G was obtained from H by *bisplitting the vertex* v. We call v_0 the *new inner vertex* and v_1 and v_2 the *new outer vertices*.

Let H be a graph. We wish to define a new graph H'' and two vertices of H''. Either $H'' = H$ and u, v are two nonadjacent vertices of H, or H'' is obtained from H by bisplitting a vertex, u is the new inner vertex of H'' and $v \in V(H'')$ is not adjacent to u, or H'' is obtained by bisplitting a vertex of a graph obtained from H by bisplitting a vertex, and u and v are the two new inner vertices of H''. Finally, let $H' = H'' + (u, v)$. We say that H' is a *linear extension* of H (see

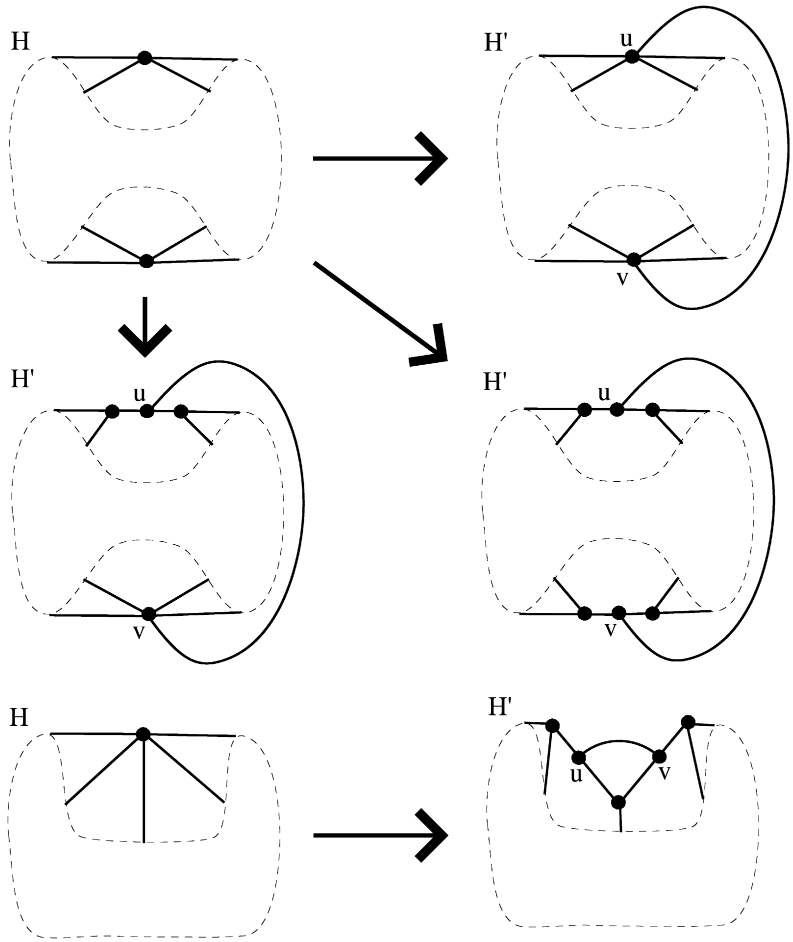

Fig. 2. Linear extensions of H.

Fig. 2). Thus H' is a linear extension of H if and only if H is a proper reduction of H'. By *the cube* we mean the graph of the 1-skeleton of the 3-dimensional cube. Notice that the cube and $K_{3,3}$ are bipartite, and hence are not bricks. Using this terminology (1.7) can be restated in a mildly stronger form. It is easy to check that if G' is obtained from a brick G by bisplitting a vertex into new outer vertices v_1 and v_2, then $\{v_1, v_2\}$ is the only set $X \subseteq V(G')$ such that $|X| \geqslant 2$ and $G' \setminus X$ has at least $|X|$ odd components. Thus a linear extension of a brick is a brick, and hence (1.10) implies (1.7).

(1.10) *Let G be a brick other than the Petersen graph, and let H be a* 3*-connected matching minor of G not isomorphic to K_4, the prism, the cube, or $K_{3,3}$. If G is not isomorphic to H and G is not a ladder, wheel, biwheel, staircase or prismoid, then a linear extension of H is isomorphic to a matching minor of G.*

The main step in the proof of (1.10) is the following.

(1.11) *Let G be a brick other than the Petersen graph, and let H be a 3-connected matching minor of G. Assume that if H is a planar ladder, then there is no strictly larger planar ladder L with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius ladders, wheels, lower biwheels, upper biwheels, staircases, lower prismoids and upper prismoids. If H is not isomorphic to G, then some matching minor of G is isomorphic to a linear extension of H.*

It is routine to verify that if G is a ladder, wheel, biwheel, staircase or prismoid, G' is a linear extension of G, and H is a 3-connected matching minor of G not isomorphic to K_4, the prism, the cube, or $K_{3,3}$, then G' has a matching minor isomorphic to a linear extension of H. Thus (1.11) implies (1.10), and we omit the details. The proof of (1.11) will occupy the rest of the paper. However, assuming (1.11) we can now deduce (1.8).

Proof of (1.8), assuming (1.11). Let G be a brick not isomorphic to K_4, the prism or the Petersen graph. By [8, Theorem 5.4.11], G has a matching minor M isomorphic to K_4 or the prism. Since M is not bipartite, it is not a biwheel, a planar ladder on $4k$ vertices, or a Möbius ladder on $4k+2$ vertices. Thus if a prismoid, wheel, ladder or staircase larger than M is isomorphic to a matching minor of G, then G has a matching minor as required for (1.8). Thus we may assume that the hypothesis of (1.11) is satisfied, and hence a matching minor of G is isomorphic to a linear extension of M. But K_4 does not have any linear extensions, and the prism has, up to isomorphism, exactly one, namely the graph obtained from it by adding an edge. This proves (1.8). □

Here is an outline of the paper. First we need to develop some machinery; that is done in Sections 2–4. In Section 5 we prove a first major step toward (1.11), namely that the theorem holds provided a graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G. Then in Section 6 we reformulate our key lemma in a form that is easier to apply, and introduce several different types of extensions. In Section 7 we use the 3-connectivity of G to show that at least one of those extensions of H is isomorphic to a matching minor of G, and in Sections 8–10 we gradually eliminate all the additional extensions. Theorem (1.11) is proved in Section 10. Finally, in Section 11 we state a strengthening of (1.11) that can be obtained by following the proof of (1.11) with minimal changes. We delegate the strengthening to the last section, because the statement is somewhat cumbersome and perhaps of lesser interest. Its applications include (1.11), (1.9) and a generation theorem for a subclass of factor-critical graphs.

A word about notation. If H is a graph, and $u, v \in V(H)$ are distinct nonadjacent vertices, then $H + (u, v)$ or $H + uv$ denotes the graph obtained from H by adding an edge with ends u and v. Now let $u, v \in V(H)$ be adjacent. By *bisubdividing* the edge uv we mean replacing the edge by a path of length three, say a path with vertices u, x, y, v, in order. Let H' be obtained from H by this operation. We say that x, y (in that order) are the *new vertices*. Thus y, x are the new vertices resulting from subdividing the edge vu (we are conveniently exploiting the notational asymmetry for edges). Now if $w \in V(H) - \{u\}$, then by $H + (w, uv)$ we mean the graph $H' + (w, x)$. Notice that the graphs $H + (w, uv)$ and $H + (w, vu)$ are different. In the same spirit, if $a, b \in V(H)$ are adjacent vertices of H with $\{u, v\} \neq \{a, b\}$, then we define $H + (uv, ab)$ to be the graph $H' + (x, ab)$.

2. Octopi and frames

Let H be a graph with a perfect matching, and let $X \subseteq V(H)$ be a set of size k. If $H \setminus X$ has at least k odd components, then X is called a *barrier* in H. The following is easy and well known.

(2.1) *A brick has no barrier of size at least two.*

Now if H and X are as above and H is a subgraph of a brick G, then X cannot be a barrier in G. If H is a central subgraph of G, then we get the following useful outcome. In the application $R_1, R_2, \dots, R_k$ will be the components of $G \setminus X$.

(2.2) *Let G be a brick and let H be a subgraph of G. Let M be a perfect matching of $G \setminus V(H)$ and let $V(H)$ be a disjoint union of X, $R_1, R_2, \dots, R_k$, where $k \geqslant 2$, $|X| \leqslant k$ and $|R_i|$ is odd for every $i \in \{1, 2, \dots, k\}$. Then there exist distinct integers $i, j \in \{1, 2, \dots, k\}$ and an M-alternating path joining a vertex in R_i to a vertex in R_j.*

Proof. Suppose for a contradiction that the lemma is false, and let H be a maximal subgraph of G that satisfies the hypothesis of the lemma, but not the conclusion.

By (2.1) there exists an edge $e_1 \in E(G)$ with one end $v \in R_i$ for some $i \in \{1, 2, \dots, k\}$ and the other end $u \in V(G) - R_i - X$. Without loss of generality we may assume that $i = 1$. If $u \in V(H)$ then the path with edge-set $\{e_1\}$ is as required. Thus $u \notin V(H)$, and hence u is incident with an edge $e_2 \in M$. Let w be the other end of e_2; then clearly $w \notin V(H)$. Let $X' = X \cup \{u\}$, $R_{k+1} = \{w\}$, $M' = M - \{e_2\}$ and construct H' by adding the vertices u and w and edges e_1 and e_2 to H. By the maximality of H the graph H', matching M' and sets X', $R_1, R_2, \dots, R_{k+1}$ satisfy the conclusion of the lemma. Thus for some distinct integers $i, j \in \{1, 2, \dots, k+1\}$ there exists an M'-alternating path P joining a vertex in R_i to a vertex in R_j. Since H does not satisfy the conclusion of the lemma we may assume that $j = k + 1$. Let P' be the graph obtained from P by adding the edges e_1 and e_2. If $i > 1$, then P' is a path and satisfies the conclusion of the lemma.

Thus we may assume that $i = 1$. Let $H'' = H \cup P'$, $M'' = M - E(P')$ and $R_1' = R_1 \cup V(P')$. Then the graph H'', matching M'' and sets X, R_1', $R_2, \dots, R_k$ also satisfy the conclusion of the lemma by the maximality of H. Thus we may assume that there is an M''-alternating path Q joining a vertex in R_1' to a vertex in R_j for some $j \in \{2, 3, \dots, k\}$. If neither of the ends of Q lies in $V(P')$ then Q is a required path for H. If one of them, say x, is in $V(P')$, we add to Q one of the subpaths of P' with end x to obtain a required path. □

To motivate the next definitions, let us consider the following example. Let G be a brick that has a matching minor isomorphic to K_4. Later in the proof there will come a step when one will be able to deduce that G has a matching minor isomorphic to the graph H depicted in Fig. 3. Unfortunately, H is not a brick, because the set $\{a, b, c\}$ is a barrier. So we try to apply (2.2). More precisely, G has a central subgraph J isomorphic to a graph obtained from H by repeatedly bisubdividing edges of H; let $a', b', c', x', y' \in V(J)$ correspond to a, b, c, x, y, respectively. By applying (2.2) we deduce that one of a number of outcomes holds, including a possibility that G has a matching minor isomorphic to $J + (x', y'a')$. The latter graph, however, is not a brick and the only brick matching minor it contains is K_4. Thus we need a strengthening of (2.2) in the case when each R_i has a restricted structure, what we call an octopus. Let us introduce the necessary definitions.

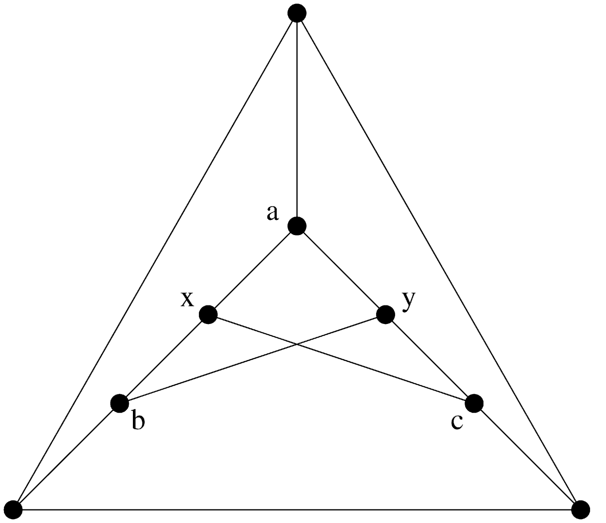

Fig. 3. A graph H, containing K_4 as a matching minor.

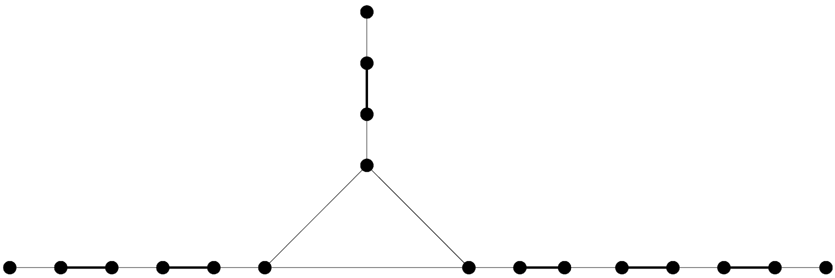

Fig. 4. An octopus Ω and an Ω-compatible matching.

Let H be a graph, let C be a subgraph of H with an odd number of vertices, and let $P_1, P_2, \ldots, P_k$ be odd paths in H. For $i = 1, 2, \ldots, k$ let u_i and v_i be the ends of P_i. If for distinct $i, j = 1, 2, \ldots, k$ we have $V(P_i) \cap V(C) = \{u_i\}$ and $V(P_i) \cap V(P_j) \subseteq \{u_i, v_i\}$, then we say that $\Omega = (C, P_1, P_2, \ldots, P_k)$ is an *octopus* in H. We say that the paths $P_1, P_2, \ldots, P_k$ are the *tentacles* of Ω, C is the *head* of Ω and v_i are the *ends* of Ω. We define the *graph of* Ω to be $C \cup P_1 \cup P_2 \cup \cdots \cup P_k$, and by abusing notation slightly we will denote this graph also by Ω. We say that a matching M in G is *Ω-compatible* if every tentacle is M-alternating and no vertex of C is incident to an edge of M. See Fig. 4. Then in the example above each component of $J \setminus \{a', b', c'\}$ can be turned into an octopus Ω, and the perfect matching in $G \setminus V(J)$ can be extended to an Ω-compatible matching in G.

Let G be a graph, and let $k \geqslant 1$ be an integer. We say that the pair $(\mathcal{F}, X)$ is a *frame* in G if $X \subseteq V(G)$ and $\mathcal{F} = \{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ satisfy

(1) $\Omega_1, \Omega_2, \ldots, \Omega_k$ are octopi,
(2) for $i = 1, 2, \ldots, k$ the ends and only the ends of Ω_i belong to X,
(3) for distinct $i, j \in \{1, 2, \ldots, k\}$, $V(\Omega_i) \cap V(\Omega_j) \subseteq X$,
(4) $|X| \leqslant k$.

We say that $\Omega_1, \Omega_2, \ldots, \Omega_k$ are the *components* of $(\mathcal{F}, X)$. We define the graph of $(\mathcal{F}, X)$ to be $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$, and denote it by $\mathcal{F}$, again abusing notation. Thus in the above example G has a frame $(\mathcal{F}, \{a', b', c'\})$ with three components. The following is the main result of this section. We say that a graph H is M*-covered* if a subset of M is a perfect matching of H.

(2.3) *Let G be a brick, let M be a matching in G, and let $(\mathcal{F}, X)$ be a frame in G such that $G \setminus (V(\mathcal{F}) \cup X)$ is M-covered and M is Ω-compatible for each $\Omega \in \mathcal{F}$. Then there exists an M-alternating path P joining vertices of the heads of two different components Ω_1, Ω_2 of $(\mathcal{F}, X)$. Moreover, there is an edge $e \in E(P) - M$ such that the two components of $P \setminus e$ can be numbered P_1 and P_2 in such a way that $V(P_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ for $i = 1, 2$.*

Proof. We say that a subpath Q of a path P is an $\mathcal{F}$*-jump* in P if the ends of Q belong to different components of $\mathcal{F}$ and Q is otherwise disjoint from $\mathcal{F}$. Let $\mathcal{F} = \{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ and let C_i denote the vertex-set of the head of Ω_i. By (2.2) applied to $X, C_1, C_2, \ldots, C_k$ there exists an M-alternating path joining vertices of the heads of two different components of $(\mathcal{F}, X)$. Choose such path P with the minimal number of $\mathcal{F}$-jumps in it. We prove that P satisfies the requirements of the theorem.

Let $v_1 \in C_1$ and $v_2 \in C_2$ be the ends of P. Since P is M-alternating and M is Ω_i-compatible for all $i = 1, 2, \ldots, k$, it follows that no internal vertex of P belongs to C_i. Suppose that $P \cap T \neq \emptyset$ for some tentacle T of Ω_i, where $i \geqslant 3$. Let $\{v_0\} = V(T) \cap C_i$ and let $v \in V(P) \cap V(T)$ be chosen so that $T[v, v_0]$ is minimal. For some $j \in \{1, 2\}$ the path $P[v_j, v] \cup T[v, v_0]$ is M-alternating and contradicts the choice of P. Thus $V(P) \cap V(\mathcal{F}) \subseteq V(\Omega_1) \cup V(\Omega_2)$.

Define a linear order on $V(P)$ so that $v \succ v'$ if and only if $v' \in P[v_1, v]$. Let P_0 be an $\mathcal{F}$-jump in P with ends $u_1 \in V(\Omega_1)$ and $u_2 \in V(\Omega_2)$ chosen so that $u_1 \succ u_2$ and $P[v_1, u_2]$ is minimal. Equivalently we can define P_0 as a second $\mathcal{F}$-jump we encounter if we traverse P from v_1 to v_2. If such an $\mathcal{F}$-jump P_0 in P does not exist then P contains a unique $\mathcal{F}$-jump. Let $e \notin M$ be an edge of this unique $\mathcal{F}$-jump; then P and e satisfy the requirements of the theorem. Therefore we may assume the existence of P_0.

For $i \in \{1, 2\}$ let T_i be the tentacle of Ω_i such that $u_i \in V(T_i)$ and let $\{w_i\} = V(T_i) \cap C_i$. Let $s_1 \in V(T_1) \cap V(P)$ be chosen so that $s_1 \succ u_1$ and $T_1[s_1, w_1]$ is minimal. Note that $s_1 \neq w_1$, because the only vertex in $V(P) \cap C_1$ is v_1 and $s_1 \succ u_1 \succ v_1$. Let s_1t_1 be the edge of M incident to s_1. We have $s_1t_1 \in E(T_1 \cap P)$ as both T_1 and P are M-alternating, $s_1 \in T_1[t_1, w_1]$ by the choice of s_1 and $s_1 \succ t_1$ as otherwise the path $T_1[w_1, s_1] \cup P[s_1, v_2]$ contradicts the choice of P. Let $s_2 \in V(T_2) \cap V(P)$ be chosen so that $s_2 \prec s_1$ and $T_2[s_2, w_2]$ is minimal. Let s_2t_2 be the edge of M incident to s_2. We again have $s_2t_2 \in T_2 \cap P$, $s_2 \in T_2[t_2, w_2]$ and $s_2 \prec t_2$, as otherwise the path $P[v_1, s_2] \cup T_2[s_2, w_2]$ contradicts the choice of P.

Consider $P' = P[s_2, s_1]$. By the choice of s_1 we have $V(P[u_2, s_1]) \cap V(T_1[s_1, w_1]) = \{s_1\}$. Also if $s_2 \prec u_2$ we have $V(P[s_2, u_2]) \cap V(\Omega_1) = \emptyset$ by the choice of P_0. It follows that $V(P') \cap V(T_1[s_1, w_1]) = \{s_1\}$. By the choice of s_2 we have $V(P') \cap V(T_2[s_2, w_2]) = \{s_2\}$. Therefore $T_2[w_2, s_2] \cup P' \cup T_1[w_1, s_1]$ is an M-alternating path contradicting the choice of P. □

3. Two paths meeting

In this section we study the following problem. Let G be a graph, let M be a matching, and let P_1 and P_2 be two M-alternating paths. In the applications we will be permitted to replace the matching M by a matching M' saturating the same set of vertices, and to replace the paths P_1 and P_2 by a pair of M'-alternating paths with the same ends. Thus we are interested in graphs that are minimal in the sense that there is no replacement as above upon which an edge of G may be deleted. The main result of this section, Theorem (3.3) below, asserts that there are exactly four types of minimally intersecting pairs of M-alternating paths, three of which are depicted in Fig. 5. We start with two auxiliary lemmas.

(3.1) *Let M be a matching in G, let P be an M-alternating path with ends x and y, let C be an M-alternating cycle such that x and y have degree at most two in $P \cup C$ and let $M' = M \triangle E(C)$. Then there exists an M'-alternating path Q with ends x and y satisfying $E(Q) \subseteq E(P) \triangle E(C)$.*

Proof. Let H be the subgraph of G with vertex-set $V(G)$ and edge-set $E(P) \triangle E(C)$. Then x, y have degree one in H, every other vertex of H has degree zero or two, and if it has degree two, then it is incident with an edge of M'. Thus some component of H is an M'-alternating path joining x and y, as desired. $\square$

(3.2) *Let M be a matching in G, let P be an M-alternating path with ends w and v, and let R be a path with ends v and z such that $R \setminus v$ is M-covered, v is incident with no edge of M, and $w \notin V(R)$. Let $M' = M \triangle E(R)$. Then there exists an M'-alternating path Q with ends w and z satisfying $E(Q) \subseteq E(P) \triangle E(R)$.*

Proof. This follows similarly as (3.1) by considering the graph with edge-set $E(P) \triangle E(R)$. $\square$

Let G be a graph, let M be a matching in G, and let P and Q be two M-alternating paths in G. For the purpose of this definition let a *segment* be a maximal subpath of $P \cap Q$, and let an *arc* be a maximal subpath of Q with no internal vertex or edge in P. We say that P and Q *intersect transversally* if either they are vertex-disjoint, or there exist vertices $q_0, q_1, \ldots, q_7 \in V(Q)$ such that

(1) $q_0, q_1, \ldots, q_7$ occur on Q in the order listed, and q_0 and q_7 are the ends of Q,
(2) $q_2, q_1, q_3, q_4, q_6, q_5$ all belong to P and occur on P in the order listed,
(3) if $q_0 \in V(P)$, then $q_0 = q_1 = q_2 = q_3$, and otherwise $Q[q_0, q_1]$ is an arc,
(4) if $q_7 \in V(P)$, then $q_7 = q_6 = q_5 = q_4$, and otherwise $Q[q_6, q_7]$ is an arc,
(5) $Q[q_3, q_4]$ is a segment,
(6) either $q_1 = q_2 = q_3$, or q_1, q_2, q_3 are pairwise distinct, $Q[q_1, q_2]$ is a segment, $Q[q_2, q_3]$ is an arc and q_2 is not an end of P, and
(7) either $q_4 = q_5 = q_6$, or q_4, q_5, q_6 are pairwise distinct, $Q[q_5, q_6]$ is a segment, $Q[q_4, q_5]$ is an arc and q_5 is not an end of P.

It follows that the definition is symmetric in P and Q. There are four cases of transversal intersection depending on the number of components of $P \cap Q$; the three cases when P and Q intersect are depicted in Fig. 5, where matching edges are drawn thicker. We shall prove the following lemma.

(3.3) *Let M be a matching in a graph G and let P_1 and P_2 be two M-alternating paths, where P_i has ends s_i and t_i. Assume that s_1, s_2, t_1 and t_2 have degree at most two in $P_1 \cup P_2$. Then there exist a matching M' saturating the same set of vertices as M and two M'-alternating paths Q_1 and Q_2 such that $M \triangle M' \subseteq E(P_1) \cup E(P_2)$, Q_i has ends s_i and t_i and Q_1 and Q_2 intersect transversally.*

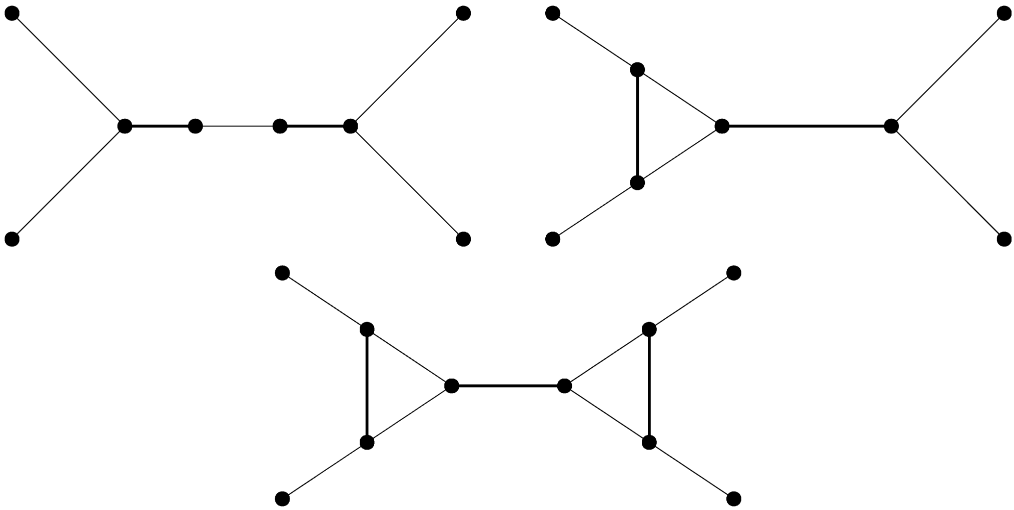

Fig. 5. Three cases of transversal intersection.

Unfortunately, for later application we need a more general, but less nice result, the following. Please notice that it immediately implies (3.3) on taking $r = t_2$.

(3.4) *Let M be a matching in a graph G and let P_1 and P_2 be two M-alternating paths, where P_i has ends s_i and t_i. Assume that s_1, s_2, t_1 and t_2 have degree at most two in $P_1 \cup P_2$. Let $r \in V(P_2)$, and let $P_2' = P_2[s_2, r]$. Then one of the following conditions holds*:

(1) *There exist a matching M' saturating the same set of vertices as M and two M'-alternating paths Q_1 and Q_2 such that Q_i has ends s_i and t_i, $M \triangle M' \subseteq E(P_1) \cup E(P_2')$, $Q_1 \subseteq P_1 \cup P_2'$, and $Q_1 \cup Q_2$ is a proper subgraph of $P_1 \cup P_2$*;
(2) *$r \neq t_2$, and there exists an M-alternating path $R \subseteq P_1 \cup P_2'$ with ends s_2 and t_1 such that R and P_1 intersect transversally*;
(3) *P_2' intersects P_1 transversally.*

Proof. We may assume that $G = P_1 \cup P_2$ and (1) does not hold. We shall refer to this as the *minimality of G*.

We claim that $P_1 \cup P_2'$ contains no M-alternating cycles. Suppose for a contradiction there exists an M-alternating cycle $C \subseteq P_1 \cup P_2'$. Let $M' = M \triangle E(C)$ and let Q_1, Q_2 be the two M'-alternating paths obtained by applying (3.1) to P_1 and P_2, respectively. Since P_1 and P_2 are M-alternating and their union includes C, they either share an edge of $M \cap E(C)$, say e, or P_1 and P_2 have the same ends. In the later case replacing P_2 by P_1 contradicts the minimality of G, and so we may assume the former. Now $Q_1 \subseteq P_1 \cup P_2'$ and $Q_1 \cup Q_2$ is a subgraph of $(P_1 \cup P_2) \setminus e$, contradicting the minimality of G.

For the purpose of this proof let us define an *arc* as a maximal subpath of P_2' that has at least one edge or contains an end of P_2' and has no internal vertex or edge in P_1. Define *segment* as a maximal subpath of $P_1 \cap P_2$. We say that two vertices of P_1 have the *same biparity* if their distance on P_1 is even, and otherwise we say they have *opposite biparity*. We claim that the ends of every arc have the same biparity. To see that, let $P_2'[s, t]$ be an arc with ends of opposite biparity. There are two cases. Either both end-edges of $P_1[s,t]$ belong to M, or both of them do not. If they do, then $P_1[s,t] \cup P_2'[s,t]$ is an M-alternating cycle, and if they do not, then P_1', P_2 contradict the minimality of G, where P_1' is obtained from P_1 by replacing the interior of $P_1[s,t]$ by $P_2'[s,t]$. (Notice that the edge of $P_1[s,t]$ incident with s does not belong to P_1' or P_2.) This proves our claim that the ends of every arc have the same biparity.

We may assume that there is an arc with both ends on P_1, for otherwise (3) holds. Let $P_2'[u_0, v_0]$ be such an arc. Since u_0, v_0 have the same biparity, exactly one end-edge of $P_1[u_0, v_0]$ belongs to M, say the one incident with u_0. Then the unique segment incident with u_0, say $P_1[u_0, v_1] = P_2'[u_0, v_1]$ has the property that v_1 lies between u_0 and v_0 on P_1. Let $P_2'[v_1, u_1]$ be the unique arc incident with v_1. Then either u_1 is an end of P_2', or u_1, v_1 have the same biparity, opposite to the biparity of u_0, v_0.

We claim that either u_1 is an end of P_2', or u_1 lies between v_1 and v_0 on P_1. To prove this claim we need to prove that neither u_0 nor v_0 lie between u_1 and v_1 on P_1. To this end suppose first that u_0 lies between u_1 and v_1 on P_1. Then P_1'' and P_2 contradict the minimality of G, where P_1'' is obtained from P_1 by replacing the interior of $P_1[u_1, v_0]$ by $P_2'[u_1, v_0]$ (the edge of $P_1[v_1, v_0]$ incident with v_1 does not belong to $P_1'' \cup P_2$). Suppose now that v_0 lies between u_1 and v_1 on P_1. Then $P_2'[v_0, u_1] \cup P_1[u_1, v_0]$ is an M-alternating cycle, a contradiction. This proves that either u_1 is an end of P_2', or u_1 lies between v_1 and v_0 on P_1.

Now assume that $P_2'[u_0, v_0]$ is chosen so that $P_1[u_0, v_0]$ is maximal, and let u_1, v_1 be as in the previous paragraph. If u_1 is an end of P_2' we stop, and so assume that it is not. Recall that u_1, v_1 have opposite biparity from u_0, v_0. Thus the unique segment incident with u_1, say $P_1[u_1, v_2] = P_2'[u_1, v_2]$ has the property that v_2 lies between v_1 and u_1 on P_1. Now let $P_2'[v_2, u_2]$ be the unique arc incident with v_2. By the result of the previous paragraph either u_2 is an end of P_2', or u_2 lies between v_2 and v_1 on P_1. By arguing in this manner we arrive at a sequence of vertices $u_0, v_0, \dots, u_{k+1}, v_{k+1}$ such that

(i) $u_0, v_1, u_2, v_3, \dots, v_{k+1}, \dots, u_3, v_2, u_1, v_0$ occur on P_1 in the order listed,
(ii) u_{k+1} is an end of P_2',
(iii) $P_2'[u_i, v_i]$ are arcs for $i = 0, 1, \dots, k+1$, and
(iv) $P_1[u_i, v_{i+1}]$ are segments for $i = 0, 1, \dots, k$.

It follows that u_i, v_i have the same biparity and that their biparity depends on the parity of i. Let $P_1[v_0, v_0']$ be the unique segment incident with v_0. Then v_0 lies between v_0' and u_0 on P_1. Let $P_2'[v_0', u_0']$ be the unique arc incident with v_0'. The maximality of $P_1[u_0, v_0]$ and the result of the previous paragraph imply that either u_0' is an end of P_2', or that u_0, v_0, v_0', u_0' occur on P_1 in the order listed. In the latter case by an analogous argument there exists a sequence of vertices $u_0', v_0', \dots, u_{k'+1}', v_{k'+1}'$ such that

(i) $u_0', v_1', u_2', v_3', \dots, v_{k'+1}', \dots, u_3', v_2', u_1', v_0'$ occur on P_1 in the order listed,
(ii) $u_{k'+1}'$ is an end of P_2,
(iii) $P_2'[u_i', v_i']$ are arcs for $i = 0, 1, \dots, k'+1$, and
(iv) $P_1[u_i', v_{i+1}']$ are segments for $i = 0, 1, \dots, k'$.

Suppose $r = t_2$. Then $k = 0$, for otherwise P_1 and the path obtained from P_2 by replacing the interior of $P_2[v_0, u_1]$ by $P_1[v_0, u_1]$ contradict the minimality of G. Similarly, either u_0' is an end of P_2 or $k' = 0$. Thus (3) holds.

Therefore we may assume $r \neq t_2$. Suppose $s_2 \neq u_0'$. Then without loss of generality we assume $s_2 = u_{k+1}$. We define $R_1 = P_2'[s_2, u_k] \cup P_1[u_k, t_1]$ and $R_2 = P_2'[s_2, v_k] \cup P_1[v_k, t_1]$. For some $i \in \{1, 2\}$ $R_i \subseteq P_1 \cup P_2'$ is an M-alternating path with ends s_2 and t_1 such that R_i and P_1 intersect transversally. Thus (2) holds.

It remains to consider the case when $s_2 = u_0'$ and $u_{k+1} = r$. If $k = 0$, then (3) holds, and so we may assume that $k \geqslant 1$. We claim that $E(P_1[v_{k+1}, v_k] \cap P_2) = \emptyset$. Suppose for a contradiction $P_2[x, y] \subseteq P_1[v_{k+1}, v_k]$ is a segment, and let $P_2[x, y]$ be chosen so that $P_2[y, t_2]$ is minimal. If $x \in V(P_1[v_k, y])$ define $Q_2 = P_2[s_2, v_k] \cup P_1[v_k, x] \cup P_2[x, t_2]$, and otherwise define $Q_2 = P_2[s_2, v_{k+1}] \cup P_1[v_{k+1}, x] \cup P_2[x, t_2]$. As $E(P_1[v_{k+1}, v_k] \cap P_2') = \emptyset$ we see that Q_2 is an M-alternating path. We replace P_2 with Q_2 to contradict the minimality of G.

Now we claim $E(P_1[v_{k-1}, u_k] \cap P_2) = \emptyset$. Again suppose $P_2[x, y] \subseteq P_1[v_{k-1}, u_k]$ is a segment, and let $P_2[x, y]$ be chosen so that $P_2[y, t_2]$ is minimal. If $x \in V(P_1[v_{k-1}, y])$ define $Q_2 = P_2[s_2, v_{k-1}] \cup P_1[v_{k-1}, x] \cup P_2[x, t_2]$, and otherwise define $Q_2 = P_2[s_2, v_k] \cup P_1[v_k, x] \cup P_2[x, t_2]$. As $E(P_1[v_{k+1}, v_k] \cap P_2) = \emptyset$ we see that Q_2 is an M-alternating path. Again we replace P_2 with Q_2 to contradict the minimality of G.

Now let $Q_2 = P_2[s_2, v_{k-1}] \cup P_1[v_{k-1}, u_k] \cup P_2[u_k, t_2]$. As $E(P_1[v_{k-1}, u_k] \cap P_2) = \emptyset$ we see that Q_2 is an M-alternating path and replacing P_2 with Q_2 we once again contradict the minimality of G. □

We deduce several corollaries of (3.4). Let Ω be an octopus in a graph G, where Ω consists of two tentacles and a head C with $V(C) = \{v\}$. Then the graph of Ω is a path. We say that Ω is a *path octopus* with head v. The head of a path octopus can be moved along Ω in the sense that if $v' \in V(\Omega)$ is at even distance from v in Ω, then there is another path octopus with the same graph and head v'. The next lemma will use this fact.

(3.5) *Let G be a graph, let Ω be a path octopus in G with head v and ends v_1 and v_2, let z be the neighbor of v_1 in Ω, let M be an Ω-compatible matching, and let P be an M-alternating path in $G \setminus v_1 \setminus v_2$ with ends v and $w \notin V(\Omega)$. Then there exist a path octopus Ω' with head z and ends v_1 and v_2, an Ω'-compatible matching M', and a path P' with ends z and w such that $E(\Omega') \subseteq E(\Omega \cup P)$, $zv_1 \in E(\Omega')$, $v_1 \notin V(P')$, M coincides with M' on $G \setminus (V(P) \cup V(\Omega))$, $\Omega \cup P \setminus V(\Omega' \cup P')$ is M'-covered, and P' intersects $\Omega' \setminus v_1$ transversally.*

Proof. Since M is Ω-compatible, v is incident with no edge of M. Let $R = \Omega[z, v]$, let $M' = M \triangle E(R)$, and let Ω' be the octopus with graph Ω and head z. Then M' is an Ω'-compatible matching. By (3.2) there exists an M'-alternating path P' with ends z and w such that $E(P') \subseteq E(P) \triangle E(R)$. By (3.3) we may assume, by replacing the tentacle $\Omega'[z, v_2]$ and path P', that P' intersects $\Omega' \setminus v_1 = \Omega'[z, v_2]$ transversally, as desired. □

Let P_1, P_2, P_3 be odd paths in a graph H. For $i = 1, 2, 3$ let u_i and v_i be the ends of P_i. If $u_1 = u_2 = u_3$ and otherwise P_1, P_2, P_3 are pairwise disjoint, then we say that the octopus with tentacles P_1, P_2 and P_3 and a head the graph with vertex-set $\{u_1\}$ is a *triad* in H. Assume now that P_1, P_2, P_3 are pairwise disjoint, and let Q_1, Q_2, Q_3 be three odd paths such that for $\{i, j, k\} = \{1, 2, 3\}$ the ends of Q_k are u_i and u_j. Assume further that $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are pairwise disjoint, except for common ends in the set $\{u_1, u_2, u_3\}$. In those circumstances we say that an octopus with tentacles P_1, P_2 and P_3 and head $Q_1 \cup Q_2 \cup Q_3$ is a *tripod* in H.

(3.6) *Let G be a graph. Let T be a triad or tripod in G with ends v_1, v_2 and v_3. Let M be a T-compatible matching, and let P be an M-alternating path in $G \setminus v_1 \setminus v_2$ with one end in the head of T and another end $w \notin V(T)$. Assume that the edge of P incident with w does not belong to M. Then there exist a triad or tripod $T' \subseteq T \cup P$ with ends v_1, v_2 and w and a T'-compatible matching M' such that M is identical to M' on $G \setminus V(P \cup T)$ and $(T \cup P) \setminus V(T')$ is M'-covered.*

Proof. If T is a triad then the result follows immediately from (3.5). If T is a tripod, then for $i \in \{1, 2, 3\}$ let P_i, Q_i, u_i, v_i be as in the definition of tripod. Extend M to Q_1, Q_2 and Q_3 in such a way that $Q_1 \cup Q_2 \cup Q_3 \setminus u_1$ is M-covered. Let T'' be the path octopus with tentacles P_1 and $P_2 \cup Q_1 \cup Q_2$. Extend P along $Q_1 \cup Q_2 \cup Q_3$ to a path P'' so that P'' is an M-alternating path with ends w and u_1. It remains to apply (3.5) to P'' and T''. □

Let Q be an even path with ends u_1 and u_3, let $u_2 = u_1$ and $u_4 = u_3$, and for $i = 1, 2, 3, 4$ let P_i be an odd path with ends u_i and v_i, disjoint from Q except for u_i, and such that the paths P_i are pairwise disjoint, except that P_1 and P_2 share a common end $u_1 = u_2$ and P_3 and P_4 share a common end $u_3 = u_4$. In those circumstances we say that the octopus with head Q and tentacles P_1, P_2, P_3, P_4 is a *quadropod*.

Now let $P_1, P_2, P_3, Q_1, Q_2, Q_3$ be as in the definition of tripod, except that Q_2 and Q_3 are allowed to intersect beyond the vertex u_1. Suppose there exists a perfect matching M of $Q_2 \cup Q_3 \setminus u_1 \setminus u_2 \setminus u_3$ such that Q_2 and Q_3 are M-alternating and intersect transversally. Then we say that the octopus Ω with tentacles P_1, P_2 and P_3 and a head $Q_1 \cup Q_2 \cup Q_3$ is a *quasi-tripod* in H. Clearly every tripod is a quasi-tripod. It follows from the definition of transversal intersection that $Q_2 \cap Q_3$ consists of one or two paths, one of which contains the vertex u_1. By shortening both Q_2 and Q_3 and extending P_1 we may assume that one of the components of $Q_2 \cap Q_3$ has vertex-set $\{u_1\}$. If that is the only component of $Q_2 \cap Q_3$, then Ω is a tripod; otherwise Ω looks as depicted in Fig. 6.

(3.7) *Let G be a graph. Let T be a triad or tripod in G with ends v_1, v_2 and v_3. Let M be a T-compatible matching, and let P be an M-alternating path in $G \setminus \{v_1, v_2, v_3\}$ with one end in the head of T and another end $w \notin V(T)$. Assume that the edge of P incident with w does not*

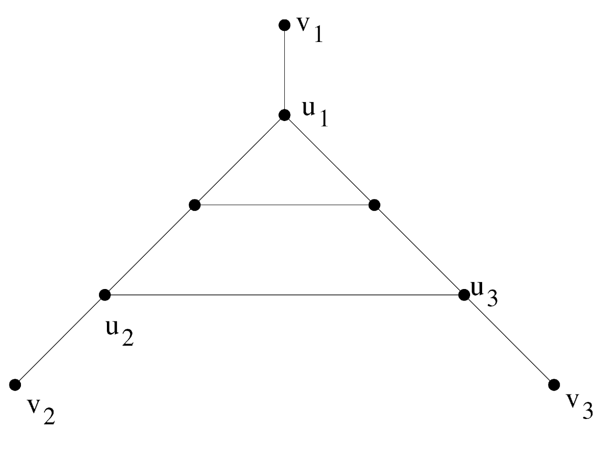

Fig. 6. A quasi-tripod.

belong to M. Then there exist an octopus $T' \subseteq T \cup P$ and a T'-compatible matching M' such that M is identical to M' on $G \setminus V(P \cup T)$, the graph $(T \cup P) \setminus V(T')$ is M'-covered and either T' is a quasi-tripod with ends v_i, v_j and w, for some distinct indices $i, j \in \{1, 2, 3\}$, or T' is a quadropod with ends v_1, v_2, v_3 and w.

Proof. We may assume that $G = T \cup P$ and that there do not exist a triad or tripod T' with ends v_1, v_2 and v_3, a T'-compatible matching M' and an M'-alternating path P' in $G \setminus \{v_1, v_2, v_3\}$ with one end in the head of T' and the other end w such that $w \notin V(T')$, $(T \cup P) \setminus V(T' \cup P')$ is M'-covered and $P' \cup T'$ is a proper subgraph of G. We refer to this as the *minimality of G*.

Let the tentacles of T be P_1, P_2, P_3, where P_i has one end v_i, and let u_i be the other end of P_i. If T is a tripod, then let Q_i be as in the definition of tripod, and otherwise let Q_i be the null graph. We say that a vertex v of P_i is *inbound* if $P_i[v, u_i]$ is even and we say that v is *outbound* otherwise.

Let $u_0 \in V(P \cap T)$ be chosen to minimize $P[w, u_0]$. If T is a triad and u_0 is inbound, then $T \cup P[w, u_0]$ is a required quadropod. If T is a tripod and $u_0 \in V(P_i)$ is inbound then by replacing $P_i[v_i, u_0]$ by $P[w, u_0]$ in T we obtain a required quasi-tripod. If T is a tripod and $u_0 \in V(Q_i)$, then we may assume from the symmetry that $Q_i[u_0, u_j]$ is even, in which case by replacing P_j by $P[w, u_0]$ we obtain a required quasi-tripod.

Therefore for the rest of the proof we may assume that $u_0 \in V(P_1)$ and that u_0 is outbound. Let $r \in V(T) \cap V(P) - V(P_1)$ be chosen to minimize $P[w, r]$ and if no such r exists let $r \neq w$ be the end of P. Apply (3.4) to P_1 and P with $s_1 = v_1$, $t_1 = u_1$ and $s_2 = w$. Outcome (3.4)(1) does not hold by the minimality of G. If (3.4)(2) holds, then by considering the path guaranteed therein we obtain a desired quasi-tripod or quadropod. Thus we may assume (3.4)(3) holds, and hence P_1 intersects $P[w, r]$ transversally.

Let v_0 be such that $P[v_0, u_0]$ is a component of $P \cap P_1$, and let u be such that $P[v_0, u]$ is a maximal path with no internal vertex or edge in T. If $u \in V(P_1)$, then by the definition of transversal intersection the vertices v_1, v_0, u_0, u, u_1 occur on P_1 in the order listed and u is inbound. By considering $T \cup P[w, u]$ and deleting $P_3 \setminus u_3$ and the interior of Q_3 we obtain a required quasi-tripod. Thus we may assume that $u \notin V(P_1)$, and hence $u = r$. If r is not outbound, then a similar argument gives a required quasi-tripod.

It follows that for the remainder of the proof we may assume that $r \in V(P_2)$, and that r is outbound. Let M_1 be the unique perfect matching of $Q_1 \cup Q_2 \cup Q_3 \setminus u_1$, and let $M^+ = M \cup M_1$. We can extend P along $Q_1 \cup Q_2 \cup Q_3$ to an M^+-alternating path P^+ so that u_1 is an end of P^+. Apply (3.2) to P^+ and $P_1[v_0, u_1]$ to produce an M'-alternating path P' with ends w and v_0, where $M' = M^+ \triangle P_1[u_1, v_0]$. Let T' be obtained from $T \cup P[v_0, r]$ by deleting the interiors of $P_2[r, u_2]$ and Q_2; then T' is a triad with ends v_1, v_2, v_3. But now T' and P' contradict the minimality of G. $\square$

4. Embeddings and main lemma

In this section we first formalize the notion of a matching minor by introducing the concept of an embedding, and show in (4.2) below that a graph H has a matching minor isomorphic to a graph G if and only if there is an embedding $H \hookrightarrow G$. Then we study the following question. Suppose that $\eta : H \hookrightarrow G$ is an embedding, G is a brick, and $v_0 \in V(H)$ has degree two. Since bricks have no vertices of degree two, there is a subgraph of G that "fixes" this violation of being a brick. What can we say about this subgraph? The answer leads to the notion of v_0-augmentation of η. We define this concept formally, and then prove two results about its existence. The second, (4.4), will be used when some graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G; otherwise we will use (4.3), the first of these results. Finally, we classify all "minimal" v_0-augmentations into one of four types.

Let T' be a tree, and let T be obtained from T' by subdividing every edge an odd number of times. Then $V(T') \subseteq V(T)$. The vertices of T that belong to $V(T')$ will be called *old* and the vertices of $V(T) - V(T')$ will be called *new*. We say that T is a *barycentric tree*. Please note that the partition into old and new vertices depends on T' (there is an ambiguity concerning vertices of degree two). We shall assume that each barycentric tree has a fixed partition into new and old vertices. By a *branch* of a barycentric tree T we mean a subpath of T with ends old vertices and all internal vertices new.

We need to formalize the concept of matching minor. Let H and G be graphs. A *weak embedding of H to G* is a mapping η with domain $V(H) \cup E(H)$ such that for $v, v' \in V(H)$ and $e, e' \in E(H)$,

(1) $\eta(v)$ is a barycentric subtree in G,
(2) if $v \neq v'$, then $\eta(v)$ and $\eta(v')$ are vertex-disjoint,
(3) $\eta(e)$ is an odd path with no internal vertex in any $\eta(v)$ or $\eta(e')$ for $e' \neq e$,
(4) if $e = u_1u_2$, then the ends of $\eta(e)$ can be denoted by x_1, x_2 in such a way that x_i is an old vertex of $\eta(u_i)$, and
(5) $G \setminus \bigcup_{x \in V(H) \cup E(H)} V(\eta(x))$ has a perfect matching.

The next lemma will show that H is isomorphic to a matching minor of G if and only if there is a weak embedding of H to G. Then we will show that such a weak embedding can be chosen with two additional properties. Thus we say that a weak embedding from H to G is an *embedding* if, in addition, it satisfies

(6) if v has degree one then $\eta(v)$ has exactly one vertex,
(7) if $v \in V(H)$ has degree two and e_1, e_2 are its incident edges, then $\eta(v)$ is an even path with ends x_1, x_2, say, and $\eta(e_1)$, $\eta(e_2)$ both have length one, one has x_1 as its end and the other has x_2 as its end, and
(8) if v has degree at least three and x is an old vertex of $\eta(v)$ of degree d, then x is an end of $\eta(e)$ for at least $3 - d$ distinct edges e.

For every subgraph H' of H define $\eta(H') = \bigcup_{x \in V(H) \cup E(H)} \eta(x)$. We denote the fact that η is an embedding of H into G by writing $\eta : H \hookrightarrow G$.

Let $T \subseteq H$ be a barycentric tree, and let (X, Y) be the unique partition of $V(T)$ into two independent sets with X including all the old vertices. The vertices of X will be called *protected* and the vertices of Y will be called *exposed*.

(4.1) *Let H and G be graphs. There exists a weak embedding of H to G if and only if H is isomorphic to a matching minor of G.*

Proof. If $\eta : H \hookrightarrow G$ then a graph isomorphic to H can be obtained from the central subgraph $\eta(H)$ of G by repeatedly bicontracting exposed vertices of $\eta(v)$ and internal vertices of $\eta(e)$ for $v \in V(H)$ and $e \in E(H)$. Thus H is a matching minor of G.

To prove the converse we may assume that H is a matching minor of G. Thus there exist graphs $H_1, H_2, \ldots, H_k$ such that $H_1 = H$, H_k is a central subgraph of G, and for $i = 2, 3, \ldots, k$

the graph H_{i-1} is obtained from H_i by bicontracting a vertex. We define $\eta_k : H_k \hookrightarrow G$ by saying that if $v \in V(H_k)$, then $\eta_k(v)$ is the graph with vertex-set $\{v\}$, and if $e \in E(H_k)$, then $\eta_k(e)$ is the graph consisting of e and its ends. It is clear that η_k satisfies (1)–(5). We now construct a sequence of mappings satisfying (1)–(5). Assuming that η_i has been defined we define η_{i-1} as follows. Let v be the vertex of H_i whose bicontraction produces H_{i-1}, let x, y be the neighbors of v, and let w be the new vertex of H_{i-1}. For $z \in V(H_{i-1}) \cup E(H_{i-1}) - \{w\}$ let $\eta_{i-1}(z) = \eta_i(z)$, and let $\eta_{i-1}(w) = \eta_i(x) \cup \eta_i(y) \cup \eta_i(v) \cup \eta_i(xv) \cup \eta_i(yv)$. This completes the construction. It is clear that η_1 satisfies (1)–(5). □

We now show that if there is a weak embedding of H to G, then there is an embedding of H to G.

(4.2) *Let H and G be graphs. There exists an embedding of H to G if and only if H is isomorphic to a matching minor of G.*

Proof. By (4.1) it suffices to show that if η is a weak embedding of H to G, then there exists an embedding of H to G.

It is easy to modify η so that it satisfies conditions (6) and (7). Thus we may choose a mapping η with domain $V(H) \cup E(H)$ satisfying (1)–(7) such that the total number of old vertices in $\eta(v)$ over all vertices $v \in V(H)$ of degree at least three is minimum. We claim that η satisfies (8) as well.

To prove that η satisfies (8) let $v \in V(H)$ have degree at least three, let x be an old vertex of $\eta(v)$, and let d be the degree of x in $\eta(v)$. If $d = 2$ and x is not an end of $\eta(e)$ for any $e \in E(G)$, then we change the barycentric structure of $\eta(v)$ by declaring x to be a new vertex. The new embedding thus obtained contradicts the minimality of η. If $d = 0$, then x is the unique vertex of $\eta(v)$, and it is an end of $\eta(e)$ for all the (at least three) edges e incident with v by (4). Thus we may assume that $d = 1$. If x is not an end of any $\eta(e)$, then we remove from $\eta(v)$ the vertex x and all internal vertices of Q, where Q is the unique subpath of $\eta(v)$ between x and the nearest old vertex. Then the set of vertices removed has a perfect matching, because Q is even by the definition of barycentric subdivision, and hence the new embedding satisfies (5). Thus the new embedding contradicts the minimality of η. To complete the proof we may therefore suppose for a contradiction that x is incident with $\eta(e)$ for exactly one $e \in E(H)$. By (4) one end of e is v; let u be the other end. If u has degree at most two, then we define a new embedding by moving x and the internal vertices of Q from $\eta(v)$ to $\eta(u)$, and changing $\eta(e)$ accordingly. If u has degree at least three, then we move x and all internal vertices of Q from $\eta(v)$ to $\eta(e)$. In either case the new embedding contradicts the minimality of η. Thus η satisfies (8), and hence it is an embedding $H \hookrightarrow G$, as desired. □

Let T be an even subpath of a graph H, and let T be regarded as a barycentric tree, with its ends designated as old and all internal vertices designated as new. Let us recall that the notions of protected and exposed were defined prior to (4.1). Let P be a path with one end, say v, in the interior of T and no other vertex in T. If v is exposed, then let Q be the null graph, and if v is protected, then let Q be a path with ends exposed vertices $q_1, q_2 \in V(T)$ and otherwise disjoint from $H \cup P$ such that v lies on T between q_1 and q_2. In those circumstances we say that *Q is a cap for P at v with respect to T and H.*

Let $\eta: H \hookrightarrow G$. For every edge $e = uv \in E(H)$ the path $\eta(e)$ is odd. Let P_e denote its interior (that is, the path obtained by deleting the ends), and let M_e be the unique perfect matching of P_e (possibly $M_e = \emptyset$). We define $M(\eta)$ to be the union of M_e over all $e \in E(H)$.

Now let $v_0 \in V(H)$ have degree two, and let v_1, v_2 be its neighbors. For $i = 1, 2$ let E_i be the set of edges of H incident with v_i, except for the edge v_0v_i, and let $E_1 \cap E_2 = \emptyset$. Let M_1 be a perfect matching of $G \setminus V(\eta(H))$, and let $M = M_1 \cup M(\eta)$. Let P be an M-alternating path with one end $x \in V(\eta(v_0))$ and the other end u in $\bigcup\{\eta(v)\colon v \in V(H) - \{v_0, v_1, v_2\}\}$ with the property that if P intersects $\eta(e)$ for some $e \in E(H)$ not incident with v_0, v_1, or v_2, then P and $\eta(e)$ intersect in a path and have a common end. Let S denote the path $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_0v_2)$; then S is obtained from $\eta(v_0)$ by appending two edges, one at each end. Let Q be an M_1-alternating cap for P at x with respect to S and $\eta(H)$. We say that the pair (P, Q) is a v_0*-augmentation of* η. It follows that P and Q have no internal vertices in $\bigcup_{v \in V(H)} \eta(v)$. We say that x is the *origin* and u is the *terminus* of P. See Figs. 7–9 for example.

Our first result about augmentations is the following.

(4.3) *Let H be a graph on at least four vertices, let v_0 be a vertex of H that has exactly two neighbors v_1 and v_2, and let v_1 and v_2 be not adjacent. Let G be a brick and let $\eta: H \hookrightarrow G$ be an embedding such that both $\eta(v_1)$ and $\eta(v_2)$ have exactly one vertex. Then there exist an embedding $\eta': H \hookrightarrow G$ and a v_0-augmentation of η'.*

Proof. Define E_1, E_2 and M as in the definition of v_0-augmentation. The path $\eta(v_0) \cup \eta(v_0v_1) \cup \eta(v_0v_2)$ is even and can therefore be regarded as path octopus, which we denote by Ω_1. Let Ω_2 be the octopus with the set of tentacles $\{\eta(e)\colon e \in E_1 \cup E_2\}$ and head $\eta(H \setminus v_0 \setminus v_1 \setminus v_2)$. The head of Ω_2 is non-null, because H has at least four vertices. We can convert M to a matching M^+ so that M^+ is Ω_i-compatible for $i = 1, 2$. We apply (2.3) to the frame $(\{\Omega_1, \Omega_2\}, V(\eta(v_1)) \cup V(\eta(v_2)))$ and denote the resulting path by R. Let R have ends $r_1 \in V(\Omega_1)$ and $r_2 \in V(\Omega_2)$ and let $e \in E(R)$ be such that each of the components $R_i = R[s_i, r_i]$ of $R \setminus e$ intersects only one of the octopi Ω_1 and Ω_2.

By (3.5) we may assume, by changing M^+, R_1, and $\eta(v_0)$, that there exist an M^+-alternating path P_1 with ends $p_1 \in V(\eta(v_0))$ and s_1, and an M^+-alternating cap Q_1 for P_1 at p_1 with respect to Ω_1 and $\eta(H)$ such that $P_1 \cup Q_1 \subseteq R_1$. We may also assume that r_2 is the only vertex of R in the head of Ω_2. If $r_2 \in \eta(v)$ for some $v \in V(H)$, then let R_2' be the null graph, and if $r_2 \in \eta(e)$ for some $e \in E(H)$, then let R_2' be an M^+-alternating subpath of $\eta(e)$ with one end r_2 and the other in $\eta(v)$ for some $v \in V(H)$. Then $(P_1 \cup R_2 \cup R_2', Q_1)$ is a desired v_0-augmentation of η. $\square$

In the next section we will need the following lemma.

(4.4) *Let H be a graph, and let v be a vertex of H of degree at least four, let G be a brick, and let $\eta: H \hookrightarrow G$ be such that $\eta(v)$ has at least two vertices. Then either*

(1) *there exist a graph H_1 obtained from H by bisplitting v, an embedding $\eta_1: H_1 \hookrightarrow G$ and a v_0-augmentation of η_1, where v_0 is the new inner vertex of H_1, or*
(2) *there exist an embedding $\eta_2: H \hookrightarrow G$, a path P with ends p_1 and p_2 in the interiors of different branches, say B_1 and B_2, of $\eta_2(v)$ and otherwise disjoint from $\eta_2(H)$ and for $i = 1, 2$ there exists a cap Q_i for P at p_i with respect to B_i and $\eta_2(H)$ such that Q_1 and Q_2 are disjoint.*

Proof. Denote the branches of $\eta(v)$ by $B_1, B_2, \dots, B_n$. They can be considered as octopi, which we denote by $\Omega_1, \Omega_2, \dots, \Omega_n$, respectively. Let Ω_0 be the octopus with the set of tentacles $\{\eta(e)\colon e \text{ is incident to } v\}$ and head $\eta(H \setminus v)$, let X be the set of old vertices of $\eta(v)$, and let $\mathcal{F} = \{\Omega_0, \Omega_1, \Omega_2, \dots, \Omega_n\}$. We can extend a perfect matching of $G \setminus \eta(H)$ to a matching M so that M is Ω-compatible for every $\Omega \in \mathcal{F}$. Clearly $|X| = n + 1$. Therefore $(\mathcal{F}, X)$ is a frame. We apply (2.3) to it and denote the resulting path by R. Furthermore, there is an edge $e \in E(R)$ such that each of the components $R_i = R[s_i, r_i]$ of $R \setminus e$ intersects only one of the octopi of $\mathcal{F}$.

If for some $i \in \{1, 2\}$ the path R_i intersects Ω_j for $j \geqslant 1$ we may assume, by changing M, and Ω_j, that there exist an M-alternating path P_i with ends $p_i \in V(B_j)$ and s_i, and an M-alternating cap Q_i for P_i at p_i with respect to B_j and $\eta(H)$ such that $P_i \cup Q_i \subseteq R_i \cup B_j$. If this happens for both R_1 and R_2 define $P = P_1 \cup P_2 + e$ and outcome (2) holds.

Therefore we may assume that R_2 intersects Ω_0 and R_1 intersects Ω_j for some $j \geqslant 1$, and furthermore that r_2 is the only vertex of R in the head of Ω_0. If $r_2 \in \eta(v)$ for some $v \in V(H)$, then let R_2' be the null graph, and if $r_2 \in \eta(e)$ for some $e \in E(H)$, then let R_2' be an M-alternating subpath of $\eta(e)$ with one end r_2 and the other in $\eta(v)$ for some $v \in V(H)$.

Let T_1 and T_2 be the two components of the graph obtained from $\eta(v)$ by removing the internal vertices of B_j. Let H_1 be obtained from H by splitting v into new outer vertices v_1 and v_2 and new inner vertex v_0 in such a way that v_i is adjacent to a neighbor u of v in H if $\eta(uv_i)$ has an end in T_i. Let $\eta_1(v_i) = T_i$, let $\eta_1(v_0)$ be B_1 minus its ends, let $\eta_1(v_1v_0)$ and $\eta_1(v_2v_0)$ be the two end-edges of B_1 and let $\eta_1(x) = \eta(x)$ for all other $x \in V(H_1) \cup E(H_1)$. Then $(P_1 \cup R_2' \cup \{e\}, Q_1)$ is a v_0-augmentation of η_1 and outcome (1) holds. □

Let H and G be graphs, let $\eta : H \hookrightarrow G$, let v_0 be a vertex of H of degree two, and let (P, Q) be a v_0-augmentation of η. We say that η is *minimal* if there exists no embedding $\eta' : H \hookrightarrow G$ and a v_0-augmentation (P', Q') of η' such that $\eta'(H) \cup P' \cup Q'$ is a proper subgraph of $\eta(H) \cup P \cup Q$. In applications we may assume that our v_0-augmentations are minimal. The next lemma will classify minimal augmentations into four types, which we now introduce.

Let $\eta : H \hookrightarrow G$, let $v_0 \in V(H)$ have degree two, let $v_1, v_2 \in V(H)$ be its neighbors, and let E_1, E_2 be as in the definition of v_0-augmentation. Let $i \in \{1, 2\}$ and $e \in E_i$. Let x_e be the end of $\eta(e)$ that belongs to $V(\eta(v_i))$. We say that an internal vertex $x \in V(\eta(e))$ is an *inbound vertex* if it is at even distance from x_e in $\eta(e)$, and otherwise we say that it is an *outbound vertex*.

Let M be a matching containing $M(\eta)$, let P be an M-alternating path with ends x_0 and x_5, and let the vertices $x_0, x_1, x_2, x_3, x_4, x_5$ appear on P in the order listed. Assume that $P[x_1, x_2]$ and $P[x_3, x_4]$ are subpaths of $\eta(e)$, and that otherwise P is disjoint from $\bigcup\{\eta(e)\colon e \in E_1 \cup E_2\}$. Assume also that x_1 is an inbound vertex of $\eta(e)$, that x_2 and x_3 are outbound, and that either $x_2 = x_3 = x_4$, or x_1, x_2, x_4, x_3, x_e are pairwise distinct and occur on $\eta(e)$ in the order listed. In those circumstances we say that P intersects $\eta(e)$ *regularly from x_0 to x_5*.

Let (P, Q) be a v_0-augmentation of η and let P have ends a and b where $a \in V(\eta(v_0))$. We say that (P, Q) is of type A if whenever P intersects $\eta(e)$ for some $e \in E_1 \cup E_2$, then P and $\eta(e)$ intersect in a path whose one end is a common end of P and $\eta(e)$. Thus P intersects at most one $\eta(e)$, because the common end must be b, and b does not belong to $\eta(v_1) \cup \eta(v_2)$. See Fig. 7.

We say that (P, Q) is of type B if there exist a vertex $x \in V(P)$, an index $i \in \{1, 2\}$, and an edge $e \in E_i$ such that the vertex v_i has degree at most three, the path $P[a, x]$ intersects $\eta(e)$ regularly from a to x, and if $P[x, b] \setminus x$ intersects $\eta(e')$ for some $e' \in E(H)$, then $P[x, b] \setminus x$ and $\eta(e')$ intersect in a path and have a common end. Moreover, if $e = e'$, then we require that $P[a, x] \cap \eta(e)$ be a path. We say that (P, Q) *crosses* $\eta(e)$. See Fig. 8.

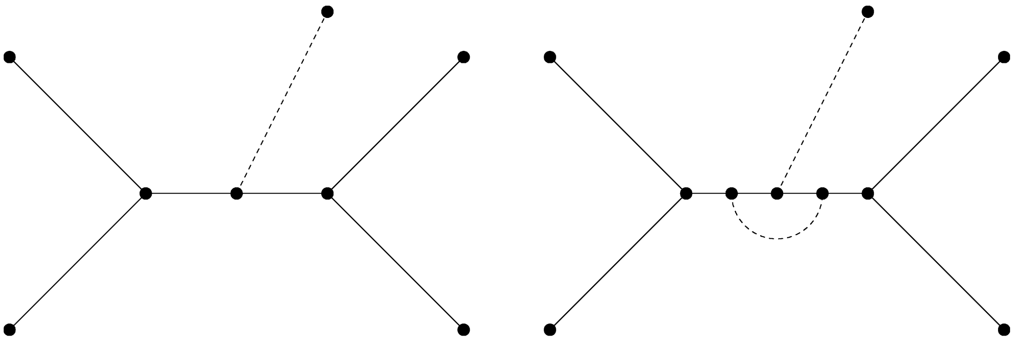

Fig. 7. Augmentations of type A.

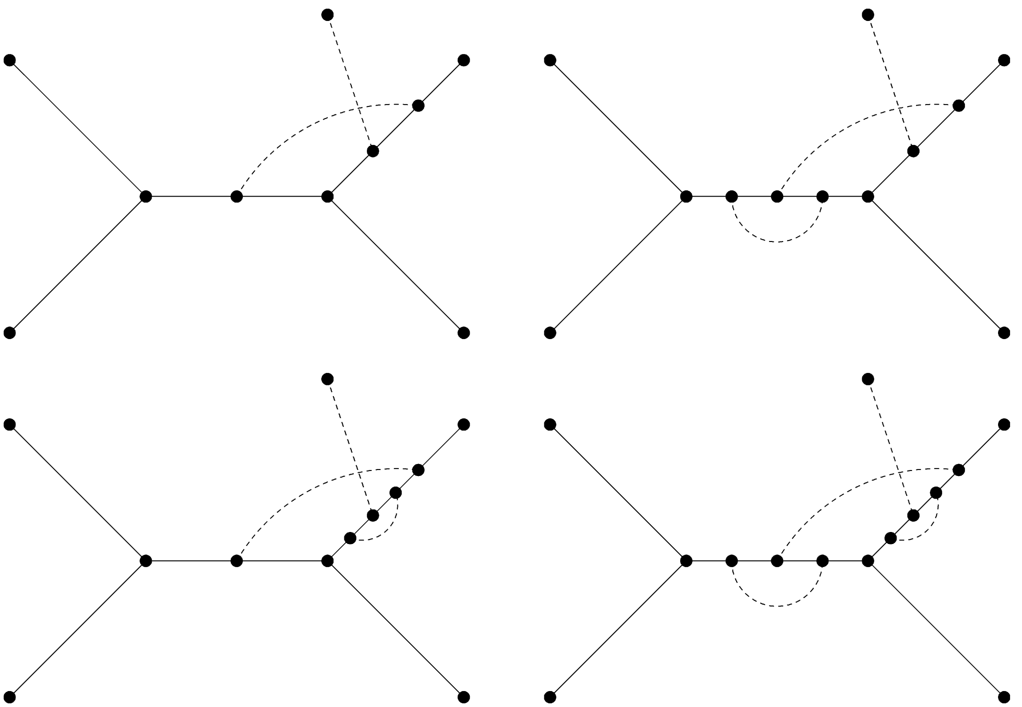

Fig. 8. Augmentations of type B.

We say that (P, Q) is of type C if there exist vertices $x_1, x_2 \in V(P)$ such that a, x_1, x_2, b occur on P in the order listed, and there exist distinct edges e_1, e_2, one in E_1 and the other in E_2, such that the end of e_1 in $\{v_1, v_2\}$ has degree at most three, $P[a, x_1]$ intersects $\eta(e_1)$ regularly from a to x_1, $P[x_1, x_2]$ has no internal vertices in $\eta(H)$ and x_2 is an inbound vertex of $\eta(e_2)$. We say that (P, Q) *crosses* $\eta(e_1)$. See Fig. 9.

We say that (P, Q) is of type D if for some $i \in \{1, 2\}$ and some $e \in E_i$ the vertex v_i has degree at least four and there exists an inbound vertex x of $\eta(e)$ such that $x \in V(P)$ and $P[a, x]$ has no internal vertex in $\eta(H)$.

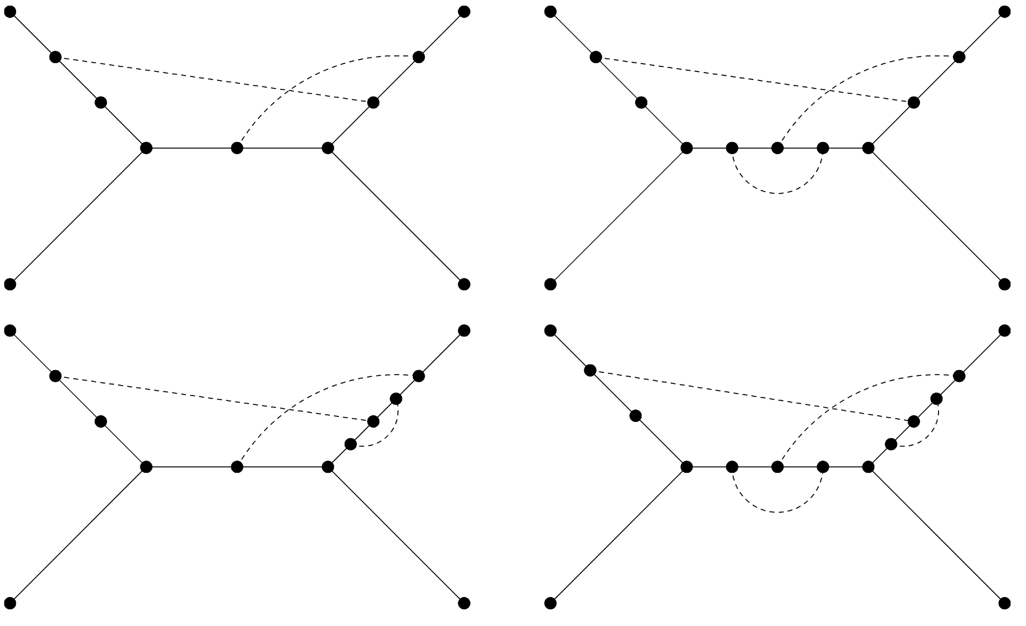

Fig. 9. Augmentations of type C.

The following classification of minimal v_0-augmentations is the third main result of this section.

(4.5) *Let H and G be graphs, and let $\eta : H \hookrightarrow G$. Let $v_0 \in V(H)$ have degree two, and let v_1, v_2 be its neighbors. Assume that v_1 is not adjacent to v_2. Then every minimal v_0-augmentation of η is of type* A, B, C, *or* D.

Proof. Let (P, Q) be a minimal v_0-augmentation of η, let x_0 be the end of P in $\eta(v_0)$, and let b be the other end of P. We wish to think of P as being directed away from x_0; thus language such as "the first vertex of P in a set Z" will mean the vertex of $V(P) \cap Z$ that is closest to x_0 on P. Let E_1 and E_2 be as in the definition of v_0-augmentation.

Let us assume for a moment that P includes an internal vertex of some $\eta(e)$, where $e \in E(H)$ is not incident with v_0, v_1, or v_2. Let z be the first such vertex on P. The vertex z divides $\eta(e)$ into two subpaths, one even and one odd. Let R be the even one. Then $(P[x_0, z] \cup R, Q)$ is a v_0-augmentation, and hence the minimality of (P, Q) implies that $R = P[z, b]$. If $e \in E_1 \cup E_2$ and z is an outbound vertex, then the same conclusion holds. This will be later referred to as the *confluence property of P*.

If P includes an internal vertex of $\eta(e_1)$ for no $e_1 \in E_1 \cup E_2$, then (P, Q) is of type A. Thus we may assume that P includes such a vertex, and let x_1 be the first such vertex on P. From the symmetry we may assume that $e_1 = v_1 v_3 \in E_1$. If x_1 is an outbound vertex, then the confluence property of P implies that (P, Q) is of type A. Thus we may assume that x_1 is inbound. If v_1 has degree at least four, then (P, Q) is of type D, and so we may assume that v_1 has degree at most three. It follows from axiom (7) in the definition of an embedding that v_1 has degree exactly three.

Let x_2 be the first vertex on P that belongs to $\eta(z)$ for some $z \in V(H) \cup E(H)$ not equal, incident or adjacent to v_0 and not equal to e_1. Then x_1 lies on P between x_0 and x_2. Let $P_1 = \eta(e_1)$. By (3.4) applied to P_1, $P_2 = P$, $r = x_2$, $s_2 = x_0$, $t_2 = b$ and the ends of P_1 numbered so that $s_1 \in V(\eta(v_0))$ and $t_1 \in V(\eta(v_3))$ we deduce that (1), (2), or (3) of (3.4) holds. But (1) does not hold by the minimality of (P, Q), and if (2) holds, then (R, Q) is a v_0-augmentation of type A or B. Thus we may assume that (3) of (3.4) holds. Since x_1 is an inbound vertex, this implies that either there exist vertices $y_1, y_2 \in V(P_1)$ such that y_1 and y_2 are outbound, $P[x_1, y_1] \subseteq P_1$, $x_1 \in P_1[y_1, y_2]$ and $P[y_1, y_2]$ has no internal vertices in $\eta(H)$, or $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ regularly from x_0 to x_2. In the former case $(P[x_0, y_2] \cup P_1[y_2, t_1], Q)$ is a v_0-augmentation of η of type B, and hence we may assume that the latter case holds. Thus $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ regularly from x_0 to x_2, and if $x_2 = t_2$, then $P[x_0, x_2] \setminus x_2$ intersects $\eta(e_1)$ in a path.

If $x_2 \in \bigcup\{V(\eta(v))\colon\ v \in V(H) - \{v_0, v_1, v_2\}\}$, then (P, Q) is of type B. Therefore we may assume that $x_2 \in V(\eta(e_2))$ for some $e_2 \in E(H \setminus v_0) - \{e_1\}$. By the confluence property of P we may assume that $e_2 \in E_1 \cup E_2$ and that x_2 is inbound, for otherwise (P, Q) is of type B.

If $e_2 \in E_2$, then (P, Q) is of type C, and the lemma holds. Thus we may assume that $e_2 \in E_1 - \{e_1\}$. Let y be such that $\eta(e_2)[x_2, y]$ is a component of $\eta(e_2) \cap P$. For simplicity of notation assume that Q is empty. The argument in the other case is similar. As v_1 has degree three, axiom (8) in the definition of an embedding implies that the tree $\eta(v_1)$ consists a single vertex, say u_1. Since x_2 is inbound it follows that y lies between u_1 and x_2 in $\eta(e_2)$. Let C be the cycle $P[x_0, y] \cup \eta(e_2)[y, u_1] \cup S$, where $S = \eta(v_0)[x_0, u_1]$. The subgraph of G with edge-set $E(P) \triangle E(C)$ includes a path with ends x_0 and b, say P'. Let f be the edge of $P[x_0, x_1]$ incident to x_1. We define a new embedding $\eta' : H \hookrightarrow G$ by $\eta'(e_1) = \eta(e_1)[x_1, t_1]$, $\eta'(e_2) = P[x_1, x_2] \cup \eta(e_2)[x_2, z]$ (where $z \neq u_1$ is the other end of $\eta(e_2)$), $\eta'(v_1)$ is the graph with vertex-set $\{x_1\}$, we define $\eta'(v_0v_1)$ to be the path with edge-set $\{f\}$, we define $\eta'(v_0)$ to be the path obtained from $\eta(v_0)$ by replacing $\eta(v_0)[x_0, u_1]$ by $P[x_0, x_1] \setminus x_1$, and we define $\eta'(x) = \eta(x)$ for all other $x \in V(H) \cup E(H)$. It follows that (P', Q) is a v_0-augmentation of η', contrary to the minimality of (P, Q), because $P' \cup Q \cup \eta'(H)$ does not include the edge of $\eta(e_2)[y, x_2]$ incident with x_2. □

5. Disposition of bisplits

The purpose of this section is to prove (1.11) under the additional hypothesis that a graph, say H', obtained from H by bisplitting some vertex is isomorphic to a matching minor of G. If that is the case we apply (4.4) and (4.5). We handle the four possible outcomes of (4.5) separately.

(5.1) *Let H and G be graphs, where H has minimum degree at least three. Let H' be obtained from H by bisplitting a vertex v, and let v_0 be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and assume that there exists a v_0-augmentation of η of type* A. *Then a linear extension of H is isomorphic to a matching minor of G.*

Proof. Let v_1 and v_2 be the new outer vertices of H', let (P, Q) be a v_0-augmentation of η of type A, and let a and b be the ends of P, where $a \in V(\eta(v_0))$. Let $b \in \eta(u)$, where $u \in V(H) - \{v_0, v_1, v_2\}$. Let us assume first that b is protected. If Q is null, then $H' + (v_0, u)$ is isomorphic to a matching minor of G, and otherwise (by ignoring Q and bicontracting its ends) we see that $H + (v, u)$ is isomorphic to a matching minor of G and is a linear extension of H unless $vu \in E(H)$. If $vu \in E(H)$ we assume without loss of generality that $uv_1 \in E(H')$. Then $\eta(H' \setminus uv_1) \cup P \cup Q$ is isomorphic to a bisubdivision of a linear extension of H.

Now let us assume that b is exposed. Let T_1, T_2 be the two components of $\eta(u) \setminus b$. For each neighbor w of u in H the path $\eta(uw)$ has exactly one end in $\eta(u)$; that end is an old vertex by axiom (4) in the definition of embedding, and hence belongs to either T_1 or T_2. For $i = 1, 2$ let N_i be the set of all neighbors w of u such that the end of $\eta(uw)$ in $\eta(u)$ belongs to T_i. Let H_1 be obtained from H by bisplitting u so that one of the new outer vertices is adjacent to every vertex of N_1, and the other new outer vertex is adjacent to every vertex of N_2. (Here we use that u has degree at least three.) Let u_0 be the new inner vertex of H_1. Let H_1' be defined similarly, but starting from H' rather than H, and let the new inner vertex be also u_0. If Q is null, then $H_1' + (v_0, u_0)$ is isomorphic to a matching minor of G; otherwise $H_1 + (v, u_0)$ is isomorphic to a matching minor of G, as desired. □

(5.2) *Let H and G be graphs, let $\eta : H \hookrightarrow G$ be an embedding, let v_0 be vertex of H of degree two, and let v_1 be a neighbor of v_0 of degree three with neighbors v_0, v_1', v_1''. Let (P, Q) be a v_0-augmentation of η of type* B *or* C *that crosses $\eta(v_1 v_1')$. Then there exists an embedding $\eta' : H \hookrightarrow G$ and a v_0-augmentation (P', Q') of η' of the same type as (P, Q) that crosses $\eta'(v_1 v_1'')$ such that $\eta'(H) \cup P' \cup Q' \subseteq \eta(H) \cup P \cup Q$ and P and P' have the same terminus.*

Proof. We first define η'. Let x_0 be the end of P in $\eta(v_0)$, let x_6 be the other end of P, let $x_5 \in V(P)$, and let $x_0, x_1, \ldots, x_5$ be as in the definition of regular intersection, witnessing that $P[x_0, x_5]$ intersects $\eta(v_1 v_1')$ regularly from x_0 to x_5. We define $\eta'(v_1) = x_1$, we define $\eta'(v_1 v_1')$ to be the subpath of $\eta(v_1 v_1')$ with one end x_1 and the other end in $\eta(v_1')$, we define $\eta'(v_1 v_1'')$ to be the union of the complementary subpath of $\eta(v_1 v_1')$ and $\eta(v_1 v_1'')$, we define $\eta'(v_0)$ to be a suitable subgraph of $\eta(v_0) \cup P \cup Q$, define $\eta'(v_0 v_1)$ to be the edge of $P[x_0, x_1]$ incident with x_1, and we define $\eta'(x) = \eta(x)$ for all other $x \in V(H) \cup E(H)$. Then $\eta' : H \hookrightarrow G$.

It is now easy to find subpaths Q' and P'' of $\eta(v_0) \cup \eta(v_0 v_1) \cup \eta(v_1 v_1') \cup P \cup Q$ such that $(P'' \cup P[x_4, x_6], Q')$ is the desired v_0-augmentation of η'. □

(5.3) *Let H and G be graphs, where H has minimum degree at least three. Let H' be obtained from H by bisplitting a vertex v, and let v_0 be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and assume that there exists a v_0-augmentation of η of type* B. *Then a linear extension of H is isomorphic to a matching minor of G.*

Proof. Let v_1 and v_2 be the new outer vertices of H'. Let (P, Q) be a v_0-augmentation of η of type B, let x_0, x_6 be the ends of P, where $x_0 \in V(\eta(v_0))$ and $x_6 \in V(\eta(u))$, and let P cross $\eta(e_1)$, where $e_1 = v_1 v_1'$ and $v_1' \neq v_0$, is a neighbor of v_1 in H'. Let $x_5 \in V(P)$ be such that $P[x_0, x_5]$ intersects $\eta(e_1)$ regularly from x_0 to x_5, and let the vertices $x_0, x_1, x_2, x_3, x_4, x_5$ be as in the definition of regular intersection. Notice that v_1 has degree three; thus $\eta(v_1)$ consists of a unique vertex by condition (8) in the definition of embedding. Let v_1'' be the third neighbor of v_1. By (5.2) we may assume that $u \neq v_1'$.

Assume first that x_2, x_3, x_4 are pairwise distinct. The path $P[x_4, x_6]$ proves that a linear extension of H is isomorphic to a matching minor of G, unless x_6 is a protected vertex of $\eta(u)$ and u is adjacent to v in H. Let $i \in \{1, 2\}$ be such that u is adjacent to v_i in H'. Consider the graph obtained from $\eta(H) \cup P[x_4, x_0]$ by deleting the interior of $\eta(v_i u)$; the path $P[x_2, x_3]$ proves that the linear extension $H'' + (v_0', v_1')$ of H is isomorphic to a matching minor of G, where H'' is obtained from H by bisplitting of the vertex v so that one of the new outer vertices is adjacent to v_1' and u, the other outer vertex is adjacent to all other neighbors of v and v_0' is the new inner vertex.

Thus we may assume that $x_2 = x_3 = x_4$. Again the path $P[x_4, x_6]$ proves that a linear extension of H is isomorphic to a matching minor of G, unless x_6 is a protected vertex of $\eta(u)$ and u is adjacent to v_1' in H. Thus we may assume that x_6 is a protected vertex of $\eta(u)$ and u is adjacent to v_1' in H. If v_1' has degree at least four, then let H'' be obtained from H' by bisplitting v_1' in such a way that one of the new vertices is adjacent to v_1 and u, and let z be the new vertex. Then $H'' + (v_0, z)$ is a linear extension of H and is clearly isomorphic to a matching minor of G. If v_1' has degree three we replace $\eta(v_1'u)$ by $P[x_4, x_6]$ and notice that (P, Q) can be easily converted to a v_0-augmentation (P', Q') of type A of the embedding thus obtained. (Notice that the terminus of P' does not belong to $\eta(v_2)$, because H' is obtained from H by bisplitting v.) Hence the theorem follows from (5.1). □

For the next lemma we need the following generalization of v_0-augmentations. Let $v_0 \in V(H)$ have degree two, and let v_1, v_2 be its neighbors. For $i = 1, 2$ let E_i be the set of edges of H incident with v_i, except for the edge v_0v_i, and let $E_1 \cap E_2 = \emptyset$. Let R be the interior of $\eta(v_0)$, let M_1 be a perfect matching of $G \setminus V(\eta(H))$, let $x \in V(R)$, let M_2 be a perfect matching of $R \setminus x$, and let $M = M_1 \cup M_2 \cup M(\eta)$. Let P be an M-alternating path with one end x and the other end u in $\bigcup\{\eta(v)\colon v \in V(H) - \{v_0, v_1, v_2\}\}$. We say that P is a *weak v_0-augmentation of* η. It follows that P has no internal vertex in $\bigcup_{v \in V(H)-\{v_0\}} \eta(v)$. This is indeed a generalization of v_0-augmentation. For let (P, Q) be a v_0-augmentation of η. If Q is null, then P is a weak v_0-augmentation of η, and otherwise $Q \cup S \cup P$ is a weak v_0-augmentation of η, where S is a subpath of $\eta(v_0)$ with one end the end of P and the other end an end of Q.

(5.4) *Let H, G be graphs, let $\eta : H \hookrightarrow G$ be an embedding, let v_0 be a vertex of H of degree two belonging to no triangle of H, and let R be a weak v_0-augmentation of η. Then there exist an embedding $\eta' : H \hookrightarrow G$ and a v_0-augmentation (P, Q) of η' such that $P \cup Q \cup \eta'(H) \subseteq R \cup \eta(H)$.*

Proof. We may assume that R is minimal in the sense that there does not exist an embedding $\eta' : H \hookrightarrow G$ and a weak v_0-augmentation R' of η' such that $R' \cup \eta'(H)$ is a proper subgraph of $R \cup \eta(H)$. It follows that R has the confluence property introduced in the proof of (4.5). Let v_1, v_2 be the neighbors of v_0, and let E be the set of all edges of H incident with a neighbor of v_0, but not with v_0 itself.

Let a, b be the ends of R, where $a \in V(\eta(v_0))$ and let z_1, z_2 be the ends of $\eta(v_0)$. Assume first that R has a vertex x such that $R[a, x]$ includes an internal vertex of $\eta(e)$ for no edge $e \in E$, and $R[x, b]$ includes no vertex of $\eta(v_0)$. Let Ω be a path octopus with head a and graph $\eta(v_0)$. We apply (3.5) to Ω and $R[a, x]$ to produce a path octopus Ω' with head z and ends z_1 and z_2 and a path R' with ends z and x. Define η' so that $\eta'(v_0)$ is the graph of Ω' and otherwise η' coincides with η. Let P be a maximal subpath of $R' \cup R[x, b]$ with no internal vertex in $\eta'(v_0)$ containing b and let Q be a maximal nonempty subpath of $R' \setminus V(P)$ with no internal vertex in $\eta'(v_0)$ if such a path exists, and otherwise let Q be the null graph. It is easy to check that (P, Q) is a v_0-augmentation of η'.

Thus we may assume that the assumption of the previous paragraph does not hold. Thus there exists an edge $e \in E$ such that when following R starting from a at some point we encounter an internal vertex of $\eta(e)$, and later an internal vertex of $\eta(v_0)$, say t. Let T be the component of $R \cap \eta(v_0)$ containing t. Let the ends of e be v_1 and v_1', where v_1 is adjacent to v_0, and let the ends of $\eta(e)$ be u_1 and u_1', where u_1 belongs to $\eta(v_1)$ and u_1' belongs to $\eta(v_1')$. Let S be the component of $R[a, t] \cap \eta(e)$ that is closest to u_1' on $\eta(e)$. Let t_1, t_2 be the ends of T, where

a, t_1, t_2, b occur on R in the order listed, and let s_1, s_2 be the ends of S chosen similarly. If t_2 lies at an even distance from a on $\eta(v_0)$, then $R[t_2, b]$ is a weak v_0-augmentation of η, contrary to the minimality of R. Thus t_1 lies at an even distance from a on $\eta(v_0)$. It follows from the confluence property that s_1 is an inbound vertex of $\eta(e)$ (that is, its distance from u_1 on $\eta(e)$ is even). Thus s_2 is an outbound vertex, and hence $R[t_1, s_2] \cup \eta(e)[s_2, u_1']$ is a weak v_0-augmentation of η, contrary to the minimality of R. $\square$

Let H and G be graphs, let H' be obtained from H by bisplitting a vertex v, and let v_0 be the new inner vertex. Let $\eta : H' \hookrightarrow G$, and let (P, Q) be a v_0-augmentation of η. We say that (P, Q) is *strongly minimal* if there exists no graph H'' obtained from H by bisplitting v, an embedding $\eta'' : H'' \hookrightarrow G$ and (letting v_0'' denote the new inner vertex of H_0'') a v_0''-augmentation (P'', Q'') of η'' such that $\eta''(H'') \cup P'' \cup Q''$ is a proper subgraph of $\eta(H') \cup P \cup Q$.

(5.5) *Let H and G be graphs. Let H' be obtained from H by bisplitting a vertex v, let v_0 be the new inner vertex, and let $\eta : H' \hookrightarrow G$. Then no v_0-augmentation of η of type* C *is strongly minimal.*

Proof. Let v_1, v_2 be the new outer vertices of H', let (P, Q) be a v_0-augmentation of η of type C, let a, b be the ends of P with $a \in V(\eta(v_0))$, and let x_1, x_2, e_1, e_2 be as in the definition of augmentation of type C. The vertex v_1 has degree three; let $e_1' \notin \{e_1, v_1v_0\}$ be the third incident edge. Let H'' be obtained from H by bisplitting v into new outer vertices v_1'', v_2'' and new inner vertex v_0'', where v_1'' is incident with e_1 and e_2, and v_2'' is incident with all the remaining edges of H incident with v. The embedding η can be modified to produce an embedding $\eta'' : H'' \hookrightarrow G$ with $\eta''(H) \subseteq P \cup \eta(H)$ by defining $\eta''(v_2'') = \eta(v_2)$, by defining $\eta''(v_1'')$ to be the graph with vertex set $\{x_2\}$, by letting $\eta''(e_2)$ be a subpath of $\eta(e_2)$ with end x_2, by letting $\eta''(e_1)$ be the union of a subpath of $P[x_2, a]$ with a suitable subpath of $\eta(e_1)$, and by letting $\eta''(e_1') = \eta(v_0) \cup \eta(v_1v_0) \cup \eta(v_2v_0) \cup \eta(e_1')$. Now $P[x_2, b] \setminus x_2$ is a weak v_0''-augmentation of η''. By (5.4) there exists an embedding $\xi : H'' \hookrightarrow G$ and a v_0''-augmentation (P'', Q'') of η'' such that

$$P'' \cup Q'' \cup \xi(H'') \subseteq P[x_2, b] \cup \eta''(H'') \subseteq P \cup \eta(H),$$

but $P'' \cup Q'' \cup \xi(H'')$ does not use the edge of P incident with a, contrary to the weak minimality of (P, Q). $\square$

(5.6) *Let H and G be graphs, let H' be obtained from H by bisplitting a vertex v, let v_0 be the new inner vertex, and let $\eta : H' \hookrightarrow G$. Then no v_0-augmentation of η of type* D *is strongly minimal.*

Proof. Let v_1, v_2 be the new outer vertices of H', let (P, Q) be a v_0-augmentation of η of type D, let a, b be the ends of P with $a \in V(\eta(v_0))$, and let i, e, x be as in the definition of augmentation of type D. We may assume that $i = 1$. Let H'' be obtained from H by bisplitting v into new outer vertices v_1'', v_2'' and new inner vertex v_0'', where v_1'' is incident with all the edges of H incident with v_1 in H' except e (note that $v_0v_1 \notin E(H)$), and v_2'' is incident with all the remaining edges of H incident with v. The embedding η can be modified to produce an embedding $\eta'' : H'' \hookrightarrow G$ with $\eta''(H) \subseteq P \cup \eta(H)$ by defining $\eta''(v_1'') = \eta(v_1')$ and letting $\eta''(v_2'')$ be a suitable subgraph of $\eta(v_2) \cup \eta(v_0) \cup \eta(v_0v_2) \cup P[a, x] \cup Q$. Now $P[x, b] \setminus x$ includes a weak v_0''-augmentation of η''. By (5.4) there exists an embedding $\xi : H'' \hookrightarrow G$ and a v_0''-augmentation (P'', Q'') of η'' such that

$$P'' \cup Q'' \cup \xi(H'') \subseteq P[x,b] \cup \eta''(H'') \subseteq P \cup \eta(H),$$

but $P'' \cup Q'' \cup \xi(H'')$ does not use one of the edges of $\eta(v_0)$ incident with a, contrary to the weak minimality of (P, Q). $\square$

We summarize (5.1), (5.3), (5.5), and (5.6) into the following.

(5.7) *Let H and G be graphs, where H has minimum degree at least three, let H' be obtained from H by bisplitting a vertex v, let v_0 be the new inner vertex, let $\eta : H' \hookrightarrow G$ be an embedding and assume that there exists a v_0-augmentation of η. Then a linear extension of H is isomorphic to a matching minor of G.*

Proof. We may assume that the v_0-augmentation is strongly minimal. By (4.5) it is of type A, B, C, or D. By (5.5) and (5.6) it is of type A or B, and so the result holds by (5.1) and (5.3). $\square$

We say that an embedding $\eta : H \hookrightarrow G$ is a *homeomorphic embedding* if $\eta(v)$ has exactly one vertex for every $v \in V(H)$ of degree at least three. The next lemma motivates this definition.

(5.8) *Let H and G be graphs. Then there exists an embedding $\eta : H \hookrightarrow G$ which is not a homeomorphic embedding if and only if a graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G.*

Proof. Suppose that $\eta : H \hookrightarrow G$ and that for some vertex $v \in V(H)$ of degree at least three its image $\eta(v)$ has more than one vertex. Then there exists a branch B of $\eta(v)$ with length greater than zero. The argument from the last paragraph of the proof of (4.4) applied to $\eta(v)$ and B, provides us with an embedding into G of a graph H_1 obtained from H by bisplitting v and therefore by (4.2) the graph H_1 is isomorphic to a matching minor of G.

On the other hand let a graph H', obtained from H by bisplitting some vertex v into new outer vertices v_1 and v_2 and new inner vertex v_0, be isomorphic to a matching minor of G. Then by (4.2) there exists an embedding $\eta' : H' \hookrightarrow G$. Let J be the subgraph of H induced by $\{v_0, v_1, v_2\}$. Define an embedding $\eta : H \hookrightarrow G$ by saying that $\eta(v) = \eta'(J)$, $\eta(vu) = \eta'(v_i u)$ for $i \in \{1, 2\}$ and all neighbors $u \neq v_0$ of v_i, and otherwise η coincides with η'. Clearly $\eta(v)$ has more than one vertex and therefore η is not a homeomorphic embedding. $\square$

The following theorem and its corollary are the main results of this section.

(5.9) *Let G be a brick, let H be a graph of minimum degree at least three, and let $\eta : H \hookrightarrow G$. If η is not a homeomorphic embedding, then a linear extension of H is isomorphic to a matching minor of G.*

Proof. Let v be a vertex of H of degree at least three such that $\eta(v)$ has at least two vertices. By axiom (8) in the definition of an embedding the vertex v has degree at least four. We apply (4.4) to H, G, η and v. If outcome (4.4)(1) holds then (5.9) holds by (5.7).

Therefore we may assume that (2) of (4.4) holds, and let η_2, P, p_1, p_2, B_1, B_2, Q_1 and Q_2 be as in (4.4). Let G' be the graph obtained from $\eta_2(H) \cup P \cup Q_1 \cup Q_2$ by bicontracting all exposed vertices, except those in $B_1 \cup B_2$. Note that G' is a matching minor of G and therefore it suffices to prove that a linear extension of H is isomorphic to a matching minor of G'. If both Q_1 and Q_2 are null, then the graph G' is isomorphic to a bisubdivision of a graph obtained from H by two bisplits and adding an edge joining the two new inner vertices. Thus a linear extension of H is isomorphic to a matching minor of G.

Therefore we may assume that Q_2 is not null. Let u be the common end of B_1 and B_2 in G' and let u_1 and u_2 be the other ends of B_1 and B_2 correspondingly. If Q_1 is not null, denote its ends by q and q' so that $q \in B_1[p_1, u_1]$ and let $q = q' = p_1$ otherwise. If u has degree at least four in G' then the graph G'' obtained from G' by deleting the interiors of $B_1[u, q']$, $B_1[p_1, q]$ and Q_2 can be bicontracted to a graph obtained from H by two bisplits and Q_2 can be bicontracted to an edge joining the two new inner vertices. Thus again a linear extension of H is isomorphic to a matching minor of G.

Therefore we may assume that u has degree three in G'. Hence there exists a unique vertex $w \in V(H)$ such that $u \in \eta_2(vw)$. Now G'' can be bicontracted to a graph obtained from H by bisplitting v and Q_2 can be bicontracted to an edge joining the new inner vertex to w. We deduce that a linear extension of H is isomorphic to a matching minor of G, as desired. □

The next result follows immediately from (5.8) and (5.9).

(5.10) *Let G be a brick, let H be a graph of minimum degree at least three, and assume that a graph obtained from H by bisplitting a vertex is isomorphic to a matching minor of G. Then a linear extension of H is isomorphic to a matching minor of G.*

6. The hierarchy of extensions

For the sake of exposition let us define a *split extension* of a graph H to be any graph obtained from H by bisplitting a vertex. We have seen in the previous section that if a split extension of H is isomorphic to a matching minor of G, then the conclusion of Theorem (1.11) holds. The purpose of this short section is to define other types of extensions and to give an ordering on these extensions, and to reformulate (4.5). The ordering reflects the order in which these extensions will be dealt with. We will be proving theorems of the form "if such an such extension is isomorphic to a matching minor of G, then an extension that is higher on our list of priorities is also isomorphic to a matching minor of G." Of course, the highest priority extensions are linear extensions.

Let us begin the definitions. The lowest on our list will be the following. Let H be a graph, let $v \in V(H)$ be a vertex of degree at least three, and let v_1, v_2 be two distinct neighbors of v in H. Let H' be obtained from H by bisubdividing the edge vv_1, and let x, y be the new vertices numbered so that x is adjacent to v. We say that the graph $H + (y, v_2v)$ is a *vertex-parallel extension of H*. We say that $H + (y, v_2)$ is an *edge-parallel extension of H*.

Let v be a vertex of degree 3 in a graph H and let v_1, v_2 and v_3 be its neighbors. We say that K is obtained from H by *replacing v by a triangle* if K is obtained from H by deleting the vertex v and adding the vertices u_1, u_2, u_3 and edges $u_1u_2, u_2u_3, u_3u_1, u_1v_1, u_2v_2$ and u_3v_3.

Let H be a graph, let v be a vertex of H of degree at least three, and let v_1 and v_2 be two neighbors of v. Let K be obtained from H by bisubdividing the edges v, v_1 and v, v_2 and let x_1, y_1, x_2, y_2 be the new vertices numbered so that $v_1y_1x_1vx_2y_2$ is a path in K. Let $K' = K + (x_1, y_2) + (x_2y_1)$, and let $J = K'$, or let J be obtained from K' by replacing one or both of the vertices x_1, x_2 by triangles. We say that J is a *cross extension* of H, and that v is its *hub*. See Fig. 10.

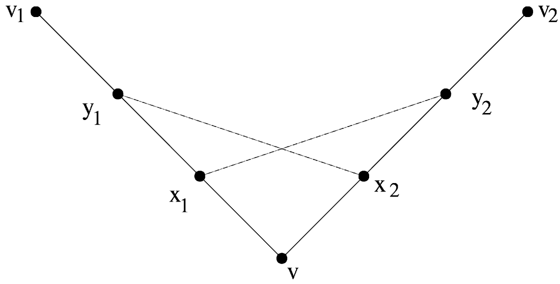

Fig. 10. A cross extension.

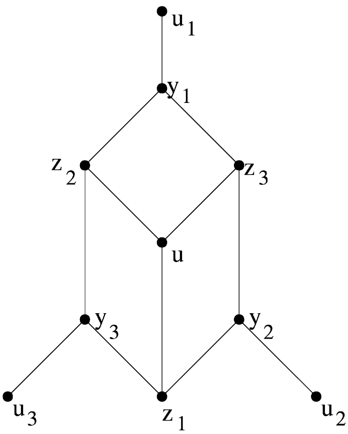

Fig. 11. A cube extension.

Let u be a vertex of H of degree three and let u_1, u_2 and u_3 be its neighbors. Let H_0 be obtained from H by bisubdividing each of the edges uu_1, uu_2 and uu_3. Let the new vertices be y_1, y_2, y_3 and z_1, z_2, z_3 in such a way that $u_1y_1z_3u$, $u_2y_2z_1u$ and $u_3y_3z_2u$ are paths. Let $H_1 := H_0 + (y_1, z_2) + (y_2, z_3) + (y_3, z_1)$, let H_2 be obtained from H_1 by replacing z_1 by a triangle, let H_3 be obtained from H_2 by replacing z_2 by a triangle, and let H_4 be obtained from H_3 by replacing z_3 by a triangle. Then each of the graphs H_1, H_2, H_3, H_4 is called a *cube extension of H*. See Fig. 11.

Let H be a graph, let $uv \in E(H)$, and let H' be obtained from H by bisubdividing uv, where the new vertices x, y are such that x is adjacent to u and y. Let $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ be not necessarily distinct vertices such that not both belong to $\{u, v\}$. In those circumstances we say that $H' + (x, x') + (y, y')$ is a *quadratic extension of H*. Now let $ab \in E(H) - \{uv\}$ be such that $a \neq v$ and $u \neq b$, let H'' be obtained from H' by bisubdividing ab, and let x', y' be the new vertices. Then the graph $H'' + (x, x') + (y, y')$ is called a *quartic extension of H*.

We are now ready to define the promised linear order on extensions. We define that linear extensions are better than quartic extensions, quartic extensions are better than quadratic extensions, which in turn are better than cross extensions, which are better than cube extensions, which are better than edge-parallel extensions, and those are better than vertex-parallel extensions. The linear order is depicted on Fig. 12.

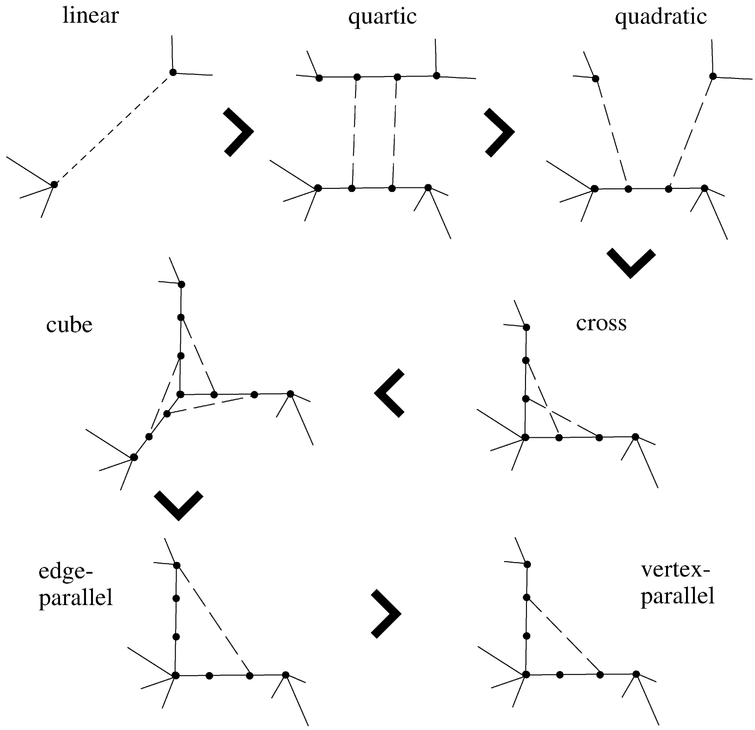

Fig. 12. The linear order on extensions.

For later convenience we reformulate (4.5) in a form more suitable for applications. To do so we will need a definition, but before we can state it, we need to introduce a convention. Let G be a graph, let $w \in V(G)$, and let uv be an edge of G not incident with w. Then the graph $G' = G + (w, uv)$ has two new vertices, and it will be convenient to have a default notation for them. We shall use τ_1 and τ_2 to denote the new vertices, so that τ_1 is adjacent to u, w and τ_2 in G'. We shall extend this convention naturally to more complicated scenarios, as exemplified by the following illustration. For instance, if $ab \in E(G) - \{uv\}$, then $G'' = G + (w, uv) + (\tau_2, ab)$ means the graph $G' + (\tau_2, ab)$, and its new vertices are denoted by τ_3 and τ_4 so that τ_3 is adjacent to a, τ_2 and τ_4 in G''. In general, the new vertices will be numbered $\tau_1, \tau_2, \tau_3, \ldots$ in the order they arise as the input graph is read from left to right. Sometimes we will use $\rho_1, \rho_2, \ldots$ rather than $\tau_1, \tau_2, \ldots$ in order to avoid confusion.

Now we are ready for the definition. Let J, G be graphs, let v_0 be a vertex of J of degree two, and let v_1, v_2 be the neighbors of v_0. We wish to reformulate the outcomes of (4.5). Let $v \in V(J) - \{v_0, v_1, v_2\}$, let $i \in \{1, 2\}$, and for $j = 1, 2$ let v'_j be a neighbor of v_j other than v_0. We define the following graphs:

- $A_1(v) = J + (v_0, v)$,
- $A_2(v) = J + (v_0, v_1 v_0) + (\tau_2, v)$,
- $B_1(v'_i v_i, v) = J + (v_0, v'_i v_i) + (\tau_2, v)$,
- $B_2(v'_i v_i, v) = J + (v_0, v'_i v_i) + (\tau_2, v_i \tau_2) + (\tau_4, v)$,
- $B_3(v'_i v_i, v) = J + (v_0, v_i v_0) + (\tau_2, v'_i v_i) + (\tau_4, v)$,

- $B_4(v_i'v_i, v) = J + (v_0, v_iv_0) + (\tau_2, v_i'v_i) + (\tau_4, v_i\tau_4) + (\tau_6, v)$,
- $C_1(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_i'v_i) + (\tau_2, v_{3-i}'v_{3-i})$,
- $C_2(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_i'v_i) + (\tau_2, v_i\tau_2) + (\tau_4, v_{3-i}'v_{3-i})$,
- $C_3(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_iv_0) + (\tau_2, v_i'v_i) + (\tau_4, v_{3-i}'v_{3-i})$,
- $C_4(v_i'v_i, v_{3-i}'v_{3-i}) = J + (v_0, v_iv_0) + (\tau_2, v_i'v_i) + (\tau_4, v_i\tau_4) + (\tau_6, v_{3-i}'v_{3-i})$.

Sometimes we will omit the arguments when they will be clear from the context and write, e.g., B_3 instead of $B_3(v_i'v_i, v)$. The graphs $A_1, \dots, C_4$ correspond to augmentations of types A, B and C, shown in Figs. 7–9. The following lemma gives the promised reformulation of the outcomes of (4.5).

(6.1) *Let J be a graph, let G be a brick, let v_0 be a vertex of J of degree two, let v_1, v_2 be the neighbors of v_0, for $i = 1, 2$ let $v_i' \neq v_0$ be a neighbor of v_i, assume that v_1 is not adjacent to v_2, assume that every vertex $v \in V(J) - \{v_0\}$ has a neighbor in $V(J) - \{v_1, v_2\}$, and assume that there exists an embedding $J \hookrightarrow G$. Then one of the following holds*:

(A) *there exists a vertex $v \in V(J) - \{v_0, v_1, v_2\}$ such that $A_1(v) \hookrightarrow G$ or $A_2(v) \hookrightarrow G$,*
(B) *there exist a vertex $v \in V(J) - \{v_0, v_1, v_2\}$ and indices $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$ such that v_i has degree three and $B_j(v_i'v_i, v) \hookrightarrow G$,*
(C) *there exist indices $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$ such that v_1, v_2 have degree three and $C_j(v_i'v_i, v_{3-i}'v_{3-i}) \hookrightarrow G$,*
(D) *some split extension of J is isomorphic to a matching minor of G.*

Proof. Let $\eta : J \hookrightarrow G$. We may assume that η is a homeomorphic embedding, for otherwise (D) holds by (5.8). By changing η we may assume that $\eta(v_1)$ and $\eta(v_2)$ each have exactly one vertex, even if v_1 or v_2 has degree less than three. By (4.4) there exists an embedding $\eta' : J \hookrightarrow G$ and a v_0-augmentation (P, Q) of η'. We may assume that (P, Q) is minimal, and hence by (4.5) it is of type A, B, C or D. Similarly as above, we may assume that η' is a homeomorphic embedding. Let P have origin $a \in V(\eta'(v_0))$ and terminus $b \in V(\eta'(u))$. We say that (P, Q) is *good* if either u has degree not equal to two, or u has degree two and b is at even distance from either end of $\eta'(u)$ (recall that $\eta'(u)$ is an even path when u has degree two, and otherwise $\eta'(u)$ has exactly one vertex).

Suppose (P, Q) is not good. Then u has degree two and b is at odd distance from the ends of $\eta(u)$. Let u' be a neighbor of u in $V(J) - \{v_1, v_2\}$ and let b' and b'' be the ends of $\eta(u)$, such that $b' \in V(\eta(uu'))$. Let G' be obtained from $\eta(H) \cup P \cup Q$ by contracting the even path $\eta(u)[b, b'] \cup \eta(uu')$. Define $\eta' : J \hookrightarrow G'$ as follows. Let $\eta'(u) = \eta(u)[b'', b] \setminus b$, $\eta'(uu')$ is a length one subpath of $\eta(u)[b, b'']$ with one end at b and η' is otherwise equal to η. Note that (P, Q) is a good augmentation of η'. Note also that G' is a matching minor of G.

Therefore we may assume that (P, Q) is a good augmentation of η of type A, B, C or D. Now if (P, Q) is of type A, then outcome (A) holds, and similarly for type D, and, by (5.2), for type B. Thus we may assume that (P, Q) is of type C. From the symmetry we may assume that P crosses an edge incident with v_1, and by (5.2) we may assume that it crosses the edge v_1v_1'. In particular, v_1 has degree at most three. But it has degree exactly three by axiom (7) in the definition of an embedding, because $\eta(v_1v_1')$ has at least one internal vertex. The existence of (P, Q) implies, by the same argument as above, that there is an integer $j \in \{1, 2, 3, 4\}$ such that $C_j(v_1'v_1, v_2''v_2) \hookrightarrow G$ for some neighbor v_2'' of v_2 other than v_0. Let $L = C_1(v_1'v_1, v_2''v_2) \setminus v_0v_2 \setminus \tau_1\tau_2$ if $j = 1$, and let it be defined analogously for $j \geqslant 2$. If v_2 has degree at least four, then L

is isomorphic to a bisubdivision of a split extension of H, and hence the lemma holds. Thus we may assume that v_2 has degree at most three, but it has degree exactly three by the same reason as v_1. If $v_2' = v_2''$, then (C) holds, and so we may assume not. If $j = 1$, then by considering L and the edges $\tau_1\tau_2$ and v_0v_2 we deduce that $C_1(v_1'v_1, v_2'v_2) \hookrightarrow G$. An analogous argument works for $j = 4$, while for $j \in \{2, 3\}$ the analogous argument proves that $C_{5-j}(v_1'v_1, v_2'v_2) \hookrightarrow G$. Thus (C) holds, as desired. □

7. Using 3-connectivity

A graph G is *matching covered* if it is connected and every edge of G belongs to a perfect matching of G. Thus every brick is matching covered.

(7.1) *Let H and G be graphs such that H has minimum degree at least three, G is matching covered, and H is isomorphic to a matching minor of G. If H is not isomorphic to G, then either a linear or split extension of H is isomorphic to a matching minor of G, or there exists a homeomorphic embedding $\eta' : H \hookrightarrow G$ such that $\eta'(e)$ has at least three edges for some $e \in E(H)$.*

Proof. By (4.2) there exists an embedding $\eta : H \hookrightarrow G$. We may assume that η is a homeomorphic embedding, for otherwise the lemma holds by (5.8). We may also assume that $\eta(e)$ has exactly one edge for each $e \in E(H)$. Thus $\eta(H)$ is isomorphic to H. But G is not isomorphic to H, and hence there exists an edge e of G with exactly one end in $\eta(H)$. Let M_1 be a perfect matching of G containing e, and let M_2 be a perfect matching of $G \setminus V(\eta(H))$. (This exists, because $\eta(H)$ is a central subgraph of G.) The component of $M_1 \triangle M_2$ containing e is a path with both ends in $\eta(H)$; let $u, v \in V(H)$ be such that P has one end in $\eta(v)$ and the other end in $\eta(u)$. If u and v are not adjacent in H, then by letting $\eta(uv) = P$ the embedding η can be extended to an embedding $H + uv \hookrightarrow G$, and hence a linear extension of H is isomorphic to a matching minor of G. On the other hand, if u and v are adjacent in H, then P has at least three edges, because in that case the unique edge of G between $\eta(u)$ and $\eta(v)$ belongs to $\eta(uv)$. Thus we obtain the desired homeomorphic embedding by replacing $\eta(uv)$ by P. □

Let G be a graph, let $A, B \subseteq V(G)$, let M be a perfect matching of $G \setminus (A \cup B)$, and let $k \geqslant 0$ be an integer. We say that the sequence of paths $(P, Q_1, Q_2, \dots, Q_k)$ is an (A, B)-*hook with respect to M* if the following conditions hold:

(1) P has ends $p_0 \in A - B$ and $p_{k+1} \in B - A$, and is otherwise disjoint from $A \cup B$,
(2) for $i = 1, 2, \dots, k$, Q_i is an even path with ends $p_i \in V(P) - \{p_0, p_{k+1}\}$ and $q_i \in A \cap B$, and is otherwise disjoint from $A \cup B \cup V(P)$,
(3) $V(Q_i) \cap V(Q_j) \subseteq \{q_i, q_j\}$ for all distinct indices $i, j \in \{1, 2, \dots, k\}$,
(4) the graph $P \cup Q_1 \cup Q_2 \cup \dots \cup Q_k \setminus (A \cup B)$ is M-covered, and
(5) the vertices $p_0, p_1, p_2, \dots, p_k, p_{k+1}$ are distinct and occur on P in the order listed.

See Fig. 13.

(7.2) *Let G be a matching covered graph, let $A, B \subseteq V(G)$, and let M be a perfect matching of $G \setminus (A \cup B)$. If $A - B$ and $B - A$ are both nonempty and belong to the same component of $G \setminus (A \cap B)$, then there exists an (A, B)-hook with respect to M.*

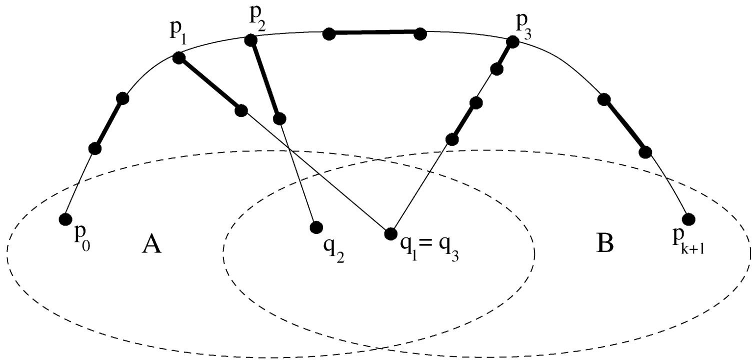

Fig. 13. An (A, B)-hook.

Proof. Suppose for a contradiction that the graph G does not satisfy the lemma, and choose (A, B) violating the lemma with $A \cup B$ maximum. Let e be an edge of G with one end in $A - B$ and the other end in $V(G) - A$. Let M' be a perfect matching of G containing e, and let P_0 be the component of $M \triangle M'$ containing e. Then P_0 is a path with one end in $A - B$, the other end in $A \cup B$, and otherwise disjoint from $A \cup B$. If the other end of P_0 is in $B - A$, then the sequence with sole term P_0 is a required (A, B)-hook, and so we may assume that the other end of P_0 is in A. Let $A' := A \cup V(P_0)$. Then $A' \cap B = A \cap B$. By the maximality of $A \cup B$ there exists an (A', B)-hook $h = (P, Q_1, Q_2, \ldots, Q_k)$.

Let $p_0 \in A'$ be an end of P. If $p_0 \in A$, then h is an (A, B)-hook, and the lemma holds. Thus we may assume that p_0 is an internal vertex of P_0. Let P_1 and P_2 be the two subpaths of P_0 with common end p_0 and union P_0. Exactly one of them, say P_1, has the property that $P_1 \cup P \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k \setminus (A \cup B)$ is M-covered. If the other end of P_1 is in $A - B$, then $(P \cup P_1, Q_1, Q_2, \ldots, Q_k)$ is a desired (A, B)-hook. Thus we may assume that P_1 has one end in $A \cap B$, in which case $(P \cup P_2, P_1, Q_1, Q_2, \ldots, Q_k)$ is a desired (A, B)-hook. $\square$

(7.3) *Let H and G be graphs, where H has minimum degree at least three and is isomorphic to a matching minor of G and G is a brick. If H is not isomorphic to G, then a vertex-parallel, edge-parallel or a linear extension of H is isomorphic to a matching minor of G.*

Proof. By (4.2) and (5.9) we may assume that there exists a homeomorphic embedding $\eta : H \hookrightarrow G$. By (7.1) we may assume that there exists an edge $uv \in E(H)$ such that $\eta(uv)$ has at least three edges. Let $A = V(\eta(uv))$ and let B consist of $V(\eta(H))$, except the internal vertices of $\eta(uv)$. Then $A - B$ and $B - A$ are nonempty and $|A \cap B| = 2$. Thus $A - B$ and $B - A$ belong to the same component of $G \setminus (A \cap B)$, because G is 3-connected. We have $A \cup B = V(\eta(H))$, and hence $G \setminus (A \cup B)$ has a perfect matching, say M, because $\eta(H)$ is a central subgraph of G. By (7.2) there exists an (A, B)-hook $h = (P, Q_1, Q_2, \ldots, Q_k)$ with respect to M. We may choose η, uv and h so that k is minimum. If $k = 0$, then by considering the path P we conclude that a required extension is isomorphic to a matching minor of G.

Thus we may assume that $k > 0$. Let the notation be as in the definition of (A, B)-hook. Thus p_0 is an internal vertex of $\eta(uv)$, and from the symmetry we may assume it is located at even distance from $\eta(v)$ on $\eta(uv)$. We have $q_i \in \{\eta(u), \eta(v)\}$ for all $i = 1, 2, \ldots, k$. We properly two-color the graph $\eta(uv) \cup P$ using the colors black and white so that $\eta(v)$ is black and $\eta(u)$ is white. For convenience let $q_0 := p_0$. We will show that $q_0, q_1, q_2, \ldots, q_k$ all have the same color.

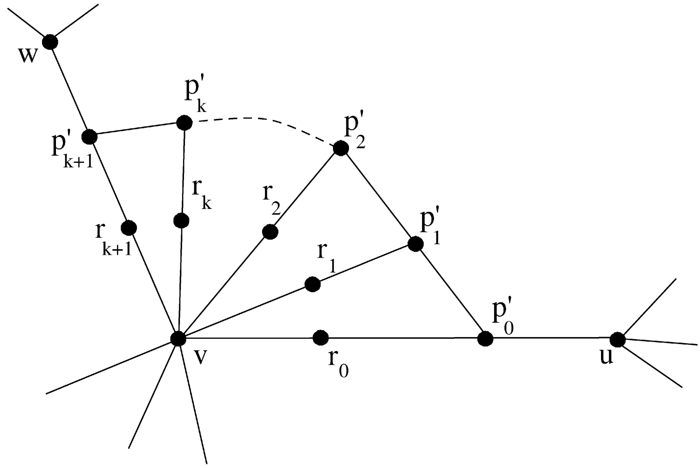

Fig. 14. An extension J of H.

Indeed, suppose for some $i \in \{0, 1, \dots, k-1\}$ the vertices q_i and q_{i+1} have different color. We replace $\eta(uv)[q_i, q_{i+1}]$ by $Q_i \cup P[p_i, p_{i+1}] \cup Q_{i+1}$ to obtain an embedding $\eta'' : H \hookrightarrow G$. Then the sequence $h' = (P[p_{i+1}, p_{k+1}], Q_{i+2}, Q_{i+3}, \dots, Q_k)$ is an (A', B')-hook, where A' and B' are defined in the same way as A and B but relative to η''. But h' contradicts the minimality of k. This proves our claim that $q_0, q_1, q_2, \dots, q_k$ all have the same color; in particular, $q_1 = q_2 = \cdots = q_k = \eta(v)$.

The graph $\eta(H) \cup Q_k \cup P[p_k, p_{k+1}]$ has a matching minor isomorphic to a desired extension of H, unless p_{k+1} belongs to $\eta(vw)$ for some neighbor w of v other than u. By using the argument of the previous paragraph we deduce that p_{k+1} is an internal vertex of $\eta(vw)$ located at even distance from $\eta(v)$ on $\eta(vw)$. Let J be obtained from H as follows. First we bisubdivide the edges uv and vw; let the new vertices be p'_0,r_0 and p'_{k+1},r_{k+1} correspondingly, where p'_0 is adjacent to u and p'_{k+1} is adjacent to w. Denote resulting graph by H'. Then we add new vertices $p'_1, p'_2, \dots, p'_k$ and $r_1, r_2, \dots, r_k$ in such a way that $p'_0p'_1 \dots p'_kp'_{k+1}$ is a path, and p'_ir_iv is a path for all $i = 1, 2, \dots, k$, and there are no other edges incident with the new vertices. This completes the definition of J. See Fig. 14. Now η can be converted to an embedding $\eta' : J \hookrightarrow G$ in a natural way; thus, for instance, $\eta'(p'_i)$ is the graph with vertex-set $\{p_i\}$.

We apply (6.1) to the graphs J and G and the vertex r_0; let J' be the resulting graph, and let $\eta'' : J' \hookrightarrow G$. Suppose outcome (D) of (6.1) holds. Then either a split extension of H is isomorphic to a matching minor of G, in which case the desired result follows from (5.9), or J' is obtained from J by splitting v. Let v_1 and v_2 be the new outer vertices and v_0 the new inner vertex. As we may assume that no split extension of H is isomorphic to a matching minor of G, we have that $|N_{J'}(v_i) \cap N_{H'}(v)| \geqslant 2$ for at most one $i \in \{1, 2\}$, where $N_{J'}(v_i)$ and $N_{H'}(v)$ denote the neighborhoods of v_i and v in J' and H' correspondingly. Without loss of generality let $|N_{J'}(v_1) \cap N_{H'}(v)| \leqslant 1$. Assume first $N(v_1) \cap N_{H'}(v) = \emptyset$, then we can choose $1 \leqslant i < i' \leqslant k$ such that $r_i, r_{i'} \in N(v_1)$ and $r_j \notin N(v_1)$ for every j such that $1 \leqslant j < i$ or $i' < j \leqslant k$. The image of the hook $h' = (p'_0p'_1 \dots p'_ir_iv_1r_{i'}p'_{i'}p'_{i'+1} \dots p'_{k+1}, p'_1r_1v_2, \dots, p'_{i-1}r_{i-1}v_2, v_1v_0v_2, p'_{i'+1}r_{i'+1}v_2, \dots)$ under η'' contradicts the minimality of k. Assume now $|N_{J'}(v_1) \cap N_{H'}(v)| = 1$. From the symmetry between r_0 and r_{k+1} we may assume $r_0 \in N(v_2)$. Let i be minimal such that $r_i \in N(v_1)$ then $i \leqslant k$

and the image of the hook $h' = (p'_0 p'_1 \dots p'_i r_i v_1, p'_1 r_1 v_2, \dots, p'_{i-1} r_{i-1} v_2)$ under η'' contradicts the minimality of k. We assume now that one of the outcomes (A), (B) or (C) of (6.1) holds.

Throughout the rest of the proof let $z \in V(J) - \{v, p'_0, r_0\}$. Outcome (C) cannot hold, because v has degree at least four in J. Assume next that either $J' = A_1(z)$, in which case we put $\tau_1 = \tau_2 = v_0$, or that $J' = A_2(z) = J + (r_0, p'_0 r_0) + (\tau_2, z)$, in which case τ_1 and τ_2 have their usual meaning. If $z \in (V(H) - \{u\}) \cup \{r_{k+1}, p'_{k+1}\}$, then $J + (\tau_2, z)$ is isomorphic to a bisubdivision of a suitable extension of H. If $z = u$ we replace $\eta(uv)$ by $\eta''(u\tau_2 r_0 \tau_1 p'_0 p'_1 r_1 v)$ and the hook $h' = (P[p_1, p_{k+1}], Q_2, Q_3, \dots, Q_k)$ contradicts the minimality of k. If $z = r_i$ for some $1 \leqslant i \leqslant k$ then the hook $h' = (\eta''(\tau_2 r_i p'_i) \cup P[p_i, p_{k+1}], Q_{i+1}, Q_{i+2}, \dots, Q_k)$ contradicts the minimality of k. Finally if $z = p'_i$ for some $1 \leqslant i \leqslant k$ we replace $\eta(uv)$ by $\eta''(up'_0 \tau_1 r_0 \tau_2 p'_i r_i v)$ and the hook $h' = (P[p_i, p_{k+1}], Q_{i+1}, Q_{i+2}, \dots, Q_k)$ contradicts the minimality of k. This completes the case $J' = A_i$.

Since v has degree at least four in J we assume that $J' = B_i(p'_1 p'_0, z)$ for some $i \in \{1, 2, 3, 4\}$. Note that J' contains $J'' = A_j(z) \setminus p'_1 p'_0$ as a matching minor for some $j \in \{1, 2\}$, and unless $z = u$ the argument from the previous paragraph provides us with a suitable extension or a contradiction. If $z = u$ we replace $\eta(uv)$ by $\eta'(u\tau\tau'\tau'' p'_1 r_1 v)$, where $\tau = \tau' = \tau'' = \tau_{2i}$ if $i \in \{1, 2\}$ and $\tau = \tau_{2i-2}$, $\tau' = \tau_{2i-3}$, $\tau'' = \tau_{2i-4}$ if $i \in \{3, 4\}$. The hook $h' = (P[p_1, p_{k+1}], Q_2, Q_3, \dots, Q_k)$ now contradicts the minimality of k. □

8. Vertex-parallel and edge-parallel extensions

The purpose of this section is to replace vertex-parallel and edge-parallel extensions in the statement of (7.3) by extensions that are closer to linear extensions. Our first goal is to prove that if a brick G has a matching minor isomorphic to a vertex-parallel extension of a 2-connected graph H, then it also has a matching minor isomorphic to a better extension of H. We will proceed in two steps; in the intermediate step we will produce a better extension or one that is "almost better," the following. Let H be a graph, let u be a vertex of degree at least three, let u_1 and u_2 be distinct neighbors of u, and let $H' = H + (u_1, uu_1) + (\tau_2, u_2 u)$. We say that H' is a *semi-edge-parallel* extension of H.

(8.1) *Let H be a graph of minimum degree at least three, and let G be a brick. If a vertex-parallel extension of H is isomorphic to a matching minor of G, then an edge-parallel, a semi-edge-parallel, a linear, a cross, or a split extension of H is isomorphic to a matching minor of G.*

Proof. Let u_0 be the vertex of H with neighbors u_1 and u_2 such that the graph H_2 defined below is isomorphic to a matching minor of G. Let H_1 be obtained from H by bisubdividing the edges $u_0 u_1$ and $u_0 u_2$ exactly once, and let x_1, y_1, x_2, y_2 be the new vertices, numbered so that $u_2 y_2 x_2 u_0 x_1 y_1 u_1$ is a path. The graph H_2 is defined as $H_1 + (y_1, y_2)$. By (6.1) applied to $J = H_2$ and the vertex x_1 there exists a graph $J' \hookrightarrow G$ satisfying (A), (B), (C) or (D) of that lemma. If J' is a split extension of J, then the graph obtained from $J' \setminus y_1 y_2$ by bicontracting y_1 and y_2 is a split extension of H. Thus if (D) holds, then the theorem holds, and so we may assume that (A), (B) or (C) holds. Throughout this proof let $v \in V(J) - \{u_0, x_1, y_1\}$. The symbols $\tau_1, \tau_2, \dots$ will refer to the new vertices of J' according to the convention introduced prior to (6.1).

Assume first that $J' = A_1 = J + (x_1, v)$. If $v = u_1$, then J' is isomorphic to a semi-edge-parallel extension of H. If $v = x_2$, then $H + (u_1, u_0 u_2) \hookrightarrow G$; if $v = y_2$, then $H + (u_1, u_2 u_0) \hookrightarrow G$; and in all other cases $H + (v, u_0 u_1) \hookrightarrow G$. In the last case, if v is not adjacent to u_1, then $H + (v, u_1)$ is a linear extension of H, and otherwise $H + (v, u_0 u_1)$ is an edge-parallel extension of H. The same argument will be used later. We will also use later the fact that the inclusions above did not use the edge y_1y_2. This completes the case $J' = A_1$.

Now we assume that $J' = A_2 = J + (x_1, x_1u_0) + (\tau_2, v)$. If $v = x_2$, then $H + (u_2, u_1u_0) \hookrightarrow G$; if $v = y_2$, then by deleting the edge y_1y_2 and bicontracting y_1 we see that a semi-edge-parallel extension of H is isomorphic to a minor of G; if $v = u_1$, then the graph $A_1 \setminus x_1 \setminus y_1$ is isomorphic to a bisubdivision of H, and by considering the path $y_2y_1x_1\tau_1$ we deduce that $H + (u_1, u_2u_0) \hookrightarrow G$; and in all other cases $H + (v, u_1u_0) \hookrightarrow G$. This completes the case $J' = A_2$.

Let $j \in \{1, 2, 3, 4\}$ and let $J' = B_j(y_2y_1, v)$. We have $A_1(v) \setminus y_1y_2 \hookrightarrow B_j(y_2y_1, v)$ for $j = 1, 2$ and $A_2(v) \setminus y_1y_2 \hookrightarrow B_j(y_2y_1, v)$ for $j = 3, 4$ (if $j = 1$ we delete the edges $y_2\tau_1$ and $y_1\tau_2$ and analogously for $j \geqslant 2$). Since the arguments of the previous two paragraphs did not use the edge y_1y_2, except for the cases of $A_1(u_1)$ and $A_2(u_1)$, we may assume that $J' = B_j(y_2y_1, u_1)$, for some $j \in \{1, 2, 3, 4\}$. But $H + (u_1, u_2u_0) \hookrightarrow B_j(y_2y_1, u_1)$ (consider the path $u_1\tau_2\tau_1y_2$ when $j = 1$). This completes the cases $J' = B_j(y_2y_1, v)$.

Our next step is to handle the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$. If $j \leqslant 2$, then $H + (u_1, u_2u_0) \hookrightarrow G$, and if $j \geqslant 3$, then $H_1 + (x_1, x_1u_0) + (\rho_2, x_2u_0) \hookrightarrow G$ and $H_1 + (x_1, x_1u_0) + (\rho_2, x_2u_0)$ after bicontraction of y_1 and y_2 becomes isomorphic to a semi-edge-parallel extension of H. (We are using "ρ" instead of "τ," because the "τ" notation is reserved for vertices of J'.)

Thus the only remaining cases are $J' = C_j(y_2y_1, x_2u_0)$. If $j = 1$, then by considering the path $x_1\tau_1\tau_2\tau_3$ we deduce that $H + (u_1, u_2u_0) \hookrightarrow G$; for $j = 2$ the argument is analogous. For $j = 3$ notice that $C_3(y_2y_1, x_2u_0) \setminus \tau_1y_1 \setminus y_2\tau_3 \setminus x_1\tau_2 \setminus \tau_4\tau_5$ is isomorphic to a bisubdivision of H. By considering the edge $\tau_4\tau_5$ we see that $H + (u_1, u_2u_0) \hookrightarrow G$. Finally, $C_4(y_2y_1, x_2u_0)$ has a matching minor isomorphic to a semi-edge-parallel extension of H. To see that, consider the edge $x_1\tau_1$ and path $\tau_2\tau_3\tau_4\tau_5\tau_6\tau_7$. (The last argument applies to $j = 3$ as well, but for the sake of the next proof we wish to avoid semi-parallel extensions as much as possible.) □

(8.2) *Let H be a 2-connected graph of minimum degree at least three, and let G be a brick. If a semi-edge-parallel extension of H is isomorphic to a matching minor of G, then an edge-parallel, a linear, a cross, a cube or a split extension of H is isomorphic to a matching minor of G, unless H is isomorphic to K_4 and G has a matching minor isomorphic to the Petersen graph.*

Proof. By hypothesis there exists a vertex u_0 of H with distinct neighbors u_1 and u_2 such that the graph H_3 is isomorphic to a matching minor of G, where $H_1, H_2, x_1, y_1, x_2, y_2$ are defined as in the proof of (8.1), and $H_3 = H_2 + (x_2, u_2)$. We may assume that u_0 has degree exactly three, for otherwise $H_3 \setminus u_2y_2 \setminus x_2u_0$ is isomorphic to a bisubdivision of a split extension of H, and hence a split extension of H is isomorphic to a matching minor of G. Let u_3 be the third neighbor of u_0. Since $H_3 \hookrightarrow G$, either a split extension of H is isomorphic to a matching minor of G, or one of the graphs H_3, $H_4 = H_2 + (x_2, y_2u_2)$, $H_5 = H_2 + (x_2, u_2'u_2)$, where $u_2' \neq u_0$ is a neighbor of u_2, has a homeomorphic embedding into G. Graphs H_3, H_4 and H_5 are shown on Fig. 15. Let J denote that graph, and let it be chosen so that $J \neq H_3$, if possible. This choice implies that if a split extension of J is isomorphic to a matching minor of G, then so is a split extension of H. Let x_2', y_2' be the new vertices of H_4 and H_5. We apply (6.1) to J and the vertex x_1, and so we may assume that (A), (B), or (C) holds, for otherwise the theorem holds. Let J' be the graph satisfying (A), (B) or (C). The symbols $\tau_1, \tau_2, \ldots$ will again refer to the new vertices of J'.

Let us assume first that either $J = H_3$, or that y_2' has degree two in J'. Then by deleting the edge x_2u_2 (and bicontracting y_2' if $J \neq H_3$) we may use the proof of (8.1). By that argument

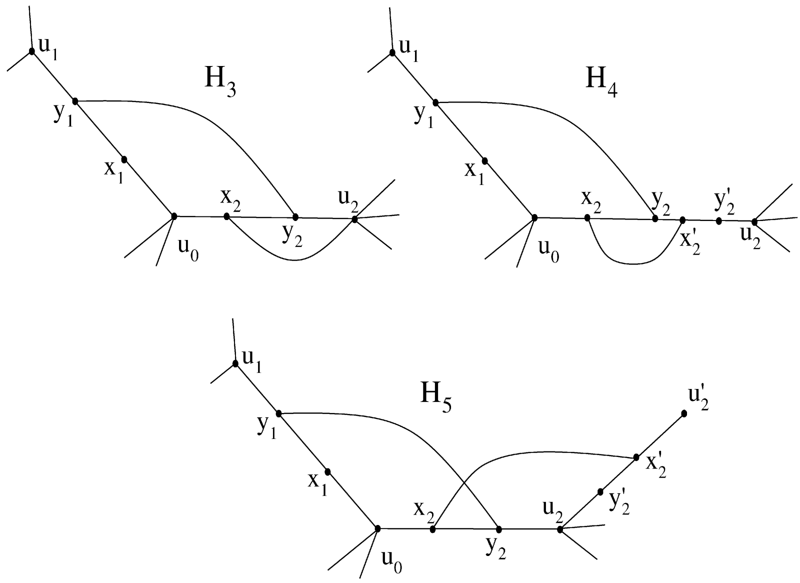

Fig. 15. The graphs H_3, H_4 and H_5 used in (8.2).

the theorem holds, unless $J' = A_1(u_1)$, $J' = A_2(y_2)$, $J' = B_j(y_2y_1, y_2)$, $J' = B_j(x_2u_0, v)$, $J' = C_j(x_2u_0, y_2y_1)$ or $J' = C_4(y_2y_1, x_2u_0)$ for some $j \in \{3, 4\}$ and $v \in V(J) - \{x_1, y_1, u_0\}$.

If $J' = A_1(u_1)$, then $J' \setminus u_1y_1 \setminus x_1u_0 \setminus x_2u_2$ is isomorphic to a bisubdivision of H, and by considering the edge u_2x_2 we deduce that $H + (u_2, u_0u_3) \hookrightarrow G$. If $J' = A_2(y_2)$ we delete the edge y_1y_2, bicontract the vertex y_1 and apply the previous argument.

Next, let $J' = B_3(y_2y_1, y_2)$. The graph obtained from J' by deleting the edges $y_1\tau_4$ and τ_3y_2 and bicontracting the vertices y_1 and τ_4 is isomorphic to $A_2(y_2)$. Thus $H + (u_2, u_0u_3) \hookrightarrow G$. Similarly if $J' = B_4(y_2y_1, y_2)$ we delete the edges $y_1\tau_5$, $\tau_4\tau_6$ and τ_3y_2 and bicontract the vertices y_1, τ_4 and τ_6 to demonstrate that $H + (u_2, u_0u_3) \hookrightarrow G$.

Our next step is to handle the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$. Assume first that $j = 3$. If $v \notin \{u_2, x_2, y_2\}$, then by considering the edge τ_4v we deduce that $H + (v, u_0u_2) \hookrightarrow B_3(x_2u_0, v) \hookrightarrow G$, and similarly $H + (u_2, u_1u_0) \hookrightarrow C_3(x_2u_0, y_2y_1) \hookrightarrow G$. For the cases $v \in \{u_2, x_2, y_2\}$ let $L_3 = B_3(x_2u_0, v) \setminus y_1y_2 \setminus x_1\tau_2 \setminus \tau_1u_0 \setminus \tau_4v \setminus x_2u_2$. By considering the edge τ_4v we deduce that $H + (u_2, u_0u_3) \hookrightarrow G$ if $v \in \{u_2, x_2\}$ and $H + (u_3, u_2u_0) \hookrightarrow G$ if $v = y_2$. Now assume $j = 4$. If $v \notin \{u_2, x_2, y_2\}$, then by considering the edge τ_6v we deduce that $H + (v, u_2u_0) \hookrightarrow B_4(x_2u_0, v) \hookrightarrow G$. If $v = u_2$ then let $L_4 = B_4(x_2u_0, u_2) \setminus x_2 \setminus y_2 \setminus x_1\tau_1 \setminus \tau_4\tau_6 \setminus \tau_5u_0$. By considering the edge $x_1\tau_1$ we deduce that $H + (u_3, u_0u_1) \hookrightarrow G$. If $v = x_2$ we get the same result by considering the graph obtained from L_4 by adding the path $x_2y_2u_2$, and if $v = y_2$ we add the path $y_2x_2u_2$ instead. The graph $C_4(x_2u_0, y_2y_1)$ has a matching minor isomorphic to a cross extension of H (delete the edges τ_7y_2 and x_2u_2; the cross extension has two vertices replaced by triangles). This concludes the cases $J' = B_j(x_2u_0, v)$ and $J' = C_j(x_2u_0, y_2y_1)$.

The graph $C_4(y_2y_1, x_2u_0)$ also has a matching minor isomorphic to a cross extension of H. To see that, delete the edges u_2y_2 and $x_2\tau_7$; the cross extension has two vertices replaced by triangles.

We may therefore assume that $J = H_4$ or $J = H_5$, and that y_2' has degree three in J'. Thus $J' = A_j(y_2')$ or $J' = B_j(y_2y_1, y_2')$ or $J' = B_j(x_2u_0, y_2')$ for some j. Assume first that $J' =$

$A_j(y_2')$. If $J = H_4$, then J' is isomorphic to a cross extension of H (with one or two vertices replaced by triangles depending on the value of j), and so we may assume that $J = H_5$. If $j = 2$, then by considering the edge $\tau_2 y_2'$ we deduce that $H + (u_0, u_2u_2') \hookrightarrow G$, and so we may assume that $j = 1$. We may assume that $u_2' = u_1$, for otherwise by considering the edge $x_1 y_2'$ we deduce that $H + (u_1, u_2u_2') \hookrightarrow G$. Now there is symmetry among u_0, u_1, u_2, and since we could assume u_0 had degree three, we may also assume u_1 and u_2 have degree three in H. The graph $K := J' \setminus u_0x_1 \setminus x_2y_2 \setminus u_2y_2'$ is isomorphic to a bisubdivision of H. If u_2 is not adjacent to u_3, then let u_2'' be the third neighbor of u_2; by considering K and the edge x_2y_2 we see that $H + (u_3, u_2u_2'') \hookrightarrow G$, as desired. Thus we may assume that u_2 is adjacent to u_3, and by symmetry we may also assume that u_1 is adjacent to u_3. But H is 2-connected, and hence u_3 is not a cutvertex; thus H is isomorphic to K_4. It follows that J' is isomorphic to the Petersen graph, as desired. This completes the case $J' = A_j(y_2')$.

Now let $J' = B_j(y_2y_1, y_2')$ or $J' = B_j(x_2u_0, y_2')$. If $J = H_4$, then J' is isomorphic to a cube extension of H, and so we may assume that $J = H_5$. If $J' = B_j(y_2y_1, y_2')$ and $j = 1$, then by considering the path $y_2\tau_1\tau_2y_2'$ we deduce that $H + (u_2', u_2u_0) \hookrightarrow G$. The argument for $j > 1$ is analogous. Thus we may assume that $J' = B_j(x_2u_0, y_2')$. If $j = 1$, then by considering the path τ_2y_2' we deduce that $H + (u_2', u_0u_2) \hookrightarrow G$. The argument is analogous for $j > 1$ with the proviso that when j is even the conclusion is $H + (u_2', u_2u_0) \hookrightarrow G$. □

We now turn our attention to edge-parallel extensions. Let us recall that G/v denotes the graph obtained from the graph G by bicontracting the vertex v.

(8.3) *Let H be a graph of minimum degree at least three, and let G be a brick. If an edge-parallel extension of H is isomorphic to a matching minor of G, then a cross, cube, linear, quadratic, quartic or split extension of H is isomorphic to a matching minor of G.*

Proof. By hypothesis there exists a vertex $u_0 \in V(H)$ of degree at least three with neighbors u_1 and u_2 such that the graph $H_2 := H + (u_2, u_1u_0)$ is isomorphic to a matching minor of G. Let y_1, x_1 be the new vertices of H_2; thus $u_0x_1y_1u_1$ is a path of H_2. Let $H_1 := H_2 \setminus u_2y_1$. Since $H_2 \hookrightarrow G$, either a split extension of H is isomorphic to a matching minor of G, or one of the graphs H_2, $H_3 = H_1 + (y_1, u_0u_2)$, $H_4 = H_1 + (y_1, u_2'u_2)$, where $u_2' \neq u_0$ is a neighbor of u_2, has a homeomorphic embedding into G. Graphs H_2, H_3 and H_4 are shown on Fig. 16. Let J denote that graph, and let it be chosen so that $J \neq H_2$, if possible. This choice implies that if a split extension of J is isomorphic to a matching minor of G, then so is a split extension of H. Let x_2, y_2 be the new vertices of H_3 and H_4. If $J = H_2$ let $x_2 := u_2$ and let y_2 be undefined. We apply (6.1) to J and the vertex x_1, and so we may assume that (A), (B), or (C) holds, for otherwise the theorem holds. Let J' be the graph satisfying (A), (B) or (C). Throughout this proof let $v \in V(J) - \{x_1, y_1, u_0\}$ and once again the symbols $\tau_1, \tau_2, \ldots$ will refer to the new vertices of J'.

We first notice that if u_0 has degree at least four, then $H_2 \setminus u_0u_2$ is isomorphic to a split extension of H, and so we may and will assume that u_0 has degree three. Let u_3 be the third neighbor of u_0. We now show that we may assume that if $J = H_4$, then u_2 has degree three. Indeed, if $J = H_4$ and u_2 has degree at least four then $H_4 \setminus u_0u_2/x_1$ is isomorphic to a split extension of H. So in the case $J = H_4$ let u_2'' be the third neighbor of u_2. Let L be obtained from J' by deleting u_0u_2 and all the "new" edges. Thus, for instance, if $J' = A_2(v)$, then $L = J' \setminus u_0u_2 \setminus x_1\tau_1 \setminus \tau_2v$. Then $L/u_0/y_2$ is isomorphic to H.

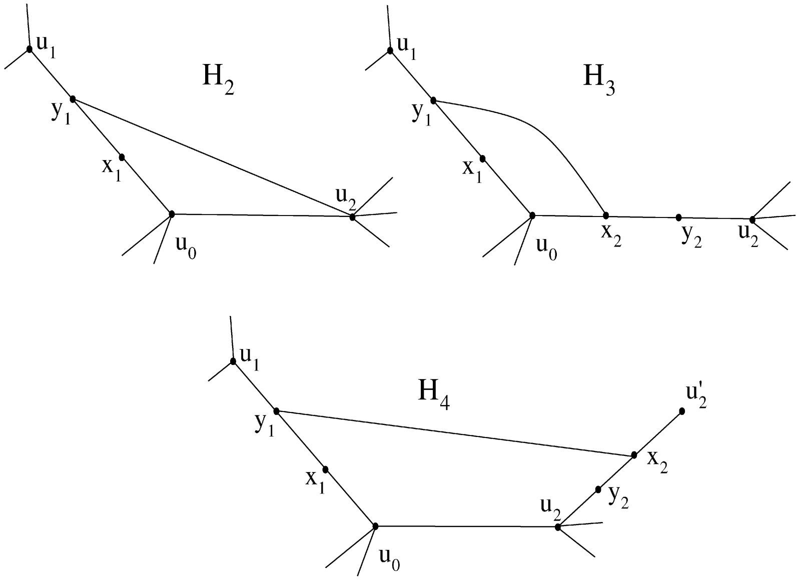

Fig. 16. The graphs H_2, H_3 and H_4 used in (8.3).

Assume first that $J' = A_1(v) = J + (x_1, v)$. If $v = y_2$, then $J \in \{H_3, H_4\}$, and J' is a cross extension of H if $J = H_3$, and a quartic or cross extension of H if $J = H_4$. Thus we may assume that $v \neq y_2$, and hence we may assume (by bicontracting y_2) that $J = H_2$. It follows that J' is a quadratic extension of H, as desired. This completes the case $J' = A_1$.

Next we assume that $J' = A_2(v) = J + (x_1, u_0x_1) + (\tau_2, v)$. Assume first that $v = y_2$. If $J = H_3$, then J' is a cross extension of H, and so we may assume that $J = H_4$. But then $J' \setminus x_1\tau_1/x_1/\tau_1$ is isomorphic to a quadratic extension of H. Thus we may assume that $v \neq y_2$, and hence, by bicontracting y_2, we may assume that $J = H_2$. If $v \neq u_1$, then $J' \setminus y_1u_2/y_1$ is a quadratic extension of H, and so we may assume that $v = u_1$. But then by considering the graph L/u_0 and edges $x_1\tau_1$ and τ_2u_1 we deduce that a quadratic extension of H is isomorphic to a matching minor of G. This completes the case $J' = A_2$.

Next we handle the cases $J' = B_j(x_2y_1, v)$. We start by assuming that $v = y_2$. If $J = H_3$, then J' is isomorphic to a cube extension of H, and so we may assume that $J = H_4$. Recall the definition of L and that u_2 has degree three. If $j = 1$, then by considering L and the edges $x_1\tau_1$ and τ_2y_2 we deduce that a quadratic extension of H, namely $H + (u_2'', u_0u_2) + (\rho_2, u_3)$, is isomorphic to a matching minor of G. If $j = 2$, then by considering the edges $\tau_2\tau_3$ and τ_4y_2 we deduce that the quadratic extension $H + (u_2'', u_2u_0) + (\rho_2, u_2)$ is isomorphic to a matching minor of G. An analogous argument applies when $j = 4$. If $j = 3$ then by deleting the edge $x_1\tau_1$ and bicontracting x_1 and τ_1 we deduce that $H + (u_2'', u_0u_2) + (\rho_2, u_0) \hookrightarrow G$, as desired. Thus we may assume that $v \neq y_2$, and hence, by bicontracting y_2, we may assume that $J = H_2$. If $j = 1$, then by considering L and the edges $x_1\tau_1$ and τ_2v we deduce that the quadratic extension $H + (u_3, u_2u_0) + (\rho_2, v)$ is isomorphic to a matching minor of G. Let $j = 2$. If $v \neq u_2$, then by considering L and the edges $\tau_2\tau_3$ and τ_4v we deduce that the quadratic extension $H + (v, u_2u_0) + (\rho_2, u_2)$ is isomorphic to a matching minor of G. If $v = u_2$ then by considering the graph obtained from L by replacing the edge x_1y_1 by τ_1x_1 and considering the edges $\tau_2\tau_3$ and τ_4u_2 we deduce that the quadratic extension $H + (u_2, u_1u_0) + (\rho_2, u_1)$ is isomorphic to a matching minor of G.

Thus we may assume $j \in \{3,4\}$. Let us assume that $v = u_1$. Then we may assume that u_1 is adjacent to u_2, for otherwise $H + (u_1, u_2) \hookrightarrow G$ (consider the path $u_1\tau_4\tau_3u_2$ when $j = 3$ and the analogous path for $j = 4$). If $j = 3$, then by replacing the edge u_1u_2 by the path $u_1\tau_4\tau_3u_2$ we obtain a graph isomorphic to a bisubdivision of H, and by considering the edges $y_1\tau_4$ and $\tau_2\tau_3$ we deduce that a quadratic extension of H, namely $H + (u_0, u_2u_1) + (\rho_2, u_0)$, is isomorphic to a matching minor of G. If $j = 4$ then by replacing the edge u_1u_2 by the path $u_1\tau_6\tau_5\tau_4\tau_3u_2$, by considering the edges $\tau_4\tau_6$ and $y_1\tau_5$ and by bicontracting x_1 and τ_3 we deduce that a quadratic extension of H, namely $H + (u_0, u_2u_1) + (\rho_2, u_2)$, is isomorphic to a matching minor of G. Thus we may assume that $v \neq u_1$. If $j = 3$, then by considering the edge $x_1\tau_1$ and path $\tau_2\tau_3\tau_4v$ we see that the quadratic extension $H + (v, u_1u_0) + (\rho_2, u_1)$ is isomorphic to a matching minor of G; an analogous argument gives the same conclusion when $j = 4$.

The cases $J' = B_j(u_2u_0, v)$ can be reduced to the cases just handled by noticing that $J \setminus u_0u_2$ is isomorphic to a bisubdivision of H, and hence J is isomorphic to the edge-parallel extension $H + (u_2, u_3u_0)$. Similarly the cases $J' = C_j(u_2y_1, u_2u_0)$ can be reduced to $J' = C_j(u_2u_0, u_2y_1)$, and so it remains to handle the cases $J' = C_j(u_2u_0, u_2y_1)$. But in all four of those cases a cross extension of H is isomorphic to a matching minor of G. □

The results of this section allow us to strengthen (7.3) as follows.

(8.4) *Let H and G be graphs, where H is 2-connected, has minimum degree at least three and is isomorphic to a matching minor of G, and G is a brick. Assume that if H is isomorphic to K_4, then G has no matching minor isomorphic to the Petersen graph. If H is not isomorphic to G, then a cross, cube, linear, quadratic or quartic extension of H is isomorphic to a matching minor of G.*

Proof. By (7.3) we may assume that a vertex-parallel or an edge-parallel extension of H is isomorphic to a matching minor of G. Thus the result follows from (8.1)–(8.3). □

9. Cube and cross extensions

In this section we strengthen (8.4) by eliminating cube and cross extensions from the conclusion.

(9.1) *Let H be a graph, let u be a vertex of H of degree three, and let u_1 and u_2 be two neighbors of u. Let H_1 be obtained from H by bisubdividing the edges uu_1 and uu_2 once, and let x_1, y_1, x_2, y_2 be the new vertices so that $u_1y_1x_1ux_2y_2u_2$ is a path. Let $H_2 := H_1 + (x_2, y_2x_2) + (\tau_2, x_1)$, let $H_3 := H_1 + (x_2, y_2x_2) + (\tau_2, x_1y_1) + (\tau_4, x_1)$, and let H_4 be obtained from H_2 or H_3 by replacing exactly one of the vertices x_2, τ_1, τ_2 by a triangle. Then each of H_2, H_3, H_4 has a matching minor isomorphic to a quadratic extension of H.*

Proof. Throughout this proof let τ_1, τ_2 denote the new vertices of H_2, and let $\tau_1, \tau_2, \tau_3, \tau_4$ denote the new vertices of H_3 with the usual numbering convention. We can naturally embed H into H_2. By bicontracting y_1 and y_2 and considering edges $x_2\tau_1$ and $x_1\tau_2$, we see that H_2 is isomorphic to a bisubdivision of a quadratic extension of H. The graph $H_3 \setminus \tau_1x_2 \setminus x_1u \setminus \tau_3\tau_4$ is isomorphic to a bisubdivision of H and by bicontracting y_1, τ_3 and τ_4 and considering edges τ_1x_2 and x_1u we deduce that H_3 has a matching minor isomorphic to a quadratic extension of H. This completes the proof for H_2 and H_3.

Suppose H_4 is obtained from H_2 by replacing τ_2 with a triangle, then $H_4 \setminus x_2\tau_1/x_2/\tau_1/y_1$ is isomorphic to a quadratic extension $H + (u_1, uu_2) + (\rho_2, u)$ of H. Similarly if H_4 is obtained from H_2 by replacing x_2 or τ_1 with a triangle then $H_4 \setminus x_1u/x_1/u/y_1$ is isomorphic to a quadratic extension of H.

It remains to consider the case when H_4 be obtained from H_3 by replacing exactly one of the vertices x_2, τ_1, τ_2 by a triangle. We need to make the following easy observation. If a graph G_1 is obtained from a graph G by replacing a vertex $t \in V(G)$ of degree three with a triangle T and G_2 is obtained from G_1 by replacing one of the vertices of T by a triangle, then G is isomorphic to a matching minor of G_2. Let $H_2' = H_1 + (x_1, y_1x_1) + (\rho_2, x_2)$. Clearly a graph obtained from H_3 by contracting a triangle with vertex set $\{x_2, \tau_1, \tau_2\}$ is isomorphic to H_2'. Therefore, by the observation above, H_4 contains H_2' as a matching minor and $H_2'/y_1/y_2$ is isomorphic to a quadratic extension of H. $\square$

(9.2) *Let H be a graph of minimum degree at least three, and let G be a brick. If a cube extension of H is isomorphic to a matching minor of G, then a linear, cross or quadratic extension of H is isomorphic to a matching minor of G.*

Proof. Let u be a vertex of H of degree three and let u_1,u_2 and u_3 be its neighbors. Let H_0 be obtained from H by bisubdividing each of the edges uu_1, uu_2 and uu_3. Let the new vertices be y_1, y_2, y_3 and z_1, z_2, z_3 in such a way that $u_1y_1z_3u$, $u_2y_2z_1u$ and $u_3y_3z_2u$ are paths. Let $H_1 := H_0 + (y_1, z_2) + (y_2, z_3) + (y_3, z_1)$, and let J be obtained from H_1 by replacing a subset of $\{z_1, z_2, z_3\}$ by triangles. If z_i is replaced by a triangle, then let the triangle be Z_i; otherwise, let Z_i denote the graph with vertex-set $\{z_i\}$. By hypothesis the vertex u and graph J may be selected so that J is isomorphic to a matching minor of G. Let $\eta : J \hookrightarrow G$. We may assume that η is a homeomorphic embedding, for otherwise a split extension of H is isomorphic to a matching minor of G and the result holds by (5.9).

When $v \in V(J)$ we will abuse notation and use $\eta(v)$ to denote the unique vertex of the graph $\eta(v)$. With that in mind let $J' = \eta(J)$, let $u_i' = \eta(u_i)$, $u' = \eta(u)$ and $z_i' = \eta(z_i)$. For $i = 1, 2, 3$ let P_i denote the path $\eta(u_iy_i)$. We may assume that J and η are chosen so that $|V(P_1)| + |V(P_2)| + |V(P_3)|$ is minimum.

Let Ω_1 be the octopus with head $\eta(Z_1)$ and tentacles the paths of $\eta(J)$ joining u', y_2' and y_3' to Z_1, and let Ω_2 and Ω_3 be defined analogously. Let Ω_4 be the octopus with head $\eta(J \setminus V(Z_1) \setminus V(Z_2) \setminus V(Z_3) \setminus \{y_1, y_2, y_3, u\})$ and tentacles P_1, P_2, P_3, let $\mathcal{F} = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}$, and let $Y' = \{y_1', y_2', y_3', u'\}$. Then $(\mathcal{F}, Y')$ is a frame in G. Let M be a perfect matching of $G \setminus V(\eta(J))$; then M has a unique extension to a matching M' that is Ω_i-compatible for all $i = 1, 2, 3, 4$. By (2.3) there exist distinct integers $i, j \in \{1, 2, 3, 4\}$ and an M'-alternating path S joining vertices v_i and v_j, where v_i belongs to the head of Ω_i and v_j belongs to the head of Ω_j, such that for some edge $e \in E(S) \setminus M'$ the two components of $S \setminus e$ may be denoted by S_i and S_j so that $V(S_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ and $V(S_j) \cap V(\mathcal{F}) \subseteq V(\Omega_j)$.

Assume first that $j = 4$. Then from the symmetry we may assume that $i = 2$. In this case it will be convenient to allow v_4 to be an internal vertex of a tentacle of Ω_4. By doing so we may assume (by replacing S by its subpath) that v_4 is the only vertex of $S \cap \Omega_4$. If for some $l \in \{1, 2, 3\}$ we have $v_4 \in V(P_l)$ and $P_l[u_l', v_4]$ is even, then let $v = u_l$; if $v_4 \in V(P_l)$ and $P_l[u_l', v_4]$ is odd, then v is undefined. If v_4 belongs to $V(\eta(z))$ for some $z \in V(J)$, then let $v = z$. Finally, if $v_4 \in V(\eta(zz'))$ for some edge $zz' \in E(H \setminus u)$, then v_4 is at even distance on $\eta(zz')$ from exactly one of $\eta(z)$, $\eta(z')$, say from $\eta(z)$. In that case we put $v = z$. Notice that if v is defined, then

$v \in V(H) - \{u\}$. From the symmetry we may assume $v \neq u_1$ and $v_4 \notin V(P_1)$. By (3.6) the graph $\Omega_2 \cup S_2 + e$ includes a triad or tripod T with ends y_1', u', v_4.

We claim that if v_4 belongs to P_3, then the path $P_3[v_4, u_3']$ is even. Indeed, otherwise by making use of T, Ω_1 and Ω_3 we obtain contradiction to the minimality of $|V(P_1)| + |V(P_2)| + |V(P_3)|$. This proves that if v is undefined then $v_4 \in V(P_2)$. In that case by deleting the path of $\eta(J)$ joining y_2' and Z_1 and by considering the path of $\eta(J)$ joining y_1' and Z_3 and using T we deduce that a cross extension of H is isomorphic to a matching minor of G. If v is defined, then one of the following graphs is isomorphic to a matching minor of G:

- $H + (v, uu_1) + (\tau_2, uu_2)$, if T is a triad and $Z_3 = \{z_3\}$,
- $H + (v, uu_1) + (\tau_2, u_2u)$, if T is a triad and Z_3 is a triangle,
- $H + (v, u_1u) + (\tau_2, \tau_1u_1)$, if T is a tripod.

But each of the above graphs has a matching minor isomorphic to a quadratic extension of H. This completes the case $j = 4$.

Thus we may assume that $i = 1$ and $j = 2$. By (3.6), $\Omega_1 \cup S_1 + e$ includes a triad or tripod T_1 with ends y_3', u', s_2 and $\Omega_2 \cup S_2 + e$ includes a triad or tripod T_2 with ends y_1', u', s_1, where $s_1 \in V(S_1), s_2 \in V(S_2)$ are the ends of e. If either T_1 or T_2 is a tripod then the required result follows from (9.1) by deleting the path of $\eta(J)$ joining y_1' and Z_3 and making use of T_1 and T_2. If both T_1 and T_2 are triads then one of the following graphs is isomorphic to a matching minor of G:

- $H + (uu_3, uu_1) + (\tau_4, uu_2)$, if Z_3 is not a triangle,
- $H + (uu_3, uu_1) + (\tau_4, u_2u)$, if Z_3 is a triangle.

Both of these graphs have matching minors isomorphic to quadratic extensions of H. □

(9.3) *Let H be a graph, let J be a cross extension of H and let v be the hub of J. If the degree of v in H is at least four then a split extension of H is isomorphic to a matching minor of J.*

Proof. Let x_1, y_1, x_2, y_2 and K' be as in the definition of cross extension. If $J = K'$ then $J \setminus vx_1 \setminus x_2y_1/x_1$ is isomorphic to a split extension of H. If $J \neq K'$ the argument is analogous. □

(9.4) *Let H be a graph of minimum degree at least three, and let G be a brick. If a cross extension of H is isomorphic to a matching minor of G, then a linear or quadratic extension of H is isomorphic to a matching minor of G.*

Proof. Let u be a vertex of H of degree three and let u_1, u_2 and u_3 be its neighbors. Let H_1 be a cross extension of H obtained by deleting the vertex u and adding the vertices x_1, x_2, y_1, y_2, y_3 and edges y_ju_j and y_jx_i for all $i = 1, 2$ and $j = 1, 2, 3$. Let H_2 be obtained from H_1 by replacing x_1 by the triangle X_1, and let H_3 be obtained from H_2 by replacing x_2 by the triangle X_2. Let the vertices of X_1 be a_1, a_2, a_3 such that a_i is adjacent to y_i, and let the vertices of X_2 be b_1, b_2, b_3 such that b_i is adjacent to y_i. By hypothesis, (9.3) and (5.10) we may assume that there exist a vertex u of H of degree three, a graph $J \in \{H_1, H_2, H_3\}$, and an embedding $\eta : J \hookrightarrow G$. If $J \neq H_3$ we define X_2 to be the subgraph of J with vertex-set $\{x_2\}$ and let $b_1 = b_2 = b_3 = x_2$, and if $J = H_1$ we define X_1 to be the subgraph of J with vertex-set $\{x_1\}$ and let $a_1 = a_2 = a_3 = x_1$. By (5.9) we may assume that η is a homeomorphic embedding. Let $J' = \eta(J)$, let $u_i' = \eta(u_i)$, and $y_i' = \eta(y_i)$. Let P_i denote the path $\eta(u_i y_i)$. We may assume that J and η are chosen so that $|V(P_1)| + |V(P_2)| + |V(P_3)|$ is minimum.

Let Ω_1 be the octopus with head $\eta(X_1)$ and tentacles $\eta(a_j y_j)$, where $j = 1, 2, 3$, and let Ω_2 be defined analogously. Let Ω_3 be the octopus with head $\eta(J \setminus V(X_1) \setminus V(X_2) \setminus \{y_1, y_2, y_3\})$ and tentacles P_1, P_2, P_3, let $\mathcal{F} = \{\Omega_1, \Omega_2, \Omega_3\}$, and let $Y' = \{y_1', y_2', y_3'\}$. Then $(\mathcal{F}, Y')$ is a frame in G. Let M be a perfect matching of $G \setminus V(\eta(J))$; then M has a unique extension to a matching M' that is Ω_i-compatible for all $i = 1, 2, 3$. By (2.3) there exist distinct integers $i, j \in \{1, 2, 3\}$ and an M'-alternating path S joining vertices v_i and v_j, where v_i belongs to the head of Ω_i and v_j belongs to the head of Ω_j, and an edge $e \in E(S) \setminus M'$ such that the components of $S \setminus e$ may be denoted by S_i and S_j so that $V(S_i) \cap V(\mathcal{F}) \subseteq V(\Omega_i)$ and $V(S_j) \cap V(\mathcal{F}) \subseteq V(\Omega_j)$.

Assume first that $j = 3$. In this case it will be convenient to allow v_3 to be an internal vertex of a tentacle of Ω_3. By doing so we may assume (by replacing S by its subpath) that v_3 is the only vertex of $S \cap \Omega_3$. If $v_3 \in V(P_i)$, then let $v = u_i$. If v_3 belongs to $V(\eta(z))$ for some $z \in V(J)$, then let $v := z$. Finally, if $v \in V(\eta(zz'))$ for some edge $zz' \in E(J)$, then v_3 is at even distance on $\eta(zz')$ from exactly one of $\eta(z)$, $\eta(z')$, say from $\eta(z)$. In that case we put $v := z$. We may assume that $v \in V(H) - \{u, u_1, u_2\}$, and that if $v_3 \in V(P_1 \cup P_2 \cup P_3)$ then $v_3 \in V(P_3)$. By (3.6) we may assume that $S \cup \Omega_i$ includes a triad or tripod T with ends y_1', y_2', v_3. We claim that if v_3 belongs to P_3, then the path $P_3[v_3, u_3']$ is even. Indeed, otherwise by making use of T and Ω_{3-i} we obtain contradiction to the minimality of $|V(P_1)| + |V(P_2)| + |V(P_3)|$. We deduce that one of the following graphs is isomorphic to a matching minor of G:

- $H_1 \setminus x_1 y_3 + (x_1, v)$,
- $H_2 \setminus x_2 y_3 + (x_2, v)$,
- $H_2 \setminus a_3 y_3 + (a_3, v)$,
- $H_3 \setminus a_3 y_3 + (a_3, v)$.

But each of the above graphs has a matching minor isomorphic to a quadratic extension of H (in the first case we bicontract y_3 and consider the edges $x_1 v$ and $y_1 x_2$). In the second case delete $a_2 a_3$, bicontract its ends and consider the edges $y_1 a_1$ and $x_2 v$; in the third case delete $y_1 x_2$, bicontract its ends, and consider the edges $a_1 a_2$ and $a_3 v$; and in the fourth case consider the same two edges, delete $y_1 b_1$ and $b_2 b_3$ and bicontract their ends. This completes the case $j = 3$.

Thus we may assume that $i = 1$ and $j = 2$. Let $s_1 \in V(S_1)$ and $s_2 \in V(S_2)$ be the ends of e. We apply (3.7) to $S_2 \cup \Omega_2$ to conclude that $\Omega_2 \cup S_2 + e$ has a central subgraph T_2 such that T_2 is either a quadropod with ends y_1', y_2', y_3', s_1, or a quasi-tripod, in which case we may assume by symmetry that its ends are y_1', y_2', s_1. By (3.6) the graph $\Omega_1 \cup S_1 + e$ has a central subgraph T_1 that is a triad or tripod with ends y_1', y_3', s_2. If T_2 is a quasi-tripod then the theorem holds by (9.1). If T_2 is a quadropod with ends y_1', y_2', y_3', s_1, then one of the following graphs is isomorphic to a matching minor of G:

- $H_1 \setminus x_1 y_2 + (x_1, x_2)$,
- $H_2 \setminus a_2 y_2 + (a_2, x_2)$.

Both of these graphs have a matching minor isomorphic to a suitable extension of H. In the first case we get a quadratic extension by bicontracting y_2 and considering the edges $x_2 y_1$ and $x_2 x_1$. In the second case we get a quadratic extension by deleting $x_2 y_1$, bicontracting y_1 and y_2 and considering the edges $x_2 a_2$ and $a_1 a_3$. $\square$

Using (9.2) and (9.4) we can upgrade (8.4) to the following statement.

(9.5) *Let H and G be graphs, where H is 2-connected and has minimum degree at least three, G is a brick and H is isomorphic to a matching minor of G. Assume that if H is isomorphic to K_4, then G has no matching minor isomorphic to the Petersen graph. If H is not isomorphic to G, then a linear, quadratic or quartic extension of H is isomorphic to a matching minor of G.*

Proof. This follows immediately from (8.4), (9.2) and (9.4). □

10. Exceptional families

We now handle quadratic extensions. The next lemma will show that a quadratic extension gives rise to a linear extension, unless it is of one of the following two types. Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extension; that is, H is a graph, $uv \in E(H)$, H' is obtained from H by bisubdividing uv, where the new vertices x, y are such that x is adjacent to u and y. Further, $x' \in V(H) - \{u\}$ and $y' \in V(H) - \{v\}$ do not both belong to $\{u, v\}$. Let $H_1 = H' + (x, x') + (y, y')$ be a quadratic extension of H. If $y' = u$, x' is adjacent to v, and v has degree three, then we say that H_1 is an *alpha extension of H*. If $x', y' \in V(H) - \{u, v\}$, x' is adjacent to v, y' is adjacent to u and both u and v have degree three, then we say that H_1 is a *prism extension of H*. An alpha and a prism extension are shown in Fig. 17.

(10.1) *Let H be a graph of minimum degree at least three, and let K be a quadratic extension of H. Then K has a matching minor isomorphic to a linear, alpha or prism extension of H. Furthermore, if $H, u, v, x, y, x', y', H'$ are as in the definition of quadratic extension and $x', y' \in V(H) - \{u, v\}$, then K has a matching minor isomorphic to a linear or prism extension of H.*

Proof. Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extension, and let $K = H' + (x, x') + (y, y')$ be a quadratic extension of H. By symmetry we may assume that $y' \neq u$. If y' is not adjacent to u, then $H + (u, y') \hookrightarrow K$, as desired. Thus we may assume that y' is adjacent to u. If u has degree at least four, then $K \setminus uy'$ is isomorphic to a linear extension of H, as desired. Thus we may assume that u has degree three. If $x' \neq v$, then by symmetry K is a prism extension of H, and if $x' = v$, then K is an alpha extension of H, as desired. □

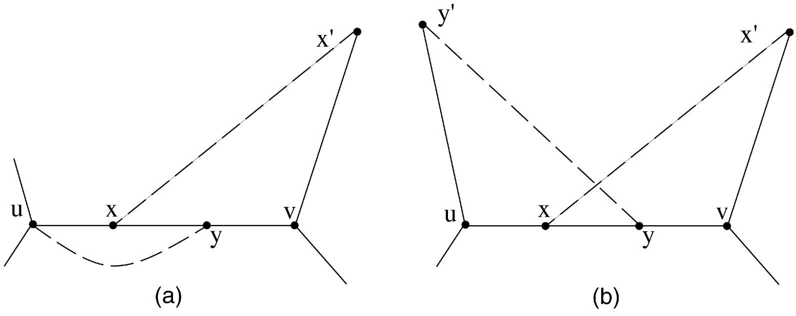

Fig. 17. (a) An alpha extension; (b) a prism extension.

(10.2) *Let K be an alpha extension of a graph H of minimum degree at least three. Then K has a matching minor isomorphic to a linear or prism extension of H.*

Proof. Let $H, u, v, x, y, x', y', H'$ be as in the definition of quadratic extension, and let $K = H' + (x, x') + (y, y')$ be an alpha extension of H, where $y' = u$. Thus v has degree three and is adjacent to x'. There exists a homeomorphic embedding $\eta : H \hookrightarrow K$ with $\eta(v) = x$ and $\eta(z) = z$ for $z \in V(H) - \{v\}$, and by considering $\eta(H)$ and the edges vx' and uy we deduce that K is isomorphic to a quadratic extension of H that satisfies the second statement of (10.1). Thus the lemma holds by that statement. $\square$

Let H be a graph. By a *fan* in H we mean a sequence of vertices $(x, y, u_1, u_2, \dots, u_k)$ such that these vertices are pairwise distinct, except that possibly $x = y$, and further $k \geqslant 2$, $u_1, u_2, \dots, u_k$ all have degree three and form a path in H in the order listed, and for $i = 1, 2, \dots, k$ the vertex u_i is adjacent to x if i is even, and otherwise it is adjacent to y.

(10.3) *Let K be a prism extension of a 3-connected graph H. If K is not a prismoid, a wheel or a biwheel, then K has a matching minor isomorphic to a linear extension of H.*

Proof. By hypothesis there exists a fan (x, y, u_1, u_2) in H such that $K = H + (x, u_1u_2) + (y, \tau_2)$. Let t_1, t_2 denote the new vertices τ_1, τ_2 of K, respectively. Let us choose a maximum integer k such that H has a fan $(x, y, u_1, u_2, \dots, u_k)$ such that $H + (x, u_1u_2) + (y, \tau_2) \hookrightarrow K$. Let u_0 be the neighbor of u_1 other than u_2 and y. Now $u_0 \neq u_k$, for otherwise H is a wheel or a biwheel (depending on whether x and y are distinct or not). Assume first that $u_0 \neq x$. There exists an embedding $\eta : H \hookrightarrow K$ such that $\eta(u_1) = t_2$. By considering the edges u_1y and xt_1 we deduce that $H + (y, u_0u_1) + (x, \tau_2) \hookrightarrow K$, and by using the proof of (10.1) we deduce that either a linear extension of H is isomorphic to a matching minor of K, or that x is adjacent to u_0 and that u_0 has degree three. But then the fan $(y, x, u_0, u_1, \dots, u_k)$ contradicts the maximality of k. Thus we may assume that $u_0 = x$, and by symmetry we may assume that u_k is adjacent to both x and y. It follows from the 3-connectivity of H that K is a prismoid, as desired. $\square$

We now turn to quartic extensions. Again, we will show that a quartic extension gives rise to a linear extension, unless it is of two special types, the following ones. Let H be a graph, and let u, v, H', x, y, a, b be as in the definition of a quartic extension. That is, $uv \in E(H)$, H' is obtained from H by bisubdividing uv, where the new vertices are x, y numbered so that x is adjacent to u and y, and let $K = H + (x, ab) + (\tau_2, y)$ be a quartic extension of H. If $b = v$ and the vertices u and a are adjacent and both have degree three, then we say that K is a *staircase extension of H*. If a, b, u, v are pairwise distinct, all have degree three, a is adjacent to u and b is adjacent to v, then we say that K is a *ladder extension of H*. We also say that the extension is *based at u, v, b, a* (in that order). A staircase and a ladder extension are shown on Fig. 18.

(10.4) *Let H be a graph of minimum degree at least three, and let K be a quartic extension of H. Then K has a matching minor isomorphic to a linear, staircase or ladder extension of H.*

Proof. If a and u are not equal or adjacent, then $H + au \hookrightarrow K$ (delete $x\tau_1$ and bicontract its ends), and hence the theorem holds. Assume now that a and u are adjacent. If both u and a have degree at least four, then $K \setminus au$ is a linear extension of H. If exactly one of a, u has degree three, say a does, then the graph obtained from $K \setminus au$ by bicontracting a is isomorphic to a

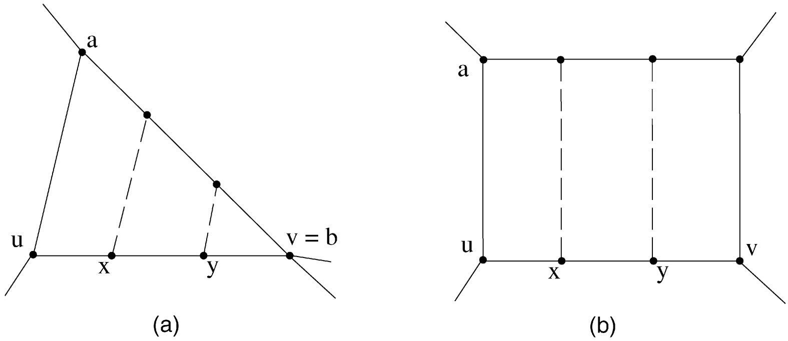

Fig. 18. (a) A staircase extension; (b) a ladder extension.

linear extension of H. Thus if $a \neq u$, and either they are not adjacent or one of them has degree at least four, then a linear extension of H is isomorphic to a matching minor of K. By symmetry the same conclusion holds about the vertices v and b, and the lemma follows. □

(10.5) *Let K be a staircase extension of a 3-connected graph H. If H has at least five vertices, then a linear or ladder extension of H is isomorphic to a matching minor of K.*

Proof. Let $K = H' + x_1x_2 + y_1y_2$, where H' is obtained from H by bisubdividing the edges vv_1 and vv_2 so that $v_1y_1x_1vx_2y_2v_2$ is a path of H', and assume that v_1, v_2 have degree three and are adjacent to each other. Let v_1', v_2' be the third neighbors of v_1 and v_2, respectively. If v_1' and v_2' are not equal or adjacent, then $H + v_1'v_2' \hookrightarrow K$ (bicontract v_1 and v_2 in $K \setminus v_1v_2$), and so the lemma holds. If v_1' and v_2' are adjacent, then K can be regarded as a ladder extension of H, and if $v_1' = v_2'$, then the 3-connectivity of H implies that it is isomorphic to K_4, contrary to hypothesis. □

A *fence* in a graph H is a sequence $(u_1, v_1, u_2, v_2, \dots, u_k, v_k)$ of distinct vertices of H such that $k \geqslant 2$, each of theses vertices has degree three, $u_1u_2 \dots u_k$ and $v_1v_2 \dots v_k$ are paths and u_i is adjacent to v_i for all $i = 1, 2, \dots, k$.

(10.6) *Let K be a ladder extension of a 3-connected graph H on an even number of vertices. If K is not a ladder or a staircase, then K has a matching minor isomorphic to a linear extension of H.*

Proof. By hypothesis there exists a fence $(u_1, v_1, u_2, v_2, \dots, u_k, v_k)$ in H such that $K = H' + x_1y_1 + x_2y_2$, where H' is obtained from H by bisubdividing u_1u_2 and v_1v_2 and x_1, x_2, y_1, y_2 are the new vertices numbered so that $u_1x_1x_2u_2v_2y_2y_1v_1$ is a cycle in H'. We may assume that the fence is chosen with k maximum. Let u_0, v_0 be the third neighbors of u_1, v_1, respectively. Assume first that $u_0 \neq v_0$. Since the quartic extension of H based at u_0, u_1, v_1, v_0 is isomorphic to K, the argument in the proof of (10.4) shows that either a linear extension of H is isomorphic to a matching minor of K, or that u_0 and v_0 are adjacent and both have degree three. We may assume the latter, for otherwise the lemma holds. By the maximality of k the sequence $(u_0, v_0, u_1, v_1, \dots, u_k, v_k)$ is not a fence in H, and hence we may assume that $u_0 = u_k$ or $u_0 = v_k$. But H is 3-connected, and so in the former case K is a planar ladder, and in the latter case it is a Möbius ladder. Thus we may assume that $u_0 = v_0$. The ladder extension of H based at $u_{k-1}u_kv_kv_{k-1}$ is clearly isomorphic to K, and hence the above argument shows that we may assume that the third neighbors of u_k and v_k are equal. Since H is 3-connected and has an even number of vertices, it is a staircase. □

The following result summarizes the previous lemmas.

(10.7) *Let K be a quadratic or quartic extension of a 3-connected graph H on an even number of vertices, and assume that K is not a prismoid, wheel, biwheel, ladder or staircase. Then a linear extension of H is isomorphic to a matching minor of K.*

Proof. If H is isomorphic to K_4, then K is not a staircase extension of H, because K is not a staircase. Thus the lemma follows from the results of this section. □

We are now ready to prove Theorem (1.11).

Proof of (1.11). Let H and G be as stated therein, and assume that they are not isomorphic. Assume first that either H is not isomorphic to K_4, or G has no matching minor isomorphic to the Petersen graph. By (9.5) we may assume that a quadratic or quartic extension K of H is isomorphic to a matching minor of G. It follows from the hypothesis of (1.11) that K is not a prismoid, wheel, biwheel, ladder or staircase. Thus K has a matching minor isomorphic to a linear extension of H by (10.7), and hence so does G, as desired. Thus we may assume that H is isomorphic to K_4 and G has a matching minor isomorphic to the Petersen graph. But G is not isomorphic to the Petersen graph by hypothesis. Since we have already shown that (1.11) holds when H is the Petersen graph, we may now apply it to deduce that G has a matching minor isomorphic to a linear extension of the Petersen graph. The Petersen graph has, up to isomorphism, a unique linear extension, and this linear extension has a matching minor isomorphic to the staircase on eight vertices. But the latter graph has a matching minor isomorphic to K_4, the staircase on four vertices, contrary to hypothesis. □

11. A generalization

In this section we state a generalization of (1.11), and point out how it follows from the theory that we developed. Let G be a graph with a perfect matching. Let us recall that a barrier in G is a set $X \subseteq V(G)$ such that $G \setminus X$ has at least $|X|$ odd components, and that bricks are 3-connected graphs with perfect matchings and no barriers of size at least two. Braces almost have no barriers, either, for if X is a barrier in a brace and X has at least two elements, then X is one of the two color classes of G. We use this fact to weaken the condition on bricks. Let $s \geqslant 0$ be an integer. We say that a set $X \subseteq V(G)$ is an s-barrier in G if $G \setminus X$ has $|X| - 1$ odd components such that the union of the remaining components of $G \setminus X$ has at least s vertices. We say that a graph is an *s-brick* if it is 3-connected and has no s-barrier of size at least two. Thus bricks are 1-bricks and braces are 2-bricks. Our proof of (1.11) actually proves the following more general theorem. A *pinched staircase* is a graph obtained from a staircase by contracting the edge v_1v_2, where the vertices v_1 and v_2 are as in the definition of a staircase.

(11.1) *Let $s \geqslant 0$ be an integer, G be an s-brick other than the Petersen graph, and let H be a 3-connected matching minor of G on at least $s + 1$ vertices. Assume that if H is a planar ladder, then there is no strictly larger planar ladder L with $H \hookrightarrow L \hookrightarrow G$, and similarly for Möbius* *ladders, wheels, lower biwheels, upper biwheels, staircases, pinched staircases, lower prismoids and upper prismoids. If H is not isomorphic to G, then some matching minor of G is isomorphic to a linear extension of H.*

Proof. The proof follows the proof of (1.11), with the following minor modifications. In (2.2) the set R_k is not required to be odd, but instead must have at least s vertices. The proof goes through with the obvious changes. Then the definition of octopus needs to be changed to permit heads with even number of vertices, and in the definition of frame we need to add a condition guaranteeing that the heads of $\Omega_1, \Omega_2, \ldots, \Omega_{k-1}$ are odd and that the head of Ω_k has at least s vertices. The assumption that H has at least $s+1$ vertices will guarantee that this additional condition is satisfied whenever (2.3) is applied. Finally, in (10.6) the assumption that H has an even number of vertices can be replaced by assuming that K is not a pinched staircase. $\square$

Clearly (11.1) implies (1.11) on taking $s=1$. Let us now turn to braces. Let L be a linear extension of a brace H. Then L need not be a brace, but if L is bipartite, then it is a brace. Furthermore, if L is isomorphic to a matching minor of a bipartite graph, then L itself is bipartite. Thus (11.1) implies (1.9) by taking $s=2$. The third application of (11.1) is to factor-critical graphs. A graph G is *factor-critical* if $G \setminus v$ has a perfect matching for every vertex $v \in V(G)$. It is easy to see that every 1-brick on an odd number of vertices is factor-critical. Thus the following immediate corollary of (11.1) gives a generation theorem for a subclass of factor-critical graphs.

(11.2) *Let G be a 1-brick on an odd number of vertices, and let H be a 3-connected matching minor of G. Assume that if H is a wheel, then there is no strictly larger wheel W with $H \hookrightarrow W \hookrightarrow G$, and similarly for pinched staircases, lower prismoids and upper prismoids. If H is not isomorphic to G, then some matching minor of G is isomorphic to a linear extension of H.*

Unfortunately, a linear extension of a 1-brick need not be a 1-brick.

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William McCuaig informed us in September 2003 that he obtained a theorem similar to (1.11).

References

[1] M.H. de Carvalho, C.L. Lucchesi, U.S.R. Murty, How to build a brick, Discrete Math. 306 (2006) 2383–2410.
[2] J. Edmonds, Maximum matching and a polyhedron with (0, 1) vertices, J. Res. Nat. Standards B 69 (1965) 125–130.
[3] J. Edmonds, L. Lovász, W.R. Pulleyblank, Brick decompositions and matching rank of graphs, Combinatorica 2 (1982) 247–274.
[4] P.W. Kasteleyn, Graph theory and crystal physics, in: F. Harary (Ed.), Graph Theory and Theoretical Physics, Academic Press, New York, 1967, pp. 43–110.
[5] C.H.C. Little, F. Rendl, Operations preserving the Pfaffian property of a graph, J. Aust. Math. Soc. Ser. A 50 (1991) 248–257.
[6] L. Lovász, Some algorithmic problems on lattices, in: L. Lovász, E. Szemerédi (Eds.), Theory of Algorithms, Proc. Colloq. Pécs, Hungary, 1984.
[7] L. Lovász, Matching structure and the matching lattice, J. Combin. Theory Ser. B 43 (1987) 187–222.
[8] L. Lovász, M. Plummer, Matching Theory, Ann. Discrete Math., vol. 29, North-Holland, Amsterdam, 1986.
[9] W. McCuaig, Brace generation, J. Graph Theory 38 (2001) 124–169.
[10] W. McCuaig, Pólya's permanent problem, Electron. J. Combin. 11 (2004), 83 pp.

[11] W. McCuaig, N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits (Extended abstract), Proceedings of the Twenty-Ninth Annual ACM Symposium on Theory of Computing, May 4–6, 1997, El Paso, Texas, United States, pp. 402–405.
[12] S. Norine, Drawing Pfaffian graphs, in: Graph Drawing, 12th International Symposium, in: Lecture Notes in Comput. Sci., vol. 3383, 2005, pp. 371–376.
[13] N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, Math. Ann. 150 (1999) 929–975.
[14] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory Ser. B 28 (1980) 305–359.
[15] W.T. Tutte, A theory of 3-connected graphs, Indag. Math. 23 (1961) 441–455.
[16] V.V. Vazirani, M. Yannakakis, Pfaffian orientations, 0–1 permanents, and even cycles in directed graphs, Discrete Appl. Math. 25 (1989) 179–190.