Vector-Valued Probability Measures on Semigroups

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The problem of defining vector-valued probability measures on a compact semitopological semigroup $S$ is considered in this paper. The set $PW(S, A)$ of all such $A$-valued measures, analogous to the scalar-valued measures, is a compact semitopological semigroup with respect to the convolution operation and the weak operator topology. Supports of idempotents and the limits of averaged convolution sequences in $PW(S, A)$ are determined.

1. INTRODUCTION

For a compact semitopological semigroup $S$, the set of scalar-valued probability measures (i.e., non-negative and normalized regular Borel measures) on $S$ is denoted by $P(S)$. Then $P(S)$ becomes a compact semitopological semigroup when endowed with the weak* topology and convolution multiplication. Idempotents in $P(S)$ are studied, for example, by Collins [8], Pym [14], and Choy [4].

Let $A$ be a Banach algebra with identity $e$ and with a closed cone $K$ which is closed under multiplication such that its norm is additive on $K$, i.e., $\|x + y\| = \|x\| + \|y\|$ if $x, y \in K$. The algebra of continuous functions from $S$ to $A$ endowed with the uniform norm is denoted by $C(S, A)$. If $A = \mathbb{C}$, the set of all complex numbers, we simply write $C(S)$.

Linear operators $T: C(S) \to A$ are said to be positive with respect to $K$ or simply positive, if $T(ff^*) \in K$ for every $f \in C(S)$, where $f^*$ is defined by $f^*(s) = (f(s))^*$ for every $s \in S$. Let $\mathcal{B}(S)$ be the $\sigma$-algebra of Borel subsets...
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of $S$ and let $\pi(S)$ be the Borel partition of $S$. If $T: C(S) \rightarrow A$ is weakly compact then its representing measure $m$ takes its value in $A$. The set of all such weakly compact measures $m: \mathcal{B}(S) \rightarrow A$ is denoted by $MW(S, A)$ and $m$ is a positive measure if $T$ is a positive operator. It is shown by Choy [7, Theorem 3.1] that if $m$ is a positive measure then, for $E \in \mathcal{B}(S)$,

$$S_c(m, E) = v(m)(E) = \| m(E) \| ;$$

where $v(m)$ is the variation of $m$ defined by

$$v(m)(E) = \sup \left\{ \sum_{i=1}^{n} \| m(e_i) \| : \{ e_i \} \in \pi(E) \right\}$$

and $S_c(m, E)$ is the semivariation of $m$ defined by

$$S_c(m, E) = \sup \left\{ \left\| \sum_{i=1}^{n} \alpha_i m(e_i) \right\| : \{ e_i \} \in \pi(E) \right\} \text{ and } \alpha_i \in \mathbb{C}, \quad | \alpha_i | \leq 1 \right\}.$$

Hence every positive measure $m$ is a dominated measure and $v(m)$ is a scalar measure associated with $m$ by White [16, Theorem 1.10]. Convolution $\mu * v$ of weakly compact measures $\mu, v: \mathcal{B}(S) \rightarrow A$ is defined by

$$\int f \, d\mu * v = \int d\mu(s) \int f(st) \, dv(t) \quad \quad (f \in C(S))$$

in [16].

Supports $\text{supp } \mu$ of weakly regular measures $\mu$ are defined in [3]. In Section 2 we define $A$-valued probability measures on $S$. The set of all such probability measures is denoted by $PW(S, A)$. We then show that $\text{supp } \mu$ of idempotents $\mu$ in $PW(S, A)$, as in the scalar case, are closed topologically simple subsemigroups of $S$ (i.e., $\text{supp } \mu = \bar{I}$ for any ideal $I$ of $\text{supp } \mu$, where the bar denotes closure). Convolution of measures in $MW(S, A)$ is clearly continuous in the $S_c(m, E)$-norm topology. However, it is frequently desirable to consider topologies weaker than the norm topology. In Section 3 we show that $PW(S, A)$ is a compact semitopological semigroup with respect to the convolution operation and the weak operator topology. In particular it is shown that the limit of an averaged convolution sequence in $PW(S, A)$ is an idempotent in $PW(S, A)$.

2. IDEMPOTENT PROBABILITY MEASURES

**Definition 2.1.** A measure $m: \mathcal{B}(S) \rightarrow A$ in $MW(S, A)$ is called a probability measure if $m$ is positive and $v(m)(S) = 1$. The set of all these probability measures is denoted by $PW(S, A)$. 
It is clear that if $A = \mathbb{C}$, then $MW(S, A) = M(S)$ (the algebra of all bounded regular Borel measures) and $PW(S, A) = P(S)$ [7]. If $m$ is a positive idempotent in the unit ball of $MW(S, A)$,

$$v(m)(S) = \|m(S)\| = \|m \ast m(S)\|$$

$$= \|(m(S))^2\| \leq \|m(S)\|^2$$

$$= (v(m)(S))^2$$

and so $v(m)(S) = 0$ or $v(m)(S) \geq 1$. That is, a non-zero positive idempotent in the unit ball of $MW(S, A)$ must be a probability measure.

**Theorem 2.2.** If $\mu, \nu$ are positive in $MW(S, A)$ then $v(\mu \ast \nu) = v(\mu) \ast v(\nu)$.

**Proof:** For $f \in C(S)$ the map $f_c : S \to A$ defined by

$$f_c(s) = \int f(st) f^*(st) \, dv(t) \quad (s \in S)$$

is continuous [16, Theorem 2.1] and clearly $f_c(s) \, d\mu(s) \in K$, since $\mu, \nu$ are positive [7, Remark 2.3]. It follows that $\mu \ast \nu$ is a positive measure in $MW(S, A)$. Then, since $\mu \ast \nu$ is dominated by $v(\mu) \ast v(\nu)$ (see the proof of [16, Theorem 4.2]) and $v(\mu \ast \nu)$ is a scalar measure associated with $\mu \ast \nu$ which is the least measure dominating $\mu \ast \nu$ [16, Theorem 1.10], we see $v(\mu \ast \nu) \leq v(\mu) \ast v(\nu)$.

On the other hand, since $v(\mu), v(\nu)$ are scalar measures associated with $\mu, \nu$, it is implied in [16, Theorem 2.2] that $v(\mu) \ast v(\nu)$ is a scalar measure associated with $\mu \ast \nu$. Therefore, for $E \in \mathcal{B}(S)$,

$$0 \leq v(\mu) \ast v(\nu)(E) \leq S_C(\mu, \nu, E) = v(\mu \ast \nu)(E).$$

We conclude that $v(\mu \ast \nu) = v(\mu) \ast v(\nu)$.

Before we come to the next corollary recall first that supports of weakly compact measures and their associated scalar measures are the same [6] and that the support of a scalar-valued idempotent probability measure is a closed topologically simple semigroup [14, Lemma 2(iii)].

**Corollary 2.3.** For any idempotent $\mu$ in $PW(S, A)$, $\text{supp } \mu$ is a closed topologically simple subsemigroup.

**Proof:** Since $\mu$ is an idempotent, $v(\mu) \ast v(\mu) = v(\mu \ast \mu) = v(\mu)$ and so $v(\mu)$ is a scalar-valued idempotent measure. Therefore, since $\text{supp } \mu = \text{supp } v(\mu)$, $\text{supp } \mu$ is a closed topologically simple subsemigroup.
Lemma 2.4. Let $\mu$ be an idempotent in $PW(S, A)$. Then, for $v \in PW(S, A)$ with $\text{supp} \ v \subseteq \text{supp} \ \mu$, $v(\mu * v * \mu) = v(\mu)$.

Proof. Let $f \in C_0^+ (S)$. Then, since $v(\mu)$ is an idempotent, the map $v(\mu)f_{v(\mu)}: S \to \mathbb{R}$ defined by $v(\mu)f_{v(\mu)}(x) = \int f(xy) \ dv(\mu)(x) \ dv(\mu)(z)$ is constant on $\text{supp} \ \mu$ [14, Lemma 2(ii)]. Let $v_{v(\mu)}(\text{supp} \ \mu) = \{ k \}$ (say). Then, by Theorem 2.2,

$$v(\mu * v * \mu)(f) = v(\mu) * v(v) * v(\mu)(f) = v(v)(v(\mu)f_{v(\mu)}) = v(v)(k) = k,$$

which is independent of $v$. Thus $v(\mu * v * \mu) = v(\mu * \mu * \mu) = v(\mu)$.

Theorem 2.5. If the support $\text{supp} \ \mu$ of a measure $\mu$ in $PW(S, A)$ is a subsemigroup and $v(\mu * v * \mu) = v(\mu)$ for every $v \in PW(S, A)$ with $\text{supp} \ v \subseteq \text{supp} \ \mu$, then $v(\mu)$ is an idempotent.

Proof. We note first, since the norm of $A$ is additive on $K$, that $v(\mu + v) = v(\mu) + v(v)$ and that $\|(\mu + \mu^2)/2(S)\| = 1$. Clearly $v(\mu^2) = v(\mu * \mu * \mu) = v(\mu)$ and so

$$v(\mu * \left(\frac{\mu + \mu^2}{2}\right) * \mu) = \frac{v(\mu) + v(\mu^2)}{2} = v\left(\frac{\mu + \mu^2}{2}\right).$$

Now, since $\text{supp} \ \mu$ is a subsemigroup, from [14, Lemma 1],

$$\text{supp} \left(\frac{\mu + \mu^2}{2}\right) \subseteq \text{supp} \ \mu \cup \text{supp} \ \mu = \text{supp} \ \mu.$$

Therefore,

$$v(\mu * \left(\frac{\mu + \mu^2}{2}\right) * \mu) = v(\mu).$$

Hence $v((\mu + \mu^2)/2) = v(\mu)$, and so $v(\mu^2) = v(\mu)$. We conclude, from Theorem 2.2, that $v(\mu)$ is an idempotent.

Examples 2.6. (i) Let $S$ be a locally compact semitopological semigroup and let $m$ be a positive measure in $M(S)$ such that $m * m \ll m$. Then $M(S)$ and the subspace $L^1(m)$ of measures in $M(S)$ absolutely continuous with respect to $m$ are Banach algebras (with convolution operation) such that their norms are additive on their usual positive cones $K$ [15, Theorem 2.3], which are clearly closed under convolution.

(ii) We show in this example a vector-valued probability measure. Let $A = l_1 = \{ \{ a_n \}_{n \geq 0}: a_n \in \mathbb{C}, \sum |a_n| < \infty \}$ with convolution. Then $l_1$ is a
Banach algebra with identity \( \{1, 0, 0, \ldots\} \). Let \( a = \{a_n\} \in l_1 \) where \( a_n = 6/(\pi n)^2 \). Then \( \|a\|_1 = 1 \). Let \( T: c \to l_1 \), where \( c \) is the algebra of all absolutely convergent sequences, be defined by

\[
T(x) = \sum a_n x_n e_n \quad (x = (x_n) \in c),
\]

where \( \{e_n\} \) is the canonical basis of \( l_1 \). Clearly \( T \) is a bounded linear operator. Actually \( \|T\| = 1 \) and \( T(xx^*) = \sum a_n x_n x_n^* e_n \geq 0 \); i.e., \( T \) is positive. The representing measure \( m \) of \( T \) is defined by \( m(E) = \sum_{n \in E} a_n e_n \) and \( m \) is a measure from the semigroup of all non-negative integers with discrete topology to \( l_1 \). Therefore \( T \) is weakly compact (see, for example, [13, p. 493] or [16, Theorem 1.41]) and \( m \) is an \( A \)-valued probability measure.

(iii) We show in this example that Corollary 2.3 is best possible in the sense that the support of an idempotent in \( PW(S, A) \) may not be simple.

Let \( A = \mathbb{C} \) and let \( I = [0, 1] \). Take \( S = I \times I \times I \) with the usual topology. Let \( f: I \times I \to I \) be defined by

\[
f(x, z) = 0, \quad \text{if } x = y = 0
\]

\[
= \frac{2xz}{x^2 + z^2}, \quad \text{otherwise}.
\]

Then \( f \) is separately continuous and \( S \) endowed with the multiplication

\[
(x, y, z)(x', y', z') = (x, f(x, z'), z')
\]

is a compact semitopological semigroup. It is known that the minimal ideal \( K(S) \) of the compact semitopological semigroup \( S \) always exists (see, for example, [2]). In fact

\[
K(S) = \{(x, f(x, z), z): x, z \in I\}.
\]

For \( a \in K(S) \), let \( \delta_a \) be the unit point mass at \( a \). Let \( \mu \) be the restriction to \( S \) of the Lebesgue measure on \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \). Then \( \mu \) is in \( PW(S, A) \) with \( \text{supp } \mu = S \). Now, since

\[
\delta_a \ast \mu \ast \delta_a(f) = \int f(ata) \, d\mu(t)
\]

\[
= \int f(a) \, d\mu(t) = f(a) = \delta_a(f)
\]

for all \( f \in C(S) \), we have \( \delta_a \ast \mu \ast \delta_a = \delta_a \). It follows that \( (\mu \ast \delta_a \ast \mu) \ast (\mu \ast \delta_a \ast \mu) = \)
\[ \mu \ast \delta_x \ast \mu \] and that \( v = \mu \ast \delta_x \ast \mu \) is an idempotent with \( \text{supp} \ v = \overline{S \delta S} = K(S) \), which is only topologically simple but not simple.

3. Convolution Semigroup of Probability Measures

Throughout this section we identify \( MW(S, A) \) as a subset of \( L[C(S), A] \) endowed with the weak operator topology. We shall show that convolution of probability measures is separately continuous.

**Lemma 3.1.** The set \( PW(S, A) \) of probability measures is closed in \( MW(S, A) \) endowed with weak operator topology.

**Proof.** We note first that, since \( PW(S, A) \) is convex in \( MW(S, A) \), the closures of \( PW(S, A) \) in the weak operator and the strong operator topologies are the same [13, Corollary VI.15]. Let \( m \in MW(S, A) \) be such that there are \( m_n \in PW(S, A) \) with \( m_n \to m \) in the strong operator topology. We identify \( m, m_n \) with their corresponding operators in \( L[C(S), A] \). Then \( m_n(f) \to m(f) \) for each \( f \in C(S) \). Therefore \( S_c(m, S) = 1 \). Furthermore, since \( m_n(ff^*) \to m(ff^*) \) for every \( f \in C(S) \) and \( K \) is closed, \( m_n \) is positive implies that \( m \) is positive.

**Theorem 3.2.** The set \( PW(S, A) \) is a compact semitopological semigroup endowed with the weak operator topology and with the convolution operation.

**Proof.** Let \( \mu, \nu \) be in \( PW(S, A) \) and let \( \mu_n \to \mu \) in the weak operator topology. Then for \( f \in C(S), x^* \in A^* \),

\[
 x^*(\mu_n(f)) - x^*(\mu(f))
\]

Clearly for \( x^* \in A^* \) and \( x \in A \), the mapping \( x^*, A \to C \) defined by \( x^*(y) = x^*(xy) \) is a bounded linear functional. Since \( \mu_n \to \mu \) in the weak operator topology,

\[
 x^*(\mu_n(f \cdot x)) = x^*(x\mu_n(f)) = x^*(\mu_n(f)) = x^*(\mu(f \cdot x)),
\]

for \( f \in C(S), x \in A \). Recall that the functions \( \sum_{i=1}^n f_i x_i \) with \( f_i \in C(S) \) and \( x_i \in A \) are dense in \( C(S, A) \) (see, for example, [11, p. 375]). We see

\[
 x^*(\mu_n \ast v(f)) = x^*(\mu_n(f_v)) - x^*(\mu(f_v)) = x^*(\mu \ast v(f))
\]

for \( f \in C(S) \). That is, convolution is separately continuous in \( PW(S, A) \).

Next we consider \( MW(S, A) \subseteq MW(S, A^{**}) \) and identify \( MW(S, A^{**}) \) as a subset in \( L[C(S, A^*), C] \) (see, for example, [3]). The rest of the proof follows immediately by noting that, since \( C \) is reflexive, the unit ball of \( L[C(S, A^*), C] \) is compact in the weak operator topology [13, p. 512].
Notation 3.3. For a subsemigroup $A$ of $PW(S, A)$, the closure of $\bigcup \{\text{supp } \mu : \mu \in A\}$ is denoted by $\text{supp } A$. If $A(\mu) = \{\mu^i : i = 1, 2, \ldots\}$ then $\text{supp } A = \Gamma(\text{supp } \mu)$, the closed semigroup generated by $\text{supp } \mu$.

The proof of the following limit theorem on averaged convolution sequences is similar to that of the scalar-valued case, and we include the proof here for completeness.

**Theorem 3.4.** For $\mu \in PW(S, A)$, let $v_n = (1/n) \sum_{i=1}^{n} \mu^i$. Then $\{v_n\}$ converges to an idempotent $L(\mu) \in PW(S, A)$. Moreover $L(\mu)\mu = \mu L(\mu) = L(\mu)$ and $\text{supp } L(\mu) = \overline{K}(\Gamma(\text{supp } \mu))$ where $\overline{K}(\Gamma(\text{supp } \mu))$ is the minimal ideal of $\Gamma(\text{supp } \mu)$.

**Proof.** Since $PW(S, A)$ is compact, the sequence $\{v_n\}$ has a cluster point $v$. Regarding $\mu v_n - v_n = (1/n)(\mu^{n+1} - \mu)$ as elements of the set $MW(S, A)$, we see, since for each $f \in C(S)$ and $x^* \in A^*$

$$\left| x^* \left[ \frac{1}{n} (\mu^{n+1}(f) - \mu(f)) \right] \right| \leq \frac{2\|x^*\|}{n},$$

that $(1/n)(\mu^{n+1} - \mu)$ converges to the zero measure in the weak operator topology. Thus $\mu v = v$ and similarly $v \mu = v$. It follows that $v_n v = v = vv_n$ for all $n$, whence $v^2 = v$. If there is any other cluster point $v'$ of $\{v_n\}$, we have $v v' = v' v = v$ and $v' v = vv' = v'$ implying $v = v'$. Hence $v_n \rightarrow v$ which we denote by $L(\mu)$, so that $L(\mu)\mu = \mu L(\mu) = L(\mu)$.

Next we note first that $\text{supp } \overline{CO} A(\mu) = \text{supp } CO A(\mu) = \Gamma(\text{supp } \mu)$, where $A(\mu) = \{\mu^n : n = 1, 2, \ldots\}$. In view of $L(\mu) \in \overline{CO} A(\mu)$, we see $\text{supp } L(\mu) \subseteq \Gamma(\text{supp } \mu)$. Since supports of measures in $PW(S, A)$ are equal to the supports of their associated scalar measures, we see from Theorem 2.2 and [14, Lemma 1] that

$$\text{supp } L(\mu)(\text{supp } \mu)^n = (\text{supp } \mu)^n \text{supp } L(\mu)$$

$$= \text{supp } L(\mu)$$

for all $n$. Therefore $\text{supp } L(\mu)$ is an ideal of $\Gamma(\text{supp } \mu)$ and so $\text{supp } L(\mu) = \overline{K}(\Gamma(\text{supp } \mu))$. From Corollary 2.3, $\text{supp } L(\mu) = \overline{K}(\Gamma(\text{supp } \mu))$ and this completes the proof of the theorem.

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