Spreads in \(H(q)\) and 1-systems of \(Q(6, q)\)

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In this paper we prove that the projections along reguli of a translation spread of the classical generalized hexagon \(H(q)\) are translation ovoids of \(Q(4, q)\). As translation ovoids of \(Q(4, 2')\) are elliptic quadrics, this forces that all translation spreads of \(H(2')\) are semi-classical. By representing \(H(q)\) as a coset geometry, we obtain a characterization of a translation spread in terms of a set of points of \(PG(3, q)\) which belong to imaginary chords of a twisted cubic and we construct a new example of a semi-classical spread of \(H(2')\). Finally, we study the semi-classical locally Hermitian 1-systems of \(Q(6, q)\) which are spreads of \(Q^*(5, q)\).

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1. INTRODUCTION

Let \(Q(2n, q)\) be a nonsingular quadric in \(PG(2n, q)\) with linear automorphism group \(PGO(2n + 1, q)\). A \(t\)-system of \(Q(2n, q)\) \((t = 0, 1, \ldots, n - 1)\) is a set \(S\) of \(q^n + 1\) \(t\)-dimensional projective subspaces of \(Q(2n, q)\) such that any \((n - 1)\)-dimensional projective subspace of \(Q(2n, q)\) containing a member of \(S\) shares no point with any other member of \(S\) (see [13]). A 0-system is an ovoid of \(Q(2n, q)\).

An ovoid \(O\) of \(Q(4, q)\) is a translation ovoid with respect to \(x \in O\) if there is a subgroup of \(PGO(5, q)\) of order \(q^2\) fixing \(O\) and all the lines of \(Q(4, q)\) incident with \(x\) and acting transitively on the points of \(O\) other than \(x\). Using a result of Glynn [7, in Section 2], we prove that all translation ovoids of \(Q(4, q)\), \(q\) even, are elliptic quadrics.

Let \(S\) be a 1-system of \(Q(6, q)\), and let \(L\) be a fixed line of \(S\). For each line \(M\) of \(S\), the subspace \((L, M)\) has dimension 3 and intersects \(Q(6, q)\) in a nonsingular hyperbolic quadric. Let \(R_{L, M}\) be the regulus of \((L, M)\) \(\cap Q(6, q)\) containing the lines \(L\) and \(M\). \(S\) is locally Hermitian with respect to \(L\) if \(R_{L, M}\) is contained in \(S\) for all lines \(M\) of \(S\) different from \(L\). If \(S\) is locally Hermitian with respect to all the lines of \(S\), then the 1-system is called Hermitian (see [11]).

Let \(x\) be a fixed point of \(Q(6, q)\) and denote by \(\Gamma_x\) the polar space whose points are the lines of \(Q(6, q)\) incident with \(x\) and whose lines are the planes of \(Q(6, q)\) incident with \(x\). By construction, \(\Gamma_x \simeq Q(4, q)\). If \(S\) is a locally Hermitian 1-system with respect to \(L\) and \(x\) is a point of \(L\), then the set of lines \(\mathcal{O}_x\), whose elements are either \(L\) or the transversals through \(x\) of the reguli of \(S\) containing \(L\), is an ovoid of \(\Gamma_x \simeq Q(4, q)\) (see [11, Section 8]). We call \(\mathcal{O}_x\) a projection along reguli of \(S\). If \(\mathcal{O}_x\) is an elliptic quadric for all \(x\) in \(L\), the 1-system is called semi-classical.

If \(H(q)\) is the classical generalized hexagon of order \(q\), then a spread of \(H(q)\) is a set of \(q^3 + 1\) mutually opposite lines of \(H(q)\). We note that a spread \(S\) of \(H(q)\) is a 1-system of \(Q(6, q)\) (see [13]). \(S\) is a locally Hermitian spread of \(H(q)\) when \(S\) defines a locally Hermitian 1-system of \(Q(6, q)\).

In Section 3, we prove that the projections along reguli of translation spreads of \(H(q)\) are always translation ovoids of \(Q(4, q)\). As translation ovoids of \(Q(4, q)\), \(q = 2'\), are elliptic quadrics, this implies that all translation spreads of \(H(2')\) are semi-classical.

In Section 4, using the construction of \(H(q)\) as a coset geometry, we obtain a characterization of translation spreads in terms of a set of points of \(PG(3, q)\) which lie on imaginary chords of a twisted cubic, and we construct a new example of a semi-classical spread of \(H(q), q = 2'\).
Finally, in Section 5 we study semi-classical locally Hermitian 1-systems of $Q(6, q)$ which are spreads of $Q^-(5, q)$. In [15] it has been noted that each spread of $PG(3, q)$ defines a locally Hermitian spread of $Q^-(5, q)$ and that each locally Hermitian spread of $Q^-(5, q)$ can be constructed in this way. If the locally Hermitian 1-system is Hermitian, then the associated spread of $PG(3, q)$ is regular, but the converse is not true. Indeed, we prove that if $q > 2$ there are regular spreads of $PG(3, q)$ whose associated locally Hermitian 1-systems are not Hermitian.

2. Translation Ovoids in $Q(4, q)$

A spread $S$ of $PG(3, q)$ is a semifield spread with respect to the line $L$ of $S$ if there is a collineation group of $PG(3, q)$ fixing $L$ pointwise and acting sharply transitively on $S \setminus \{L\}$, i.e., the translation plane associated with $S$ is a semifield plane (see, e.g., [6]).

An ovoid $O$ of $Q(4, q)$ defines, via the Klein correspondence, a symplectic spread $S(O)$ of $PG(3, q)$, i.e., a spread of $PG(3, q)$ all lines of which are totally isotropic with respect to a nonsingular symplectic polarity of $PG(3, q)$. Let $L$ be the line of $S(O)$ corresponding to the point $x$ of $O$ in the Klein correspondence. Then $O$ is a translation ovoid with respect to $x$ if and only if $S(O)$ is a semifield spread with respect to $L$.

Theorem 1. If $q$ is even, then a translation ovoid of $Q(4, q)$ is an elliptic quadric.

Proof. By projection from the nucleus of $Q(4, q)$, the ovoid defines an ovoid $O'$ of the symplectic space $W(3, q)$ with a collineation group $G$ of order $q^2$ fixing a point $x'$ of $O'$ and all tangents to $O'$ at $x'$. By [7], $O'$ is either a Suzuki–Tits ovoid or an elliptic quadric. Then, there is a collineation of $W(3, q)$ fixing $O'$ and moving $x'$. As each collineation of $W(3, q)$ defines exactly one collineation of $Q(4, q)$, we conclude that there is a collineation fixing $Q(4, q)$, the ovoid $O$, and moving the point $x$. Hence, the spread $S(O)$ is a semifield spread with respect to each line of $S(O)$, i.e., $S(O)$ is a Desarguesian spread. Since the representation on the Klein quadric of a Desarguesian spread of $PG(3, q)$ is an elliptic quadric intersection with a three-dimensional (3D) subspace $T$ (see, e.g., [8]), we conclude that $O = Q(4, q) \cap T$ is an elliptic quadric.

If $Q(4, q)$ has equation $X_2^2 = X_0 X_3 + X_1 X_4$, then an ovoid $O$ containing $(0, 1, 0, 0, 0)$ of $Q(4, q)$ is a translation ovoid with respect to the point $(0, 0, 0, 1)$ if and only if

$$O \simeq \{(a, 1, b, f(a, b), b^2 - af(a, b)) \mid a, b \in GF(q)\} \cup \{(0, 0, 0, 0, 1)\}$$

where $f$ is an additive map (see, e.g., [10]). By Corollary 5.11 of [3], $O$ is a classical ovoid if and only if $f$ is a $GF(q)$-linear map.

3. Translation Spreads in $H(q)$

Let $Q(6, q)$ have equation $X_2^2 = X_0 X_4 + X_1 X_5 + X_2 X_6$. Then, Tits [16] defines the generalized hexagon $H(q)$ as follows. The points are all the points of $Q(6, q)$. The lines are those lines of $Q(6, q)$ whose Grassmann coordinates satisfy

$$p_{34} = p_{12}, \quad p_{35} = p_{20}, \quad p_{36} = p_{01}, \quad p_{36} = p_{56}, \quad p_{13} = p_{64}, \quad p_{23} = p_{45}.$$ 

The lines of $H(q)$ incident with a fixed point of $H(q)$ belong to a plane contained in $Q(6, q)$. The points of $H(q)$ at distance less than 6 from a fixed point $x$ are all the points of $H(q)$ in the tangent hyperplane to $Q(6, q)$ at $x$. Two elements of $H(q)$ are opposite if they are at
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distance 6 in the incidence graph. An ovoid (respectively, a spread) of \(H(q)\) is a set of \(q^3 + 1\) mutually opposite points (respectively, lines). Therefore, two points of \(H(q)\) are at distance 6 if and only if they are not collinear on \(Q(6, q)\). This implies that an ovoid of \(H(q)\) is an ovoid of \(Q(6, q)\), and a spread of \(H(q)\) defines a 1-system of \(Q(6, q)\).

Let \(L\) be a fixed line of \(H(q)\) and denote by \(E^L\) the group of automorphisms of \(H(q)\) generated by all the collineations fixing \(L\) pointwise and stabilizing all the lines through a common point of \(L\). The group \(E^L\) has order \(q^5\) and acts regularly on the set of the lines of \(H(q)\) at distance 6 from \(L\) (see, e.g., [2] or [17]).

Let \(Q^-(5, q)\) be an elliptic quadric intersection of \(Q(6, q)\) with a five-dimensional space \(T\). Let \(S\) be the set of the lines of \(H(q)\) contained in \(Q^-(5, q)\). By [14], \(S\) is a Hermitian spread. Moreover, all Hermitian spreads of \(H(q)\) are isomorphic to \(S\) (see, e.g., [4] or [12]).

For any line \(M\) of \(H(q)\), denote by \(\Gamma_4(M)\) the set of the lines of \(H(q)\) at distance 4 from \(M\). Let \(S\) be a spread of \(H(q)\), and \(L\) a fixed line of \(S\). Fix a point \(x\) of \(L\). For each line \(M \neq L\) incident with \(x\), there are exactly \(q^2\) elements in \(S_M = \Gamma_4(M) \cap S\). The spread \(S\) is a translation spread with respect to the flag \((x, L)\) if there is a subgroup \(G_{x, L}\) of \(E^L\), which preserves \(S\) and acts transitively on the lines of \(S_M\), for all lines \(M\) incident with \(x\) and different from \(L\). \(S\) is a translation spread with respect to \(L\) if \(S\) is a translation spread with respect to the flag \((x, L)\) for each point \(x\) of \(L\). The following results are known about translation spreads.

(a) ([4, Theorem 6]) A translation spread of \(H(q)\) with respect to \(L\) is locally Hermitian with respect to \(L\).

(b) ([12, Theorem 3.6]) A spread \(S\) of \(H(q)\) is a translation spread with respect to a line \(L\) if and only if the stabilizer of \(S\) in \(E^L\) has order \(q^3\), i.e., it acts regularly on \(S \setminus \{L\}\).

For more details on generalized hexagons and locally Hermitian spreads see [4, 12].

The lines

\[
[\infty] = \{(1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 1)\}
\]

\[
M = \{(0, 0, 1, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0)\}
\]

of \(Q(6, q)\) are lines of \(H(q)\) at distance 6. The subgroup \(Z\) of \(E^{[\infty]}\) fixing the regulus \(R_{[\infty], M}\) and acting sharply transitively on \(R_{[\infty], M} \setminus \{[\infty]\}\) is the subgroup

\[
Z = \{\theta(0, 0, l', 0, 0) \mid l' \in GF(q)\}
\]

where \(\theta(0, 0, l', 0, 0)\) is the collineation mapping \((x_0, x_1, \ldots, x_6)\) to

\[
\begin{pmatrix}
1 & 0 & l' & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -l' & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix}
\]

(cf. [4, Section 3]). The group \(Z\) is contained in the centre of \(E^{[\infty]}\) when \(q = 3^r\), and it is the centre of \(E^{[\infty]}\) when \(q \neq 3^r\).

**Lemma 1.** For each line \(N\) at distance 6 from \([\infty]\), the group \(Z\) fixes the regulus \(R_{[\infty], N}\) and acts sharply transitively on the lines of \(R_{[\infty], N}\) other than \([\infty]\).
PROOF. There is exactly one element \( g \) of \( E[∞] \) such that \( M^g = N \). Therefore, \( g^{-1}Zg \) fixes \( R_{[∞],N} \) and acts sharply transitively on the lines of \( R_{[∞],N} \) other than \([∞]\). As \( Z \) is in the centre of \( E[∞] \), \( Z = g^{-1}Zg \).

**Theorem 2.** Let \( S \) be a translation spread of \( H(q) \) with respect to the line \( L \). For each point \( x \) of \( L \), the projection along reguli \( O_x \) of \( S \) is a translation ovoid with respect to the point \( L \) of \( Γ_x \).

**Proof.** We may suppose \( L = [∞] \), since the automorphism group of \( H(q) \) is transitive on the lines of \( H(q) \). By [12, Theorem 3.6], the stabilizer \( G \) of \( S \) in \( E[∞] \) has order \( q^3 \) and by [4, Theorem 6], \( S \) is locally Hermitian with respect to \([∞]\), i.e., there are exactly \( q^2 \) reguli in \( S \) containing \([∞]\).

Let \( N \) be a line of \( S \) different from \([∞]\). By Lemma 1, \( Z \) acts sharply transitively on \( R_{[∞],N} \setminus \{[∞]\} \). As \( E[∞] \) is sharply transitive on the lines at distance 6 from \([∞]\) and \( G \) is sharply transitive on \( S \setminus \{[∞]\} \), \( Z \) is a subgroup of \( G \) and the group \( G \) is transitive on the reguli of \( S \) containing \([∞]\). Therefore, \( G \) acts transitively on the elements of \( O_x \setminus \{[∞]\} \), for \( x \) a point of \([∞]\), and fixes the \( q + 1 \) planes of \( (6, q) \) containing \([∞]\) and a line of \( H(q) \) incident with a point of \([∞]\). This implies that \( G \) induces on \( Γ_x \simeq Q(4, q) \) a group \( H \) fixing the point \([∞] \) of \( Γ_x \) and all the lines of \( Γ_x \) incident with \([∞]\), and acts transitively on \( O_x \setminus \{[∞]\} \). By Lemma 1, \( Z \) fixes all the reguli of \( S \) containing the line \([∞]\). This implies that \( Z \) fixes all the elements of \( O_x \), because it acts as the identity on \([∞]\). Hence, \( H \) has order \( q^2 \), i.e., \( O_x \) is a translation ovoid with respect to \([∞]\).

**Corollary 1.** All translation spreads \( S \) of \( H(2') \) are semi-classical.

**Proof.** The proof follows from Theorems 1 and 2.

### 4. Coset Geometries

Fix the two lines

\[
[∞] = \langle (1, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1) \rangle,
\]

\[
M = \langle (0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 1, 0) \rangle,
\]

of \( H(q) \), which are at distance 6.

Denote by \( x_∞ = (1, 0, 0, 0, 0, 0) \) and \( x_t = (t, 0, 0, 0, 0, 1) \), \( t \in GF(q) \), the points of the line \([∞]\). For each \( t \in \tilde{F} = GF(q) \cup \{∞\} \), there is a unique chain, say \( (x_t, N_t, y_t, R_t, z_t, M) \), in \( H(q) \) of length 5 joining \( x_t \) and \( M \). Put

\[
A_4(t) = \{ g \in E[∞] \mid N_t \}^g = N_t \}
\]

\[
A_3(t) = \{ g \in E[∞] \mid y_t \}^g = y_t \}
\]

\[
A_2(t) = \{ g \in E[∞] \mid R_t \}^g = R_t \}
\]

\[
A_1(t) = \{ g \in E[∞] \mid z_t \}^g = z_t \}
\]

Then, \( A_1(t) < A_2(t) < A_3(t) = A_4(t) \) and \( A_3(t) = A_2(t)Z \), where \( Z \) is the group defined in Section 3. Moreover, \( |A_1(t)| = q, |A_2(t)| = q^2, |A_3(t)| = q^3 \) and \( |A_4(t)| = q^4 \) (see, e.g., [2] or [17]).

Define a point-line geometry \( H = (P, L, I) \) as follows:

\[
P = \{ (t), A_3(t)g, A_1(t)g : g \in E[∞], t \in \tilde{F} \}
\]

\[
L = \{ [∞], A_4(t)g, A_2(t)g : g \in E[∞], t \in \tilde{F} \}
\]
where \([\infty]\) and \((t)\) are symbols, and the incidences are: \([\infty]I(t), A_{1}(t)I(t), gI_{A_{1}(t)}g\) for all \(t \in \tilde{F}\) and \(g \in E^{[\infty]}\), whereas \(A_{1}(t)gI_{A_{1}}(v)h\) if and only if \(t = v\) and \(g \in A_{i+1}(v)h\) with \(i = 1, 2, 3\) and for all \(g, h \in E^{[\infty]}\) and \(t, v \in \tilde{F}\). Then, \(H\) is isomorphic to \(H(q)\), via the isomorphism \(\theta\) from \(H\) to \(H(q)\) defined by

\[
\begin{align*}
\theta : [\infty] & \mapsto [\infty], \\
\theta : (t) & \mapsto x_{t}, \\
\theta : A_{4}(t)g & \mapsto N_{g}, \\
\theta : A_{3}(t)g & \mapsto y_{g}, \\
\theta : A_{2}(t)g & \mapsto R_{g}, \\
\theta : A_{1}(t)g & \mapsto z_{g}, \\
\theta : g & \mapsto M_{g}
\end{align*}
\]

(see, e.g., [2]). In the following, we identify any line at distance 6 from \([\infty]\) with the element of \(E^{[\infty]}\) which is mapped onto it by \(\theta\).

**Theorem 3.** Let \(G\) be a subgroup of \(E^{[\infty]}\) of order \(q^{3}\). Then \(S = G \cup \{[\infty]\}\) is a translation spread of \(H \cong H(q)\) if and only if \(G \cap A_{1}(t)A_{1}(u) = 1\) and \(G \cap A_{2}(t) = 1\) for all \(t, u \in \tilde{F}\), with \(t \neq u\).

**Proof.** As \(G\) acts sharply transitively on \(S \setminus \{[\infty]\}\), if \(S\) is a spread of \(H\), then it is a translation spread by [12, Theorem 3.6]. Since \([\infty] \in S\), all the other lines of the spread are of type \(g\).

Two lines \(g\) and \(h\) of \(H\) are at distance 2 if and only if the chain joining \(g\) and \(h\) is of type \((g, A_{1}(t)h, h)\) for some \(t \in \tilde{F}\). This is equivalent to saying that \(gh^{-1} \in A_{1}(t) \prec A_{2}(t)\).

Two lines \(g\) and \(h\) of \(H\) are at distance 4 if and only if the chain joining \(g\) and \(h\) is either of type

\[(g, A_{1}(t)g, k, A_{1}(u)h, h)\]

with \(k \in G\) and \(t, u \in \tilde{F}, t \neq u,\) or of type

\[(g, A_{1}(t)g, A_{2}(t)g, A_{1}(t)h, h)\]

with \(t \in \tilde{F}\). This is equivalent to saying either \(gh^{-1} \in A_{1}(t)A_{1}(u)\) with \(t \neq u\) or \(gh^{-1} \in A_{2}(t)\).

Therefore, \(S\) is a spread if and only if \(G \cap A_{1}(t)A_{1}(u) = 1\) and \(G \cap A_{2}(t) = 1\) for all \(t, u \in \tilde{F}, t \neq u\).

**Corollary 2.** If \(G\) is a subgroup of \(E^{[\infty]}\) of order \(q^{3}\) such that \(G \cap A_{1}(t)A_{1}(u) = 1\), and \(G \cap A_{2}(t) = 1\) for all \(t, u \in \tilde{F}, t \neq u\), then \(Z\) is a subgroup of \(G\).

**Proof.** Since \(S = G \cup \{[\infty]\}\) is a translation spread of \(H\) and \(G\) acts sharply transitively on the lines of \(S\) different from \([\infty]\), the proof follows from Lemma 1.

The group

\[
\tilde{E} = E^{[\infty]} / Z = \{ (x, y, z, t) \mid x, y, z, t \in GF(q) \}
\]

is elementary abelian (for more details see [1]), and can be regarded as a four-dimensional vector space over \(GF(q)\).

For each element \(g\) of \(E^{[\infty]}\), let \(g^{*}\) be the preimage in \(E^{[\infty]}\) of the 1-space of \(\tilde{E}\) spanned by \(\tilde{g} = gZ\). If \(\tilde{g}, \tilde{h}\) are elements of \(\tilde{E}\), then \((\tilde{g}, \tilde{h}) = [g, h]\) defines an alternating \(GF(q)\)-bilinear
form on \( \tilde{E} \): if \( q \) is even, \( \tilde{g} \mapsto g^2 \) defines a quadratic form associated with \((\ , \ )\). Thus, \( \tilde{E} \) is endowed with a symplectic or an orthogonal geometry.

If \([g, h] = 1\), then \([g^*, h^*] = 1\). Thus, maximal elementary abelian subgroups of \( E^{[\infty]} \) are preimages of maximal totally isotropic (or singular) 2-spaces of \( \tilde{E} \).

Let \( PG(3, q) \) be the 3D projective space associated with the \( GF(q)\)-vector space \( \tilde{E} \). As \( A_3(t) = A_2(t)Z \), \( A_3(t) \) is a maximal elementary abelian subgroup of \( E^{[\infty]} \) for each \( t \in \tilde{F} \). Thus, \( L_t = A_3(t)/Z \) is a totally isotropic (or singular) line of the projective space \( PG(3, q) \).

Denote by \( p_t, L_t, \alpha_t \), respectively, the point \( A_1(t)/Z \) of \( E^{[\infty]} \), the line \( A_3(t)/Z \) and the plane \( A_4(t)/Z \) of \( PG(3, q) \). The set \( \Sigma = \{ p_t : t \in \tilde{F} \} \) is a twisted cubic of \( PG(3, q) \) and \( L_t \) is the tangent line to \( \Sigma \) at \( p_t \) and \( \alpha_t \) is the osculating plane to \( \Sigma \) at \( p_t \). Moreover, we may suppose \( p_\infty = (0, 0, 1) \) and \( p_t = (1, -t, t^2, t^3), t \in GF(q) \) (see [1]).

If \( \tau \) is an automorphism of \( H(q) \) fixing \([\infty]\) then \( \tau E^{[\infty]}/\tau^{-1} = E^{[\infty]} \) (see [2, 9]). Since the automorphism of \( E^{[\infty]} \) defined by \( g \mapsto \tau g \tau^{-1} \) preserves the families \( \{ A_1(t) \mid t \in \tilde{F} \} \) \((i = 1, 2, 3, 4)\), the automorphisms \( \tau \) induce, by conjugation, a collineation group of \( PGL(4, q) \) fixing the twisted cubic \( \Sigma \), and acting 3-transitively on \( \Sigma \). Therefore, the subgroup \( K^* \) of \( Aut H(q) \) fixing the line \([\infty]\) and isomorphic to \( PSL(2, q) \) induces a subgroup \( K \simeq PSL(2, q) \) of \( PGL(4, q) \) acting 2-transitively on \( \Sigma \).

Let \( \Sigma \) be the twisted cubic of \( PG(3, \tilde{F}) \) defined by \( \tilde{\Sigma} \), where \( \tilde{F} \) is the algebraic closure of \( GF(q) \). A line of \( PG(3, q) \) is a chord of \( \Sigma \) if it contains two points of \( \tilde{\Sigma} \). There are three possibilities: the two points belong to \( \Sigma \), or they are coincident, or they are conjugate over \( GF(q^2) \) (for more details, see [5]). This is called a real chord, a tangent or an imaginary chord, respectively. By [5, Lemma 1], every point off \( \Sigma \) lies exactly on one chord.

**Theorem 4.** Let \( G \) be a subgroup of order \( q^3 \) of \( E^{[\infty]} \) containing \( Z \). Then \( \tilde{S} = G \cup \{ [\infty] \} \) is a spread of \( H \simeq H(q) \) if and only if all the points of \( S = \{ (\tilde{g} \mid \tilde{g} \in \tilde{G}) \} \subset PG(3, q) \) lie on imaginary chords of \( \Sigma \).

**Proof.** As \( Z \) is a subgroup of \( G \), then \( g \in G \cap A_1(t)A_1(u) (g \neq 1) \) if and only if \( \tilde{g} \) belongs to the real chord \( \{ p_t, p_u \} \) of \( \tilde{\Sigma} \), and \( G \cap A_2(t) \neq 1 \) if and only if the tangent \( L_t \) is not skew with \( S \). The proof now follows from Theorem 3.

**Corollary 3.** Let \( G \) be a subgroup of order \( q^3 \) of \( E^{[\infty]} \) containing \( Z \), \( \tilde{S} = G \cup \{ [\infty] \} \) is a semi-classical spread of \( H \simeq H(q) \) if and only if \( \tilde{G} \) defines a line of \( PG(3, q) \) whose points belong to imaginary chords of \( \tilde{\Sigma} \).

**Proof.** By the proof of Theorem 4.5 of [12] we can suppose

\[
G = \{ \theta(a, b, c, f(a, b), g(a, b)) \mid a, b, c \in GF(q) \}
\]

where \( f \) and \( g \) are additive maps. \( S \) is semi-classical if and only if \( f \) and \( g \) are \( GF(q)\)-linear maps. Therefore, \( G/Z \) is a two-dimensional vector space over \( GF(q) \).

Let \( G \) be a subgroup of order \( q^3 \) of \( E^{[\infty]} \) containing \( Z \) and such that \( \tilde{S} = G \cup \{ [\infty] \} \) is a spread of \( H(q) \). If \( q \) is even, then \( S \) is semi-classical by Theorem 1. Thus, \( G/Z \) is a line whose points lie on imaginary chords of \( \Sigma \).

**Theorem 5.** Let \( G \) be a subgroup of order \( q^3 \) of \( E^{[\infty]} \) containing \( Z \) such that \( \tilde{S} = G \cup \{ [\infty] \} \) is a spread of \( H \). The spread \( \tilde{S} \) is Hermitian if and only if \( G/Z \) is an imaginary chord of \( \Sigma \).

**Proof.** Suppose \( q \) is odd. Let \( \gamma \) be a fixed nonsquare in \( GF(q) \) and

\[
G_0 = \{ \theta(k, b, k', yk, -\gamma b) \mid k, b, k' \in GF(q) \}.
\]
Then $\mathcal{S}^* = G_0 \cup \{[\infty]\}$ is the Hermitian spread of $H(q)$ defined by the elliptic quadric $X_5 - \gamma X_1 = 0 = X_2^2 - \gamma X_1^2 - X_0 X_4 - X_2 X_6$ (see [4]). Then $G_0/\mathbb{Z}$ is an imaginary chord of $\Sigma$.

Suppose $q$ is even. Let $\gamma$ be a fixed element of $GF(q)$ such that the polynomial $X^2 + X + \gamma$ has no root in $GF(q)$, i.e., $tr(\gamma) = 1$. If $G_0 = \{\theta(k, b, k', b + \gamma k, (1 + \gamma)b + \gamma k) \mid k, b, k' \in GF(q)\}$, then we can prove, by a direct calculation, that $\mathcal{S}^* = G_0 \cup \{[\infty]\}$ is the Hermitian spread of $H$ defined by the elliptic quadric $X_5 - \gamma X_1 = 0 = X_2^2 - X_3 X_1 - \gamma X_1^2 - X_0 X_4 - X_2 X_6$ and that $G_0/\mathbb{Z} = \{(k, b, b + \gamma k, (1 + \gamma)b + \gamma k) \mid k, b \in GF(q)\}$ is an imaginary chord of $\Sigma$.

Let $\mathcal{S} = G \cup \{[\infty]\}$ be a Hermitian spread of $H \cong H(q)$ containing the line $[\infty]$. Hence, there is an automorphism $\tau$ of $K^*$ such that $\mathcal{S}^* = \mathcal{S}\tau^{-1}$. So, $G_0 = \tau G\tau^{-1}$. Since $G_0/\mathbb{Z}$ is an imaginary chord, $G/\mathbb{Z}$ is an imaginary chord.

If $G/\mathbb{Z}$ is an imaginary chord, then there is an element $\tau$ of $K^*$ such that $G_0 = \tau G\tau^{-1}$ because the imaginary chords of $\Sigma$ form a line orbit under the action of the subgroup of $PGL(4, q)$ defined by $K^*$. Therefore $\mathcal{S}^* = \mathcal{S}\tau^{-1}$, i.e., $\mathcal{S}$ is a Hermitian spread of $H$.

We conclude this section by giving a description of the known non-Hermitian translation spreads of $H(q)$ in terms of $G/\mathbb{Z}$.

The spread $\mathcal{S}_{[q]} ([4])$.

Let $q \equiv 1 \pmod{3}$, $q$ odd, and $\gamma$ be a fixed nonsquare in $GF(q)$. If $G = \{\theta(k, b, k', -\gamma k, 9\gamma b) \mid k, b, k' \in GF(q)\}$, then $\mathcal{S}_{[q]} = G \cup \{[\infty]\}$ is a semi-classical spread, which is not Hermitian. The line $G/\mathbb{Z}$ is not an imaginary chord.

The spread $\mathcal{S}_{-\gamma^{1/3}} ([4])$.

Let $q = 3^e > 3$ and $\gamma$ be a fixed nonsquare in $GF(q)$. If $G = \{\theta(k, -\gamma b - \gamma^{1/3}k^{1/3}, k', \gamma^{-1}k - \gamma^{1/3}b^{1/3}, b) \mid k, b, k' \in GF(q)\}$, then $\mathcal{S}_{-\gamma^{1/3}} = G \cup \{[\infty]\}$ is a locally Hermitian spread whose projections along reguli are isomorphic to the Payne–Thas ovoid (see [4]).

For $q$ odd, a semi-classical spread is isomorphic either to the Hermitian spread or to the spread $\mathcal{S}_{[q]}$ (see [4, Section 6]). For $q = 3^e$ a locally Hermitian spread is either Hermitian or isomorphic to $\mathcal{S}_{-\gamma^{1/3}}$ (see [12]).

5. A New Example of Semi-classical Spread of $H(q)$

Suppose $q = p^t$ with $p$ a prime different from 3.

Denote by $\omega$ the nonsingular symplectic polarity of $PG(3, q)$ interchanging a point $x$ of $\Sigma$ with the osculating plane to $\Sigma$ at $x$. We call an axis of $\Sigma$ any line $l$ of $PG(3, q)$ whose polar line with respect to $\omega$ is a chord. We say that $l$ is a real axis or an imaginary axis if and only if $l^\omega$ is a real chord or an imaginary chord, respectively. It is known (see, e.g., [5]) that the subgroup $K \cong PSL(2, q)$ of $PGL(4, q)$ acting two transitively on $\Sigma$ is transitive on the set of imaginary chords and on the set of imaginary axes. Moreover, if $l$ is either an imaginary chord or an imaginary axis, then the stabilizer in $K$ of $l$ has order $q + 1$, and it acts transitively on the points of $l$ only if $q \equiv 1 \pmod{3}$ (see [5]).

If $q \equiv -1 \pmod{3}$, then the tangents to $\Sigma$, the imaginary chords of $\Sigma$ and the imaginary axes of $\Sigma$ define a line spread of $PG(3, q)$ [5]. Therefore, the points belonging to imaginary axes cannot be incident with any tangent or with any imaginary chord.
If \( q \equiv 1 \pmod{3} \), then the tangents to \( \Sigma \) and the imaginary chords of \( \Sigma \) define a maximal partial spread of order \((q^2 + q + 2)/2\) (see [5]). Therefore if \( l \) is an imaginary axis, then \( l \) intersects at least one imaginary chord in a point \( x \). As all the imaginary chords belong to the same line-orbit of \( K \), and the stabilizer of \( l \) in \( K \) is transitive on the points of \( l \), all the points on \( l \) belong to some imaginary chord. If \( l = \{ \bar{g} \mid \bar{g} \in G/Z \} \), then the subgroup \( G \) defines a semi-classical spread \( S_l = G \cup \{ \infty \} \) of \( H(q) \) which is not Hermitian. We note that, for \( q \) odd, \( S_l \) is isomorphic to \( S_{[q]} \). For \( q \) even, \( S_l \) is a new semi-classical spread of \( H(q) \).

6. **LOCALLY HERMITIAN SPREADS OF** \( Q^{-}(5, q) \)

Let \( Q^{-}(5, q) = \Sigma \cap Q(6, q) \) where \( \Sigma \) is a nonsingular hyperplane of \( PG(6, q) \). A spread of \( Q^{-}(5, q) \) is a partition of the pointset of \( Q^{-}(5, q) \) into lines. Note that a spread of \( Q^{-}(5, q) \) is a 1-system of \( Q(6, q) \). A spread of \( Q^{-}(5, q) \) is called **locally Hermitian** (resp. **Hermitian**) if it defines a locally Hermitian (resp. Hermitian) 1-system of \( Q(6, q) \).

Suppose that \( S \) is a locally Hermitian spread of \( Q^{-}(5, q) \) with respect to the line \( L \). This implies that the projections along reguli are classical ovoids because, for each point \( x \) of \( L \), \( \mathcal{O}_x \) is contained in the tangent hyperplane to \( Q^{-}(5, q) \) at \( x \). Hence a locally Hermitian spread of \( Q^{-}(5, q) \) defines a semi-classical 1-system of \( Q(6, q) \).

We recall that a 1-system \( S \) of \( Q(6, q) \) is a **Hermitian** 1-system if it is locally Hermitian with respect to all lines of \( S \). If \( S \) is a Hermitian 1-system of \( Q(6, q) \), then \( S \) is a Hermitian spread of \( Q^{-}(5, q) \) (see [4]).

The generalized quadrangle \( Q^{-}(5, q) \) is the dual of the dual of the generalized quadrangle \( H(3, q^2) \) and a regulus of \( Q^{-}(5, q) \) is the dual, in \( H(3, q^2) \), of a Baer-subline intersection of \( H(3, q^2) \) with a nonsingular line (see e.g., [8]). A Hermitian spread of \( Q^{-}(5, q) \) is the dual of a nonsingular Hermitian curve \( H(2, q^2) \) intersection of \( H(3, q^2) \) with a nonsingular plane (see [15, Section 8]). This implies that any two Hermitian spreads of \( Q^{-}(5, q) \) are isomorphic.

Let \( S \) be a locally Hermitian spread of \( Q^{-}(5, q) \) with respect to the line \( L \). Denote by \( \perp \) the polarity of \( PG(5, q) \) defined by \( Q^{-}(5, q) \). If \( \Lambda = L^\perp \), then \( \Lambda \) is a 3D subspace of \( PG(5, q) \) containing \( L \) and \( \Lambda \cap Q^{-}(5, q) = L \). If \( M \) is a line of \( S \) different from \( L \) and \( R_{L,M} \) is the regulus of \( S \) containing \( L \) and \( M \), then \( m_{L,M} = \langle L, M \rangle^\perp \) is a line of \( \Lambda \) disjoint from \( \langle L, M \rangle \).

This implies that \( m_{L,M} \) is disjoint from \( L \). If \( M \) and \( N \) are lines of \( S \) different from \( L \) such that \( \langle L, M \rangle \neq \langle L, N \rangle \), then \( m_{L,M} \) and \( m_{L,N} \) are disjoint because \( \langle L, M \rangle \cap \langle L, N \rangle = L \).

Hence

\[
S_\Lambda = \{ m_{L,M} \mid M \in S, M \neq L \} \cup \{ L \}
\]

is a set of \( q^2 + 1 \) mutually disjoint lines, i.e., \( S_\Lambda \) is a spread of \( \Lambda \). It is easy to prove that for each spread \( F \) of \( \Lambda \) containing \( L \), there is a locally Hermitian spread \( S(F, L) \) of \( Q^{-}(5, q) \) with respect to \( L \) such that \( S(F, L)_\Lambda = F \). Note that if \( S \) is a locally Hermitian spread of \( Q^{-}(5, q) \) with respect to \( L \), then \( S = S(F, L) \) where \( F = S_\Lambda \).

The construction given above is the dual of the one given in [15, Section 8] Case d. \( F \) is the dual of the spread \( S_\Lambda \), i.e., there is a correlation from \( \Lambda \) to \( T \) which maps \( S_\Lambda \) onto \( F \).

For any mutually disjoint lines \( L_1, L_2, L_3 \) of \( PG(3, q) \), there is a unique regulus \( R(L_1, L_2, L_3) \) of \( PG(3, q) \) containing \( L_1, L_2 \) and \( L_3 \). A spread \( F \) of \( PG(3, q) \) is **regular** if the regulus \( R(L_1, L_2, L_3) \) is contained in \( F \) whenever \( L_1, L_2, L_3 \) are mutually distinct lines of \( F \). For \( q = 2 \), all spreads of \( PG(3, q) \) are regular. We note that the dual of a regular spread is regular.

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1This (private communication) told us that Offer has constructed a new example of translation spread of \( H(q) \), \( q \) even, by using the coordinates of \( H(q) \). By Corollaries 1 and 3, Offer’s example is equivalent to a line all points of which are on imaginary chords.
Spreads in $H(q)$ and 1-systems of $Q(6, q)$

If $S$ is a Hermitian spread of $Q^-(5, q)$, then for each line $L$ of $S$ the spread $S_L$ of $\Lambda = L^\perp$ is regular (see [15]), but the converse is not true as shown in the following theorem:

**Theorem 6.** Let $L$ be a fixed line of $Q^-(5, q)$. If $q > 2$, then there is a regular spread $\mathcal{F}$ of $\Lambda = L^\perp$ containing $L$ such that $S(\mathcal{F}, L)$ is not Hermitian.

**Proof.** The generalized quadrangle $Q^-(5, q)$ is the dual of the generalized quadrangle $H(3, q^2)$ and, as already remarked, a Hermitian spread of $Q^-(5, q)$ is the dual of a nonsingular Hermitian curve $H(2, q^2)$ intersecting $H(3, q^2)$ with a nonsingular plane. Therefore the number of Hermitian spreads of $Q^-(5, q)$ containing a fixed line $L$ is equal to the number of nonsingular planes of $PG(3, q^2)$ intersecting $H(3, q^2)$ and incident with a fixed point $x$ of $H(3, q^2)$, i.e., to the number of points of the polar plane of $x$ with respect to $H(3, q^2)$, which do not belong to $H(3, q^2)$. Hence, there are exactly $q^3(q - 1)$ Hermitian spreads of $Q^-(5, q)$ containing $L$.

Embed $\Lambda = PG(3, q)$ in $\Lambda^* = PG(3, q^2)$. Then, for each point $x$ of $\Lambda^*$ not in $\Lambda$ there is a unique line $L(x)$ incident with $x$ and containing $q + 1$ points of $\Lambda$. If $m$ is an imaginary line of $\Lambda^*$ (i.e., a line disjoint from $\Lambda$), then $\mathcal{F}_m = \{L(x) \mid x \in m\}$ is a regular spread of $\Lambda$ and for each regular spread $\mathcal{F}$ of $\Lambda$ there are exactly two imaginary lines $m$ and $n$ such that $\mathcal{F} = \mathcal{F}_m = \mathcal{F}_n$ (see, e.g., [8]). Therefore, the number of regular spreads of $\Lambda$ containing the fixed line $L$ is equal to the number of imaginary lines of $\Lambda^*$ intersecting $L$ divided by two.

We note that there are $q^2 - q$ planes of $\Lambda^* \setminus \Lambda$ which intersect $\Lambda$ exactly in $L$ and $q + 1$ planes of $\Lambda^*$ which intersect $\Lambda$ in a plane through $L$. If $\alpha^*$ is a plane of $\Lambda^* \setminus \Lambda$, then $\alpha$ is a Baer subplane of $\alpha^*$, i.e., each line of $\alpha^*$ contains a point of $\alpha$. If $\beta^*$ is a plane of $\Lambda^* \setminus \Lambda$, then $\beta^* \cap \Lambda$ is not a plane of $\Lambda$, then $\beta^*$ contains exactly $q^3(q - 1)$ imaginary lines. Hence the number of imaginary lines of $\Lambda^*$ incident with $L$ is $q^4(q - 1)^2$. Therefore the number of regular spreads of $\Lambda = PG(3, q)$ containing $L$ is $\frac{1}{4}q^4(q - 1)^2$.

If $q > 2$ then $\frac{1}{4}q^4(q - 1)^2 > q^3(q - 1)$, i.e., there is a regular spread $\mathcal{F}$ containing $L$ such that $S(\mathcal{F}, L)$ is not Hermitian. \qed

In [11, Section 6] a non-Hermitian semi-classical locally Hermitian 1-system of $Q(6, q)$, $q$ odd, has been constructed using a cubic scroll of $PG(4, q)$. In the proof of Theorem 8.1 of [11] it has been proved that such a 1-system is the unique semi-classical Hermitian 1-system which is not a spread of $Q^-(5, q)$. Therefore a semi-classical locally Hermitian 1-system of $Q(6, q)$, $q$ odd, is isomorphic either to the 1-system constructed by Luyckx and Thas in [11, Section 6] or to $S(\mathcal{F}, L)$ for some spread $\mathcal{F}$ of $PG(3, q)$.

In Section 5 we have constructed a semi-classical locally Hermitian spread $S$ of $H(q)$, which is not a spread of $Q^-(5, q)$. When $q$ is odd, such a spread is a particular case of the construction given in Section 6 of [11], because it is isomorphic to $S_{9q}$. When $q$ is even, $S$ is the first known example of semi-classical locally Hermitian 1-system of $Q(6, q)$ which is not a spread of $Q^-(5, q)$.

**References**


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