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Positively regular vague matrices

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Abstract

Positive regularity is a common attribute of inaccurate square matrices which can be used in linear equation systems that provide only nonnegative solutions. It is studied within the framework of vague matrices which can be considered as a generalization of interval matrices. Criteria of positive regularity are derived and a method of verifying them is outlined. The exposition concludes with a characterization of the radius of positive regularity. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

The problem of solving systems of linear equations and inequalities with inaccurate data has been drawing attention for more than 30 years. It has been treated within the framework of both the interval analysis and the optimality theory. Dantzig [2] introduced the concept of the generalized linear programming problem (GLPP), the columns of which were convex polyhedral sets. In GLPP, the so-called *optimistic approach* is used: a solution is considered feasible if it is feasible for at least one realization of the data. The opposite, *pessimistic approach* to inaccuracy of the entries is used in the semi-infinite programming [3], the inexact programming [14,17] and the inclusive programming [14,15]: a solution is required to satisfy all possible realizations of the data. (Cf. [16,17].)

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The interval analysis uses mostly the optimistic approach. Its significant part deals with square systems of inaccurate linear equations [1,8,9,11–13]. One of the currently discussed topics is the problem of checking regularity of a square interval matrix. Poljak and Rohn [10] proved that this problem is NP-hard.

In this paper, we deal with a more specific concept of regularity that can be considered as a common attribute of inaccurate square matrices which are suitable for models the solutions of which are supposed to be nonnegative. Such an assumption is usually accepted in many applications. On the other hand, we study a more general type of inaccurate matrices. As it is shown in Section 2, the procedures given below can be effectively applied to several interesting types of these matrices. Besides the simplest case of interval matrices, we discuss the octaedric matrices defined by using simple polyhedra of the well-known type and the elliptic matrices, the columns of which can move in n-dimensional ellipsoids. A matrix of the latter type can be used as a deterministic equivalent of a random matrix with the n-dimensional normal distribution of columns [6].

The following definition was introduced in [5].

Definition 1. Let $A^1, \ldots, A^n \subset \mathbb{R}^m$ be compact convex sets. The set of matrices

$$A^{\mathrm{V}} = \left\{A \mid A = \left(a^1, \ldots, a^n\right),\ a^j \in A^j,\ j = 1, \ldots, n\right\}$$

is called a *vague matrix*.

A vague matrix A^{V} could be equivalently defined as an ordered n-tuple of vague columns, i.e., $A^{\mathrm{V}} = (A^1, \ldots, A^n)$.

In this paper, we deal with square $n \times n$ vague matrices only. A square vague matrix A^{V} is *singular* if there exists a singular $A \in A^{\mathrm{V}}$. Otherwise, A^{V} is *regular*. The solution set of a vague linear equation system $A^{\mathrm{V}}x = b$ is defined consistently with the optimistic approach, i.e.,

$$X\left(A^{\mathrm{V}}, b\right) = \left\{x \mid \exists A \in A^{\mathrm{V}}\colon\ Ax = b\right\}.$$

Further let

$$X_+\left(A^{\mathrm{V}}, b\right) = X\left(A^{\mathrm{V}}, b\right) \cap \mathbb{R}^n_+ = \left\{x \mid x \in X\left(A^{\mathrm{V}}, b\right),\ x \geqslant 0\right\}.$$

If A^{V} is regular, then $X(A^{\mathrm{V}}, b)$, being a continuous map of a convex compact set $A^{\mathrm{V}} \subset \mathbb{R}^{n^2}$, is a connected compact set.

2. Basic properties of positively regular vague matrices

Definition 2. A square vague matrix A^{V} is called *positively regular* if there exists a $b \in \mathbb{R}^n$ such that

$$Ax = b \text{ has a solution for each } A \in A^{\mathrm{V}}, \tag{1}$$

$$x \in X\left(A^{\mathrm{V}}, b\right) \quad \Rightarrow \quad x > 0. \tag{2}$$

We can formulate a few plausible assertions which follow immediately from this definition.

Proposition 1. *A positively regular vague matrix is regular.*

Proof. Consider a b satisfying (1). If the assertion did not hold, then there would exist a nontrivial affine subspace $\mathscr{L} \in X(A^{\mathrm{V}}, b)$, which would contradict condition (2). □

Proposition 2. *A 'one-point' vague matrix $A^{\mathrm{V}} = \{A\}$ is positively regular if and only if A is nonsingular.*

Proposition 3. *Let A_0^{V} be positively regular. Then there exists an open set $\Omega \supset A_0^{\mathrm{V}}$ in $\mathbb{R}^{n \times n}$ such that any vague matrix A^{V} contained in Ω is positively regular.*

Thus, the requirement of positive regularity is not too restricting for a set of small perturbations of a given nonsingular matrix.

Proposition 4. *Let A^{V} be regular and let there exists a b such that* (2) *holds. Then A^{V} is positively regular.*

Condition (2) can be expressed as $X(A^{\mathrm{V}}, b) \subset \operatorname{int} \mathbb{R}^n_+$, where int denotes the interior of the respective set. The problem of verifying this condition is solved in the following section. Condition (1) itself, however, can be hardly verified in an operative way. Therefore, we are going to give a more transparent equivalent of (1), (2).

Consider vector functions

$$a^j(t) : [0, \infty) \to \mathbb{R}^n, \quad j \in J = \{1, \ldots, n\}$$

and a variable convex cone

$$K(t) = \left\{u = \lambda_1 a^1(t) + \cdots + \lambda_n a^n(t),\ \lambda_j > 0\ \ \forall j \in J\right\}. \tag{3}$$

Lemma 1. *Assume that*

(i) $a^j(t)$ *are continuous in a* $t_* > 0$ *and*

(ii) $a^j(t_*) \neq 0\ \forall j \in J$.

Then

$$\bigcap_{t \in [0, t_*)} K(t) = \bigcap_{t \in [0, t_*]} K(t). \tag{4}$$

Proof. Let us choose a $b \notin K(t_*)$. According to the well-known separation theorem, there exists a vector v such that $v^{\mathrm{T}} b > 0$ and $v^{\mathrm{T}} u < 0\ \forall u \in K(t_*)$. Due to condition (ii), the latter relation is equivalent to $v^{\mathrm{T}} a^j(t_*) < 0\ \forall j \in J$. Thus, $v^{\mathrm{T}} a^j(t) <$

$0\ \forall j \in J\ \forall t \in [t_* - \varepsilon, t_*]$ must hold for a sufficiently small $\varepsilon > 0$, which implies $b \notin \bigcap_{t\in[0,t_*)} K(t)$. □

Theorem 1. *A^{V} is positively regular if and only if the following conditions are satisfied*:

$$0 \notin A^j \quad \forall j \in J, \tag{5}$$

$$\exists b\colon\ \emptyset \neq X\big(A^{\mathrm{V}}, b\big) \subset \operatorname{int} \mathbb{R}^n_+. \tag{6}$$

Proof. 'If part'. Choose a b satisfying (6). There exist a nonsingular matrix $A_0 = ({}^0a^1, \dots, {}^0a^n) \in A^{\mathrm{V}}$ and an $x \in \mathbb{R}^n$ such that $A_0x = b$. Choose another matrix $A = (a^1, \dots, a^n) \in A^{\mathrm{V}}$ arbitrarily and denote $A(t) = (a^1(t), \dots, a^n(t)) = (1-t)A_0 + tA$. Further, let

$$t_* = \sup\{t \mid A(\tau) \text{ is nonsingular for } \tau \in [0, t)\}. \tag{7}$$

Let us suppose that $t_* < 1$. Then $b \in K(t)$ for $0 \leqslant t < t_*$ due to (6). According to Lemma 1, we have $b \in K(t_*)$, which means that $A(t_*)x = b$ has a solution. Consequently, $A(t_*)$ is nonsingular due to (6) again. Relation (7), however, implies that $A(t_*)$ is singular because the set of all nonsingular matrices is open. This contradiction yields $t_* > 1$, which means that A is nonsingular and, consequently, A^{V} is regular. Then the positive regularity of A^{V} follows from Proposition 4.

'Only if' part follows immediately from Definition 1. □

The natural assumption (5) that none of the matrices of A^{V} contains a zero column can be easily verified. If there exists a right-hand side vector that produce only positive solutions, we have an interesting equivalence:

Theorem 2. *If condition* (6) *holds, then the following properties are equivalent*:
(i) $0 \notin A^j\ \forall j \in J$;
(ii) *A^{V} is positively regular*;
(iii) *A^{V} is regular.*

Proof. (i) ⇒ (ii) follows from Theorem 1, (ii) ⇒ (iii) holds due to Proposition 1. The rest is plausible. □

Corollary 1. *Assume that* (6) *holds and let $X(A^{\mathrm{V}}, b)$ be bounded. Then A^{V} is positively regular.*

Let us define

$$Y\big(A^{\mathrm{V}}\big) = \big\{y \mid X\big(A^{\mathrm{V}}, y\big) \subset \mathbb{R}^n_+\big\}. \tag{8}$$

Provided $X(A^{\mathrm{V}}, b) \subset \operatorname{int} \mathbb{R}^n_+$ holds for a $b \in \mathbb{R}^n$, the same relation is kept for sufficiently small perturbations of b. Hence, condition (6) is equivalent to

$$\operatorname{int} Y\left(A^{\mathrm{V}}\right) \neq \emptyset. \tag{9}$$

Thus, we can reformulate Theorem 1 as follows:

Corollary 2. *A^{V} is positively regular if and only if* (5) *and* (9) *hold.*

3. Basic problems

Let a vague matrix A^{V} and vectors $b, c \in \mathbb{R}^n$ be given. We are going to discuss two related problems:

Problem I. Maximize $f(x) = c^{\mathrm{T}}x$ subject to $x \in X(A^{\mathrm{V}}, b)$.

Problem II. Find a pair (A_*, z^*), $A_* \in A^{\mathrm{V}}$, $z^* \in \mathbb{R}^n$ such that

$$A_*^{\mathrm{T}} z^* = c, \tag{10}$$

$$A^{\mathrm{T}} z^* \geqslant c \quad \forall A \in A^{\mathrm{V}}. \tag{11}$$

Proposition 5. *If (A_*, z^*) is a solution of Problem* II, *then*

$$c^{\mathrm{T}}x \leqslant b^{\mathrm{T}}z^* \quad \forall x \in X_+\left(A^{\mathrm{V}}, b\right)$$

holds for each $b \in \mathbb{R}^n$.

Proof. Condition (11) implies

$$c^{\mathrm{T}}x \leqslant (z^*)^{\mathrm{T}}Ax = b^{\mathrm{T}}z^* \quad \forall x \geqslant 0 \text{ satisfying } Ax = b. \qquad \square$$

Theorem 3. *Let A^{V} be positively regular and (A_*, z^*) be a solution of Problem* II. *Then $b^{\mathrm{T}}z^*$ is the exact upper bound of $f(x) = c^{\mathrm{T}}x$ on $X(A^{\mathrm{V}}, b)$ for any $b \in Y(A^{\mathrm{V}})$.*

Proof. We have $X(A^{\mathrm{V}}, b) = X_+(A^{\mathrm{V}}, b)$ for $b \in Y(A^{\mathrm{V}})$. Using Proposition 5, it is sufficient to realize that the upper bound $b^{\mathrm{T}}z^*$ of $c^{\mathrm{T}}x$ is actually achieved for $x^* = A_*^{-1}b$. $\square$

Provided a $b \in Y(A^{\mathrm{V}})$ is given, we can construct a supporting half-space of $X(A^{\mathrm{V}}, b) = X_+(A^{\mathrm{V}}, b)$ for an arbitrary given normal vector.

It was proved in [5] that there exists a solution of Problem II for any regular vague matrix A^{V}. Assuming that A^{V} is positively regular, we can present a much more transparent proof. Before doing that let us formulate a lemma.

Lemma 2. *Let $A_0, A_1 \in A^{\mathrm{V}}$, $c \in \mathbb{R}^n$ and $b \in \operatorname{int} Y(A^{\mathrm{V}})$ be given such that*

(i) *A_0, A_1 are nonsingular and*

(ii) *$c \neq A_1^{\mathrm{T}}z \geqslant c$ holds for $z = (A_0^{\mathrm{T}})^{-1}c$.*

Then

$$c^{\mathrm{T}}x^1 < c^{\mathrm{T}}x^0 \quad \textit{for } x^i = A_i^{-1}b,\ i = 0, 1.$$

Proof. Since $x^1 > 0$, we have

$$c^{\mathrm{T}}x^1 < z^{\mathrm{T}}A_1x^1 = z^{\mathrm{T}}b = z^{\mathrm{T}}A_0x^0 = c^{\mathrm{T}}x^0. \qquad \square$$

Theorem 4. *Let A^{V} be positively regular. Then*

(i) *there exists a solution (A_*, z^*) of Problem* II *for each $c \in \mathbb{R}^n$ and*
(ii) *z^* is determined uniquely.*

Proof. (i) Choose a $b \in \operatorname{int} Y(A^{\mathrm{V}})$ and let x^* be an optimal solution of Problem I. Such a vector does exist because $X(A^{\mathrm{V}}, b)$ is compact. Let $A_* = (a^{*1}, \dots, a^{*n}) \in A^{\mathrm{V}}$ be such that $A_*x^* = b$ holds and z^* be defined as $z^* = A_*^{-1}b$. We are going to prove (11) by contradiction. Let us assume that

$$\exists k \in J = \{1, \dots, n\}:\ \ \exists a \in A^k:\ \ a^{\mathrm{T}}z^* < c_k. \tag{12}$$

Consider $A = A_* + d(e^k)^{\mathrm{T}}$, where $d = a - a^{*k}$ and e^k is the kth unit vector. Using the formula for one column change inversion, we have

$$A^{-1} = A_*^{-1} - \beta A_*^{-1}d\big(e^k\big)^{\mathrm{T}}A_*^{-1},$$

where $\beta > 0$ holds due to the regularity of A^{V}. Hence, we obtain

$$c^{\mathrm{T}}x = c^{\mathrm{T}}A^{-1}b = c^{\mathrm{T}}x^* - \beta(z^*)^{\mathrm{T}}\,\mathrm{d}x_k^* \quad \text{for } x = A^{-1}b \in X\big(A^{\mathrm{V}}, b\big). \tag{13}$$

Since $x_k^* > 0$ and $d^{\mathrm{T}}z^* < 0$ due to (10), formula (13) yields $c^{\mathrm{T}}x < c^{\mathrm{T}}x^*$. This inequality, however, contradicts the assumption that x^* solves Problem I. Hence, (12) is not true.

(ii) Let (A_0, z^0), (A_1, z^1) be solutions of Problem II and let $A_0x^0 = A_1x^1 = b$ for a $b \in \operatorname{int} Y(A^{\mathrm{V}})$. Suppose for a moment that $A_1^{\mathrm{T}}z^0 \neq c$. Since $A_1^{\mathrm{T}}z^0 \geqslant c$,

$$c^{\mathrm{T}}x^1 < c^{\mathrm{T}}x^0 \leqslant \big(z^1\big)^{\mathrm{T}}A_0x^0 = \big(z^1\big)^{\mathrm{T}}b = \big(z^1\big)^{\mathrm{T}}A_1x^1 = c^{\mathrm{T}}x^1$$

must hold according to Lemma 2. This contradiction proves the equality $A_1^{\mathrm{T}}z^0 = A_1^{\mathrm{T}}z^1 = c$, which yields $z^0 = z^1$ due to regularity of A_1. $\square$

Problem II can be solved effectively by an iterative method which consists in solving n elementary optimization problems in each step. This method has been described in [5] as the method of simultaneous optimization (SO).

Algorithm SO.

1° Choose $A_0 \in A^{\mathrm{V}}$. Set $z^0 := 0$, $s := 1$.
2° Compute $z^s = (A_{s-1}^{\mathrm{T}})^{-1}c$.

3° Find $A_s = ({}^s a^1, \ldots, {}^s a^n) \in A^{\mathrm{V}}$ such that

$$\left(z^s\right)^{\mathrm{T}} {}^s a^j = \min\left\{\left(z^s\right)^{\mathrm{T}} \xi \;\middle|\; \xi \in A^j\right\}, \quad j \in J. \tag{14}$$

4° If $z^s \neq z^{s-1}$, then set $s := s + 1$ and go to 2°.

5° If $z^s = z^{s-1}$, then END.

A simplified convergence proof of this algorithm in comparison with that given in [5] is demonstrated below. (Cf. also [7].)

Theorem 5. *Let A^{V} be positively regular. Then the following assertions hold*:

(i) *The sequence $\{z^s\}$ produced by Algorithm SO converges for any $A_0 \in A^{\mathrm{V}}$.*

(ii) *If $z^* = \lim_{k\to\infty} z^s$ and A_* is an accumulation point of $\{A_s\}$, then (A_*, z^*) is a solution of Problem* II.

Proof. (i) The sequence $\{z^s\}$ is bounded due to the regularity of A^{V}. We want to show that it has a unique accumulation point: choose a $b \in \operatorname{int} Y(A^{\mathrm{V}})$ and denote $x^s = A_s^{-1} b \geqslant 0$. Due to 2° and 3° we have $(z^{s+1})^{\mathrm{T}} b = (z^{s+1})^{\mathrm{T}} A_s x^s = c^{\mathrm{T}} x^s \geqslant (z^s)^{\mathrm{T}} A_s x^s = (z^s)^{\mathrm{T}} b$ or

$$\left(z^{s+1} - z^s\right)^{\mathrm{T}} b \geqslant 0, \quad s = 0, 1, \ldots \tag{15}$$

For an arbitrary pair $z^*,\ z^{**}$ of accumulation points of $\{z^s\}$, relation (15) implies

$$\left(z^{**} - z^*\right) = 0 \quad \forall b \in \operatorname{int} Y\left(A^{\mathrm{V}}\right).$$

This condition, however, can be fulfilled only if $z^{**} = z^*$. Consequently, $z^s \to z^*$.

(ii) The assertion follows from (14). □

Apparently, the vector $b \in Y(A^{\mathrm{V}})$ mentioned in the proof is not used in the algorithm. If Algorithm SO fails, it means that such a vector does not exist and, consequently, A^{V} is not positively regular. Positive regularity, however, is not a necessary condition of convergence. In the case of positively regular polyhedral vague matrix A^{V}, the solution of (14) can be found among the vertices of A^j. Then, Algorithm SO is finite since there is a finite number of vertices.

The form of elementary optimization problems

$$\min\left\{z^{\mathrm{T}} \xi \;\middle|\; \xi \in A^j\right\}, \tag{16}$$

which are to be solved in step 3°, depends on the way in which the vague matrix A^{V} is defined. Let us consider a few alternatives:

Interval matrix:

$$A^{\mathrm{V}} = \left\{A \;\middle|\; \underline{D} \leqslant A \leqslant \overline{D}\right\}, \quad \underline{D} = \left(\underline{d}_{i,j}\right), \quad \overline{D} = \left(\overline{d}_{i,j}\right). \tag{17}$$

Then (16) takes on the form

$$\text{minimize} \ \ z^{\mathrm{T}} \xi \quad \text{subject to} \ \ \underline{d}_{ij} \leqslant \xi_i \leqslant \overline{d}_{ij}, \ i \in J. \tag{18}$$

The solution ξ^* is evident:

$$\xi_k^* = \begin{cases} \underline{d}_{ij} & \text{if } z_i \geqslant 0, \\ \overline{d}_{ij} & \text{if } z_i < 0. \end{cases} \tag{19}$$

In this specific case, Algorithm SO is a close analogy of the so-called sign-accord algorithm proposed by Rohn [13].

Octaedric matrix:

$$A^{\mathrm{V}} = \left\{ A = (a_{ij}) \,\middle|\, \sum_i \frac{|a_{ij} - \tilde{a}_{ij}|}{d_{ij}} \leqslant 1,\ j \in J \right\}, \quad d_{ij} > 0, \tag{20}$$

where $\tilde{A} = (\tilde{a}_{ij})$ is a given 'central' matrix. Translating the situation into the centre, we have another trivial optimization problem:

$$\text{minimize } \ z^{\mathrm{T}}\eta \quad \text{subject to } \ \sum_i |\eta_i| \Big/ d_{ij} \leqslant 1, \tag{21}$$

where $\eta_i = \xi_i - \tilde{a}_{ij}, i \in J$.

Let $k \in J$ be chosen so that $|z_k d_{kj}| \geqslant |z_i d_{ij}|\ \forall i \in J$. Then the optimal solution of (21) is determined as follows:

$$\eta_i^* = \begin{cases} -d_{kj} \ \operatorname{sgn} z_k, \\ 0, \ \text{otherwise}. \end{cases} \tag{22}$$

Elliptic matrix:

$$A^{\mathrm{V}} = \left\{ A \,\middle|\, \sum_i \frac{(a_{ij} - \tilde{a}_{ij})^2}{d_{ij}^2} \leqslant 1,\ j \in J \right\}, \quad d_{ij} > 0. \tag{23}$$

The optimal solution of the problem

$$\text{minimize } \ z^{\mathrm{T}}\eta \quad \text{subject to } \ \sum_i \eta_i^2 \Big/ d_{ij}^2 \leqslant 1 \tag{24}$$

can be easily obtained by utilizing the fact that the only constraint must be restricting in the optimum. Using the Lagrange multiplier, the relations

$$z_i + 2\lambda\eta_i/d_{ij}^2 = 0, \quad \sum_i \eta_i^2 \Big/ d_{ij}^2 = 1, \qquad \lambda > 0,$$

must be satisfied in the saddle point (η^*, λ).

Hence,

$$\eta_i^* = -\frac{1}{2\lambda} z_i d_{ij}^2, \quad \lambda^2 = \frac{1}{4} \sum_i z_i^2 d_{ij}^2,$$

and finally

$$\eta_i^* = -\beta z_i d_{ij}^2, \quad i \in J, \qquad \text{where } \beta = \left(\sum_i z_i^2 d_{ij}^2 \right)^{-1/2}. \tag{25}$$

We can summarize that for all these special types of vague matrix the solution of the auxiliary problem (16) is obtained by using very simple explicit formulae.

4. Checking positive regularity

Let (A_i, z^i) be a solution of Problem II for $c = -e^i$ and denote

$$w^i = -z^i = \left(A_i^{\mathrm{T}}\right)^{-1} e^i, \quad i = 1, \ldots, n.$$

The matrix $W = (w^1, \ldots, w^n)^{\mathrm{T}}$ satisfies the following conditions:

$$WA \leqslant E \quad \forall A \in A^{\mathrm{V}} \quad (E \text{ is the unit matrix}), \tag{26}$$

$$\forall i \in J, \ \exists A \in A^{\mathrm{V}}: \quad A^{\mathrm{T}} w^i = e^i. \tag{27}$$

Definition 3. A matrix W satisfying (26) and (27) is called the *lower inverse matrix* of A^{V} ($\underline{\mathrm{inv}}\, A^{\mathrm{V}}$).

Provided A^{V} is positively regular, $\underline{\mathrm{inv}}\, A^{\mathrm{V}}$ is determined uniquely due to Theorem 4.

Proposition 6. *Let A^{V} be positively regular. Then $\underline{x} = (\underline{\mathrm{inv}}\, A^{\mathrm{V}})b$ is the vector of the exact component-wise lower bounds of the solutions of $A^{\mathrm{V}}x = b$ for an arbitrary $b \in Y(A^{\mathrm{V}})$.*

Proof. $b^{\mathrm{T}} w^i = -b^{\mathrm{T}} z^i$ is the exact lower bound of x_i on $X(A^{\mathrm{V}}, b)$ for any $b \in Y(A^{\mathrm{V}})$ according to Theorem 3. □

Let us denote $(W)^+$ the polar cone of the nonnegative hull of $\{w^1, \ldots, w^n\}$, i.e., $(W)^+ = \{y \mid Wy \geqslant 0\}$. For a positively regular A^{V}, we have $Y(A^{\mathrm{V}}) \subset (W)^+$ due to Proposition 6.

Theorem 6. *Suppose that $W = \underline{\mathrm{inv}}\, A^{\mathrm{V}}$ exists and let $A_0 \in A^{\mathrm{V}}$ be a nonsingular matrix. Then the following assertions are equivalent*:

(i) $\exists y^0 \in \mathrm{int}(W)^+$: $A_0^{-1} y^0 \geqslant 0$;

(ii) $Y(A^{\mathrm{V}}) = (W)^+$.

Proof. (i) $\Rightarrow$ (ii): Choose a $y \in \mathrm{int}\,(W)^+$, consider $y(t) = (1-t)y^0 + ty$ and denote $t_* = \sup\{t \mid A_0^{-1} y(t) \geqslant 0,\ t \in [0, 1]\}$. Evidently, $X_+(A^{\mathrm{V}}, y(t_*)) \neq \emptyset$ because $x^* = A_0^{-1} y(t_*) \geqslant 0$. Since $y(t_*) \in \mathrm{int}\,(W)^+$, we have $Wy(t_*) > 0$. According to Proposition 6, $x > 0$ holds for all $x \in X_+(A^{\mathrm{V}}, y(t_*))$, namely, $A_0^{-1} y(t_*) > 0$. It means that $t_* = 1$ and hence $y \in \mathrm{int}\, Y(A^{\mathrm{V}})$. We have proved that $\mathrm{int}\,(W)^+ \subset \mathrm{int}\, Y(A^{\mathrm{V}})$ which implies $(W)^+ \subset Y(A^{\mathrm{V}})$ because $Y(A^{\mathrm{V}})$ is a closed convex set. On the other hand, Proposition 6 implies $Wy \geqslant 0\ \forall y \in Y(A^{\mathrm{V}})$ or $Y(A^{\mathrm{V}}) \subset (W)^+$.

(ii) $\Rightarrow$ (i): This implication follows from the defining relation (8). $\square$

The convex cone $(W)^+$ has a nonempty interior if and only if W is nonsingular. Thus, Theorem 6 implies:

Corollary 3. *If A^{V} is positively regular, then $W = \underline{\mathrm{inv}}\, A^{\mathrm{V}}$ is nonsingular and $Y(A^{\mathrm{V}}) = (W)^+$.*

Theorem 7. *Assume that*:
(a) $0 \notin A^j\ \forall j \in J$;
(b) *there exists a nonsingular $A \in A^{\mathrm{V}}$*;
(c) $W = \underline{\mathrm{inv}}\, A^{\mathrm{V}}$ *exists.*
Then the following assertions are equivalent:
(i) $A^{-1}b \geqslant 0\ \forall b \in \mathrm{int}(W)^+$;
(ii) A^{V} *is positively regular.*

Proof. (i) $\Rightarrow$ (ii): According to Theorem 6, assumptions (a)–(c) (i) imply $\mathrm{int}(W)^+ = \mathrm{int}\, Y(A^{\mathrm{V}}) \neq \emptyset$. Thus, we can apply Corollary 2. The converse implication is plausible. $\square$

Since $b \in \mathrm{int}(W)^+$ can be chosen arbitrarily, we can take $b = W^{-1}e$, where $e = \{1\}^n$.

Corollary 4. *Let conditions* (a)–(c) *of Theorem* 7 *be satisfied. Then A^{V} is positively regular if and only if WA is nonsingular and*

$$(WA)^{-1}e \geqslant 0. \tag{28}$$

We now can recommend an operative procedure for checking positive regularity:

Procedure CPR1.
1° Verify that $0 \notin A^j\ \forall j \in J$.
2° Find $W = \underline{\mathrm{inv}}\, A^{\mathrm{V}}$ by using Algorithm SO.
3° Choose an arbitrary $A \in A^{\mathrm{V}}$.
4° Verify (28).
If Algorithm SO fails or WA is singular, then the process ends: A^{V} is not positively regular.

Let us recall a few concepts of the theory of special matrices [4]. The matrix classes $\mathscr{Z}$ and $\mathscr{P}_0$ are defined as follows:

$$\begin{array}{ll} G = (g_{ij}) \in \mathscr{Z} & \text{if } g_{ij} \leqslant 0\ \forall i \neq j, \\ G \in \mathscr{P}_0 & \text{if all the principal minors of } G \text{ are nonnegative.} \end{array}$$

A matrix $G \in \mathscr{Z} \cap \mathscr{P}_0$ is called an M-matrix. G is a nonsingular M-matrix if and only if

1° $G \in \mathscr{Z}$,

2° there exists a $p \geqslant 0$, $p \in \mathbb{R}^n$ such that $p^{\mathrm{T}}G > 0$.

This property is invariant with respect to transposition. If G is a nonsingular M-matrix, then $G^{-1} \geqslant 0$.

Definition 4. G^{V} is called a *vague* M-*matrix* if each $G \in G^{\mathrm{V}}$ is an M-matrix.

Proposition 7. *If G^{V} is a regular vague M-matrix, then there exists a $p \geqslant 0$ such that $p^{\mathrm{T}}G > 0$ for each $G \in G^{\mathrm{V}}$.*

Proof. It is necessary to prove that p can be chosen independently of the choice of $G \in G^{\mathrm{V}} = \{G^1, \ldots, G^n\}$. Let us denote

$$\mathscr{G} = \left\{g \mid g = t_1 g^1 + \cdots + t_n g^n,\ g^j \in G^j,\ t_j > 0,\ j \in J\right\}.$$

Being a nonnegative hull of a system of convex sets, $\mathscr{G}$ is convex and, in addition, $0 \notin \mathscr{G}$ due to regularity of G^{V}. It means that $\mathscr{G}$ lies in a homogeneous half-space $H = \{x \mid p^{\mathrm{T}}x > 0\}$. Then $y = G^{\mathrm{T}}p > 0$ holds for an arbitrary $G \in G^{\mathrm{V}}$. Since G is a nonsingular M-matrix, $p^{\mathrm{T}} = y^{\mathrm{T}}G^{-1} > 0$ holds due to $G^{-1} \geqslant 0$. □

Proposition 8. *Let U be a matrix such that*

$$G^{\mathrm{V}} = UA^{\mathrm{V}} = \left\{G \mid G = UA,\ A \in A^{\mathrm{V}}\right\}$$

is a regular vague M-matrix. Then A^{V} is positively regular and

$$(U)^+ \subset Y\left(A^{\mathrm{V}}\right).$$

Proof. Choose a $z > 0$ and $A \in A^{\mathrm{V}}$ arbitrarily. Since $(UA)^{-1} \geqslant 0$, the equation system

$$Ax = b, \quad \text{where } b = U^{-1}z,$$

has the only solution $x = (UA)^{-1}z > 0$. It means that $b \in \operatorname{int}(U)^+$ implies $b \in \operatorname{int} Y(A^{\mathrm{V}})$. In addition, b is the vector required in Definition 2. □

Theorem 8. *A^{V} is positively regular if and only if $(\underline{\mathrm{inv}}\ A^{\mathrm{V}})A^{\mathrm{V}}$ is a regular vague M-matrix.*

Proof. 'Only if' part. Let A^{V} be positively regular. According to Theorem 4, $W = \underline{\mathrm{inv}}\ A^{\mathrm{V}}$ exists. For an arbitrary $A \in A^{\mathrm{V}}$, $G = WA \in \mathscr{Z}$ holds due to (26). For a $y \in \operatorname{int} Y(A^{\mathrm{V}})$ and $x = A^{-1}y$ we have

$$Gx = WAx = Wy = \underline{x} > 0, \tag{29}$$

which implies $x > 0$. Hence, G is an M-matrix.

The converse implication follows from Proposition 8. □

Thus, we have another operative criterion of positive regularity:

Theorem 9. *A^{V} is positively regular if and only if there exist vectors $u^1, \ldots, u^n$ satisfying the following system of inequalities*:

$$a^{\mathrm{T}} u^k \leqslant 0 \quad \forall a \in A^j,\ j \neq k,\ k \in J, \tag{30}$$

$$a^{\mathrm{T}} \sum_k u^k \geqslant 1 \quad \forall a \in A^j,\ j \in J. \tag{31}$$

Proof. 'If' part. Assume that $U = (a^1, \ldots, a^n)$ is a solution of (30) and (31). Then $G = UA \in \mathscr{Z}$ for each $A \in A^{\mathrm{V}}$ and, in addition, $G^{\mathrm{T}} e \geqslant e > 0$. Hence, $G^{\mathrm{V}} = UA^{\mathrm{V}}$ is a regular vague M-matrix and A^{V} is positively regular according to Proposition 8.

'Only if' part. Let A^{V} be positively regular. According to Proposition 7, there exists a $p \geqslant 0$ such that

$$p^{\mathrm{T}} W A > e \ \ \forall A \in A^{\mathrm{V}}, \quad \text{where } W = \underline{\mathrm{inv}}\, A^{\mathrm{V}}. \tag{32}$$

If we denote $u^j = p_j w^j$, $j \in J$, then (32) is equivalent to (31). Furthermore, (30) follows from the fact that $WA \in \mathscr{Z}\ \forall A \in A^{\mathrm{V}}$. □

Positive regularity of a polyhedral vague matrix can be verified in such a way that all the vertices of the polyhedra A^j are substituted for the vector a into (30), (31). Such a procedure, however, is not of a high practical value. Evidently, it is very laborious when applied to interval matrices.

Theorem 10. *If A^{V} is positively regular, then $W = \underline{\mathrm{inv}}\, A^{\mathrm{V}}$ is nonsingular and the following implication holds for any $h \in \mathbb{R}^n$*:

$$\left(\exists A \in A^{\mathrm{V}}\colon h^{\mathrm{T}} A > 0\right) \quad \Rightarrow \quad h^{\mathrm{T}} W^{-1} > 0. \tag{33}$$

Proof. (a) If W were singular, then $(W)^+$ would be a linear subspace and consequently $\mathrm{int}\, Y(A^{\mathrm{V}}) = \mathrm{int}(W)^+ = \emptyset$ would hold. Therefore, W must be nonsingular.

(b) $h^{\mathrm{T}} A > 0$ yields $p^{\mathrm{T}} G = y^{\mathrm{T}} > 0$ for $p^{\mathrm{T}} = h^{\mathrm{T}} W^{-1}$ and $G = WA$. Since G is a regular M-matrix, $G^{-1} \geqslant 0$ holds and hence $p^{\mathrm{T}} = y^{\mathrm{T}} G^{-1} \geqslant 0$. The matrix G^{-1}, however, cannot have any zero row. Therefore, $y > 0$ yields $p > 0$. □

Let (A_*, h^*) be the solution of Problem II for $c = e$. According to Proposition 5, the scalar product $(h^*)^{\mathrm{T}} y$ gives an upper bound of the sum $\sum x_i$ for any $x \in X_+(A^{\mathrm{V}}, y)$. Let us call h^* the *upper-bounding vector* and denote it $\mathrm{ub}(A^{\mathrm{V}})$. If A^{V} is positively regular, $\mathrm{ub}(A^{\mathrm{V}})$ is determined uniquely due to Theorem 4.

Now, we can formulate an analogy of Theorem 8:

Theorem 11. *A^V is positively regular if and only if the following conditions hold*:

(i) $h^* = \mathrm{ub}(A^V)$ *and* $W = \underline{\mathrm{inv}}\, A^V$ *exist*;

(ii) *W is nonsingular*;

(iii)

$$\left(W^{-1}\right)^{\mathrm{T}} h^* > 0. \tag{34}$$

Proof. 'If' part. We have $WA \in \mathscr{Z}\ \forall A \in A^V$. In addition, $A^{\mathrm{T}}h^* \geqslant e$ implies $p^{\mathrm{T}}G \geqslant e$ for $p = (W^{-1})^{\mathrm{T}}h^*$ and $G \in WA^V$. Hence, WA^V is a regular vague M-matrix and, therefore, A^V is positively regular due to Theorem 8.

'Only if' part follows from Theorem 10. □

Thus, Procedure CPR1 can be modified as follows:

Procedure CPR2.

1° Find $h^* = \mathrm{ub}(A^V)$ and $W = \underline{\mathrm{inv}}\, A^V$ by using Algorithm SO.

2° Verify (34).

Proposition 9. *Let A_0^V, A_1^V be positively regular and let $A_1^V \subset A_0^V$. Then*

$$(W_0)^+ \subset (W_1)^+ \quad \textit{holds for } W_i = \underline{\mathrm{inv}}\, A_i^V,\ i = 0, 1.$$

Proof. Choose $b \in Y(A_0^V)$. Then $A^{-1}b > 0\ \forall A \in A_1^V \subset A_0^V$, which yields $(W_0)^+ = Y(A_0^V) \subset Y(A_1^V) = (W_1)^+$. □

Proposition 10. *Assume that*:

(i) $A_1^V \subset A_0^V$;

(ii) $(W_0)^+ \subset (W_1)^+$ *for* $W_i = \underline{\mathrm{inv}}\, A_i^V$, $i = 0, 1$;

(iii) $\exists h\colon\ h^{\mathrm{T}}A \geqslant e\ \forall A \in A_0^V$;

(iv) A_1^V *is positively regular.*

Then A_0^V is positively regular as well.

Proof. (a) First of all we show that W_0 is nonsingular. It follows from condition (ii) because $(W_1)^+$ is an n-dimensional pointed convex cone and, therefore, $\mathscr{L} \subset (W_1)^+$ cannot hold for any nontrivial linear subspace $\mathscr{L} \subset \mathbb{R}^n$.

(b) Since $W_0^{-1}e \in (W_1)^+$, there exists a $p > 0$ such that $y = W_1^{-1}p = W_0^{-1}e$. For an arbitrary $A \in A_1^V \subset A_0^V$ we have $A^{-1}W_0^{-1}e = A^{-1}W_1^{-1}y > 0$. In addition, $0 \notin A^j\ \forall j \in J$ due to condition (iii). Thus, A_0^V is positively regular according to Corollary 4. □

Consider a nonsingular matrix A_0 and a vague matrix D^V such that $0 \in D^V$. Further let

$$A^{\mathrm{V}}(t) = A_0 + tD^{\mathrm{V}} = \left\{A \mid A = A_0 + tD,\ D \subset D^{\mathrm{V}}\right\}, \quad t \geqslant 0, \tag{35}$$

and denote

$$\mathscr{T}\left(A_0, D^{\mathrm{V}}\right) = \left\{t \mid A^{\mathrm{V}}(t) \text{ is positively regular},\ t > 0\right\}. \tag{36}$$

Lemma 3. *$\mathscr{T}(A_0, D^{\mathrm{V}})$ is an open interval.*

Proof. Follows from Proposition 3. □

Definition 5. Let A_0 be nonsingular and $0 \in D^{\mathrm{V}}$. Then

$$r\left(A_0, D^{\mathrm{V}}\right) = \sup\left\{t \mid A^{\mathrm{V}}(t) \text{ is positively regular}\right\}$$

is called the *radius of positive regularity* of the matrix A_0 with respect to the vagueness—type D^{V}.

The radius of positive regularity is an analogy of the radius of nonsingularity introduced in [10].

Theorem 12. *Let*
(i) $W_* = \underline{\mathrm{inv}}\ A^{\mathrm{V}}(t_*)$ *exist for a* $t_* > 0$;
(ii) $0 \notin A^j\ \forall j \in J$.
Then the following implication holds:

$$t_* = r\left(A_0, D^{\mathrm{V}}\right) \quad \Rightarrow \quad W_* \textit{ is singular}. \tag{37}$$

Proof. Taking into account that $W_0 = A_0^{-1}$, we have $A_0^{-1} \geqslant 0\ \forall b \in (W)^+$. According to Proposition 9, the inclusion $(W_t)^+ \subset (W_0)^+$ holds for $W_t = \underline{\mathrm{inv}}\ A^{\mathrm{V}}(t), t < t_*$. If W_* were nonsingular, then $(W_*)^+ \subset (W_0)^+$ would hold as well and consequently A^{V} would be positively regular according to Theorem 8. In such a case, however, Lemma 3 would imply $t_* < r(A_0, D^{\mathrm{V}})$, which would contradict the premise of (37). Hence, W_* must be singular. □

Theorem 13. *Assume that*
(i) $h^* = \mathrm{ub}(A^{\mathrm{V}}(t_*))$ *exists*;
(ii) $W_* = \underline{\mathrm{inv}}\ A^{\mathrm{V}}(t_*)$ *is singular*;
(iii) *all the nondiagonal elements of $W_* A_0$ are negative.*
Then $t_* = r(A_0, D^{\mathrm{V}})$.

Proof. There exists a vector $d \neq 0$ such that $d^{\mathrm{T}} W_* = 0$. Let us assume without loss of generality that $d_1 > 0$ and form a matrix U_ε as follows:

$$U_\varepsilon = W_* + \varepsilon e^1 (h^*)^{\mathrm{T}}. \tag{38}$$

Then, for any $t < t^*$ there exists an $\varepsilon > 0$ such that

$$U_\varepsilon A \in \mathscr{Z} \quad \forall A \in A^{\mathrm{V}}(t). \tag{39}$$

In addition,

$$d^{\mathrm{T}} U_\varepsilon A = d^{\mathrm{T}} W_* A + \varepsilon d_1 (h^*)^{\mathrm{T}} A > 0 \quad \forall A \in A^{\mathrm{V}}(t). \tag{40}$$

Thus, $U_\varepsilon A^{\mathrm{V}}(t)$ is a regular vague M-matrix and hence $A^{\mathrm{V}}(t)$ is positively regular for $0 \leqslant t < t_*$ according to Proposition 8. Since $A^{\mathrm{V}}(t_*)$ is not positively regular due to Theorem 10, $t_* = r(A_0, D^{\mathrm{V}})$. □

Assumption (iii) of this theorem is satisfied, for example, if $0 \in \operatorname{int} D^{\mathrm{V}}$.

The set $Y_t = Y(A^{\mathrm{V}}(t))$ of all the right-hand sides y providing only nonnegative solutions of $A^{\mathrm{V}}(t)x = y$ can be characterized as follows:

1° Y_t is an n-dimensional convex cone for $0 \leqslant t < r(A_0, D^{\mathrm{V}})$, $Y_t = (W_t)^+$.
2° Y_t is a degenerated convex cone for $t = r(A_0, D^{\mathrm{V}})$, $Y_t \subset (W_t)^+$.
3° $Y_t = \emptyset$ for $t > r(A_0, D^{\mathrm{V}})$.

Example. Let us consider

$$A_0 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 1 & 1 & 1 \end{pmatrix}, \qquad D^{\mathrm{V}} = \begin{pmatrix} 0 & [-1, 1] & [-1, 1] \\ [-1, 1] & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$A^{\mathrm{V}}(t) = A_0 + tD^{\mathrm{V}}$ is an interval matrix for $t > 0$. Owing to a small number of alternatives, we can easily prove that $A^{\mathrm{V}}(t)$ is regular for $t < 2$. For $t^* = 1$ we obtain

$$W_* = \begin{pmatrix} -1 & 0 & 1 \\ 1/4 & -1/4 & 1/4 \\ 1/4 & 1/4 & -3/4 \end{pmatrix},$$

$$h^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$W_* A_0 = \begin{pmatrix} 1 & -1 & -1 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{pmatrix}.$$

Since $\det W_* = 0$, $r(A_0, D^{\mathrm{V}}) = 1$ due to Theorem 13.

The set of all the right-hand sides b providing nonnegative solutions of $A_*^{\mathrm{V}} x = b$ is a half line defined by the following conditions:

$$W_* b = 0, \qquad A_0^{-1} b \geqslant 0. \tag{41}$$

Thus, $Y(A_*^{\mathrm{V}}) = \{b = \lambda d \mid \lambda \geqslant 0\}$, where $d = (1, 2, 1)^{\mathrm{T}}$. Let us choose, for example, $b^0 = (5, 10, 5)^{\mathrm{T}} \in Y(A_*^{\mathrm{V}})$. The exact upper bound of the sum $f(x) = e^{\mathrm{T}} x = x_1 + x_2 + x_3$ on $X_+(A_*^{\mathrm{V}}, b^0)$ is equal to $\overline{f} = (h^*) b^0 = 5$. The exact component-wise lower bounds $\underline{x}_i$ of the same set, of course, are equal to 0, because $W_* b^0 = 0$.

In order to find the lower bound $\underline{f} = \min\{f(x) \mid x \in X_+(A_*^V, b^0)\}$, let us compute $g^* = -\text{ub}(-A_*^V) = (0, 0, 1)^T$. Hence, $\underline{f} = (g^*)^T b^0 = 5$. We can conclude that for any $t \in [0, 1)$, $A^V(t)$ is positively regular and

$$X\big(A^V(t), b^0\big) \subset X\big(A_*^V, b^0\big) \subset \big\{x \,\big|\, e^T x = 5,\ x \geqslant 0\big\}. \tag{42}$$

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