Nowhere-Zero Integral Flows on a Bidirected Graph

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It is proved that every bidirected graph which can be provided with a nowhere-zero integral flow can also be provided with a nowhere-zero integral flow with absolute values less than 216. The connection between these flows and the local tensions on a graph which is 2-cell imbedded in a closed 2-manifold is explained. These local tensions will be studied in a subsequent paper.

1. INTRODUCTION

The motivation of this paper is in connexion with topological graph theory. Although the topological problems will actually be studied in a separate paper [3], they are a natural introduction to bidirected graphs and their flows. Thus we begin by stating these problems.

Let us consider a connected graph G where the edges have been directed. For each oriented circuit \( \mu \) and each edge \( e \) we put \( \mu(e) = 0 \) if \( e \not\in \mu \), \( \mu(e) = +1 \) if \( e \in \mu \) and the direction of \( e \) agrees with the orientation of \( \mu \), \( \mu(e) = -1 \) if \( e \in \mu \) and the direction of \( e \) does not agree with the orientation of \( \mu \).

We consider now 0-chains and 1-chains on \( G \) over \( \mathbb{Z} \). A tension on \( G \) is a 1-chain \( \theta \) such that

\[
\langle \theta, \mu \rangle = \sum_{e \in E(G)} \theta(e) \mu(e) = 0
\]

for every oriented circuit of \( G \). It is known (see, for instance, [11]) that \( \theta \) is a tension on \( G \) if and only if there exists a 0-chain \( p \) such that \( \theta(e) = p(y) - p(x) \) for every edge \( e \) directed from \( x \) to \( y \). This 0-chain \( p \), called a potential attached to \( \theta \), is defined up to a constant; thus \( \mathcal{F}(G) \), the chain-group constituted by all the tensions on \( G \) has a rank equal to \( |V(G)| - 1 \).

Let us now consider the case where \( G \) is 2-cell imbedded in a closed 2-manifold \( S \) without boundary which may be orientable or not. To simplify let us suppose here that each region \( r \) delimited by \( G \) in \( S \) is bounded by an
elementary cycle; thus after providing \( r \) with some orientation, the boundary of \( r \) becomes a circuit \( \mu_r \). We define a local tension on \( G \) as a 1-chain \( \theta \) such that \( \langle \theta, \bar{\mu}_r \rangle = 0 \) for every region \( r \) delimited by \( G \) in \( S \). The chain-group constituted by the local tensions will be denoted by \( \mathcal{F}(G) \).

It is clear that \( \mathcal{F}(G) \subset \mathcal{F}(G) \); thus a first problem is to determine when equality holds. We prove in [3] that \( \mathcal{F}(G) = \mathcal{F}(G) \) if and only if \( S \) is either the sphere or the projective plane. In the other cases, we have the rank formula

\[
r(\mathcal{F}(G)) - r(\mathcal{F}(G)) = 2 - \chi(S) + \text{or}(S),
\]

where \( \chi(S) \) is the Euler characteristic of \( S \), \( \text{or}(S) = 0 \) if \( S \) is orientable, \( \text{or}(S) = 1 \) if \( S \) is nonorientable.

A fundamental problem about \( \mathcal{F}(G) \) is to find a nowhere-zero \( \theta \in \mathcal{F}(G) \) (i.e., \( \theta(e) \neq 0 \) for every edge \( e \)) minimizing \( \max |\theta(e)| \). Every potential \( p \) attached to \( \theta \) gives then a proper \( q \)-coloring after reduction mod \( q \) for \( q = \max |\theta(e)| + 1 \). If \( S \) is an orientable surface then the similar problem on \( \mathcal{F}(G) \) is equivalent to a flow problem on \( G^* \), the dual graph of \( G \) in \( S \). This flow problem was stated by Tutte [8], who conjectured that every graph without an isthmus has a nowhere-zero flow with absolute values less than \( v = 5 \). Jaeger [5] found this was true with \( v = 8 \) and Seymour [7] improved it to \( v = 6 \).

The duality between local tensions and flows no longer holds in the restricted framework of directed graphs if \( S \) is a nonorientable surface. We demonstrate this fact, which is a motivation for introducing bidirected graphs and their flows.

Let us recall first that \( G^* \) is constructed by placing a central vertex \( r^* \) inside each region \( r \) delimited by \( G \) in \( S \). Whenever \( e \) is a common edge to two boundaries \( \mu_r \) and \( \mu_s \), then \( r^* \) and \( s^* \) are joined by an edge \( e^* \) which crosses \( e \). Thus \( e^* \) appears to be parted into two half-edges lying in each of the regions \( r \) and \( s \). We notice that a half-edge has a single endpoint; it is the center of the region containing it.

If \( h \) is a half-edge of \( e^* \) lying in the region \( r \), then we set \( \tau(h) = \bar{\mu}_r(e) \). Figure 1 represents the possible cases, which depend on the direction of \( e \) and the orientations of the regions whose boundaries contain \( e \). If \( \tau(h) = +1 \), then we have drawn an arrow stemming from the endpoint of \( h \); this arrow points to the endpoint of \( h \) if \( \tau(h) = -1 \). Thus \( e^* \) is bidirected by two independent arrows.

In cases 1.1 and 1.2 of Fig. 1 the two arrows are aligned; therefore we can replace them by a single arrow. More formally, \( e^* \) is directed in the usual way from its positive half-edge to its negative half-edge. We shall find a positive half-edge and a negative half-edge on each edge of \( G^* \) if and only if the orientations of the regions are always nonaligned along their common
boundaries. This condition can be fulfilled if and only if $S$ is orientable. In this case the local tension $\theta_i$ yields a flow $\theta_i^*$ defined by $\theta_i^*(e^*) = \theta_i(e)$ for every edge $e$ of $G$.

In cases 1.3 and 1.4 the two arrows are nonaligned. Thus it is no longer possible to find a direction on $e^*$ in the usual way. None the less we can still set $\theta_i^*(e^*) = \theta_i(e)$ for every edge $e$ of $G$. The 1-chain $\theta_i^*$ can still be considered as a kind of flow on $G^*$ because the equality $\langle \theta_i, \hat{\mu}, \rangle = 0$ means now that the sum of the flows along the arrows pointing to $r^*$ is equal to the sum of the flows along the arrows stemming from $r^*$.

It turns out that problems on local tensions require preliminary results on their dual flows for their solution. Therefore we shall deal only with bidirected graphs and their flows in this paper.

Bidirected graphs and their flows were introduced by Edmonds for expressing algorithms on matchings. This work is briefly reported in [6]. The notion of a bidirected graph $G$ appears also in Zaslavsky's work [11] as the orientation of a signed graph. Flows appear implicitly in this work as the kernel of the incidence matrix, which serves also for defining the matroid $M(G)$. This matroid was described by Zaslavsky [10]; we recall his results in Section 2.

The chain-group of flows on a bidirected graph is no longer regular.
However, it is possible to adapt the results obtained by Tutte [8] in the regular case; this is done in Section 3. This section ends with the construction of a nowhere-zero chain by using a covering by cobases; it is an application of the idea used in Jaeger's work [5].

Finally Section 4 gives an upper bound for $w$, the minimal integer such that every bidirected graph with a nowhere-zero flow has a nowhere-zero flow $f$ with $|f(e)| < w$ on every edge $e$. We find an upper bound equal to 216. This is the essential result of this paper in the sense that it shows the existence of an absolute upper bound for $w$ which does not depend on the bidirected graph. On the other side the value 216 is certainly too great and we conjecture that 6 is sufficient. Moreover we show that 6 is certainly the least possible value of $w$.

2. BASIC PROPERTIES OF FLOWS ON A BIDIRECTED GRAPH

We suppose that the reader is familiar with the theory of chain-groups and their matroids; otherwise he should refer either to [9] or [10]. We note that the chain-groups considered in this paper will be over $\mathbb{Z}$.

A graph will be finite and undirected, with possible loops and/or multiple edges. It will be convenient to consider that each edge $e$ is constituted of two distinct half-edges $h'$ and $h''$ each having a single endpoint; so the endpoints of $e$ consist of the endpoints of $h'$ and $h''$. If $G$ is a graph, then the sets of its vertices, edges, and half-edges will be respectively denoted by $V(G)$, $E(G)$, and $H(G)$.

A graph $G$ is called a bidirected graph if a signature of the half-edges, $\tau: H(G) \to \{+1, -1\}$, has been defined. Intuitively these signs can be interpreted by arrows on the half-edges as explained in the introduction (Fig. 1). A 1-chain (resp. 0-chain) on $G$ is a function from $E(G)$ (resp. $V(G)$) into $\mathbb{Z}$. A flow on $G$ is a 1-chain $f$ such that

$$\sum_{h} \tau(h)f(e_h) = 0$$

for all $v \in V(G)$

summed over all half-edges $h$ having $v$ as endpoint, where $e_h$ denotes the edge containing $h$.

The chain-group constituted by the flows on $G$ will be denoted by $N(G)$; the matroid whose circuits are the supports of the elementary chains of $N(G)$ will be denoted by $M(G)$. Let us define the incidence matrix $B = (b_{ve}; v \in V(G), e \in E(G))$ by the formula

$$b_{ve} = \sum_{h} \tau(h)$$

summed over those constituent half-edges $h$ of $e$ that have endpoint $v$. 
If we consider $B$ as a linear mapping from the set of the 1-chains into the set of the 0-chains, then $N(G)$ is the kernel of $B$. Therefore $M(G)$ is also the matroid of the linear dependencies on the columns of $B$; this matroid has been characterized in Zaslavsky's work on signed graphs [10, 11].

The sign of an edge $e$ consisting of two half-edges $h'$ and $h''$ is $\sigma(e) = -\tau(h')\tau(h'')$. The sign of a circle $C$ (subset of edges inducing a connected and regular subgraph of degree 2) is the product of its edge signs; $C$ is said to be balanced (resp. unbalanced) if its sign is positive (resp. negative). A subgraph $G$ is balanced if all its circles are balanced.

2.1. Theorem (Zaslavsky). A set of edges is a circuit of $M(G)$ if and only if it is either

(i) a balanced circle $C$, or

(ii) the union of two unbalanced circles $C_1$ and $C_2$ which have no common edge and one common vertex, or

(iii) the union of two vertex disjoint unbalanced circles $C_1$ and $C_2$, and a simple connecting path $P$ meeting $C_1$ and $C_2$ at its endpoints.

2.2. Corollary. The rank in $M(G)$ of a subset $A \subseteq E(G)$ is equal to $|V(G)| - b(A)$, where $b(A)$ is the number of balanced components of the subgraph consisting of $A$ and $V(G)$. [2]

2.3. Corollary. Let $A$ be the support of an elementary chain $f$ of $N(G)$. There exists a nonnull integer $\alpha$ such that $f(e)$ is equal to $\pm \alpha$ or $\pm 2\alpha$ for every $e$ in $A$. Moreover if $f(e) = \pm 2\alpha$, then $A$ falls in case (iii) of the theorem and $e$ belongs to the connecting path.

Corollary 2.2 is contained in Zaslavsky's work. A property similar to Corollary 2.3 has already been noted by Bolker [2] in a discussion of certain tripartite signed graphs.

Proof of Corollary 2.3. If we consider two edges $e'$ and $e''$ with a common endpoint of degree 2 in the subgraph defined by $A$, then the flow condition implies $|f(e')| = |f(e'')|$. Thus $|f|$ is constant on $C, C_1, C_2$, and $P$ with the notations of Theorem 2.1. In particular the proof is ended in case (i).

For $i = 1, 2$, let $\alpha_i$ be the constant value of $|f|$ over $C_i$ and $x_i$ be the vertex of $C_i$ which has a degree higher than 2 in $A$. Since $C_i$ is unbalanced, the two half-edges of $C_i$ incident to $x_i$ have the same sign. The contributions of these two half-edges to the sum expressing the flow condition at $x_i$ is equal to $\pm 2\alpha_i$. The remaining contribution comes from the half-edges of either $C_j$ ($j \neq i$ in case (ii)) or $P$ (in case (iii)). This implies $2\alpha_1 = 2\alpha_2$ in case (ii). The
constant value of \(|f|\) over \(P\) is equal to both \(2a_1\) and \(2a_2\) in case (iii). So the property is proved with \(a = a_1 = a_2\).

A signed graphic isthmus of \(G\) is an edge which belongs to every base, or equivalently to no circuit, of \(M(G)\). Equivalently it is an edge \(e\) such that \(f(e) = 0\) for every flow \(f\). A signed graphic isthmus of \(G\) will be carefully distinguished from an isthmus of \(G\), an edge whose deletion augments the number of connected components.

2.4. Lemma. If \(G\) is a bidirected graph where all the edges are positive with the exception of a single one \(e\), then \(e\) is a signed graphic isthmus of \(G\).

Proof. If it is false we can find a circuit \(\Gamma\) of \(M(G)\) which contains \(e\). This circuit contains a single negative edge, and this is impossible in each case of Theorem 2.1.

2.5. Lemma. Let \(e\) be an isthmus of a connected bidirected graph \(G\) which separates a balanced component \(G'\) from another component which is either balanced or not. The edge \(e\) is also a signed graphic isthmus of \(G\).

Proof. If it is false we can find a circuit \(\Gamma\) of \(M(G)\) which contains \(e\). The edge \(e\) cannot be contained in a cycle of \(G\); therefore it is necessarily contained in the connecting path \(P\) of \(\Gamma\) (Theorem 2.1(iii)). Thus one of the two unbalanced cycles \(C_1\) and \(C_2\) is contained in \(G'\). But this is impossible because \(G'\) is balanced.

Finally we state the following definitions about the bidirected graph \(G\) which will be used in Section 4. Reversing the direction of an edge \(e\) is the same as changing the signs of its half-edges. Switching \(G\) at a vertex \(u\) means changing the signs of the half-edges whose endpoint is equal to \(u\). The effect of one of the preceding operations is either to change the \(e\)-column signs or the \(v\)-row signs in the incidence matrix; therefore these operations leave unchanged the matroid \(M(G)\) and the signs of the circles. A flow \(f\) remains a flow after switching at \(u\); it yields a new flow by changing \(f(e)\) into \(-f(e)\) after reversing the direction of \(e\).

Contracting a positive edge \(e\) of \(G\) is done by deleting \(e\) and merging its endpoints into a single one; the bidirected graph which is so obtained will be denoted by \(G/e\). If \(f\) is a flow on \(G\), then its restriction on \(G/e\) is the 1-chain \(f'\) defined by \(f'(e') = f(e')\) for every \(e'\) in \(E(G)\backslash\{e\}\). Since \(e\) is positive in \(G\), the half-edges have opposite signs; therefore the flow conditions at the endpoints of \(e\) imply that \(f'\) is still a flow on \(G/e\).
3. WIDTHS OF INTEGRAL CHAINS

Throughout this section, \( N \) is an integral chain-group defined on a set \( E \). The matroid of \( N \) is denoted by \( M(N) \); the circuits of \( M(N) \) are the supports of the elementary chains of \( N \). An isthmus is an element of \( E \) which belongs to every base, or equivalently to no circuit, of \( M(N) \). Equivalently it is an element \( e \in E \) such that \( f(e) = 0 \) for every chain \( f \) in \( N \). The width of a chain \( f \) is the highest value of \( 1 + |f(e)| \) when \( e \) ranges in \( E \); it will be denoted by \( w(f) \). A nowhere-zero chain is a chain whose support is equal to \( E \).

3.1. PROPOSITION. There exists a nowhere-zero chain in \( N \) if and only if no element of \( E \) is an isthmus.

Proof. The condition is obviously necessary. Conversely, if there is no isthmus, we can find chains \( f_0, f_1, \ldots, f_k \) whose supports cover \( E \). Let \( w \) be the highest width of these chains; then \( \sum w f_i \) is a nowhere-zero chain.

If \( M(N) \) is without isthmus, then let \( w(N) \) be the minimal value of \( w(f) \) when \( f \) ranges among the nowhere-zero chains of \( N \). This section is devoted to finding an upper bound for \( w(N) \).

DEFINITION. A principal chain of \( N \) is an elementary chain \( f \) which is a submultiple of all the elementary chains with the same support as \( f \). For example, the primitive chains of a regular chain-group are the principal chains of this chain-group.

3.2. PROPOSITION. For every elementary chain \( f \in N \), there exists a principal chain which is a submultiple of \( f \).

Proof. Let us consider the set \( R \) of the chains of \( N \) which have the same support \( S \) as \( f \). Let \( \phi \) be a chain of \( R \) such that

\[
\forall g \in R, \sum_{e \in E} |\phi(e)| \leq \sum_{e \in E} |g(e)|.
\]

Let \( a \) be an element of \( S \). The chain \( f(a)\phi - \phi(a)f \) has support properly contained in \( C \), hence is the null chain. Therefore there exists relatively prime nonzero integers \( p \) and \( q \) such that \( p\phi = qf \). Then \( q \) divides \( \phi(e) \) for every \( e \) in \( C \). By (i), \( |q| \) must be equal to 1. Hence \( \phi \) is a submultiple of \( f \).

The following is but a weaker formulation of Corollary 2.3.

3.3. PROPOSITION. The principal chains of \( N(G) \), the chain-group of the flows on a bidirected graph \( G \), have absolute values equal to 0, 1, or 2.
DEFINITION. Let \( f \) and \( g \) be two chains in \( N \); \( f \) conforms to \( g \) if \( f(a) g(a) > 0 \) whenever \( a \in \| f \| \).

3.4. PROPOSITION. For every nonnull chain \( g \in N \) and \( a \in \| g \| \), there exists a principal chain \( f \) which conforms to \( g \) and such that \( a \in \| f \| \).

Proof. Let us consider the inclusion \( Z \subset Q \). So \( N \) is now imbedded in \( Q^k \) and we consider the vector space \( \tilde{N} \) over \( Q \) which is spanned by \( N \). For every elementary vector \( F \in \tilde{N} \) there exists a positive integer \( p \) such that \( pF \) is an elementary chain of \( N \).

We apply a result of Fulkerson [4, Eq. (1.1) p. 318] for expressing \( g \) as a sum \( F_1 + F_2 + \cdots + F_k \) of elementary vectors of \( \tilde{N} \), where each \( F_i \) conforms to \( g \). There exists an index \( i \) such that \( F_i(a) \neq 0 \) and a positive integer \( p \) such that \( pF_i \) is an elementary chain of \( N \). The chain \( pF_i \) conforms to \( g \). We consider then a principal chain \( f \) which is a submultiple of \( pF_i \) (Proposition 3.2). Either \( f \) or \( -f \) conforms to \( g \).

DEFINITION. Let \( q \geq 2 \) be an integer. Two chains \( f \) and \( f' \) in \( N \) are \( q \)-equivalent if \( f(e) \equiv f'(e) \pmod{q} \) for every \( e \in E \).

Remark. The pairs of opposite principal chains correspond \( 1 \)-\( 1 \) to their supports. Therefore, since \( E \) is finite, \( w(f) \) is bounded when \( f \) ranges among the principal chains of \( N \).

3.5. PROPOSITION. Let \( q \geq 2 \) be an integer and \( W \) be the maximum value of \( w(f) \) when \( f \) ranges among the principal chains of \( N \). Every chain \( g \in N \) is \( q \)-equivalent to a chain \( g' \in N \) such that \( w(g') \leq q(W - 1) \).

Proof. For each \( g \in N \) and each \( e \in E \), let us set

\[
\tilde{g}(e) = 0 \quad \text{if} \quad |g(e)| < q(W - 1);
\]

\[
\tilde{g}(e) = 1 + |g(e)| - q(W - 1) \quad \text{if} \quad |g(e)| \geq q(W - 1);
\]

\[
Z(\tilde{g}) = \sum_{e \in E} \tilde{g}(e).
\]

We prove the proposition by induction on \( Z(\tilde{g}) \). It is obvious for \( Z(\tilde{g}) = 0 \) since we have then \( w(g) \leq q(W - 1) \). Therefore we suppose that \( Z(\tilde{g}) > 0 \) and that the proposition is proved for every chain \( h \) such that \( Z(h) < Z(\tilde{g}) \).

Since \( Z(\tilde{g}) > 0 \), there exists \( a \in E \) such that \( \tilde{g}(a) > 0 \). We have then \( g(a) \neq 0 \), and according to Proposition 3.4 we can find a principal chain \( f \) which conforms to \( g \) and such that \( f(a) \neq 0 \). Let \( h = g - qf \). We prove that \( h(e) \leq \tilde{g}(e) \) for every \( e \in E \).
Case 1 \((f(e) = 0)\)

We have then \(h(e) = g(e)\); therefore \(\tilde{h}(e) = \tilde{g}(e)\).

Case 2 \((f(e) \neq 0)\)

Let \(e\) be equal to +1 or -1, so that \(ef(e) > 0\). Since \(f\) conforms to \(g\), we have also \(eg(e) > 0\). The inequality \(|f(e)| \leq W - 1\) is satisfied since \(f\) is a principal chain; thus

\[
0 > -qef(e) \geq -q(W - 1).
\]

We distinguish two subcases.

Subcase 2.1 \((g(e) = 0)\). We have then

\[
q(W - 1) > \varepsilon g(e) > 0.
\]

Adding (i) and (ii) yields

\[
q(W - 1) > \varepsilon (g(e) - qf(e)) > -q(W - 1).
\]

Thus we have \(\tilde{h}(e) = \tilde{g}(e) = 0\).

Subcase 2.2 \((g(e) > 0)\). We have then

\[
\varepsilon g(e) \geq q(W - 1).
\]

The inequalities (i) and (iii) imply

\[
\varepsilon (g(e) - qf(e)) \geq 0.
\]

Inequality (i) also implies

\[
\varepsilon g(e) > \varepsilon (g(e) - qf(e)).
\]

Therefore we have in this case \(\tilde{h}(e) < \tilde{g}(e)\).

Since \(a\) falls in Subcase 2.2, \(Z(\tilde{h}) < Z(\tilde{g})\). According to the induction hypothesis we can find \(g' \in N\) which is \(q\)-equivalent to \(h\) and such that \(w(g') \leq q(W - 1)\). But \(h\) is clearly \(q\)-equivalent to \(g\); therefore \(g'\) is also \(q\)-equivalent to \(g\).

**Definition.** A prime integer \(p\) is a regular value for \(N\) if for every principal chain \(f\) and every \(e \in \|f\|, p\) does not divide \(f(e)\). The other prime integers are the singular values for \(N\).

For example, \(N\) is a regular chain-group if and only if every prime integer is a regular value for \(N\).
3.6. Theorem. Let $N$ be an integral chain-group without isthmi defined on a set $E$ with the following three parameters: $P$ is the minimal regular value for $N$, $W$ is the maximal width of a principal chain of $N$, $s$ is the minimal number of cobases of $M(N)$ which can cover $E$. Then there exists in $N$ a nowhere-zero chain with a width less than or equal to $p'(W - 1)^s$.

Proof. Let us consider a covering of $E$ by cobases $B_0^*, B_1^*, ..., B_{s-1}^*$. For each cobase $B_i^*$ and each $e \in B_i^*$, there exists a circuit $C_i^e$ of the matroid $M(N)$ such that $C_i^e \cap B_i^* = \{e\}$. This circuit is by definition the support of an elementary chain of $N$; therefore it is also by Proposition 3.2 the support of a principal chain $f_i^e$. This chain is such that $f_i^e(e) \neq 0$ and $f_i^e(e') = 0$ for every other $e' \in B_i^*$. Therefore if we consider $F_i$, the sum of all the $f_i^e$'s when $e$ ranges in $B_i^*$, we have $F_i(e) = f_i^e(e) \neq 0$ for each $e \in B_i^*$. Since $f_i^e$ is a principal chain and $p$ is a regular value, the nonnull value $f_i^e(e)$ is also nonnull (mod $p$); thus $F_i(e) \neq 0 \pmod{p}$ if $e \in B_i^*$.

Let $u = p(W - 1)$, and let us apply Proposition 3.5 for each of the $F_i$'s. So we find for each $i = 0, 1, ..., s - 1$ a new chain $F'_i$ which is $p$-equivalent to $F_i$ and such that $w(F'_i) \leq u$. In particular each value $F'_i(e)$ for $e \in B_i^*$ will be nonnull since $F_i(e)$ was nonnull (mod $p$). It is then easy to verify that the chain $F' = \sum_{i=0}^{s-1} v_i F'_i$ fulfills the requirements.

Jaeger's method [5] for constructing a nowhere-zero flow of small width on a graph $G$ is based on the fact that $E(G)$ can be covered by $s = 3$ (resp. $s = 2$) cotrees when $G$ is 3-edge connected (resp. 4-edge connected). Moreover the flows on $G$ constitute a regular chain-group; then $p = W = 2$. The application of the theorem yields a nowhere-zero flow of width $\leq 8$ (resp. $\leq 4$).

We apply a similar method to bidirected graphs in the next section.

4. Nowhere-Zero Flow with a Minimal Width

Let $G$ be a bidirected graph with no signed graphic isthmi. There exists a nowhere-zero flow on $G$; thus $w(G)$, the least integer such that there exists a nowhere-zero flow $f$ satisfying $w(f) \leq w(G)$, is well defined.

For every subgraph $G'$ of a graph $G$ we shall denote by $d_{G'}(x')$ the degree in $G'$ of a vertex $x'$ in $V(G')$. We set

$$\delta(G') = \sum_{x' \in V(G')} (d_{G'}(x') - d_{G'}(x')) .$$

We shall say that $G$ is $k$-unbalanced (where $k \geq 0$ is an integer) if $\delta(G') \geq k$ for every $G'$ which is a connected and balanced subgraph of $G$.

Proposition 4.1 is proved in a fashion to Jaeger's [5] that a 3-edge (resp. 4-edge) connected graph has a co-arboricity at most equal to 3 (resp. 2).
4.1. Proposition. If $G$ is 3-unbalanced (resp. 4-unbalanced), then $E(G)$ can be covered by three (resp. two) cobases of $M(G)$.

Proof. We deal with the two cases together by supposing that $G$ is $k$-unbalanced with $k \geq 3$. Thus we assume that

(i) $\delta(G') \geq k \geq 3$ if $G'$ is a connected and balanced subgraph of $G$.

In order to reduce the notation, we set $E = E(G)$ and $V = V(G)$. Let $M^*(G)$ be the dual of $M(G)$. For every $A \subseteq E$, we denote by $r(A)$ the rank of $A$ in $M(G)$, by $r^*(A)$ the rank of $A$ in $M^*(G)$, by $b(A)$ the number of balanced components of the subgraph consisting of $A$ and $V(G)$.

We have $b(E) = 0$ since otherwise $\delta(G')$ would be equal to 0 for some balanced component $G'$ of $G$, a contradiction to (i). Let $A \subseteq E$. We have $r^*(A) = |A| - r(E) + r(E\setminus A)$. Following Corollary 2.2 we have $r(E) = |V| - b(E)$ and $r(E\setminus A) = |V| - b(E\setminus A)$. Since $b(E) = 0$ we obtain

(ii) $r^*(A) = |A| - b(E\setminus A)$.

Let us denote by $C_1, C_2, \ldots, C_q$ the connected components of the subgraph $H$ consisting of $E\setminus A$ and $V(G)$.

(iii) $\sum_{i=1}^{q} \delta(C_i) = \delta(H) = 2 |A|.$

For each of the $C_i$'s which is balanced, $\delta(C_i) \geq k$ is true by (i). Thus inequality (iii) implies $kb(E\setminus A) \leq 2 |A|$; and so (ii) yields

(iv) $\frac{k}{k-2} r^*(A) \geq |A|.$

Edmonds' covering theorem says that the support of a matroid with rank function $\rho$ can be covered by $s$ bases if $s\rho(A) \geq |A|$ is verified for every set of elements $A$ (see [10, p. 125]). Applying this theorem to $M^*(G)$, and using (iv), we see that $E$ can be covered by three bases of $M^*(G)$ when $k = 3$, and by two bases of $M^*(G)$ when $k = 4$.

The next proposition is the analog for bidirected graphs of a proposition proved by Seymour [7] for ordinary graphs.

4.2. Proposition. If the inequality $w(G) \leq k$ is satisfied for every connected and 3-unbalanced bidirected graph $G$, then it is also satisfied for every connected and unbalanced bidirected graph $G$ without signed graphic isthmis.

Proof. Let us suppose that the implication is false. Then we can find a bidirected graph $G$ with a minimum number of edges such that:

(i) $G$ is connected;

(ii) $G$ is unbalanced;
(iii) $G$ has no signed graphic isthmii;
(iv) $w(G) > k$.

The bidirected graph $G$ is not 3-unbalanced. Then there exists a connected and balanced subgraph $G'$ satisfying

$$\sum_{x' \in V(G')} (d_G(x') - d_G(x')) < 3.$$  

We can assume (using switching) that every edge of $G'$ is positive. Let us consider the set $A$ of the edges which are not in $E(G')$ and have an endpoint in $V(G')$. Following the preceding inequality, the number $m$ of the edges in $A$ is at most equal to 2.

**Case $m = 0$**

This case contradicts conditions (i) and (ii).

**Case $m = 1$**

Let $e$ be the single edge in $A$. We distinguish two subcases depending on the endpoints of $e$.

If both endpoints of $e$ belong to $V(G')$, then condition (i) implies that $e$ is the single edge in $E(G) \setminus E(G')$. It follows that $e$ is the single negative edge because $G'$ is balanced and $G$ is unbalanced. Therefore $e$ is a signed graphic isthmus of $G$ by Lemma 2.4, a contradiction with (iii).

If one of the endpoints of $e$ is in $V(G')$ and the other in $V(G) \setminus V(G')$, then $e$ is an isthmus of $G$ which separates the balanced component $G'$ from another one which is either balanced or not. Thus it is also a signed graphic isthmus of $G$ by Lemma 2.5, a contradiction with (iii).

**Case $m = 2$**

Let $e_1$ and $e_2$ be the two edges of $A$. Each of these edges has an endpoint in $V(G')$ and the other one in $V(G) \setminus V(G')$. We denote by $h_1'$ (resp. $h_2'$) the half-edge of $e_1$ (resp. $e_2$) which has its endpoint in $V(G')$.

The value of $w(G)$ will not be changed after switching $G$ or reversing the directions of some edges. So we can suppose that $e_1$ is a positive edge; otherwise we could switch $G$ at an endpoint of $e_1$. Similarly we can suppose that $\tau(h_1') = \tau(h_2') = +1$; otherwise we could reverse the direction of $e_1$ and/or $e_2$.

Let $f$ be an arbitrary flow on $G$. After summing the flow conditions on the vertices of the balanced component $G'$, we obtain $f(e_1) + f(e_2) = 0$. Since $e_1$ is positive, it is possible to contract this edge; we shall denote by $f'$ the restricted flow on $G/e_1$.

Let us suppose now that $f$ is a nowhere-zero flow. Then $f'$ is also a nowhere-zero flow on $G/e_1$; so there is no signed graphic isthmus in $G/e_1$.
The graph $G/e$, is connected and unbalanced like $G$; thus it satisfies conditions (i) to (iii). Since $G/e$, has fewer edges than $G$, it cannot satisfy (iv); therefore we can find a nowhere-zero flow $\varphi'$ on $G/e$, such that $w(\varphi') \leq k$. This flow $\varphi'$ is the restriction of a flow $\varphi$ on $G$ such that $\varphi(e) = \varphi'(e)$ when $e = e$, and $\varphi(e) = -\varphi(e_2)$. Thus $\varphi$ is a nowhere-zero flow on $G$ such that $w(\varphi) = w(\varphi') \leq k$, a contradiction of (iv).

4.3. **Theorem.** A connected bidirected graph $G$ without signed graphic isthmi can be provided with a nowhere-zero flow $f$ such that $w(f) \leq 216$.

**Proof.** Following Proposition 4.2 it is sufficient to prove the property when $G$ is 3-unbalanced. So Proposition 4.1 implies that we can cover $E(G)$ with three cobases of $M(G)$. Theorem 3.6 can then be applied with $s$, $p$, and $W = 3$, and this yields a nowhere-zero flow $f$ such that $w(f) \leq 6^3 = 216$.

**Remark.** We can find a nowhere-zero flow $f$ such that $w(f) \leq 32$ if there exists three connected bases of $M(G)$ with an empty intersection. We do not give the proof since it is not known whether such a set of bases always exists.

**Conjecture.** A connected and bidirected graph $G$ without signed graphic isthmi can be provided with a nowhere-zero flow $f$ such that $w(f) \leq 6$.

4.4. **Proposition.** The value 6 is best possible.

**Proof.** The proof depends on the results stated in the introduction. Let us consider the triangular imbedding of $K_6$ into the projective plane. So $\mathcal{F}(K_6) = \mathcal{F}(K_6)$. Let $G$ be a bidirected graph which is dual of $K_6$ (G is

![Figure 2](image-url)
depicted in Fig. 2 with a single arrow drawn on those edges directed in the usual fashion). Every nowhere-zero flow $f$ on $G$ is the dual of a nowhere-zero local tension $\theta_i$ on $K_6$. But $\theta_i$ is also a tension on $K_6$ because $F(K_6) = F_i(K_6)$. Therefore, we have $|\theta_i(e)| \geq 5$ for some edge $e$ of $K_6$, and so $|f(e^*)| \geq 5$ for the edge $e^*$ of $G$ which is dual of $e$. 

REFERENCES