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An Eigenvalue Problem for a Quasilinear Elliptic Field Equation[1]

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1. INTRODUCTION

In this paper, we are concerned with an eigenvalue problem relative to a nonlinear Schrödinger equation:

$$i\,\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi + \varepsilon^r N(\psi), \tag{1}$$

where $N(\psi)$ is a nonlinear differential operator. The standing waves

$$\psi(x,t) = u(x)e^{-i\mu t}$$

of Eq. (1) are determined by the solutions of the following nonlinear eigenvalue problem:

$$-\Delta u + V(x)u + \varepsilon^r N(u) = \mu u \tag{2}$$

provided that

$$N(u(x)e^{-i\mu t}) = e^{-i\mu t}N(u(x)). \tag{3}$$

If ψ is a scalar function and $N(\psi) = f(|\psi|)\psi$ is a nonlinear function of ψ, Eq. (2) has been widely considered.

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We are particularly concerned with Eq. (2) when the nonlinear term $N(u)$ has a more complex structure, namely

$$N(u) = -\Delta_p u + W'(u), \tag{4}$$

where $W : \mathbb{R}^{n+1} \to \mathbb{R}$ is a nonlinear function having a singularity in the point $\xi_\bigstar$ and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1} \setminus \{\xi_\bigstar\}$. Note that operator (4) can be extended to the complex functions in such a way to satisfy (3). The motivation for considering an operator such as (4) needs some explanation.

In [3] (see also [4, 5]), the authors, motivated by a conjecture of Derrick [12], proved that the equation

$$-\Delta\varphi + \varepsilon^r N(\varphi) = 0, \tag{5}$$

where $\varphi : \mathbb{R}^3 \to \mathbb{R}^4$ and the nonlinear operator N is like (4), has a family $\{\varphi_q\}_{q\in\mathbb{Z}\setminus\{0\}}$ of nontrivial solutions with the energy concentrated around the origin in a region of radius infinitesimal with ε. These solutions are characterized by a topological invariant $\mathrm{ch}(\cdot)$, called topological charge, which takes integer values (see (11)). In fact, for every $q \in \mathbb{Z}\setminus\{0\}$, we have a solution φ_q with $\mathrm{ch}(\varphi_q) = q$.

The solutions of Eq. (5) allow one to construct particular solutions of Eq. (1) when $V(x)$ is constant: $V(x) = V_0 \in \mathbb{R}$. In this case Eq. (1) admits standing waves of the form

$$\psi_q(t, x) = \varphi_q(x)e^{-i\omega t},$$

where $\omega = V_0$, and travelling solitary waves of the form

$$\psi_q(t, x) = \varphi_q(x - 2kt)e^{i(k\cdot x - \omega t)},$$

where $\omega = V_0 + k^2$. Moreover in [1], the authors proved the orbital stability of these solutions (for suitable values of k) together with some of their dynamical properties.

The orbital stability of suitable solutions of (1) implies that this equation has solutions of the form

$$\psi(t, x) = \varphi(x - Q(t))e^{i(k\cdot x - \omega t)} + \psi_1(t, x), \tag{6}$$

where ψ_1 is small compared with $\varphi(x)$.

Solutions of this type can be considered as a combination of a wave and a "particle". The region $B_\varepsilon(Q(t))$ occupied by the particle is characterized by the fact that $\mathrm{ch}(\varphi, B_\varepsilon(Q(t))) \neq 0$. In this region, the energy is highly concentrated.

If we consider standing waves in a bounded domain Ω, we are interested in solutions of (1) of the form

$$\psi(t, x) = u(x)e^{-i\mu t}$$

and the "presence" of particles is guaranteed by the fact that $\mathrm{ch}(u, \Omega) \neq 0$. Thus, we are led to the following eigenvalue problem for any assigned topological charge $q \in \mathbb{Z} \setminus \{0\}$ (see (11)):

To find solutions $\mu \in \mathbb{R}$ and u with topological charge q of the field equation

$$\begin{cases} -\Delta u + V(x)u + \varepsilon^r(-\Delta_p u + W'(u)) = \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_\varepsilon$}$$

where ε is a positive parameter, Ω is a bounded smooth domain of $\mathbb{R}^n$ with $n \geqslant 3$ and $p, r \in \mathbb{N}$ with $p > n$ and $r > p - n$. Here $\Delta u = (\Delta u_1, \Delta u_2, \ldots, \Delta u_{n+1})$, with $u = (u_1, u_2, \ldots, u_{n+1})$ and Δ the classical Laplacian operator. Moreover, $\Delta_p u$ denotes the $(n+1)$-vector, whose ith component is given by

$$(\Delta_p u)_i = \nabla \cdot (|\nabla u_i|^{p-2} \nabla u_i).$$

Finally, V is a real function $V : \Omega \to \mathbb{R}$ and W' is the gradient of the function $W : \mathbb{R}^{n+1} \setminus \{\xi_\bigstar\} \to \mathbb{R}$, where $\xi_\bigstar$ is a point of $\mathbb{R}^{n+1}$ which for simplicity we choose on the $(n+1)$th component, namely

$$\xi_\bigstar = (0, \bar{\xi}), \tag{7}$$

with $0 \in \mathbb{R}^n$ and $\bar{\xi} \in \mathbb{R}$, $\bar{\xi} > 0$.

Throughout the paper, we always assume the following hypotheses:

- $V \in L^{n+1}(\Omega, \mathbb{R})$ and V is essentially bounded from below,
- $W \in C^1(\mathbb{R}^{n+1} \setminus \{\xi_\bigstar\}, \mathbb{R})$,
- $W(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}^{n+1} \setminus \{\xi_\bigstar\}$,
- there exist two constants $c_1, c_2 > 0$ such that

$$\xi \in \mathbb{R}^{n+1},\ 0 < |\xi| < c_1 \Rightarrow W(\xi_\bigstar + \xi) \geqslant \frac{c_2}{|\xi|^q},$$

where $q = \frac{np}{p-n}$.

We state the following existence result (see Theorem 4.1):

Given $q \in \mathbb{Z} \backslash \{0\}$ *and* $k \in \mathbb{N}$, *we consider* $\xi_{\bigstar} = (0, \bar{\xi})$ *with* $0 \in \mathbb{R}^n$ *and* $\bar{\xi}$ *large enough. Then for* ε *sufficiently small and for any* $j \leqslant k$ *with* $\tilde{\lambda}_{j-1} < \tilde{\lambda}_j$, *there exist* $\mu_j(\varepsilon)$ *and* $u_j(\varepsilon)$, *respectively, eigenvalue and eigenfunction of the problem* (P_ε), *such that the topological charge of* $u_j(\varepsilon)$ *is* q.

Here $\tilde{\lambda}_j$ (see Section 4.1) are the eigenvalues of the linear problem $-\Delta u + V(x)u = \tilde{\lambda} u$ with $u \in H_0^1(\Omega, \mathbb{R}^{n+1})$.

We look for critical values of the energy functional associated to (P_ε), that is

$$J_\varepsilon(u) = \int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \frac{1}{2} V(x)|u|^2 + \frac{\varepsilon^r}{p} |\nabla u|^p + \varepsilon^r W(u) \right] dx, \tag{8}$$

in the intersection of every connected component, characterized by the topological charge, with the unitary sphere in $L^2(\Omega, \mathbb{R}^{n+1})$. It is clear that the functional J_ε is not even. Technically, we are considering a perturbation of a symmetric problem and we want to preserve critical values. Namely, we prove that some critical values $\tilde{\lambda}_j$ of the functional

$$J_0(u) = \int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \tfrac{1}{2} V(x)|u|^2 \right] dx \qquad \text{with } u \in H_0^1(\Omega, \mathbb{R}^{n+1})$$

on the unitary sphere of $L^2(\Omega, \mathbb{R}^{n+1})$ are preserved for the perturbed functional J_ε.

Perturbations of symmetric problems have been studied by several authors. The first work of this kind seems to be [2]. It would be beyond our purpose to give a complete bibliography on the subject. We only cite [7, 8, 18]. The problem (P_ϵ) has been successively studied in [6] in the case $\Omega = \mathbb{R}^n$.

The paper is organized as follows:

– Section 2 is devoted to the description of the functional setting and of some topological devices.

– In Section 3, we prove the existence of minima (see Theorem 3.1) for the functional J_ε, defined in (8), in every component of the unitary sphere, characterized by the topological charge (see (9), (10), (12)). Thus, we state:

Given $q \in \mathbb{Z}$ *and* $\xi_{\bigstar} = (0, \bar{\xi})$ (*with* $0 \in \mathbb{R}^n$ *and* $\bar{\xi} > 0$), *for any* $\varepsilon > 0$ *there exist* $\mu_1(\varepsilon)$ *and* $u_1(\varepsilon)$, *respectively, eigenvalue and eigenfunction of the problem* (P_ε), *such that the topological charge of* $u_1(\varepsilon)$ *is* q.

— Finally, in Section 4 there are some arguments of eigenvalues theory and the proof of the main result of the paper by a variational approach. We build some suitable functions G_ε^q of topological charge q (see (28)) and some suitable manifolds $\mathscr{M}_{\varepsilon,j}^q$ (see (29)). Thus, we are able to find critical values $c_{\varepsilon,j}^q$ (see (30)) of the functional J_ε in every component of the unitary sphere in $L^2(\Omega, \mathbb{R}^{n+1})$, characterized by the topological charge (see (12)). These values $c_{\varepsilon,j}^q$ are critical values of "min–max type" and tend to the eigenvalues $\tilde{\lambda}_j$ when ε tends to zero.

NOTATION

We fix the following notations:

- $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$,
- if $\xi \in \mathbb{R}^{n+1}$ some times we will use the notation $\xi = (\tilde{\xi}, \bar{\xi})$, where $\tilde{\xi} \in \mathbb{R}^n$ and $\bar{\xi} \in \mathbb{R}$,
- if $x \in \mathbb{R}^n$ and $\rho > 0$, then $B(x, \rho)$ is the open ball with centre in x and radius ρ,
- B_n is the closed ball with centre 0 and radius 1 in $\mathbb{R}^n$,
- given a Banach space B, we denote by B^* the dual of B,
- if $a : B \times B \to \mathbb{R}$ is a continuous bilinear map, we put for all $u \in B$

$$a[u] = a(u, u),$$

- if J is a C^1-functional on B, we put $J^c = \{u \in B / J(u) \leqslant c\}$.

2. FUNCTIONAL SETTING

2.1. The Space H and the Open Set Λ

Let H denote the closure of $C_0^\infty(\Omega, \mathbb{R}^{n+1})$ with respect to the following norm:

$$\|u\|_H = \|\nabla u\|_{L^2(\Omega, \mathbb{R}^{n+1})} + \|\nabla u\|_{L^p(\Omega, \mathbb{R}^{n+1})},$$

where $p > n$.

The following remark summarizes the main properties of the Banach space H:

Remark 1. In the Banach space H the norm $\|\cdot\|_H$ and the usual norm of the Banach space $W_0^{1,p}(\Omega, \mathbb{R}^{n+1})$ are equivalent. By Sobolev embedding theorem, we get that H is continuously embedded in $C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^{n+1})$ with $0 \leqslant \alpha \leqslant 1 - \frac{n}{p}$. The embedding is compact if $\alpha < 1 - \frac{n}{p}$.

By S we denote the following submanifold of class C^2 of H:

$$S = \left\{ u \in H \Big/ \int_\Omega |u(x)|^2 \, dx = 1 \right\}. \tag{9}$$

In the space H, we consider the open subset

$$\Lambda = \{ u \in H / \xi_\star \notin u(\Omega) \}. \tag{10}$$

The energy functional

$$J_\varepsilon(u) = \int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \frac{1}{2} V(x)|u|^2 + \frac{\varepsilon^r}{p} |\nabla u|^p + \varepsilon^r W(u) \right] dx$$

is real valued on Λ and of class C^1.

It is obvious that if u is a critical point for the functional J_ε restricted on $\Lambda \cap S$, there exists $\mu \in \mathbb{R}$ such that for all $v \in H$

$$\int_\Omega (\nabla u \cdot \nabla v + V(x) u \cdot v + \varepsilon^r |\nabla u|^{p-2} \nabla u \cdot \nabla v + \varepsilon^r W'(u) \cdot v) \, dx = \mu \int_\Omega u \cdot v \, dx,$$

hence u is a weak solution of (P_ε).

2.2. Topological Charge and Connected Components of Λ

We recall now the definition of topological charge introduced by Benci *et al.* [5] (we report here the definition given in [3]).

We write the $n+1$ components of a function $u \in H$ in the following way:

$$u(x) = (\tilde{u}(x), \bar{u}(x)),$$

where $\tilde{u} : \Omega \to \mathbb{R}^n$ and $\bar{u} : \Omega \to \mathbb{R}$.

DEFINITION 1. Let u be a function in $\Lambda \subset H$, then the support of u is the following set:

$$K_u = \{ x \in \Omega / \bar{u}(x) > \bar{\xi} \},$$

where $\bar{\xi}$ is defined in (7). Then the topological charge of u is the following function:

$$\mathrm{ch}(u) = \begin{cases} \deg(\tilde{u}, K_u, 0) & \text{if } K_u \neq \emptyset, \\ 0 & \text{if } K_u = \emptyset. \end{cases} \tag{11}$$

We recall that, as a consequence of the fact that u is continuous (see Remark 1) and $u|_{\partial\Omega} = 0$, K_u is an open subset of Ω and more precisely $\overline{K_u} \subset \Omega$. Since $u \in \Lambda$, if $x \in \partial K_u$, we have $\bar{u}(x) = \bar{\xi}$ and $\tilde{u}(x) \neq 0$. Therefore, the previous definition is well posed.

Moreover, the topological charge is continuous with respect to the uniform convergence (see [5]):

LEMMA 2.1. *For every $u \in \Lambda$ there exists $r = r(u) > 0$ such that, for every $v \in \Lambda$*

$$\|v - u\|_{L^\infty(\Omega,\mathbb{R}^{n+1})} \leqslant r \quad \Rightarrow \quad \mathrm{ch}(u) = \mathrm{ch}(v).$$

The space $\Lambda \subset H$ is divided into connected components by the topological charge:

$$\Lambda = \bigcup_{q \in \mathbb{Z}} \Lambda_q,$$

where

$$\Lambda_q = \{u \in \Lambda / \mathrm{ch}(u) = q\}. \tag{12}$$

We define the following subset of Λ:

$$\Lambda^* = \bigcup_{q \in \mathbb{Z} \setminus \{0\}} \Lambda_q. \tag{13}$$

PROPOSITION 2.1. *For all $q \in \mathbb{Z}$, the connected component Λ_q is not empty.*

Proof. If $q = 0$, $u \equiv 0$ is in Λ_0. Then let q be different from zero. If ρ is a positive parameter, we consider two functions $\varphi_\rho, \psi_\rho : \mathbb{R}^+ \to [0,1]$ of class C^∞ and such that

$$\varphi_\rho(r) = \begin{cases} 1 & \text{for } 0 \leqslant r \leqslant \rho^2, \\ 0 & \text{for } r \geqslant 4\rho^2, \end{cases}$$

$$\psi_\rho(r) = \begin{cases} 1 & \text{for } 0 \leqslant r \leqslant 9\rho^2, \\ 0 & \text{for } r \geqslant 16\rho^2, \end{cases} \tag{14}$$

moreover, φ_ρ and ψ_ρ take values between 0 and 1 for $\rho^2 \leqslant r \leqslant 4\rho^2$ and $9\rho^2 \leqslant r \leqslant 16\rho^2$, respectively. Let U_ρ be the following function:

$$\begin{array}{llll} U_\rho: & B(0,4\rho) \subset \mathbb{R}^n & \to & (\mathbb{R}^n \times \mathbb{R}) \setminus \{\xi_\bigstar\}, \\ & x & \mapsto & \psi(|x|^2)(x, (\bar{\xi} + C)\varphi(|x|^2)), \end{array} \tag{15}$$

where C is a positive constant. Now we choose $|q|$ points $\hat{x}_i \in \Omega$ and $|q|$ radiuses ρ_i such that $B(\hat{x}_i, \rho_i) \subset \Omega$ $(i = 1, \ldots, |q|)$ and $B(\hat{x}_i, \rho_i) \cap B(\hat{x}_j, \rho_j) = \emptyset$ for all $i \neq j$. Then, we can define

$$U^q(x) = \begin{cases} U_{\rho_i}(\gamma_q(x - \hat{x}_i)) & \text{for all } x \in B(\hat{x}_i, 4\rho_i),\ i = 1, \ldots, |q|, \\ 0 & \text{for all } x \in \Omega \setminus \bigcup_{i=1}^{|q|} B(\hat{x}_i, 4\rho_i), \end{cases} \tag{16}$$

where γ_q is the following function:

$$\gamma_q(x_1, x_2, \ldots, x_n) = \begin{cases} (x_1, x_2, \ldots, x_n) & \text{for } q > 0, \\ (-x_1, x_2, \ldots, x_n) & \text{for } q < 0. \end{cases} \tag{17}$$

The function U^q is in $C_0^\infty(\Omega, \mathbb{R}^{n+1})$ and belongs to Λ_q. ∎

Remark 2. It is immediate from the construction of the functions U^q that their norm in $L^2(\Omega, \mathbb{R}^{n+1})$ can be as small as we need. Then, for $q \in \mathbb{Z} \setminus \{0\}$, we can consider $0 < \|U^q\|_{L^2(\Omega, \mathbb{R}^{n+1})} \leqslant 1$. Because of the form of the image of U^q, it is possible to expand it of a factor $t \geqslant 1$ to obtain a function with unitary L^2-norm, without reaching the point $\xi_\bigstar$: i.e. $tU^q \in \Lambda_q$ and, in particular, the function $\dfrac{U^q}{\|U^q\|_{L^2(\Omega, \mathbb{R}^{n+1})}}$ is in $\Lambda_q \cap S$. This means that $\Lambda_q \cap S$ is not empty for all $q \in \mathbb{Z}$ (it is obvious that $\Lambda_0 \cap S \neq \emptyset$).

3. EXISTENCE OF MINIMA IN THE COMPONENTS OF $\Lambda \cap S$

The following lemma describes the behaviour of the functional J_ε near the boundary of Λ (for the proof see [5]):

LEMMA 3.1. *Let $\{u_m\}_{m \in \mathbb{N}}$ be a sequence in Λ weakly converging in H to $u \in \partial\Lambda$. Then,*

$$\lim_{m \to \infty} \int_\Omega W(u_m)\, dx = +\infty.$$

We can now state the theorem of existence of minima in the components $\Lambda_q \cap S$ of $\Lambda \cap S$:

THEOREM 3.1. *For any $q \in \mathbb{Z}$ and for any $\varepsilon > 0$ there exists a minimum for the functional J_ε in $\Lambda_q \cap S$.*

Proof. Since V is bounded from below and W is positive, the functional J_ε is bounded from below on $\Lambda \cap S$. Moreover, by positiveness of W, J_ε is coercive on $\Lambda \cap S$, that is for every sequence $\{v_m\}_{m\in\mathbb{N}} \subset \Lambda \cap S$, if $\|v_m\|_H \to +\infty$, we get $J_\varepsilon(v_m) \to +\infty$. Therefore, fixed $q \in \mathbb{Z}$ and $\varepsilon > 0$, we recall that the set $\Lambda_q \cap S$ is not empty (Remark 2) and we consider a minimizing sequence $\{u_m\}_{m\in\mathbb{N}}$ for the functional J_ε on $\Lambda_q \cap S$. The sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in H and hence weakly converging in H to u up to a subsequence. Since the sequence $\{J_\varepsilon(u_m)\}_{m\in\mathbb{N}}$ is bounded, by Lemma 3.1 u does not belong to the boundary $\partial\Lambda$.

We verify that u is the required minimizer. As $\{u_m\}_{m\in\mathbb{N}}$ is weakly converging to u, by Remark 1 we know that $\{u_m\}_{m\in\mathbb{N}}$ is uniformly converging to u; then $\int_\Omega W(u_m)$ converges to $\int_\Omega W(u)$. Since the functional $J_\varepsilon(u) - \varepsilon^r \int_\Omega W(u)$ is convex and strongly continuous, we get that the functional J_ε is weakly lower semicontinuous. Therefore, u is the minimizer because, even if Λ_q is not weakly closed, u belongs to $\Lambda_q \cap S$ by Lemma 3.1. ∎

4. A MULTIPLICITY RESULT IN THE COMPONENTS OF $\Lambda^* \cap S$

4.1. Eigenvalues of the Schrödinger Operator

In the following, we will assume, without loss of generality, that $\operatorname{essinf}_{x\in\Omega} V(x) > 0$.

We denote by

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_m \leqslant \cdots,$$

the sequence of the eigenvalues of the problem

$$-\Delta z + V(x)z = \lambda z \qquad \text{with } z \in H_0^1(\Omega, \mathbb{R}) \tag{18}$$

and by $\{e_i\}_{i\in\mathbb{N}}$ the sequence of the associated eigenvectors with $(e_i, e_j)_{L^2(\Omega,\mathbb{R})} = \delta_{ij}$.

We consider now the sequence

$$\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \tilde{\lambda}_3 \leqslant \cdots \leqslant \tilde{\lambda}_m \leqslant \cdots$$

of the eigenvalues of the problem

$$-\Delta u + V(x)u = \tilde{\lambda} u \qquad \text{with } u \in H_0^1(\Omega, \mathbb{R}^{n+1}). \tag{19}$$

If $u = (u_1, u_2, \ldots, u_{n+1})$, then (19) is equivalent to

$$-\Delta u_i + V(x)u_i = \tilde{\lambda} u_i \qquad \text{with } i = 1, 2, \ldots, n+1.$$

It is trivial that $\lambda_1 = \tilde{\lambda}_1 = \tilde{\lambda}_2 = \cdots = \tilde{\lambda}_{n+1} < \tilde{\lambda}_{n+2}$, in fact if λ is an eigenvalue of multiplicity v of problem (18), then λ is an eigenvalue of (19) of multiplicity $(n+1)v$. Moreover if $\lambda_k < \lambda_{k+1}$, then $\tilde{\lambda}_{(n+1)k} < \tilde{\lambda}_{(n+1)k+1}$.

If we set $\tilde{e}_j = (e_j, 0, \ldots, 0), \tilde{e}_{j+1} = (0, e_j, \ldots, 0), \ldots, \tilde{e}_{j+n} = (0, 0, \ldots, e_j)$, it is clear what we mean by the sequence of the eigenvectors $\{\varphi_i\}_{i\in\mathbb{N}}$ corresponding to the sequence $\{\tilde{\lambda}_i\}_{i\in\mathbb{N}}$, which is an orthonormal set in $L^2(\Omega, \mathbb{R}^{n+1})$.

We introduce the following symmetric continuous bilinear maps:

$$a(w, z) = \int_\Omega \nabla w \cdot \nabla z \, dx + \int_\Omega V(x) wz \, dx \qquad \forall w, z \in H_0^1(\Omega, \mathbb{R}), \tag{20}$$

$$A(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega V(x) u \cdot v \, dx \qquad \forall u, v \in H_0^1(\Omega, \mathbb{R}^{n+1}). \tag{21}$$

The main properties of the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ and $\{\tilde{\lambda}_i\}_{i\in\mathbb{N}}$ are summarized in the following lemma (see [11, 14]):

LEMMA 4.1. *The following properties hold*:

$$\lambda_i = \min_{\substack{w \in H_0^1(\Omega,\mathbb{R}) \\ (w, e_j)_{L^2(\Omega,\mathbb{R})} = 0 \\ \forall j = 1, \ldots, i-1}} \frac{a[w]}{\|w\|^2_{L^2(\Omega,\mathbb{R})}},$$

$$\tilde{\lambda}_i = \min_{\substack{u \in H_0^1(\Omega,\mathbb{R}^{n+1}) \\ (u, \varphi_j)_{L^2(\Omega,\mathbb{R}^{n+1})} = 0 \\ \forall j = 1, \ldots, i-1}} \frac{A[u]}{\|u\|^2_{L^2(\Omega,\mathbb{R}^{n+1})}}$$

and

$$a(e_i, e_j) = \lambda_i \delta_{ij} \qquad \forall i, j \in \mathbb{N},$$

$$A(\varphi_i, \varphi_j) = \tilde{\lambda}_i \delta_{ij} \qquad \forall i, j \in \mathbb{N}.$$

If we set $E_m = \operatorname{span}[e_1, \ldots, e_m]$ *and*

$$E_m^\perp = \{w \in H_0^1(\Omega, \mathbb{R}) / (w, e_i)_{L^2(\Omega,\mathbb{R})} = 0 \quad \textit{for } \; i = 1, \ldots, m\},$$

we get

$$w \in E_m \;\; \Rightarrow \;\; \lambda_1 \leqslant \frac{a[w]}{\|w\|^2_{L^2(\Omega,\mathbb{R})}} \leqslant \lambda_m,$$

$$w \in E_m^\perp \;\; \Rightarrow \;\; \frac{a[w]}{\|w\|^2_{L^2(\Omega,\mathbb{R})}} \geqslant \lambda_{m+1}.$$

If we set, respectively, $F_m = \text{span}\,[\varphi_1, \dots, \varphi_m]$ *and* $F_m^{\perp} = \{u \in H_0^1(\Omega, \mathbb{R}^{n+1})/ (u, \varphi_i)_{L^2(\Omega,\mathbb{R}^{n+1})} = 0$ *for* $i = 1, \dots, m\}$, *we get*

$$u \in F_m \quad \Rightarrow \quad \tilde{\lambda}_1 \leqslant \frac{A[u]}{\|u\|^2_{L^2(\Omega,\mathbb{R}^{n+1})}} \leqslant \tilde{\lambda}_m, \tag{22}$$

$$u \in F_m^{\perp} \quad \Rightarrow \quad \frac{A[u]}{\|u\|^2_{L^2(\Omega,\mathbb{R}^{n+1})}} \geqslant \tilde{\lambda}_{m+1}. \tag{23}$$

The proof is a direct consequence of classical argumentations of spectral theory.

From the theorems of regularity, we get the following lemma:

LEMMA 4.2. *If* $u \in H_0^1(\Omega, \mathbb{R}^{n+1})$ *is a solution of*

$$-\Delta u + V(x)u = \lambda u$$

with $\lambda \in \mathbb{R}$, *then* $u \in H$.

Proof. Using the regularity result of Agmon–Douglis–Nirenberg (see for example [9]) and the assumption that $V \in L^{n+1}(\Omega, \mathbb{R})$, by a bootstrap argument it follows that $u \in W^{1,s}(\Omega, \mathbb{R}^{n+1})$ for any $s \in \mathbb{N}$ and hence we obtain the claim. ∎

4.2. *The Functions* G_ε^q

Fixed an integer $k \in \mathbb{N}$, we define

$$M_k = \sup_{u \in S(k)} \|u\|_{L^\infty(\Omega,\mathbb{R}^{n+1})}, \tag{24}$$

where for any $m \in \mathbb{N}$ $S(m)$ is the following subset of H:

$$S(m) = F_m \cap S. \tag{25}$$

Then we choose the $(n+1)$th coordinate $\bar{\xi}$ of the point $\xi_\bigstar$ defined in (7) in such a way that

$$\bar{\xi} > 2M_k. \tag{26}$$

We can now introduce for any $q \in \mathbb{Z}\setminus\{0\}$ the functions G_ε^q similar to the functions U^q introduced in (16), but with some more properties. Like in the previous case, we construct a function G_ρ in the following way:

$$\begin{aligned} G_\rho : \quad B(0,4\rho) \subset \mathbb{R}^n \quad &\to \qquad (\mathbb{R}^n \times \mathbb{R})\setminus\{\xi_\bigstar\}, \\ x \qquad\quad &\mapsto \quad \psi_\rho(|x|^2)\left(\frac{\bar{\xi}}{\rho}\,x, 2\bar{\xi}\varphi_\rho(|x|^2)\right), \end{aligned} \tag{27}$$

where φ_ρ and ψ_ρ are the functions defined in (14). It is important to observe that the distance of the image of G_ρ from the point $\xi_\bigstar$ is $\bar{\xi}$.

Definition 2. If $q \in \mathbb{Z}\setminus\{0\}$ and $0<\varepsilon\leqslant 1$, we set

$$G_\varepsilon^q(x) = \begin{cases} G_{\rho_i}\left(\dfrac{\gamma_q(x-\hat{x}_i)}{\varepsilon}\right) & \text{for } x \in B(\hat{x}_i, 5\varepsilon\rho_i) \text{ and } i = 1,\dots,|q|, \\ 0 & \text{for } x \in \Omega \Big\backslash \bigcup_{i=1}^{|q|} B(\hat{x}_i, 5\varepsilon\rho_i), \end{cases} \tag{28}$$

where G_ρ is defined in (27), γ_q in (17) and the points $\hat{x}_i$ and the radiuses ρ_i are chosen in such a way that

1. $B(\hat{x}_i, \rho_i) \subset \Omega$ $(i = 1,\dots,|q|)$,
2. $B(\hat{x}_i, \rho_i) \cap B(\hat{x}_j, \rho_j) = \emptyset$ for all $i \neq j$, $i, j = 1,\dots,|q|$,
3. $\|G_1^q\|_{L^2(\Omega,\mathbb{R}^{n+1})} < 1$ (see Remark 2).

Finally, we define $G^q = G_1^q$.

Remark 3. We note that by construction the image of G_ε^q does not intersect the point $\xi_\bigstar$ and the distance of the image from the point is $\bar{\xi}$. Moreover, even if we expand the functions G_ε^q $(0<\varepsilon\leqslant 1)$ of a factor $t\geqslant 1$, their image is such that they do not meet the point $\xi_\bigstar$ and the distance is still $\bar{\xi}$. Hence $tG_\varepsilon^q \in \Lambda_q$ for all $t\geqslant 1$ and $\varepsilon \in (0,1]$.

The following lemma presents some useful properties of the functions G_ε^q which will be crucial in the sequel:

Lemma 4.3. *There exist* $\hat{\rho} > 0$ *and* $\bar{\varepsilon}$, *with* $0<\bar{\varepsilon}\leqslant 1$, *such that for all* $0<\varepsilon\leqslant\bar{\varepsilon}$ *we have*

(i) $\|G_\varepsilon^q + \hat{\rho}u\|_{L^2(\Omega,\mathbb{R}^{n+1})} \leqslant 1$ *for all* $u \in S(k)$,

(ii) $\inf_{\varepsilon\in(0,\bar{\varepsilon}],\, u\in S(k)} \|G_\varepsilon^q + \hat{\rho}u\|_{L^2(\Omega,\mathbb{R}^{n+1})} > 0$,

(iii) $\inf_{x\in\Omega,\ \varepsilon\in(0,\bar{\varepsilon}],\ u\in S(k)} \left| \dfrac{G_\varepsilon^q(x) + \hat{\rho}u(x)}{\|G_\varepsilon^q + \hat{\rho}u\|_{L^2(\Omega,\mathbb{R}^{n+1})}} - \xi_\bigstar \right| > \dfrac{\bar{\xi}}{2}$,

(iv) $\dfrac{G_\varepsilon^q + \hat{\rho}u}{\|G_\varepsilon^q + \hat{\rho}u\|_{L^2(\Omega,\mathbb{R}^{n+1})}} \in \Lambda_q \cap S$ *for all* $u \in S(k)$.

Proof. (i) For any $\rho > 0$ and $0 < \varepsilon \leqslant 1$ we have $\|G_\varepsilon^q + \rho u\|_{L^2(\Omega,\mathbb{R}^{n+1})} \leqslant \varepsilon^{\frac{n}{2}} \|G^q\|_{L^2(\Omega,\mathbb{R}^{n+1})} + \rho$ and, by 3 in Definition 2, there exists $\hat{\rho} > 0$ such that $\|G^q\|_{L^2(\Omega,\mathbb{R}^{n+1})} + \hat{\rho} \leqslant 1$.

(ii) As $\|G_\varepsilon^q + \hat{\rho} u\|_{L^2(\Omega,\mathbb{R}^{n+1})} \geqslant \hat{\rho} - \|G_\varepsilon^q\|_{L^2(\Omega,\mathbb{R}^{n+1})}$, if ε is small enough, we get $\|G_\varepsilon^q + \hat{\rho} u\|_{L^2(\Omega,\mathbb{R}^{n+1})} > 0$.

(iii) By Remark 3 we deduce that for all $u \in S(k)$

$$\inf_{x\in\Omega,\ \varepsilon\in(0,1]} \left| \frac{G_\varepsilon^q(x)}{\|G_\varepsilon^q + \hat{\rho} u\|_{L^2(\Omega,\mathbb{R}^{n+1})}} - \xi_\star \right| = \bar{\xi}.$$

To get (iii) it is sufficient to prove that there exists $\bar{\varepsilon} \in (0,1]$ such that for all $\varepsilon \leqslant \bar{\varepsilon}$

$$\sup_{u\in S(k)} \frac{\hat{\rho}\|u\|_{L^\infty(\Omega,\mathbb{R}^{n+1})}}{\|G_\varepsilon^q + \hat{\rho} u\|_{L^2(\Omega,\mathbb{R}^{n+1})}} < \frac{\bar{\xi}}{2}.$$

We observe that

$$\sup_{u\in S(k)} \frac{\hat{\rho}\|u\|_{L^\infty(\Omega,\mathbb{R}^{n+1})}}{\|G_\varepsilon^q + \hat{\rho} u\|_{L^2(\Omega,\mathbb{R}^{n+1})}} \leqslant \frac{\hat{\rho} M_k}{\inf_{u\in S(k)} \|G_\varepsilon^q + \hat{\rho} u\|_{L^2(\Omega,\mathbb{R}^{n+1})}}$$

$$\leqslant \frac{M_k}{1 - \frac{\varepsilon^{\frac{n}{2}}}{\hat{\rho}} \|G^q\|_{L^2(\Omega,\mathbb{R}^{n+1})}}.$$

Since $M_k < \frac{\bar{\xi}}{2}$, for ε sufficiently small we have (iii).

(iv) It follows immediately from (iii). ∎

4.3. *The Values* $c_{\varepsilon,j}^q$

Now we can introduce some definitions which we will use to study multiplicity of solutions.

DEFINITION 3. Fixed $k \in \mathbb{N}$, $q \in \mathbb{Z}\setminus\{0\}$ and $0 < \varepsilon \leqslant \bar{\varepsilon}$, where $\bar{\varepsilon}$ is defined in Lemma 4.3, we set

$$\mathscr{M}_{\varepsilon,j}^q = \left\{ \frac{G_\varepsilon^q + \hat{\rho} u}{\|G_\varepsilon^q + \hat{\rho} u\|_{L^2(\Omega,\mathbb{R}^{n+1})}} \Big/ u \in S(j) \right\} \tag{29}$$

with $j \leqslant k$ and $\hat{\rho}$ defined in Lemma 4.3.

Remark 4. It is trivial that for $j \leqslant k$ we have $\mathscr{M}_{\varepsilon,j-1}^q \subset \mathscr{M}_{\varepsilon,j}^q$, where $\mathscr{M}_{\varepsilon,0}^q = \emptyset$. By Lemma 4.3, we can claim that $\mathscr{M}_{\varepsilon,j}^q \subset \Lambda_q \cap S$. Moreover, $\mathscr{M}_{\varepsilon,j}^q$ is a submanifold of $\Lambda_q \cap S$ for ε sufficiently small.

DEFINITION 4. Fixed $k \in \mathbb{N}$, for all $q \in \mathbb{Z}\setminus\{0\}$, $j \leqslant k$ and $0 < \varepsilon \leqslant \bar{\varepsilon}$ ($\bar{\varepsilon}$ is defined in Lemma 4.3), we introduce the following values:

$$c^q_{\varepsilon, j} = \inf_{h \in \mathscr{H}^q_{\varepsilon, j}} \sup_{v \in \mathscr{M}^q_{\varepsilon, j}} J_\varepsilon(h(v)), \tag{30}$$

where $\mathscr{H}^q_{\varepsilon, j}$ are the following sets of continuous transformations:

$$\mathscr{H}^q_{\varepsilon, j} = \left\{ h\colon\ \Lambda_q \cap S \to \Lambda_q \cap S / h \text{ continuous},\ h|_{\mathscr{M}^q_{\varepsilon, j-1}} = \mathrm{id}_{\mathscr{M}^q_{\varepsilon, j-1}} \right\}.$$

We observe that $\mathscr{H}^q_{\varepsilon, j+1} \subset \mathscr{H}^q_{\varepsilon, j}$.

LEMMA 4.4. *Fixed $k \in \mathbb{N}$, for all $q \in \mathbb{Z}\setminus\{0\}$, $j < k$ and $0 < \varepsilon \leqslant \bar{\varepsilon}$, we have*
(i) $c^q_{\varepsilon, j} \leqslant c^q_{\varepsilon, j+1}$,
(ii) $c^q_{\varepsilon, j} \in \mathbb{R}$.

Proof. (i) It is immediate from the fact that $\mathscr{H}^q_{\varepsilon, j+1} \subset \mathscr{H}^q_{\varepsilon, j}$.

(ii) Since V is bounded from below and W is positive, we know that the functional J_ε restricted to $\Lambda_q \cap S$ is bounded from below: then $c^q_{\varepsilon, j} > -\infty$. Let us suppose that $c^q_{\varepsilon, j} = +\infty$, then $\sup_{v \in \mathscr{M}^q_{\varepsilon, j}} J_\varepsilon(v) = +\infty$. This is a contradiction, as by Definition 3 $\mathscr{M}^q_{\varepsilon, j}$ is a compact set. ∎

4.4. Main Theorem

To get some critical points of the functional J_ε on the C^2 manifold $\Lambda \cap S$, we use the following version of Palais–Smale condition. For $J_\varepsilon \in C^1(\Lambda, \mathbb{R})$, the norm of the derivative at $u \in S$ of the restriction $\hat{J}_\varepsilon = J_\varepsilon|_{\Lambda \cap S}$ is defined by

$$\|\hat{J}'_\varepsilon(u)\|_\bigstar = \min_{t \in \mathbb{R}} \|J'_\varepsilon(u) - t g'(u)\|_{H^*},$$

where $g : H \to \mathbb{R}$ is the function defined by $g(u) = \int_\Omega |u|^2\, dx$.

DEFINITION 5. The functional J_ε is said to satisfy the Palais–Smale condition in $c \in \mathbb{R}$ on $\Lambda \cap S$ (on $\Lambda_q \cap S$, for $q \in \mathbb{Z}$) if for any sequence $\{u_m\}_{m \in \mathbb{N}} \subset \Lambda \cap S$ ($\{u_m\}_{m \in \mathbb{N}} \subset \Lambda_q \cap S$) such that $J_\varepsilon(u_m) \to c$ and $\|\hat{J}'_\varepsilon(u_m)\|_\bigstar \to 0$, there exists a subsequence which converges to $u \in \Lambda \cap S$ ($u \in \Lambda_q \cap S$).

LEMMA 4.5. *The functional J_ε satisfies the Palais–Smale condition on $\Lambda \cap S$ (on $\Lambda_q \cap S$ for $q \in \mathbb{Z}$) for any $c \in \mathbb{R}$ and $0 < \varepsilon \leqslant 1$.*

Proof. It is immediate that every Palais–Smale sequence $\{u_m\}_{m \in \mathbb{N}}$ on $\Lambda \cap S$ is bounded in H. Hence, we can choose a subsequence, which for simplicity we denote again $\{u_m\}_{m \in \mathbb{N}}$, converging to a function u weakly in H and strongly in $C^0(\bar{\Omega}, \mathbb{R}^{n+1})$. As we have

$$\min_{t \in \mathbb{R}} \|J'_\varepsilon(u_m) - t g'(u_m)\|_{H^*} \to 0,$$

there is a sequence $\eta_m > 0$, with $\eta_m \to 0$ for $m \to \infty$ and a sequence $t_m \in \mathbb{R}$ such that for all $v \in H$

$$\left| \int_\Omega [\nabla u_m \cdot \nabla v + V(x) u_m \cdot v + \varepsilon^r |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla v + \varepsilon^r W'(u_m) \cdot v]\, dx \right.$$

$$\left. - 2t_m \int_\Omega u_m \cdot v\, dx \right| \leqslant \eta_m \|v\|_H. \tag{31}$$

From the substitution $v = u_m$ in (31), we obtain

$$\left| \int_\Omega [|\nabla u_m|^2 + V(x)|u_m|^2 + \varepsilon^r |\nabla u_m|^p + \varepsilon^r W'(u_m) \cdot u_m]\, dx - 2t_m \right| \leqslant \eta_m \|u_m\|_H.$$

Hence, t_m is bounded.

Substituting now $v = u_m - u$, we get

$$\left| \int_\Omega [\nabla u_m \cdot \nabla (u_m - u) + V(x) u_m \cdot (u_m - u) + \varepsilon^r |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla (u_m - u) \right.$$

$$\left. + \varepsilon^r W'(u_m) \cdot (u_m - u)]\, dx - 2t_m \int_\Omega u_m \cdot (u_m - u)\, dx \right| \leqslant \eta_m \|u_m - u\|_H.$$

Since u_m converges to u in $C^0(\bar{\Omega}, \mathbb{R}^{n+1})$, we get

$$\int_\Omega \nabla u_m \cdot \nabla (u_m - u)\, dx + \varepsilon^r \int_\Omega |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla (u_m - u)\, dx$$

$$= - \int_\Omega V(x)\, u_m \cdot (u_m - u)\, dx - \varepsilon^r \int_\Omega W'(u_m) \cdot (u_m - u)\, dx$$

$$+ 2t_m \int_\Omega u_m \cdot (u_m - u)\, dx + \tilde{\eta}_m \|u_m - u\|_H = o(1).$$

We have

$$\limsup_{m \to \infty} \langle \Delta_p u_m, u_m - u \rangle = \limsup_{m \to \infty} \int_\Omega |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla (u_m - u)\, dx$$

$$= \frac{1}{\varepsilon^r} \limsup_{m \to \infty} \left[- \int_\Omega \nabla u_m \cdot \nabla (u_m - u)\, dx + o(1) \right]$$

$$\leqslant -\frac{1}{\varepsilon^r} \liminf_{m\to\infty} \int_\Omega \nabla u_m \cdot \nabla(u_m - u)\, dx$$

$$= -\frac{1}{\varepsilon^r} \liminf_{m\to\infty} \left(\|u_m\|^2_{H^1_0(\Omega,\mathbb{R}^{n+1})} - \|u\|^2_{H^1_0(\Omega,\mathbb{R}^{n+1})} \right)$$

$$\leqslant 0.$$

Now as

$$\limsup_{m\to\infty} \langle \Delta_p u_m - \Delta_p u, u_m - u \rangle \leqslant 0,$$

by the (S_+)-property of the p-Laplacian (see [10, 15]) the Palais–Smale sequence u_m converges strongly to u in H. Therefore, we get $J_\varepsilon(u) = c$ and $u \in S$. Concluding $u \in \Lambda$, because by Lemma 3.1. if $u \in \partial\Lambda$ then $J_\varepsilon(u_m) \to +\infty$ and this is a contradiction. Moreover, if $\{u_m\}_{m\in\mathbb{N}} \subset \Lambda_q$, since u_m converges to $u \in \Lambda$ in $C^0(\bar{\Omega}, \mathbb{R}^{n+1})$ and the topological charge is continuous with respect to the uniform convergence, then $u \in \Lambda_q \cap S$. ∎

In the following, we will use the version of the deformation lemma on a C^2 manifold which we now recall (see for example [13, 16, 17]).

LEMMA 4.6 (Deformation Lemma). *Let J be a C^1-functional defined on a C^2-Finsler manifold M. Let c be a regular value for J. We assume that*

(i) *J satisfies the Palais–Smale condition in c on M,*
(ii) *there exists $k > 0$ such that the sublevel J^{c+k} is complete.*

Then there exist $\delta > 0$ and a deformation $\eta : [0,1] \times M \to M$ such that

$$\eta(0,u) = u \qquad \forall u \in M,$$

$$\eta(t,u) = u \qquad \forall t \in [0,1],\ \forall u \in J^{c-2\delta},$$

$$\eta(1, J^{c+\delta}) \subset J^{c-\delta}.$$

LEMMA 4.7. *For any $q \in \mathbb{Z}$, $\varepsilon \in (0,1]$ and $a \in \mathbb{R}$, the subset $\Lambda_q \cap S \cap J^a_\varepsilon$ of the Banach space H is complete.*

Proof. It is sufficient to observe that if $\{u_m\}_{m\in\mathbb{N}} \subset \Lambda_q \cap S \cap J^a_\varepsilon$ converges in H to u, then by Lemma 3.1 $u \notin \partial\Lambda_q$ (because $J_\varepsilon(u_m) \leqslant a$ for all $m \in \mathbb{N}$). Now by the continuity of the functional J_ε we have that $J_\varepsilon(u) \leqslant a$. ∎

We can now prove the main result:

THEOREM 4.1. *Given* $q \in \mathbb{Z}\setminus\{0\}$ *and* $k \in \mathbb{N}$, *we consider* $\xi_\star = (0, \bar{\xi})$ *with* $0 \in \mathbb{R}^n$, $\bar{\xi} > 2M_k$, *where* M_k *is defined in* (24).

Then there exists $\hat{\varepsilon} \in (0, 1]$ *such that for any* $\varepsilon \in (0, \hat{\varepsilon}]$ *and for any* $j \leqslant k$ *with* $\tilde{\lambda}_{j-1} < \tilde{\lambda}_j$, *we get that* $c^q_{\varepsilon,j}$ *is a critical value for the functional* J_ε *restricted to the manifold* $\Lambda_q \cap S$. *Moreover*, $c^q_{\varepsilon,j-1} < c^q_{\varepsilon,j}$.

Proof. We begin with some notations: if $u \in H$ we set

$$P_{F_j}u = \sum_{i=1}^{j} A(u, \varphi_i)\varphi_i \qquad \text{and} \qquad Q_{F_j}u = u - P_{F_j}u. \tag{32}$$

It is immediate that

$$A(Q_{F_j}u, \varphi_i) = \tilde{\lambda}_i (Q_{F_j}u, \varphi_i)_{L^2(\Omega,\mathbb{R}^{n+1})} = 0 \qquad \forall i = 1, \ldots, j. \tag{33}$$

In the following proof we will denote $\|\cdot\|_{L^q}$ the norm in $L^q(\Omega, \mathbb{R}^{n+1})$ and $\|\cdot\|_{W_0^{1,q}}$ the norm in $W_0^{1,q}(\Omega, \mathbb{R}^{n+1})$.

We divide the argument into five steps.

Step 1: *For any* $h \in \mathscr{H}^q_{\varepsilon,j}$, *the intersection of the set* $h(\mathscr{M}^q_{\varepsilon,j})$ *with the set* $\{u \in H / A(u, \varphi_i) = 0 \ \forall i = 1, \ldots, j-1\}$ *is not empty*: *in fact*, *there exists* $v \in \mathscr{M}^q_{\varepsilon,j}$ *such that* $P_{F_{j-1}}h(v) = 0$.

To prove this we will use the Brouwer degree of a continuous function K_h in the point 0.

If $d = (d_1, d_2, \ldots, d_{j-1}) \in B_{j-1}$, we consider the continuous map $w = w^q_{\varepsilon,j-1} : B_{j-1} \to H$ defined by

$$w(d) = G^q_\varepsilon + \hat{\rho} \sum_{i=1}^{j} d_i \varphi_i,$$

where $d_j = \sqrt{1 - \sum_{i=1}^{j-1} d_i^2}$ and $\hat{\rho}$ is defined in Lemma 4.3. By definition $\frac{w(d)}{\|w(d)\|_{L^2}} \in \mathscr{M}^q_{\varepsilon,j}$. Then for all $h \in \mathscr{H}^q_{\varepsilon,j}$, we can define the continuous map $K_h = K^q_{\varepsilon,j-1}(h) : B_{j-1} \to F_{j-1}$ such that

$$K_h(d) = \|w(d)\|_{L^2}\, P_{F_{j-1}} h\left(\frac{w(d)}{\|w(d)\|_{L^2}}\right).$$

Now to calculate the degree of the map K_h in the point 0, we construct a homotopy with the identity map $I : B_{j-1} \to F_{j-1}$ defined by $I(d) = \hat{\rho} \sum_{i=1}^{j-1} d_i \varphi_i$. Obviously, the homotopy is $tK_h + (1-t)I$ with $t \in [0, 1]$. If

$d \in \partial B_{j-1}$, then $\frac{w(d)}{\|w(d)\|_{L^2}} \in \mathcal{M}^q_{\varepsilon, j-1}$ and we have

$$tK_h(d) + (1-t)I(d) = t\, P_{F_{j-1}}\, G^q_\varepsilon + \hat{\rho} \sum_{i=1}^{j-1} d_i \varphi_i.$$

Since $\|P_{F_{j-1}}\, G^q_\varepsilon\|_{L^2} \leqslant \|G^q_\varepsilon\|_{L^2} = \varepsilon^{\frac{n}{2}} \|G^q\|_{L^2}$, for ε small enough $tK_h(d) + (1-t)I(d) \neq 0$ for $t \in [0,1]$ and $d \in \partial B_{j-1}$. Concluding

$$\deg(K_h, 0, B_{j-1}) = \deg(I, 0, B_{j-1}) = 1.$$

Then, we have the claim.

Step 2: *We prove that*

$$\sup_{v \in \mathcal{M}^q_{\varepsilon, j}} J_\varepsilon(v) \leqslant \tilde{\lambda}_j + \sigma(\varepsilon), \tag{34}$$

$$c^q_{\varepsilon, j} \leqslant \tilde{\lambda}_j + \sigma(\varepsilon), \tag{35}$$

where $\lim_{\varepsilon \to 0} \sigma(\varepsilon) = 0$.

First of all we verify that

$$\sup_{v \in \mathcal{M}^q_{\varepsilon, j}} J_0(v) \leqslant \tilde{\lambda}_j + \sup_{u \in S(j)} \frac{A[Q_{F_j} G^q_\varepsilon]}{\|P_{F_j} G^q_\varepsilon + \hat{\rho} u\|^2_{L^2} + \|Q_{F_j} G^q_\varepsilon\|^2_{L^2}}. \tag{36}$$

In fact by (21), Definition 3, (32) and (33) we have

$$\begin{aligned}
\sup_{v \in \mathcal{M}^q_{\varepsilon, j}} J_0(v) &= \sup_{u \in S(j)} A\left[\frac{G^q_\varepsilon + \hat{\rho} u}{\|G^q_\varepsilon + \hat{\rho} u\|_{L^2}}\right] \\
&= \sup_{u \in S(j)} \frac{A[P_{F_j} G^q_\varepsilon + \hat{\rho} u] + A[Q_{F_j} G^q_\varepsilon]}{\|P_{F_j} G^q_\varepsilon + \hat{\rho} u\|^2_{L^2} + \|Q_{F_j} G^q_\varepsilon\|^2_{L^2}} \\
&\leqslant \sup_{u \in S(j)} \left(\frac{A[P_{F_j} G^q_\varepsilon + \hat{\rho} u]}{\|P_{F_j} G^q_\varepsilon + \hat{\rho} u\|^2_{L^2}} + \frac{A[Q_{F_j} G^q_\varepsilon]}{\|P_{F_j} G^q_\varepsilon + \hat{\rho} u\|^2_{L^2} + \|Q_{F_j} G^q_\varepsilon\|^2_{L^2}} \right) \\
&\leqslant \tilde{\lambda}_j + \sup_{u \in S(j)} \frac{A[Q_{F_j} G^q_\varepsilon]}{\|P_{F_j} G^q_\varepsilon + \hat{\rho} u\|^2_{L^2} + \|Q_{F_j} G^q_\varepsilon\|^2_{L^2}}.
\end{aligned}$$

Now by definition of J_ε and by (36), we prove the following inequalities:

$$
\begin{aligned}
c^q_{\varepsilon,j} &= \inf_{h\in\mathscr{H}^q_{\varepsilon,j}} \sup_{v\in\mathscr{M}^q_{\varepsilon,j}} J_\varepsilon(h(v)) \\
&\leqslant \sup_{v\in\mathscr{M}^q_{\varepsilon,j}} J_\varepsilon(v) \\
&\leqslant \sup_{v\in\mathscr{M}^q_{\varepsilon,j}} J_0(v) + \varepsilon^r \sup_{v\in\mathscr{M}^q_{\varepsilon,j}} \int_\Omega \left(\frac{1}{p}|\nabla v|^p + W(v)\right) dx \\
&\leqslant \tilde{\lambda}_j + \sup_{u\in S(j)} \frac{A[Q_{F_j} G^q_\varepsilon]}{\|P_{F_j} G^q_\varepsilon + \hat{\rho} u\|^2_{L^2} + \|Q_{F_j} G^q_\varepsilon\|^2_{L^2}} \\
&\quad + \frac{\varepsilon^r}{p} \sup_{u\in S(j)} \frac{\|G^q_\varepsilon + \hat{\rho} u\|^p_{W_0^{1,p}}}{\|G^q_\varepsilon + \hat{\rho} u\|^p_{L^2}} \\
&\quad + \varepsilon^r \sup_{u\in S(j)} \int_\Omega W\left(\frac{G^q_\varepsilon + \hat{\rho} u}{\|G^q_\varepsilon + \hat{\rho} u\|_{L^2}}\right) dx.
\end{aligned} \tag{37}
$$

At this point, we note that $\lim_{\varepsilon\to 0} A[Q_{F_j} G^q_\varepsilon] = 0$, in fact by (32) and (33), we have

$$
\begin{aligned}
A[Q_{F_j} G^q_\varepsilon] &\leqslant A[G^q_\varepsilon] \\
&\leqslant \|G^q_\varepsilon\|^2_{H_0^1} + \|V\|_{L^2} \|G^q_\varepsilon\|^2_{L^4} \\
&= \varepsilon^{n-2} \|G^q\|^2_{H_0^1} + \varepsilon^{\frac{n}{2}} \|V\|_{L^2} \|G^q\|^2_{L^4}.
\end{aligned}
$$

Moreover, by (ii) of Lemma 4.3 we obtain

$$
\sup_{0<\varepsilon\leqslant\bar{\varepsilon}} \sup_{u\in S(j)} \frac{1}{\|P_{F_j} G^q_\varepsilon + \hat{\rho} u\|^2_{L^2} + \|Q_{F_j} G^q_\varepsilon\|^2_{L^2}} < +\infty.
$$

In fact, $\|P_{F_j} G^q_\varepsilon\|^2_{L^2} \leqslant \varepsilon^n \|G^q\|^2_{L^2}$ and $\|Q_{F_j} G^q_\varepsilon\|^2_{L^2} \leqslant \varepsilon^n \|G^q\|^2_{L^2}$. Therefore, the second term of the last inequality of (37) goes to zero when ε goes to zero.

Now, we observe that the following inequality holds:

$$
\varepsilon^r \|G^q_\varepsilon + \hat{\rho} u\|^p_{W_0^{1,p}} \leqslant \left(\varepsilon^{\frac{r-(p-n)}{p}} \|G^q\|_{W_0^{1,p}} + \varepsilon^{\frac{r}{p}} \hat{\rho} \|u\|_{W_0^{1,p}}\right)^p.
$$

Then by this inequality and (ii) of Lemma 4.3 (we recall that $r > p - n$), we have that the third term of the last inequality of (37) tends to zero when ε tends to zero.

As regards the last term, we verify that $\int_\Omega W\left(\frac{G_\varepsilon^q + \hat{\rho}u}{\|G_\varepsilon^q + \hat{\rho}u\|_{L^2}}\right) dx$ is bounded uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon}]$ and $u \in S(k)$. In fact, it is obvious that there exists $c \in \mathbb{R}^+$ such that $\frac{\|G_\varepsilon^q + \hat{\rho}u\|_{L^\infty}}{\|G_\varepsilon^q + \hat{\rho}u\|_{L^2}} \leqslant c$ for $\varepsilon \in (0, \bar{\varepsilon}]$ and $u \in S(k)$. Finally from (iii) of Lemma 4.3, we get the claim.

Step 3: *We prove that* $c_{\varepsilon, j}^q \geqslant \tilde{\lambda}_j$.

By Step 1 and by the positivity of W, we get

$$c_{\varepsilon, j}^q \geqslant \inf_{h \in \mathscr{H}_{\varepsilon, j}^q} \sup_{v \in \mathscr{M}_{\varepsilon, j}^q} A[h(v)]$$

$$\geqslant \inf_{h \in \mathscr{H}_{\varepsilon, j}^q} \sup_{v \in \mathscr{M}_{\varepsilon, j}^q,\ P_{F_{j-1}} h(v) = 0} A[h(v)] \geqslant \tilde{\lambda}_j.$$

In fact, by Step 1 for all $h \in \mathscr{H}_{\varepsilon, j}^q$ we have that the set $h(\mathscr{M}_{\varepsilon, j}^q)$ intersects the set $\{u \in H / A(u, \varphi_i) = 0 \ \forall i = 1, \ldots, j-1\}$ and so from (23) we get the claim.

Step 4: *If* $\tilde{\lambda}_{j-1} < \tilde{\lambda}_j$, *then for* ε *small enough we have*

$$c_{\varepsilon, j-1}^q < c_{\varepsilon, j}^q, \tag{38}$$

$$\sup_{v \in \mathscr{M}_{\varepsilon, j-1}^q} J_\varepsilon(v) < c_{\varepsilon, j}^q. \tag{39}$$

By Steps 2 and 3, we obtain for ε small enough

$$c_{\varepsilon, j-1}^q \leqslant \tilde{\lambda}_{j-1} + \sigma(\varepsilon) < \tilde{\lambda}_j \leqslant c_{\varepsilon, j}^q,$$

$$\sup_{v \in \mathscr{M}_{\varepsilon, j-1}^q} J_\varepsilon(v) \leqslant \tilde{\lambda}_{j-1} + \sigma(\varepsilon) < \tilde{\lambda}_j \leqslant c_{\varepsilon, j}^q.$$

Step 5: *If* $\tilde{\lambda}_{j-1} < \tilde{\lambda}_j$, *then* $c_{\varepsilon, j}^q$ *is a critical value for the functional* J_ε *on the manifold* $\Lambda_q \cap S$.

By contradiction we suppose that $c_{\varepsilon, j}^q$ is a regular value for J_ε on $\Lambda_q \cap S$. By Lemmas 4.5–4.7 there exist $\delta > 0$ and a deformation $\eta : [0, 1] \times \Lambda_q \cap S \to \Lambda_q \cap S$ such that

$$\eta(0, u) = u \qquad \forall u \in \Lambda_q \cap S,$$

$$\eta(t, u) = u \qquad \forall t \in [0, 1],\ \forall u \in J_\varepsilon^{c_{\varepsilon, j}^q - 2\delta},$$

$$\eta(1, j_\varepsilon^{c_{\varepsilon, j}^q + \delta}) \subset J_\varepsilon^{c_{\varepsilon, j}^q - \delta}.$$

By (39), we can suppose

$$\sup_{v \in \mathcal{M}^q_{\varepsilon, j-1}} J_\varepsilon(v) < c^q_{\varepsilon, j} - 2\delta. \tag{40}$$

Moreover, by definition of $c^q_{\varepsilon, j}$ there exists a transformation $\hat{h} \in \mathcal{H}^q_{\varepsilon, j}$ such that $\sup_{v \in \mathcal{M}^q_{\varepsilon, j}} J_\varepsilon(\hat{h}(v)) < c^q_{\varepsilon, j} + \delta$. Now by the properties of the deformation η and by (40) we get $\eta(1, \hat{h}(\cdot)) \in \mathcal{H}^q_{\varepsilon, j}$ and $\sup_{v \in \mathcal{M}^q_{\varepsilon, j}} J_\varepsilon(\eta(1, \hat{h}(v))) < c^q_{\varepsilon, j} - \delta$ and this is a contradiction. ∎

Remark 5. By Step 2 of the proof of Theorem 4.1 we have that for all $q \in \mathbb{Z} \setminus \{0\}$, $\varepsilon \in (0, 1]$ and $j \in \mathbb{N}$ there holds $c^q_{\varepsilon, j} \leqslant \tilde{\lambda}_j + \sigma(\varepsilon)$, with $\lim_{\varepsilon \to 0} \sigma(\varepsilon) = 0$. Moreover, in Step 3 we proved that $c^q_{\varepsilon, j} \geqslant \tilde{\lambda}_j$. Hence, we can conclude that the critical values $c^q_{\varepsilon, j}$ tend to the eigenvalues $\tilde{\lambda}_j$ when ε tends to zero.

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