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Finite type invariants of cyclic branched covers

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Abstract

Given a knot in an integer homology sphere, one can construct a family of closed 3-manifolds (parameterized by the positive integers), namely the cyclic branched coverings of the knot. In this paper, we give a formula for the Casson–Walker invariants of these 3-manifolds in terms of residues of a rational function (which measures the 2-loop part of the Kontsevich integral of a knot) and the signature function of the knot. Our main result actually computes the LMO invariant of cyclic branched covers in terms of a rational invariant of the knot and its signature function.

MSC: primary 57N10; secondary 57M25

Keywords: Cyclic branched covers; Signatures; Finite type invariants; Rational lift of the Kontsevich integral; Wheels

1. Introduction

1.1. History

One of the best known integer-valued concordance invariants of a knot K in an integer homology sphere M is its (suitably normalized) *σignature function* $\sigma(M,K): S^1 \to \mathbb{Z}$ defined for all complex numbers of absolute value 1, see for example [18]. The σignature function and its values at complex roots of unity are closely related to a sequence (indexed by a natural number p, not necessarily prime) of closed 3-manifolds, the *p-fold cyclic branched coverings* $\Sigma^p_{(M,K)}$, associated to the pair (M,K) and play a key role in the approach to knot theory via surgery theory.

It is an old problem to find a formula for the Casson–Walker invariant of cyclic branched covers of a knot. For two-fold branched covers, Mullins used skein theory of the Jones polynomial to

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show that for all knots K in S^3 such that $\Sigma^2_{(S^3,K)}$ is a rational homology 3-sphere, there is a linear relation between the Casson–Walker invariant (see [30]) $\lambda(\Sigma^2_{(S^3,K)})$, $\sigma_{-1}(S^3,K)$ and the logarithmic derivative of the Jones polynomial of K at -1, [24]. A different approach was taken by the first author in [9], where the above-mentioned linear relation was deduced and explained from the wider context of finite-type invariants of knots and 3-manifolds.

For $p>2$, Hoste, Davidow and Ishibe studied a partial case of the above problem for Whitehead doubles of knots, [6,16,17].

However, a general formula was missing for $p>2$. Since the map $(M,K)\to\lambda(\Sigma^p_{(M,K)})$ is not a concordance invariant of (M,K), it follows that a formula for the Casson invariant of cyclic branched coverings should involve more than just the *total p-σignature* σ_p (that is, $\sum_{\omega^p=1}\sigma_\omega$).

In [13], a conjecture for the Casson invariant of cyclic branched coverings was formulated. The conjecture involved the total signature and the sums over complex roots of unity, of a *rational function* associated to a knot. The rational function in question was the 2-loop part of a rational lift Z^{rat} of the Kontsevich integral of a knot.

In [12] the authors constructed this rational lift, combining the so-called *surgery view of knots* (see [11]) with the full apparatus of perturbative field theory, formulated by the Aarhus integral and its function-theory properties.

The goal of the present paper is to prove the missing formula of the Casson invariant of cyclic branched coverings, under the mild assumption that these are rational homology spheres. In fact, our methods will give a formula for the LMO invariant of cyclic branched coverings in terms of the σignature function and residues of the Z^{rat} invariant.

Our main Theorem 1 will follow from a formal calculation, presented in Section 2.3. This illustrates the relation between the formal properties of the Z^{rat} invariant and the geometry of the cyclic branched coverings of a knot.

1.2. Statement of the results

Let us call a knot (M,K) *p-regular* iff $\Sigma^p_{(M,K)}$ is a rational homology 3-sphere. We will call a knot *regular* iff it is p-regular for all p. It is well-known that (M,K) is p-regular iff its Alexander polynomial $\Delta(M,K)$ has no complex pth roots of unity.

Let Z denote the LMO invariant of a knot (reviewed in Section 2.1), and let τ^{rat} denote the twisting map of Definition 4.5 and Lift_p denote the lifting map of Section 5.1.

Theorem 1. *For all p and p-regular pairs (M,K) we have*

$$Z(\Sigma^p_{(M,K)})=\mathrm{e}^{\sigma_p(M,K)\Theta/16}\mathrm{Lift}_p\circ\tau^{\mathrm{rat}}_{\alpha_p}\circ Z^{\mathrm{rat}}(M,K)\in\mathscr{A}(\phi),$$

where $\alpha_p=v^{-(p-1)/p}\in\mathscr{A}(\circledast)$, $v=Z(S^3,\mathrm{unknot})$.

The proof of Theorem 1 is a formal computation, given in Section 2, that involves the rational invariant Z^{rat} and its function-theory properties, phrased in terms of operations (such as twisting and lifting) on diagrams. In a sense, the Z^{rat} invariant is defined by using properties of the universal abelian cover of knot complements. Since the universal abelian cover maps onto every cyclic branched cover, it is not too surprising that the Z^{rat} invariant appears in a formula for the LMO invariant of cyclic branched covers. The presence of the signature function is a framing defect of the branched covers. It arises because we need to normalize the 3-manifold invariants by their values at a ± 1-framed unknot. These values are some universal constants, whose ratio (for positively versus negatively unit framed unknot) is given by the signature term. We do not know of a physics explanation of the above formula in terms of anomalies.

Remark 1.1. Twisting and lifting are important operations on diagrams with beads that commute with the operation of integration, see Propositions 4.11 and 5.8. For properties of the twisting operation, see Lemma 4.8 in Section 4. For a relation between our notion of twisting and the notion of *wheeling* (introduced in [1] and studied in [3,4]), see Section 7. For properties of the lifting operation, see Section 5. Twisting and lifting are closely related to the magic formula for the Kontsevich integral of a long Hopf link [4], and to rational framings, [3].

The next corollary gives a precise answer for the value of the Casson–Walker invariant of cyclic branched covers as well as its growth rate (as $p \to \infty$), in terms of the 2-loop part of the Kontsevich integral and the σignature function. In a sense, the σignature function and the 2-loop part of the Kontsevich integral are generating function for the values of the Casson–Walker invariant of cyclic branched covers.

Corollary 1.2. (a) *For all* p *and* (M,K) *and* p*-regular, we have*

$$\lambda(\Sigma^p_{(M,K)}) = \tfrac{1}{3}\,\mathrm{Res}^{t_1,t_2,t_3}_p Q(M,K)(t_1,t_2,t_3) + \tfrac{1}{8}\,\sigma_p(M,K).$$

Note the difference between the normalization of Res_p *of* [13, *Section* 1.5] *and that of Section* 5.2.

(b) *For all regular pairs* (M,K), *we have*

$$\lim_{p\to\infty} \frac{\lambda(\Sigma^p_{(M,K)})}{p} = \frac{1}{3}\int_{S^1\times S^1} Q(M,K)(s)\,\mathrm{d}\mu(s) + \frac{1}{8}\int \sigma_s(M,K)\,\mathrm{d}\mu(s),$$

where $\mathrm{d}\mu$ *is the Haar measure.*

In other words, the Casson invariant of cyclic branched coverings grows linearly with respect to the degree of the covering, and the growth rate is given by the average of the Q function on a torus and the *total σignature* of the knot (i.e., the term $\int \sigma_s(M,K)\,\mathrm{d}\mu(s)$ above). The reader may compare this with the following theorem of Fox–Milnor, [8] which computes the torsion of the first homology of cyclic branched covers in terms of the Alexander polynomial, and the growth rate of it in terms of the *Mahler measure* of the Alexander polynomial:

Theorem 2 (Fox and Milnor [8]). (a) *Let* $\beta_p(M,K)$ *denote the order of the torsion subgroup of* $H_1(\Sigma^p_{(M,K)},\mathbb{Z})$. *Assuming that* (M,K) *is* p*-regular, we have that*

$$\beta_p(M,K) = \prod_{\omega^p=1} |\Delta(M,K)(\omega)|.$$

(b) *If* (M,K) *is regular, it follows that*

$$\lim_{p\to\infty} \frac{\log \beta_p(M,K)}{p} = \int_{S^1} \log(|\Delta(M,K)(s)|)\,\mathrm{d}\mu(s).$$

In case (M,K) is not regular, (b) still holds, as was shown by Silver and Williams [27].

1.3. Plan of the proof

In Section 2, we review the definition of Z^{rat}, and we reduce Theorem 1 to Theorem 3 (which concerns signatures of surgery presentations of knots) and Theorem 4 (which concerns the behavior of the Z^{rat} invariant under coverings of knots in solid tori).

Section 3 consists entirely of topological facts about the surgery view of knots, and shows Theorem 3.

Sections 4 and 5 introduce the notion of twisting and lifting of diagrams, and study how they interact with the formal diagrammatic properties of the Z^{rat} invariant. As a result, we give a proof of Theorem 4.

In Section 5.3, we prove Corollary 1.2.

Finally, we give two alternative versions of Theorem 1: in Section 6 in terms of an invariant of branched covers that remembers a lift of the knot, and in Section 7 in terms of the wheeled rational invariant $Z^{\text{rat},\text{¤}}$.

1.4. Recommended reading

The present paper uses at several points a simplified version of the notation and the results of [12] presented for knots rather than boundary links. Therefore, it is a good idea to have a copy of [12] available.

2. A reduction of Theorem 1

In this section, we will reduce Theorem 1 to two theorems; one involving properties of the invariant Z^{rat} under lifting and integrating, and another involving properties of the σignature function. Each will be dealt with in a subsequent section.

2.1. A brief review of the rational invariant Z^{rat}

In this section, we briefly explain where the rational invariant takes values and how it is defined. The invariant $Z^{\text{rat}}(M,K)$ is closely related to the surgery view of pairs (M,K) and is defined in several steps explained in [11] and below, with some simplifications since we will be dealing *exclusively* with knots and not with boundary links, [12, Remark 1.6]. In that case, the rational invariant Z^{rat} takes values in the subset

$$\mathscr{A}^{\text{gp},0}(\Lambda_{\text{loc}}) = \mathscr{B} \times \mathscr{A}^{\text{gp}}(\Lambda_{\text{loc}}) \quad \text{of} \quad \mathscr{A}^{0}(\Lambda_{\text{loc}}) = \mathscr{B} \times \mathscr{A}(\Lambda_{\text{loc}}),$$

where

- $\Lambda_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$, $\Lambda = \mathbb{Q}[\mathbb{Z}] = \mathbb{Q}[t^{\pm 1}]$ and $\Lambda_{\text{loc}} = \{p(t)/q(t), p,\ q \in \mathbb{Q}[t^{\pm 1}], q(1) = \pm 1\}$, the localization of Λ with respect to the multiplicative set of all Laurent polynomials of t that evaluate to 1 at $t = 1$. For future reference, Λ and Λ_{loc} are rings with involution $t \leftrightarrow t^{-1}$, selected group of units $\{t^n \mid n \in \mathbb{Z}\}$ and ring homomorphisms to $\mathbb{Z}$ given by evaluation at $t = 1$.

- $\mathrm{Herm}(\Lambda_{\mathbb{Z}} \to \mathbb{Z})$ is the set of *Hermitian matrices* A over $\Lambda_{\mathbb{Z}}$, invertible over $\mathbb{Z}$, and $B(\Lambda_{\mathbb{Z}} \to \mathbb{Z})$ (*abbreviated by* $\mathscr{B}$) denote the quotient of $\mathrm{Herm}(\Lambda_{\mathbb{Z}} \to \mathbb{Z})$ modulo the equivalence relation generated by the move: $A \sim B$ iff $A \oplus E_1 = P^*(B \oplus E_2)P$, where E_i are diagonal matrices with ± 1 on the diagonal and P is either an elementary matrix (i.e., one that differs from the diagonal at a single nondiagonal entry) or a diagonal matrix with monomials in t in the diagonal.
- $\mathscr{A}(\Lambda_{\mathrm{loc}})$ is the (completed) graded algebra over $\mathbb{Q}$ spanned by trivalent graphs (with vertex and edge orientations) whose edges are labeled by elements in Λ_{loc}, modulo the AS,IHX relations and the multilinear and vertex invariance relations of [12, Figs. 3,4, Section 3]. The multiplication is induced by the disjoint union of graphs. The degree of a graph is the number of its trivalent vertices and the multiplication of graphs is given by their disjoint union. $\mathscr{A}^{\mathrm{gp}}(\Lambda_{\mathrm{loc}})$ is the set of *group-like* elements of $\mathscr{A}(\Lambda_{\mathrm{loc}})$, that is elements of the form $\exp(c)$ for a series c of connected graphs.

So far, we have explained where Z^{rat} takes values. In order to recall the definition of Z^{rat}, we need to consider unitrivalent graphs as well and a resulting set $\mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$ explained in detail in Section 4. Then, we proceed as follows:

- Choose a *surgery presentation* L for (M,K), that is a null homotopic framed link L (in the sense that each component of L is a null homotopic curve in ST) in a *standard solid torus* $ST \subset S^3$ such that its linking matrix is invertible over $\mathbb{Z}$ and such that ST_L can be identified with the complement of a tubular neighborhood of K in M.
- Define an invariant $\check{Z}^{\mathrm{rat}}(L)$ with values in $\mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$ where X is a set in 1–1 correspondence with the components of L.
- Define an integration $\int^{\mathrm{rat}} \mathrm{d}X : \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda) \to \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})$ as follows. Consider an integrable element s, that is one of the form

$$s = \exp_{\sqcup}\left(\frac{1}{2}\sum_{i,j} \underset{x_i}{\overset{x_j}{\updownarrow}} M_{ij}\right) \sqcup R \tag{1}$$

with $M \in \mathrm{Herm}(\Lambda_{\mathbb{Z}} \to \mathbb{Z})$ and R a series of X-substantial diagrams (i.e., diagrams that do not contain a strut component). Notice that M, the *covariance matrix* of s, and R, the *X-substantial part* of s, are uniquely determined by s. We define

$$\int^{\mathrm{rat}} \mathrm{d}X(s) = \left(M, \left\langle \exp_{\sqcup}\left(-\frac{1}{2}\sum_{i,j} \underset{x_i}{\overset{x_j}{\updownarrow}} M_{ij}^{-1}\right), R \right\rangle_X\right).$$

In words, $\int^{\mathrm{rat}}$-integration is gluing the legs of the X-substantial graphs in X using the negative inverse covariance matrix.
- Finally, define

$$Z^{\mathrm{rat}}(M,K) = \frac{\int^{\mathrm{rat}} \mathrm{d}X\, \check{Z}^{\mathrm{rat}}(L)}{c_+^{\sigma_+(B)} c_-^{\sigma_-(B)}} \in \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}), \tag{2}$$

where $c_{\pm} = \int \mathrm{d}U\, \check{Z}(S^3, U_{\pm})$ are some universal constants of the unit-framed unknot $U_{\pm}$.

The following questions may motivate a bit the construction of Z^{rat}:

Question: Why is $Z^{\mathrm{rat}}(M,K)$ an invariant of (M,K) rather than of L?

Answer: Because for fixed (M,K) any two choices for L are related by a sequence of Kirby moves, as shown by the authors in [11, Theorem 1]. Even though $\check{Z}^{\mathrm{rat}}(L) \in \mathcal{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$ is not invariant under Kirby moves, it becomes so after $\int^{\mathrm{rat}}$-integration.

Question: Why do we need to introduce the $\int^{\mathrm{rat}}$-integration?

Answer: To make $L \to \check{Z}^{\mathrm{rat}}(L)$ invariant under Kirby moves on L.

Question: Why do we need to consider diagrams with beads in Λ_{loc}?

Answer: Because $\int^{\mathrm{rat}}$-integration glues struts by inverting the covariance matrix W. If W is a Hermitian matrix over Λ which is invertible over $\mathbb{Z}$, after $\int^{\mathrm{rat}}$-integration appear diagrams with beads entries of W^{-1}, a matrix defined over Λ_{loc}.

Remark 2.1. Z stands for the Kontsevich integral of framed links in S^3, extended to an invariant of links in 3-manifolds by Le et al. [22], and identified with the Aarhus integral in the case of links in rational homology 3-spheres, [2, Part III]. In this paper, we will use exclusively the Aarhus integral $\int$ and its rational generalization $\int^{\mathrm{rat}}$, whose properties are closely related to function-theoretic properties of functions on Lie groups and Lie algebras.

By convention, Z^{rat} contains *no* wheels and no Ω terms. That is, $Z^{\mathrm{rat}}(S^3,U)=1$. On the other hand, $Z(S^3,U)=\Omega$. Note that $\check{Z}^{\mathrm{rat}}(L)$ equals to the connect sum of copies of Ω (one to each component of L) to $Z^{\mathrm{rat}}(L)$.

2.2. Surgery presentations of cyclic branched covers

Fix a surgery presentation L of a pair (M,K). We begin by giving a surgery presentation of $\Sigma^p_{(M,K)}$. Let $L^{(p)}$ denote the preimage of L under the p-fold cover $ST \to ST$. It is well-known that $L^{(p)}$ can be given a suitable framing so that $\Sigma^p_{(M,K)}$ can be identified with $S^3_{L^{(p)}}$, see [5].

It turns out that the total p-σignature can be calculated from the linking matrix of the link $L^{(p)}$. In order to state the result, we need some preliminary definitions. For a symmetric matrix A over $\mathbb{R}$, let $\sigma_+(A), \sigma_-(A)$ denote the number of positive and negative eigenvalues of A, and let $\sigma(A), \mu(A)$ denote the *signature* and *size* of A. Obviously, for nonsingular A, we have $\sigma(A)=\sigma_+(A)-\sigma_-(A)$ and $\mu(A)=\sigma_+(A)+\sigma_-(A)$.

Let B (resp. $B^{(p)}$) denote the linking matrix of the framed link L (resp. $L^{(p)}$) in S^3. We will show later that

Theorem 3 (Proof in Section 3.2). *With the above notation, we have*

$$\sigma_p(M,K) = \sigma(B^{(p)}) - p\sigma(B) \quad \text{and} \quad \mu(B^{(p)}) = p\mu(B).$$

2.3. A formal calculation

Assuming the existence of a suitable maps Lift_p and τ^{rat}, take residues in Eq. (2). We obtain that

$$
\begin{aligned}
\mathrm{Lift}_p \circ \tau^{\mathrm{rat}}_{\alpha} \circ Z^{\mathrm{rat}}(M,K) &= \mathrm{Lift}_p \circ \tau^{\mathrm{rat}}_{\alpha} \left(\frac{\int^{\mathrm{rat}} \mathrm{d}X\, \check{Z}^{\mathrm{rat}}(L)}{c_+^{\sigma_+(B)} c_-^{\sigma_-(B)}} \right) \\
&= \mathrm{Lift}_p \left(\frac{\int^{\mathrm{rat}} \mathrm{d}X\, \tau^{\mathrm{rat}}_{\alpha} \check{Z}^{\mathrm{rat}}(L)}{c_+^{\sigma_+(B)} c_-^{\sigma_-(B)}} \right) \quad \text{by Theorem 4.11} \\
&= \frac{\mathrm{Lift}_p \left(\int^{\mathrm{rat}} \mathrm{d}X\, \tau^{\mathrm{rat}}_{\alpha} \check{Z}^{\mathrm{rat}}(L) \right)}{c_+^{p\sigma_+(B)} c_-^{p\sigma_-(B)}} \quad \text{by Remark 5.2} \\
&= \frac{\mathrm{Lift}_p \left(\int^{\mathrm{rat}} \mathrm{d}X\, \tau^{\mathrm{rat}}_{\alpha} \check{Z}^{\mathrm{rat}}(L) \right)}{(\sqrt{c_+/c_-})^{p\sigma(B)} (\sqrt{c_+ c_-})^{p\mu(B)}}.
\end{aligned}
$$

Adding to the above the term corresponding to the total p-σignature $\sigma_p(M,K)$ of (M,K), and using the identity $c_+/c_- = \mathrm{e}^{-\Theta/8}$ (see [3, Eq. (19), Section 3.4]) it follows that

$$
\begin{aligned}
\mathrm{Lift}_p \circ \tau^{\mathrm{rat}}_{\alpha_p} \circ Z^{\mathrm{rat}}(M,K) \mathrm{e}^{\sigma_p(M,K)\Theta/16} &= \mathrm{Lift}_p \circ \tau^{\mathrm{rat}}_{\alpha_p} \circ Z^{\mathrm{rat}}(M,K) \left(\sqrt{\frac{c_+}{c_-}} \right)^{-\sigma_p(M,K)} \\
&= \frac{\mathrm{Lift}_p \left(\int^{\mathrm{rat}} \mathrm{d}X\, \tau^{\mathrm{rat}}_{\alpha_p} \check{Z}^{\mathrm{rat}}(L) \right)}{(\sqrt{c_+/c_-})^{\sigma(B^{(p)})} (\sqrt{c_+ c_-})^{\mu(B^{(p)})}} \quad \text{by Theorem 3} \\
&= \frac{\mathrm{Lift}_p \left(\int^{\mathrm{rat}} \mathrm{d}X\, \tau^{\mathrm{rat}}_{\alpha_p} \check{Z}^{\mathrm{rat}}(L) \right)}{c_+^{\sigma_+(B^{(p)})} c_-^{\sigma_-(B^{(p)})}} \\
&= \frac{\int \mathrm{d}X^{(p)}\, \check{Z}(L^{(p)})}{c_+^{\sigma_+(B^{(p)})} c_-^{\sigma_-(B^{(p)})}} \quad \text{by Theorem 4} \\
&= Z(\Sigma^p_{(M,K)}) \quad \text{by } Z'\text{s definition.}
\end{aligned}
$$

Theorem 4 (Proof in Section 5.1). *For* $\alpha_p = v^{-(p-1)/p}$ *we have*

$$
\mathrm{Lift}_p \left(\int^{\mathrm{rat}} \mathrm{d}X\, \tau^{\mathrm{rat}}_{\alpha_p} \check{Z}^{\mathrm{rat}}(L) \right) = \int \mathrm{d}X^{(p)}\, \check{Z}(L^{(p)}).
$$

This reduces Theorem 1 to Theorems 3 and 4, for a suitable Lift_p map, and moreover, it shows that the presence of the σignature function in Theorem 1 is due to the normalization factors $c_\pm$ of Z^{rat}.

The rest of the paper is devoted to the proof of Theorems 3 and 4 for a suitable residue map Lift_p.

3. Three views of knots

This section consists entirely of a classical topological view of knots and their abelian invariants such as σignatures, Alexander polynomials and Blanchfield pairings. There is some overlap of this section with [11]; however for the benefit of the reader we will try to present this section as self-contained as possible.

3.1. The surgery and the Seifert surface view of knots

In this section, we discuss two views of knots K in integral homology 3-spheres M: the *surgery view*, and the *Seifert surface* view.

We begin with the surgery view of knots. Given a surgery presentation L for a pair (M,K), let W denote the equivariant linking matrix of L, i.e., the linking matrix of a lift $\tilde{L}$ of L to the universal cover $\widetilde{ST}$ of ST. It is not hard to see that W is a Hermitian matrix. Recall the quotient $\mathscr{B}$ of the set of Hermitian matrices, from Section 2.1. In [12, Section 2] it was shown that $W \in \mathscr{B}$ depends only on the pair (M,K) and not on the choice of a surgery presentation of it. In addition, W determines the *Blanchfield pairing* of (M,K). Thus, the natural map Knots $\to$ BP (where BP stands for the *set of Blanchfield pairings*) factors through an (onto) map Knots $\to \mathscr{B}$.

We now discuss the *Seifert surface view* of knots. A more traditional way of looking at the set BP of knots is via Seifert surfaces and their associated Seifert matrices. There is an onto map Knots $\to$ Sei, where Sei is the set of matrices A with integer entries satisfying $\det(A - A') = 1$, considered modulo an equivalence relation called S-*equivalence*, [23]. It is known that the sets Sei and BP are in 1–1 correspondence, see for example [23,29]. Thus, we have a commutative diagram

$$\begin{array}{ccc} \text{Knots} & \twoheadrightarrow & \mathscr{B} \\ \big\downarrow & & \big\downarrow \\ \text{Sei} & \xrightarrow{\sim} & \text{BP} \end{array}$$

It is well-known how to define abelian invariants of knots, such as the σignature and the *Alexander polynomial* Δ, using Seifert surfaces. Lesser known is a definition of these invariants using equivariant linking matrices, which we now give.

Definition 3.1. Let

$$\delta : \mathrm{Herm}(\Lambda_{\mathbb{Z}} \to \mathbb{Z}) \to \Lambda_{\mathbb{Z}}$$

denote the (normalized) determinant given by $\delta(W) = \det(W)\det(W(1))^{-1}$ (for all $W \in \mathrm{Herm}(\Lambda_{\mathbb{Z}} \to \mathbb{Z})$) and let

$$\varsigma : \mathrm{Herm}(\Lambda_{\mathbb{Z}} \to \mathbb{Z}) \to \mathrm{Maps}(S^1, \mathbb{Z})$$

denote the function given by $\varsigma_z(W) = \sigma(W(z)) - \sigma(W(1))$. For a natural number p, let

$$\varsigma_p : \mathrm{Herm}(\Lambda_{\mathbb{Z}} \to \mathbb{Z}) \to \mathbb{Z}$$

be given by $\sum_{\omega^p=1} \varsigma_\omega(W)$.

It is easy to see that δ and ς descend to functions on $\mathscr{B}$. Furthermore, we have that

$$\varsigma_p(W) = \sigma(W(T^{(p)})) - p\sigma(W(1)),$$

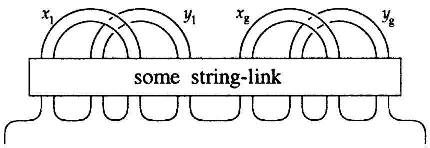
where $T^{(p)}$ is a p-cycle p by p matrix, given by example for $p = 4$

$$T^{(4)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

3.2. The clover view of knots

It seems hard to give an explicit algebraic map Sei $\to \mathscr{B}$ although both sets may well be in 1–1 correspondence. Instead, we will give a third view of knots, the *clover view* of knots, which enables us to prove Theorem 3.

Consider a standard Seifert surface Σ of genus g in S^3, which we think of as an embedded disk with pairs of bands attached in an alternating way along the disk:



Consider an additional link L' in $S^3 \setminus \Sigma$, such that its linking matrix C satisfies $\det(C) = \pm 1$ and such that the linking number between the cores of the bands and L' vanishes. With respect to a suitable orientation of the 1-cycles corresponding to the cores of the bands, a Seifert matrix of Σ is given by

$$A = \begin{bmatrix} L^{xx} & L^{xy} \\ L^{yx} - I & L^{yy} \end{bmatrix},$$

where

$$\begin{bmatrix} L^{xx} & L^{xy} \\ L^{yx} & L^{yy} \end{bmatrix}$$

is the linking matrix of the closure of the above string-link in the basis $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$. Let (M, K) denote the pair obtained from $(S^3, \partial\Sigma)$ after surgery on L'. With the notation

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

we claim that

Theorem 5. *Given (Σ, L') as above, there exists a $2g$ component link L in the complement of L' such that*:

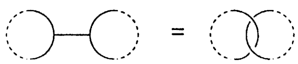
(a) *$L \cup L' \subset ST$ is a surgery presentation of (M, K) in the sense of Section 2.2.*
(b) *The equivariant linking matrix of $L \cup L'$ is represented by $W(t) \oplus C$ where*

$$W(t) = \begin{bmatrix} L^{xx} & (1-t^{-1})L^{xy} - I \\ (1-t)L^{yx} - I & (1-t-t^{-1}+1)L^{yy} \end{bmatrix}.$$

(c) *Every pair (M, K) comes from some (Σ, L') as above.*

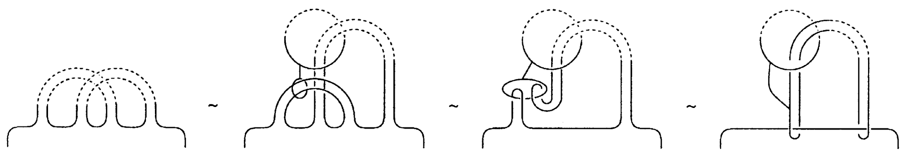
We will call such surgery presentations the *clover* view of knots.

Proof. (a) We will construct L using the calculus of *clovers with two leaves* introduced independently by Goussarov and Habiro [14,15]; see also [10, Section 3]. Clovers with two leaves is a shorthand notation (on the left) for framed links shown on the right of the following figure:



Since clovers can be thought of as framed links, surgery on clovers makes sense. Two clovers are equivalent (denoted by $\sim$ in the figures) if after surgery, they represent the same 3-manifold. By calculus on clovers (a variant of Kirby's calculus on framed links) we mean a set of moves that result to equivalent clovers. For an example of calculus on clovers, we refer the reader to [14,15] and also [10, Sections 2,3].

In figures involving clovers, L is constructed as follows:



Notice that at the end of this construction, $L \cup L' \subset ST$ is a surgery presentation for (M, K).

(b) Using the discussion of [21, Section 3.4], it is easy to see that the equivariant linking matrix of (a based representative of) $L \cup L'$ is given as stated.

(c) Finally, we show that every pair (M, K) arises this way. Indeed, choose a Seifert surface Σ for K in M and a link $L' \subset M$ such that $M_{L'} = S^3$. The link L' may intersect Σ', and it may have nontrivial linking number with the cores of the bands of Σ'. However, by a small isotopy of L' in M (which preserves the condition $M_{L'} = S^3$) we can arrange that L' be disjoint from Σ' and that its linking number with the cores of the bands vanishes. Viewed from S^3 (i.e., reversing the surgery), this gives rise to (Σ, L') as needed. □

The next theorem identifies the Alexander polynomial and the signature function of a knot with the functions δ and ς of Definition 3.1.

Theorem 6. *The maps composition of the maps δ and ς with the natural map* Knots $\to \mathscr{B}$ *is given by the Alexander polynomial and the σignature function, respectively.*

Proof. There are several ways to prove this result, including an algebraic one, which is a computation of appropriate Witt groups, and an analytic one, which identifies the invariants with $U(1)$ ρ-invariants. None of these proofs appear in the literature. We will give instead a proof using the ideas already developed.

Fix a surgery presentation $L \cup L'$ for (M,K), with equivariant linking matrix $W(t) \oplus C$ as in Theorem 5. Letting

$$P = \begin{bmatrix} (1-t)I & 0 \\ 0 & I \end{bmatrix} \oplus I$$

it follows that

$$\begin{aligned} P(W(t)\oplus C)P^{\star} &= \left(\begin{bmatrix} (1-t)I & 0 \\ 0 & I \end{bmatrix} \oplus I \right) (W(t)\oplus C) \left(\begin{bmatrix} (1-t^{-1})I & 0 \\ 0 & I \end{bmatrix} \oplus I \right) \\ &= \begin{bmatrix} ((1-t)+(1-t^{-1}))L^{xx} & ((1-t)+(1-t^{-1}))L^{xy}-(1-t)I \\ ((1-t)+(1-t^{-1}))L^{yx}-(1-t^{-1})I & ((1-t)+(1-t^{-1}))L^{yy} \end{bmatrix} \oplus C \\ &= ((1-t^{-1})A + (1-t)A') \oplus C. \end{aligned}$$

Taking signatures for any $t \in S^1$, $t \neq 1$, it follows that

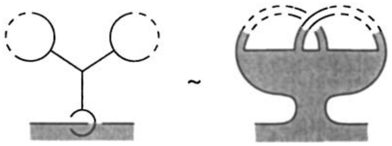
$$\begin{aligned} \sigma(W(t)) + \sigma(C) &= \sigma(W(t) \oplus C) \\ &= \sigma(((1-t^{-1})A + (1-t)A') \oplus C) \\ &= \sigma(((1-t^{-1})A + (1-t)A')) + \sigma(C) \\ &= \sigma_t(M,K) + \sigma(C), \end{aligned}$$

where the last equality follows from the definition of the σignature, see [18, p. 289] and [25]. Thus, $\sigma(W(t)) = \sigma_t(M,K)$. Since $W(1)$ is a metabolic matrix, it follows that $\sigma(W(1)) = 0$, from which it follows that $\varsigma(M,K) = \sigma(M,K)$. Taking determinants rather than signatures in the above discussion, it follows that $\delta(M,K) = \Delta(M,K)$. □

Proof of Theorem 3. Fix a surgery presentation $L \cup L'$ for (M,K), with equivariant linking matrix $W(t) \oplus C$ as in Theorem 5. Then the linking matrix B and $B^{(p)}$ of $L \cup L'$ and $L^{(p)} \cup L'^{(p)}$ are given by $W(1) \oplus C$ and $W(T^{(p)}) \oplus C \otimes I$ with an appropriate choice of basis. The result follows using Definition 3.1 and Theorem 6. □

Remark 3.2. An alternative proof of Theorem 3 can be obtained using the G-signature theorem to the 4-manifold N obtained by gluing two 4-manifolds N_1, N_2 with $\mathbb{Z}_p$ actions along their common boundary $\partial N_1 = \partial N_2 = \Sigma^p_{(M,K)}$. Here N_1 is the branched cover of D^4 branched along D^2 (obtained from adding the handles of L to D^4) and N_2 is a 4-manifold obtained from a Seifert surface construction of $\Sigma^p_{(M,K)}$.

Remark 3.3. An alternative proof of Theorem 5 can be obtained as follows. Start from a surgery presentation of (M,K) in terms of clovers with three leaves, as was explained in [10, Section 6.4] and summarized in the following figure:



Surgery on a clover with three leaves can be described in terms of surgery on a six component link L'''. It was observed by the second author in [19, Figure 3.1] that L''' can be simplified via Kirby moves to a four component link L''. It is a pleasant exercise (left to the reader) to further simplify L'' using Kirby moves to the two component link L that appears in Theorem 5.

Remark 3.4. Though we will not make use of this, we should mention that the clover presentation L of (S^3, K) appears in work of Freedman [7, Lemma 1]. Freedman starts with a knot of Arf invariant zero together with a Seifert surface and constructs a spin 4-manifold W_K with boundary $S^3_{K,0}$ (zero-surgery on K) by adding suitable 1- and 2-handles in the 4-ball. The intersection form of W_K, as Freedman computes in [7, Lemma 1] coincides with the equivariant linking matrix of L of our Theorem 5 This is not a coincidence, in fact the clover view of knots, interpreted in a four-dimensional way as addition of 1 and 2 handles to the 4-ball, gives precisely Freedman's 4-manifold.

4. Twisting

In this section, we define a notion of *twisting* $\tau_\alpha : \mathscr{A}^{\mathrm{gp}}(\star_{X\cup k}) \to \mathscr{A}^{\mathrm{gp}}(\star_{X\cup k})$ and its rational cousin $\tau_\alpha^{\mathrm{rat}} : \mathscr{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}})$. Twisting (by elements of $\mathscr{A}(\star)$) is an operation on diagrams with beads which is analogous to the "differential operator" action of $\mathscr{A}(\star)$ on $\mathscr{A}(\star)$ defined in terms of gluing all legs of the differential operator to some of the legs of a diagram.

A special case of twisting is the operation of wheeling on diagrams, studied by [1,3,4]. For a further discussion on the relation of twisting and wheeling, see Section 7.

4.1. Various kinds of diagrams

Manipulating the invariant Z^{rat} involves calculations that take values in vector spaces spanned by diagrams, modulo subspaces of relations. The notation is as follows: given a ring R with a distinguished group of units U, and (possibly empty sets) $X, Y \cup T$, $\mathscr{D}(\uparrow_X, \star_{Y\cup T}, R, U)$ is the set of

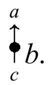
- Uni-trivalent diagrams with skeleton $\uparrow_X$, with symmetric univalent vertices labeled by $Y \cup T$.
- The diagrams have oriented edges and skeleton and each edge is labeled by an element of R, such that the edges that are part of the skeleton are labeled only by U. Moreover, the product of the labels along each component of the skeleton is 1. Labels on edges or part of the skeleton will be called *beads*.

$\mathscr{A}(\uparrow_X, \star_Y, \circledast_T, R, U)$ is the quotient of the free vector space over $\mathbb{Q}$ on $\mathscr{D}(\uparrow_X, \star_{Y\cup T}, R, U)$, modulo the relations of

- AS, IHX, *multilinearity on the beads* shown in [12, Fig. 3].

- The *Vertex Invariance Relation* shown in [12, Fig. 4].
- The *T-flavored basing relations* of [12, Appendix D].

Empty sets will be omitted from the notation, and so will U, the selected group of units of R. For example, $\mathscr{A}(\star_Y, R)$, $\mathscr{A}(R)$ and $\mathscr{A}(\phi)$ stands for $\mathscr{A}(\uparrow_\phi, \star_Y, \circledast_\phi, R, U)$, $\mathscr{A}(\uparrow_\phi, \star_\phi, \circledast_\phi, R, U)$ and $\mathscr{A}(\uparrow_\phi, \star_\phi, \circledast_\phi, \mathbb{Z}, 1)$ respectively. Univalent vertices of diagrams will often be called *legs*. Diagrams will sometimes be referred to as *graphs*. Special diagrams, called *struts*, labeled by a, c with bead b are drawn as follows:



oriented from bottom to top.

To further simplify notation, we will write $\mathscr{A}(\star), \mathscr{A}(\uparrow)$ and $\mathscr{A}(S^1)$ instead of $\mathscr{A}(\star_E), \mathscr{A}(\uparrow_E)$ and $\mathscr{A}(S^1_E)$ where E is a set of one element.

A technical variant of the vector space $\mathscr{A}(\uparrow_X, \star_Y, \circledast_T, R, U)$ of diagrams is the *set* $\mathscr{A}^{\mathrm{gp}}(\uparrow_X, \star_Y, \circledast_T, R, U)$ which is the quotient of the set of group-like elements in $\mathscr{A}(\uparrow_X, \star_Y, \circledast_T, R, U)$ (that is, exponential of a power series of connected diagrams) modulo the group-like basing relation described in [12, Section 3.3].

There is a natural map

$$\mathscr{A}^{\mathrm{gp}}(\uparrow_X, \star_Y, \circledast_T, R, U) \to \mathscr{A}(\uparrow_X, \star_Y, \circledast_T, R, U).$$

Finally, $\mathscr{A}^{\mathrm{gp},0}$ and $\mathscr{A}^0$ stand for $\mathscr{B} \times \mathscr{A}^{\mathrm{gp}}$ and $\mathscr{B} \times \mathscr{A}^{\mathrm{gp}}$, respectively.

4.2. A review of Wheels and Wheeling

Twisting is closely related to the Wheels and Wheeling Conjectures introduced in [1] and subsequently proven by [4]. See also [28]. The Wheels and Wheeling Conjectures are a good tool to study structural properties of the Aarhus integral, as was explained in [3]. In our paper, they play a key role in understanding twisting. In this section, we briefly review what Wheels and Wheeling is all about.

To warm up, recall that given an element $\alpha \in \mathscr{A}(\star)$ (such that α does not contain a diagram one of whose components is a strut $\uparrow$) we can turn it into an operator (i.e., linear map):

$$\hat{\alpha} : \mathscr{A}(\star) \to \mathscr{A}(\star)$$

such that α acts on an element x by gluing all legs of α to some of the legs of x. It is easy to see that $\widehat{\alpha \sqcup \beta} = \hat{\alpha} \circ \hat{\beta}$, which implies that if the constant term of α is nonzero, then the operator α is invertible with inverse $\hat{\alpha}^{-1} = \widehat{\alpha^{-1}}$.

Of particular interest is the following element:

$$\Omega = \exp\left(\sum_{n=1}^{\infty} b_{2n} \omega_{2n}\right) \in \mathscr{A}(\star),$$

where ω_{2n} is a wheel with $2n$ legs and

$$\sum_{n=1}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \log \frac{\sinh x/2}{x/2}.$$

The corresponding linear maps

$$\hat{\Omega}^{-1}, \hat{\Omega} : \mathscr{A}(\star) \to \mathscr{A}(\star)$$

are called, respectively, the *wheeling* and the *unwheeling* maps and are denoted by $x \to x^{\Omega}$ and $x \to x^{\Omega^{-1}}$, respectively. Due to historical reasons dating back to the days in Aarhus (where wheeling was discovered) and also due to Lie algebra reasons, wheeling was defined to be $\hat{\Omega}^{-1}$ and not $\hat{\Omega}$.

Recall the symmetrization map $\chi : \mathscr{A}(\star) \to \mathscr{A}(\uparrow)$ which sends an element $x \in \mathscr{A}(\star)$ to the average of the diagrams that arise by ordering the legs of x on a line. χ is a vector space isomorphism (with inverse σ) and can be used to transport the natural multiplication on $\mathscr{A}(\uparrow)$ (defined by joining two skeleton components of diagrams $\to \circ \to$ one next to the other to obtain a diagram on a skeleton component $\to$) to a multiplication on $\mathscr{A}(\star)$ which we denote by #. There is an additional multiplication $\sqcup$ on $\mathscr{A}(\star)$, defined using the disjoint union of graphs.

The *Wheeling Conjecture* states that the unwheeling isomorphism $\hat{\Omega} : (\mathscr{A}(\circledast_k), \sqcup) \to (\mathscr{A}(\circledast_k), \#)$ interpolates the two multiplications on $\mathscr{A}(\circledast_k)$. Namely, that for all $x, y \in \mathscr{A}(\circledast_k)$, we have

$$\hat{\Omega}(x \sqcup y) = \hat{\Omega}(x) \# \hat{\Omega}(y).$$

The *Wheels Conjecture* states that

$$Z(S^3, \text{unknot}) = \chi(\Omega).$$

The *long Hopf link formula* states that

$$Z\left(S^3, k \overset{x}{\uparrow}\!\!\circ\right) = \Omega(k) \overset{x}{\uparrow} \mathrm{e}^k \in \mathscr{A}(\uparrow_x \ \circledast_k).$$

Here and below, if $x \in \mathscr{A}(\star)$, then $x(h) \in \mathscr{A}(\star_h)$ denotes the diagram obtained from x by replacing the color of the legs of x by h.

It can be shown that the Wheels and Wheeling Conjectures are equivalent to the long Hopf link formula. In [4] the Wheels and Wheeling Conjectures and the long Hopf link formula were all proven. The identity $1 + 1 = 2$ (that is, doubling the unknot component of the long Hopf link is a tangle isotopic to connecting sum twice the long Hopf link along the vertical strand), together with the long Hopf link formula imply the following *Magic Formula*:

$$\Omega(k)\,\Omega(h) \overset{x}{\uparrow} \begin{matrix} \mathrm{e}^h \\ \mathrm{e}^k \end{matrix} = \Omega(k+h) \overset{x}{\uparrow} \mathrm{e}^{k+h} \in \mathscr{A}(\uparrow_x, \circledast_{k,h}). \tag{4}$$

Before we end this section, we should mention that for $\alpha \in \mathscr{A}(\star)$, the operator $\hat{\alpha}$ can be defined for diagrams whose legs are colored by $X \cup \{k\}$ (abbreviated by $X \cup k$), where $k \notin X$, by gluing all legs of α to some of the k-colored legs of a diagram. Furthermore, $\hat{\alpha}$ preserves Y-flavored basing relations for $Y \subset X \cup k$. In addition, if α is group-like, then $\hat{\alpha}$ sends group-like elements to group-like elements. Note finally that $\mathscr{A}(\star) = \mathscr{A}(\circledast)$; thus the operator $\hat{\alpha}$ can be defined for $\alpha \in \mathscr{A}(\circledast)$.

4.3. Twisting

Throughout this section, X denotes a (possibly empty) set disjoint from the two-element set $\{k, h\}$. Recall that given $x \in X$ and two diagrams $\alpha, \beta \in \mathscr{A}(\star_X)$ with k and l x-colored legs, respectively, the notation

$$\langle \alpha, \beta \rangle_{\{x\}} \in \mathscr{A}(\star_{X-\{x\}})$$

means either zero (if $k \neq l$) or the sum of diagrams obtained by gluing all x-colored legs of α with the x-colored legs of β. This definition can be extended to linear combination of diagrams, as a bilinear symmetric operation, and can be further extended to an operation of gluing Y-colored legs, for any $Y \subset X$.

Remark 4.1. We will often write

$$\langle \alpha(y), \beta(y) \rangle_Y$$

for the above operation, to emphasize the Y-colored legs of the diagrams. *Warning:* In [1,12], the authors used the alternative notation $\langle \alpha(\partial y), \beta(y) \rangle_Y$ for the above operation.

Given a diagram $s \in \mathscr{A}(\star_{X \cup k})$, the diagram $\phi_{k \to k+h}(s) \in \mathscr{A}(\star_{X \cup k,h})$ denotes the sum of relabelings of legs of s marked by k by either k or h.

Definition 4.2. For a group-like element $\alpha \in \mathscr{A}(\star)$, we define a map

$$\tau_\alpha : \mathscr{A}(\star_{X \cup k}) \to \mathscr{A}(\star_{X \cup k})$$

by

$$\tau_\alpha(s) = \langle \phi_{k \to k+h}(s) \Omega(h)^{-1}, \alpha(h) \rangle_h.$$

It is easy to see that τ_α maps group-like elements to group-like elements and maps Y-flavored basing relations to Y-flavored basing relations for $Y \subset X \cup k$; the latter follows from a “sweeping argument”.

The following lemma summarizes the elementary tricks about the operators $\hat{\alpha}$ that are very useful.

Lemma 4.3. *The operation $\langle \cdot, \cdot \rangle_X$ of gluing X-colored legs of diagrams satisfies the following identities*:

$$\langle A(x), B(x) \sqcup C(x) \rangle_X = \langle \hat{B}\, A(x), C(x) \rangle_X = \langle A(x + x'), B(x) \sqcup C(x') \rangle_{X,X'}$$

where X' is a set in 1–1 correspondence with the set X.

In fact, twisting can be expressed in terms of the above action.

Lemma 4.4. *We have that*

$$\tau_\alpha = \hat{\Omega}\widehat{^{-1}(\alpha)},$$

$$\tau_\alpha \circ \tau_\beta = \tau_{\alpha \# \beta}.$$

Proof. Recall that $\hat{\alpha}(y) = \langle \alpha(h), y(k+h) \rangle_h = \langle y(k+h), \alpha(h) \rangle_h$. For the first part, we have

$$\begin{aligned}\tau_\alpha(x) &= \langle x(k+h)\Omega(h)^{-1}, \alpha(h) \rangle_h \\ &= \langle x(k+h), (\hat{\Omega}^{-1}(\alpha))(h) \rangle_h \quad \text{by Lemma 4.3} \\ &= \widehat{\hat{\Omega}^{-1}(\alpha)}(x) \quad \text{by above discussion.}\end{aligned}$$

For the second part, we have

$$\begin{aligned}\tau_\alpha \circ \tau_\beta &= \widehat{\hat{\Omega}^{-1}(\alpha)} \circ \widehat{\hat{\Omega}^{-1}(\beta)} \\ &= \widehat{\hat{\Omega}^{-1}(\alpha) \sqcup \hat{\Omega}^{-1}(\beta)} \\ &= \widehat{\hat{\Omega}^{-1}(\alpha \# \beta)} \quad \text{by wheeling} \\ &= \tau_{\alpha\#\beta}. \qquad \square\end{aligned}$$

We now define a rational version

$$\phi_{t \to te^h} : \mathscr{A}^{\mathrm{gp}}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp}}(\star_{X \cup h}, \Lambda_{\mathrm{loc}})$$

of the map $\phi_{k \to k+h}$. The idea is that we substitute te^h for t (where t and h do not commute) and then replace e^h by an exponential of h-colored legs. This was explained in [12, Section 3.1] using the notion of the *Cohn localization* of the free group in two generators. We will not repeat the explanation of [12] here, but instead use the substitution map freely. The reader may either refer to the above mentioned reference for a complete definition of the $\phi_{t \to te^h}$ map, or may compromise with the following property of the $\phi_{t \to te^h}$ map:

$$\phi_{t \to te^h}(\updownarrow p(t)/q(t)) = \sum_{n=0}^{\infty} \updownarrow p(te^h)/q(t)((q(t) - q(te^h))/q(t))^n,$$

where $p, q \in \mathbb{Q}[t^{\pm 1}]$ and $q(1) = \pm 1$.

Definition 4.5. For a group-like element $\alpha \in \mathscr{A}^{\mathrm{gp}}(\star)$, we define a map

$$\tau_\alpha^{\mathrm{rat}} : \mathscr{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp},0}(\star_X, \Lambda_{\mathrm{loc}})$$

by

$$\tau_\alpha^{\mathrm{rat}}(M, s) = (M, \langle \psi(M(te^h)M(t)^{-1})\, \phi_{t \to te^h}(s), \alpha(h) \rangle_h),$$

where

$$\psi(A) = \exp\left(-\tfrac{1}{2}\,\mathrm{tr}\log(A)\right).$$

Remark 4.6. Here and below, we will be using the notation $\phi_{t \to e^k}(s)$ and $s(t \to e^k)$ to denote the substitution $t \to e^k$.

The motivation for this rather strange definition comes from the proof of Lemma 4.9 and Theorem 4.11 below.

Lemma 4.7. $\tau^{\rm rat}_\alpha$ *descends to a map*:

$$\mathscr{A}^{\rm gp,0}(\circledast_X, \Lambda_{\rm loc}) \to \mathscr{A}^{\rm gp,0}(\circledast_X, \Lambda_{\rm loc})$$

Proof. We need to show that the group-like basing relations are preserved. With the notation and conventions of [12, Section 3], there are two group-like basing relations $\beta_1^{\rm gp}$ and $\beta_2^{\rm gp}$ on diagrams. It is easy to see that the $\beta_1^{\rm gp}$ basing relation is preserved. The $\beta_2^{\rm gp}$ relation (denoted by $\overset{\beta_2^{\rm gp}}{\sim}$) is generated in terms of a move of pushing t on all legs (of some fixed color x) of a diagram. Given a diagram $s(x)$ with some x-colored legs, let $s(xt)$ denote the result of pushing t on every x-colored leg of $s(x)$. In order to show that the $\beta_2^{\rm gp}$ relation is preserved, we need to show that $\tau^{\rm rat}_\alpha(M, s(xt)) \overset{\beta^{\rm gp}}{\sim} \tau^{\rm rat}_\alpha(M, s(x))$.

Ignoring the matrix part (i.e., setting M the empty matrix), we can compute as follows:

$$\begin{aligned}
\tau^{\rm rat}_\alpha(s(xt)) &= \langle \phi_{t\to t{\rm e}^h}(s)(\phi_{t\to t{\rm e}^h}(xt)), \alpha(h)\rangle_h \\
&= \langle \phi_{t\to t{\rm e}^h}(s)(xt{\rm e}^{h'}), \alpha(h+h')\rangle_{h,h'} \quad \text{by Lemma 4.3} \\
&\overset{\beta_2^{\rm gp}}{\sim} \langle \phi_{t\to t{\rm e}^h}(s)(x{\rm e}^{h'}), \alpha(h+h')\rangle_{h,h'} \\
&\overset{\beta_1^{\rm gp}}{\sim} \langle \phi_{t\to t{\rm e}^h}(s)(x), \alpha(h+h')\rangle_{h,h'} \\
&= \langle \phi_{t\to t{\rm e}^h}(s)(x), \alpha(h)\rangle_{h,h'} \\
&= \tau^{\rm rat}_\alpha(s(x)).
\end{aligned}$$

The same calculation can be performed when we include the matrix part, to conclude that $\tau^{\rm rat}_\alpha(M, s(xt)) \overset{\beta^{\rm gp}}{\sim} \tau^{\rm rat}_\alpha(M, s(x))$. □

The next lemma about $\tau^{\rm rat}$ should be compared with Lemma 4.8 about τ.

Lemma 4.8. *We have*

$$\tau^{\rm rat}_\alpha \circ \tau^{\rm rat}_\beta = \tau^{\rm rat}_{\alpha\#\beta}.$$

Proof. Observe that

$$\langle {\rm e}^h{\rm e}^{h'}, \alpha(h) \sqcup \beta(h')\rangle_{h,h'} = \chi(\alpha)\#\chi(\beta) = \langle {\rm e}^h, \sigma(\chi(\alpha)\#\chi(\beta))\rangle_h. \tag{5}$$

In [12, Section 3], it was shown that the "determinant" function ψ is multiplicative, in the sense that (for suitable matrices A, B) we have

$$\psi(AB) = \psi(A)\psi(B). \tag{6}$$

Let us define ${\rm pr}: \mathscr{A}^{\rm gp,0} \to \mathscr{A}^{\rm gp}$ to be the projection $(M,s) \to s$. It suffices to show that ${\rm pr} \circ \tau^{\rm rat}_\alpha \circ \tau^{\rm rat}_\beta = {\rm pr} \circ \tau^{\rm rat}_{\alpha\#\beta}$. We compute this as follows:

$$\begin{aligned}
{\rm pr} \circ \tau^{\rm rat}_{\alpha\#\beta}(M,s) &= \langle \psi(M(t{\rm e}^h)M(t)^{-1})\, \phi_{t\to t{\rm e}^h}(s), (\alpha\#\beta)(h)\rangle_h \\
&= \langle \psi(M(t{\rm e}^h{\rm e}^{h'})M(t)^{-1})\, \phi_{t\to t{\rm e}^h{\rm e}^{h'}}(s), \alpha(h) \sqcup \beta(h')\rangle_{h,h'} \quad \text{by (5)}
\end{aligned}$$

$$= \langle\langle \psi(M(te^{h}e^{h'})M(te^{h'})^{-1})\psi(M(te^{h'})M(t)^{-1})$$
$$\times \phi_{t\to te^{h}e^{h'}}(s), \alpha(h)\rangle_h, \beta(h')\rangle_{h'} \quad \text{by (6)}$$
$$= \langle \psi(M(te^{h'})M(t)^{-1})\phi_{t\to te^{h'}} \langle \psi(M(te^{h})M(t)^{-1})\, \phi_{t\to te^{h}}(s), \alpha(h)\rangle_h, b(h')\rangle_{h'}$$
$$= \langle \psi(M(te^{h'})M(t)^{-1})\phi_{t\to te^{h'}} \circ \mathrm{pr} \circ \tau^{\mathrm{rat}}_{\alpha}(M,s), \beta(h')\rangle_{h'}$$
$$= \mathrm{pr} \circ \tau^{\mathrm{rat}}_{\beta}(\tau^{\mathrm{rat}}_{\alpha}(M,s)).$$

Since $\alpha\#\beta = \beta\#\alpha$, the result follows. □

Our next task is to relate the two notions $\tau, \tau^{\mathrm{rat}}$ of twisting. In order to do so, recall the map

$$\mathrm{Hair}_k\colon \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp}}(\circledast_{X\cup k})$$

of [12, Section 7.1] defined by the substitution

 $t \to \sum_{n=0}^{\infty} \frac{1}{n!}$  n h-labeled legs

and extended to a map

$$\mathrm{Hair}^{\Omega}_k : \mathscr{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp}}(\circledast_{X\cup k})$$

by

$$\mathrm{Hair}^{\Omega}_k(M,s) = \psi(M(e^{k})M(1)^{-1}) \sqcup \mathrm{Hair}_k(s) \sqcup \Omega(k).$$

Then,

Lemma 4.9. *The following diagram commutes*:

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{gp}}(\star) \times \mathscr{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}}) & \xrightarrow{\tau^{\mathrm{rat}}} & \mathscr{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}}) \\ \Big\downarrow {\scriptstyle Id\times \mathrm{Hair}^{\Omega}_k} & & \Big\downarrow {\scriptstyle \mathrm{Hair}^{\Omega}_k} \\ \mathscr{A}^{\mathrm{gp}}(\star) \times \mathscr{A}(\circledast_{X\cup k}) & \xrightarrow{\tau} & \mathscr{A}(\circledast_{X\cup k}) \end{array}$$

Proof. For $\alpha \in \mathscr{A}^{\mathrm{gp}}(\star)$ and $(M,s) \in \mathscr{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}})$, we have

$$\tau(\mathrm{Hair}^{\Omega}_k(M,x)) = \langle \mathrm{Hair}^{\Omega}_{k+h}(M,x)\Omega^{-1}(h), \alpha(h)\rangle_h$$
$$= \langle \Omega(k+h)\psi(M(e^{k+h})M(1)^{-1})x(t\to e^{k+h})\Omega^{-1}(h), \alpha(h)\rangle_h$$
$$= \langle \Omega(k+h)\psi(M(e^{k+h})M(1)^{-1})x(t\to e^{k+h}), \widehat{\Omega^{-1}}(\alpha)(h)\rangle_h \quad \text{by Lemma 4.3}$$
$$= \langle \Omega(k)\Omega(h)\psi(M(e^{k}e^{h})M(1)^{-1})x(t\to e^{k}e^{h}), \widehat{\Omega^{-1}}(\alpha)(h)\rangle_h \quad \text{by (4)}$$
$$= \langle \Omega(k)\psi(M(e^{k}e^{h})M(1)^{-1})x(t\to e^{k}e^{h}), (\hat{\Omega}\,\widehat{\Omega^{-1}})(\alpha)(h)\rangle_h \quad \text{by Lemma 4.3}$$
$$= \langle \Omega(k)\psi(M(e^{k}e^{h})M(1)^{-1})x(t\to e^{k}e^{h}), \alpha(h)\rangle_h$$

$$= \Omega(k)\psi(M(e^k)M(1)^{-1})\phi_{t\to e^k}\langle\psi(M(te^h)M(t)^{-1})x(t\to te^h),\alpha(h)\rangle_h \quad \text{by (6)}$$

$$= \Omega(k)\,\psi(M(e^k)M(1)^{-1})\,\phi_{t\to e^k}\mathrm{pr}\circ\tau_\alpha^{\mathrm{rat}}(M,x) \quad \text{by definition of } \tau^{\mathrm{rat}}$$

$$= \mathrm{Hair}_k^\Omega(\tau_\alpha^{\mathrm{rat}}(s)). \qquad \square$$

The above lemma among other things explains the rather strange definition of τ^{rat}.

Corollary 4.10. *For all $\alpha \in \mathscr{A}^{\mathrm{gp}}(\star)$ we have*

$$\mathrm{Hair}^\Omega \circ \tau_\alpha^{\mathrm{rat}} \circ Z^{\mathrm{rat}}(M,K) = \tau_\alpha \circ Z(M,K) \in \mathscr{A}^{\mathrm{gp}}(\star).$$

Proof. It follows from the above lemma, together with the fact that

$$\mathrm{Hair}^\Omega \circ Z^{\mathrm{rat}}(M,K) = Z(M,K) \in \mathscr{A}^{\mathrm{gp}}(\star),$$

shown in [12, Theorem 1.3]. $\square$

The next proposition states that τ^{rat} intertwines (i.e., commutes with) the integration map $\overset{\mathrm{rat}}{\int}$.

Proposition 4.11. *For all $X' \subset X$ and $\alpha \in \mathscr{A}^{\mathrm{gp}}(\star)$, the following diagram commutes*:

$$\begin{array}{ccc}
\mathscr{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}}) & \xrightarrow{\overset{\mathrm{rat}}{\int} dX'} & \mathscr{A}^{\mathrm{gp},0}(\circledast_{X-X'}\Lambda_{\mathrm{loc}}) \\
\tau_\alpha^{\mathrm{rat}}\downarrow & & \downarrow\tau_\alpha^{\mathrm{rat}} \\
\mathscr{A}^{\mathrm{gp},0}(\circledast_{X(p)}) & \xrightarrow{\overset{\mathrm{rat}}{\int} dX'} & \mathscr{A}^{\mathrm{gp},0}(\circledast_{X-X'}\Lambda_{\mathrm{loc}})
\end{array}$$

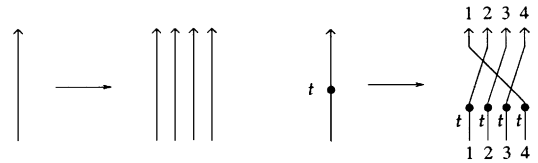
with the understanding that $\overset{\mathrm{rat}}{\int}$ is partially defined for X'-integrable elements.

Proof. This is proven in [12, Appendix E] and repeated in Appendix A. $\square$

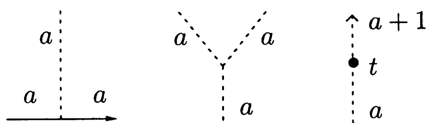
5. Lifting

5.1. *The definition of the* Lift_p *map*

The goal of this section is to define the map Lift_p and prove Theorem 4. We begin with a somewhat general situation. Consider a diagram D with skeleton $\uparrow_X$, whose edges are labeled by elements of Λ. For convenience, we express this by a diagram where there is a separate bead for each $t^{\pm 1}$. D consists of a solid part $\uparrow_X$ and a dashed part, that each have beads on them. The skeleton $X^{(p)}$ is defined by replacing each solid edge of $\uparrow_X$ by a parallel of p solid edges. The skeleton $X^{(p)}$ has beads $t^{\pm 1}$ and the connected components of $X^{(p)}-$(beads) are labeled by $\mathbb{Z}_p$ according to the figure shown below (for $p=4$)



There is a projection map $\pi_p : X^{(p)} \to X$. A *lift* of a diagram D on X is a diagram on $X^{(p)}$ whose dashed part is an isomorphic copy of the dashed part of D, where the location on $X^{(p)}$ of each univalent vertex maps under π_p to the location of the corresponding univalent vertex on X. A $\mathbb{Z}_p$*-labeling* of a diagram is an assignment of an element of $\mathbb{Z}_p$ to each of the dashed or solid edges that remain once we remove the beads of a diagram. A $\mathbb{Z}_p$-labeling of a diagram on $X^{(p)}$ is called p*-admissible* if (after inserting the beads) it locally looks like



Now, we define $\mathrm{Lift}_p(D)$ to be the sum of all diagrams on $X^{(p)}$ that arise, when all the labels and beads are forgotten, from all p-admissible labelings of all lifts of D. As usual, the sum over the empty set is equal to zero.

Remark 5.1. Here is an alternative description of $\mathrm{Lift}_p(D,\alpha)$ for a labeling α of the edges of D by monomials in t. Place a copy of (D,α) in ST in such a way that a bead t corresponds to an edge going around the hole of ST, as in [20, Section 2.1]. Look at the p-fold cover $\pi_p : ST \to ST$, and consider the preimage $\pi_p(D,\alpha) \subset ST \subset S^3$ as an abstract linear combination of diagrams without beads. This linear combination of diagrams equals to $\mathrm{Lift}_p(D,\alpha)$.

Remark 5.2. Notice that in case D has no skeleton, b connected components, and all the beads of its edges are 1, then $\mathrm{Lift}_p(D) = p^b D$.

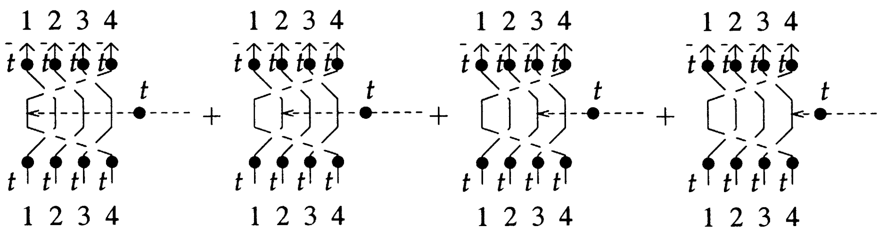
Lemma 5.3. *The above construction gives a well-defined map*

$$\mathrm{Lift}_p : \mathscr{A}(\uparrow_X, \Lambda) \to \mathscr{A}(\uparrow_{X^{(p)}}).$$

Proof. We need to show that the vertex invariance relations [12, Fig. 4] are preserved. There are two possibilities: the case that all three edges in a vertex invariance relation are dashed, and the case that two are part of the skeleton and the remaining is dashed.

In the first case, the vertex invariance relation is preserved because there is an obvious correspondence between lifts that admit an admissible labeling.

In the second case, the skeleton looks like (for $p=4$, with the convention that $\bar{t}=t^{-1}$)



and again there is a correspondence between p-admissible labelings of lifts of the two sides of the equation. □

There is a symmetrized version

$$\mathscr{A}(\star_X, \Lambda) \to \mathscr{A}(\star_{X^{(p)}})$$

of the Lift_p map, defined as follows: a lift of a diagram $D \in \mathscr{A}(\star_X, \Lambda)$ is a diagram in $\mathscr{A}(\star_{X^{(p)}}, \Lambda)$ which consists of the same dashed part as D, with each univalent vertex labeled by one of the p copies of the label of the univalent vertex of D that it corresponds to. There is an obvious notion of an admissible labeling of a diagram in $\mathscr{A}(\star_{X^{(p)}}, \Lambda)$, which is a labeling satisfying the conditions above, and also

$$\underset{x_i^{(a)}}{\vdots}\; a \in \mathbb{Z}_p \,.$$

Then, $\mathrm{Lift}_p(D)$ is defined to be the sum of all diagrams on $X^{(p)}$ that arise, when all the labels and beads are forgotten, from p-admissible labelings of lifts of D.

Lemma 5.4. (a) Lift_p *sends group-like elements to group-like elements and induces maps that fit in the commutative diagram*

$$\begin{array}{ccccc}
\mathscr{A}^{\mathrm{gp}}(\uparrow_X, \Lambda) & \xrightarrow{\sigma} & \mathscr{A}^{\mathrm{gp}}(\star_X, \Lambda) & \longrightarrow & \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda) \\
{\scriptstyle \mathrm{Lift}_p}\downarrow & & \downarrow{\scriptstyle \mathrm{Lift}_p} & & \downarrow{\scriptstyle \mathrm{Lift}_p} \\
\mathscr{A}^{\mathrm{gp}}(\uparrow_{X^{(p)}}) & \xrightarrow{\sigma} & \mathscr{A}^{\mathrm{gp}}(\star_{X^{(p)}}) & \longrightarrow & \mathscr{A}^{\mathrm{gp}}(\circledast_{X^{(p)}}).
\end{array}$$

(b) Lift_p *can be extended to a map* $\mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda^{(p)}_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp}}(\circledast_{X^{(p)}})$, *where* $\Lambda^{(p)}_{\mathrm{loc}}$ *is the subring of* Λ_{loc} *that consists of all rational functions whose denominators do not vanish at the complex pth root of unity.*

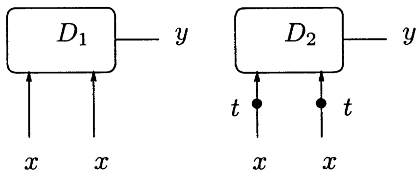
Proof. (a) Let us call an element of $\mathscr{A}(\uparrow_X, \Lambda)$ *special* if the beads of its skeleton equal to 1. Using the vertex invariance relations, it follows that $\mathscr{A}(\uparrow_X, \Lambda)$ is spanned by special elements.

It is easy to see that Lift_p maps group-like elements of $\mathscr{A}(\star_X, \Lambda)$ to group-like elements, and special group-like elements in $\mathscr{A}(\uparrow_X, \Lambda)$ to group-like elements in $\mathscr{A}(\uparrow_X, \Lambda)$. Further, it is easy to show that the diagram

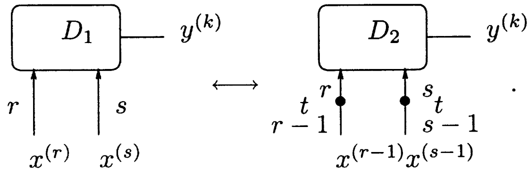
$$\begin{array}{ccc} \mathscr{A}(\uparrow_X, \Lambda) & \xrightarrow{\sigma} & \mathscr{A}(\star_X, \Lambda) \\ {\scriptstyle \mathrm{Lift}_p}\downarrow & & \downarrow{\scriptstyle \mathrm{Lift}_p} \\ \mathscr{A}(\uparrow_{X^{(p)}}) & \xrightarrow{\sigma} & \mathscr{A}(\star_{X^{(p)}}) \end{array}$$

commutes when evaluated at special elements of $\mathscr{A}(\uparrow_X, \Lambda)$. From this, it follows that the left square diagram of the lemma commutes.

For the right square, we need to show that the X-flavored *basing relations* in $\mathscr{A}^{\mathrm{gp}}(\star_X, \Lambda)$ are mapped to $X^{(p)}$-flavored basing relations in $\mathscr{A}^{\mathrm{gp}}(\star_{X^{(p)}})$. There are two kinds of X-flavored basing relations, denoted by β_1^{gp} and β_2^{gp} in [12, Section 3]. First, we consider β_2^{gp}. Take two elements s_1, s_2 such that $s_1 \overset{\beta_2^{\mathrm{gp}}}{\sim} s_2$; we may assume that s_2 is obtained from pushing t to each of the x-colored legs of s_1, for some $x \in X$. Corresponding to a diagram D_1 appearing in s_1, there exists a diagram D_2 of s_2 obtained by pushing t onto each of the x-colored legs of D_1. For example,



There is a 1–1 correspondence between admissible p-colorings of $\pi_p^{-1}(D_1)$ and those of $\pi_p^{-1}(D_2)$ (if we cyclically permute at the same time the labels $x^{(0)}, \ldots, x^{(p-1)}$), shown as follows:



Applying β_2^{gp} basing relations, the two results agree. In other words, $\mathrm{Lift}_p(D) \overset{\beta_2^{\mathrm{gp}}}{\sim} \mathrm{Lift}_p(D')$.

Now, consider the case of β_1^{gp}, (in the formulation of [12, Section 3]). Given $s_1 \overset{\beta_1^{\mathrm{gp}}}{\sim} s_2$, there exists an element $s \in \mathscr{A}^{\mathrm{gp}}(\star_{X \cup \partial h}, \Lambda)$ with some legs labeled by ∂h, such that

$$s_1 = \mathrm{con}_{\{h\}}(s),$$

$$s_2 = \mathrm{con}_{\{h\}}(s(x \to x\mathrm{e}^h))$$

for some $x \in X$, where $\mathrm{con}_{\{h\}}$ is the operation that contracts all ∂h legs of a diagram to all h legs of it. Now observe that

$$\begin{aligned} \mathrm{Lift}_p(s_2) &= \mathrm{Lift}_p(\mathrm{con}_{\{h\}}(s(x \to x\mathrm{e}^h))) \\ &= \mathrm{con}_{\{h^{(0)}, \ldots, h^{(p-1)}\}} \circ \mathrm{Lift}_p(s(x \to x\mathrm{e}^h)) \\ &= \mathrm{con}_{\{h^{(0)}, \ldots, h^{(p-1)}\}} \circ \mathrm{Lift}_p s(x^{(0)} \to x^{(0)}\mathrm{e}^{h^{(0)}}, \ldots, x^{(p-1)} \to x^{(p-1)}\mathrm{e}^{h^{(p-1)}}) \end{aligned}$$

$$\stackrel{\beta_1^{\rm gp}}{\sim} \mathrm{Lift}_p(s(h \to 0))$$
$$= \mathrm{Lift}_p(s_1).$$

(b) Notice first that Lift_p can be defined when beads are labeled by elements of $\mathbb{C}[t]/(t^p - 1)$. There is an isomorphism $\Lambda_{\rm loc}^{(p)}/(t^p - 1) \cong \mathbb{C}[t]/(t^p - 1)$ over $\mathbb{C}$ which gives rise (after composition with the projection $\Lambda_{\rm loc}^{(p)} \to \Lambda_{\rm loc}^{(p)}/(t^p - 1)$) to a map

$$\mathrm{ch}_p : \Lambda_{\rm loc}^{(p)} \to \mathbb{C}[t]/(t^p - 1). \tag{7}$$

Using this map, we can define Lift_p as before and check that the relations are preserved. □

Remark 5.5. Lift_p can also be extended to a map

$$\mathrm{Lift}_p : \mathscr{A}^{\rm gp,0}(\circledast_X, \Lambda_{\rm loc}^{(p)}) \to \mathscr{A}^{\rm gp}(\circledast_{X^{(p)}})$$

by forgetting the matrix part, i.e., by $\mathrm{Lift}_p(M, s) = \mathrm{Lift}_p(s)$.

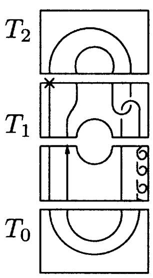
Let L be a surgery presentation of a pair (M, K) as in Section 2.2 and let $L^{(p)}$ be the lift of L to the p-fold cover of the solid torus, regarded as a link in S^3. The following proposition is a key point.

Proposition 5.6. *With the above notation, we have*

$$\check{Z}(L^{(p)}) = \mathrm{Lift}_p \circ \tau_{\alpha_p}^{\rm rat} \circ \check{Z}^{\rm rat}(L).$$

Proof. We begin by recalling first how $Z^{\rm rat}(L)$ is defined, following [12, Section 4]. The definition is given by representing L in terms of objects called *sliced crossed links* in a solid torus. Sliced crossed links are planar tangles of a specific shape that can be obtained from a generic height function of a link L in a standard solid torus ST. Each component of their corresponding link in ST is marked by a cross ($\times$). Given a null homotopic link L in ST, choose a sliced crossed link representative (T_0, T_1, T_2) where T_0 consists of local minima, T_2 consists of local maxima and T_1, thought of as a tangle in $I \times I$, equals to $I_w \sqcup \gamma$. Here w, the *gluing site*, is a sequence in $\uparrow$ and $\downarrow$, and $\bar{w}$ is the reverse sequence (where the reverse of $\uparrow\uparrow\downarrow$ is $\downarrow\uparrow\uparrow$).

For example, for $w = \downarrow\uparrow$, we may have the following presentation of a knot in ST:



where $\gamma =$ 

(and where the sliced crossed link is a tangle in an annulus). For typographical reasons, we will often say that (T_0, T_1, T_2) is the *closure* of the tangle γ.

Consider a representation of a null homotopic link L in ST by (T_0, T_1, T_2) as above. Recall that the fractional powers of v in the algebra $(A(\circledast), \#)$ are defined as follows: for integers n, m, $v^{n/m} \in A(\circledast)$ is the unique element whose constant term is 1 that satisfies $(v^{n/m})^m = v^n$.

Then, $Z^{\mathrm{rat}}(L)$ is defined as the element of $\mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$ obtained by composition of

$$(Z(T_0), I_{\bar{w}}(1) \otimes \Delta_w(v^{1/2}), I_{\bar{w}}(1) \otimes I_w(t), I_{\bar{w}}(1) \otimes Z(\gamma), I_{\bar{w}}(1) \otimes \Delta_w(v^{1/2}), Z(T_2)),$$

where $I_w(a)$ means a skeleton component that consists of solid arcs with orientations according to the arrows in w, with a (resp. $\bar{a}$) placed on each $\uparrow$ (resp. $\downarrow$), and Δ_w is the comultiplication obtained by replacing a solid segment $\uparrow$ by a w-parallel of it. After cutting the sliced crossed link at the crosses ($\times$), we consider the resulting composition of diagrams as an element of $\mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$. We claim that

Lemma 5.7. $\tau_\alpha^{\mathrm{rat}} \circ Z^{\mathrm{rat}}(L) \in \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$ *equals to the element obtained by composition of*

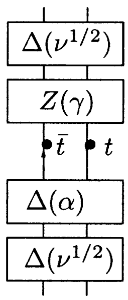
$$(Z(T_0), I_{\bar{w}}(1) \otimes \Delta_w(v^{1/2}), I_{\bar{w}}(1) \otimes \Delta_w(\alpha), I_{\bar{w}}(1) \otimes I_w(t), I_{\bar{w}}(1) \otimes Z(\gamma), I_{\bar{w}}(1) \otimes \Delta_w(v^{1/2}), Z(T_2)).$$

Proof. This follows easily from the definition of the $\tau_\alpha^{\mathrm{rat}}$ using the fact that the beads of the diagrams in $\check{Z}^{\mathrm{rat}}(L)$ appear only at the gluing site. □

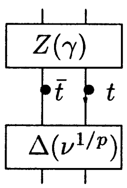
In short, we will say that $\tau_\alpha^{\mathrm{rat}} \circ Z^{\mathrm{rat}}(L)$ is obtained by the *closure* of the following sequence:

$$(\Delta_w(v^{1/2}), \Delta_w(\alpha), I_w(t), Z(\gamma), \Delta_w(v^{1/2})),$$

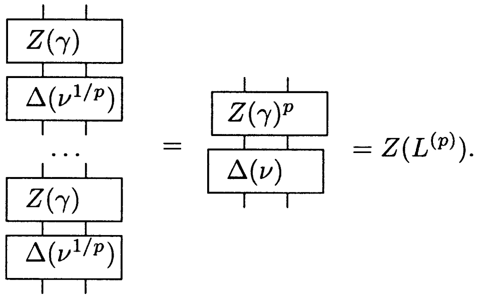
which we will draw schematically as follows:



Going back to the proof of Proposition 5.6, using $\alpha_p = v^{-(p-1)/p}$, and the group-like basing relations on $\mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda)$, it follows that we can slide and cancel the powers of v. Thus the closure of the above sequence for $\alpha = \alpha_p$, equals to the following sequence:



Now, we calculate Lift_p of the above sequence. Observe that both $Z(\gamma)$ and $v^{1/p}$ are exponentials of series of connected diagrams with symmetric legs whose dashed graphs are not marked by any nontrivial beads. Thus, one can check that Lift_p is the closure of the following diagram (there are p copies displayed):



The proposition follows for Z. The extension to the stated normalization $\check{Z}$ is trivial. □

The next proposition states that Lift_p intertwines the integration maps $\overset{\mathrm{rat}}{\int}$ and $\int$.

Proposition 5.8. *The following diagram commutes*:

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda_{\mathrm{loc}}) & \xrightarrow{\overset{\mathrm{rat}}{\int} dX} & \mathscr{A}^{\mathrm{gp}}(\Lambda_{\mathrm{loc}}) \\ \mathrm{Lift}_p \big\downarrow & & \big\downarrow \mathrm{Lift}_p \\ \mathscr{A}^{\mathrm{gp}}(\circledast_{X^{(p)}}) & \xrightarrow{\int dX^{(p)}} & \mathscr{A}^{\mathrm{gp}}(\phi) \end{array}$$

Since $\overset{\mathrm{rat}}{\int}$, $\int$ and Lift_p are partially defined maps (defined for X-integrable elements and for diagrams with nonsingular beads when evaluated that complex pth roots of unity), the maps in the above diagram should be restricted to the domain of definition of the maps, and the diagram then commutes, as the proof shows.

Proof of Proposition 5.8. Consider a pair (M,s) where s is given by

$$s = \exp\left(\frac{1}{2}\sum_{i,j} \underset{x_j}{\overset{x_i}{\updownarrow}} W_{ij}(t)\right) \sqcup R.$$

If we write

$$\mathrm{Lift}_p(s) = \exp\left(\frac{1}{2}\sum_{i,j}\sum_{r=0}^{p-1}\sum_{s=0}^{p-1} \underset{x_i^{(r)}}{\overset{x_j^{(s)}}{\uparrow}} W^{(p)}_{(i,r),(j,s)}\right) \sqcup R',$$

then, observe that

$$\mathrm{Lift}_p\left(\underset{x_j}{\overset{x_i}{\updownarrow}} W_{ij}(t)\right) = \sum_{r=0}^{p-1}\sum_{s=0}^{p-1} \underset{x_i^{(r)}}{\overset{x_j^{(s)}}{\uparrow}} W^{(p)}_{(i,r),(j,s)},$$

$$\mathrm{Lift}_p(R) = R'.$$

Recall the map $\mathrm{ch}_p : \Lambda^{(p)}_{\mathrm{loc}} \to \mathbb{C}[t]/(t^p-1)$ of Equation (7). It follows from the above that for any r we have

$$\mathrm{ch}_p(W_{ij}(t)) = \sum_{s=0}^{p-1} W^{(p)}_{(i,r),(j,s)} t^{s-r}.$$

We wish to determine $\mathrm{ch}_p(W_{ij}(t)^{-1})$, which we write as

$$\mathrm{ch}_p(W_{ij}(t)^{-1}) = \sum_{s=0}^{p-1} W^{(p)'}_{(i,r),(j,s)} t^{s-r}.$$

Since $\delta_{ij} = \sum_k W_{ik} W^{-1}_{kj}$, we can solve for $W^{(p)'}_{(i,r),(j,s)}$ in terms of $W^{(p)}_{(i,r),(j,s)}$ and obtain that

$$\mathrm{Lift}_p\left(\overset{x_i}{\underset{x_j}{\updownarrow}} W^{-1}_{ij}(t) \right) = \sum_{r=0}^{p-1}\sum_{s=0}^{p-1} \overset{x_j^{(s)}}{\underset{x_i^{(r)}}{\uparrow}} (W^{(p)})^{-1}_{(i,r),(j,s)}.$$

Observe further the following consequence of the "state-sum" definition of Lift_p: for diagrams D_1, D_2 in $\mathscr{A}(\star_X, \Lambda_{\mathrm{loc}})$, we have that

$$\mathrm{Lift}_p(\langle D_1, D_2\rangle_X) = \langle \mathrm{Lift}_p(D_1), \mathrm{Lift}_p(D_2)\rangle_{X^{(p)}} \in \mathscr{A}(\phi).$$

Now, we can finish the proof of the proposition as follows:

$$\begin{aligned}
\mathrm{Lift}_p\left(\int^{\mathrm{rat}} \mathrm{d}X(s)\right) &= \mathrm{Lift}_p\left(\left\langle \exp\left(-\frac{1}{2}\sum_{i,j} \overset{x_i}{\underset{x_j}{\updownarrow}} W^{-1}_{ij}(t)\right), R\right\rangle\right)_X \\
&= \left\langle \mathrm{Lift}_p\left(\exp\left(-\frac{1}{2}\sum_{i,j} \overset{x_i}{\underset{x_j}{\updownarrow}} W^{-1}_{ij}(t)\right)\right), \mathrm{Lift}_p(R)\right\rangle_{X^{(p)}} \\
&= \left\langle \exp\left(-\frac{1}{2}\sum_{i,j}\sum_{r=0}^{p-1}\sum_{s=0}^{p-1} \overset{x_j^{(s)}}{\underset{x_i^{(r)}}{\uparrow}} (W^{(p)})^{-1}_{(i,r),(j,s)}\right), R'\right\rangle_{X^{(p)}} \\
&= \int \mathrm{d}X^{(p)}\, \mathrm{Lift}_p(s). \qquad \square
\end{aligned}$$

Proof of Theorem 4. It follows immediately from Propositions 5.6 and 5.8. □

5.2. *The connection of* Lift$_p$ *with* mod p *residues*

Rozansky [26] considered the following vector space $\mathscr{A}^R$ to lift the Kontsevich integral,

$$\mathscr{A}^R = (\oplus_\Gamma A^R_{\Gamma,\mathrm{loc}} \cdot \Gamma)/(\mathrm{AS}, \mathrm{IHX}),$$

where the sum is over trivalent graphs Γ with oriented vertices and edges, and where

$$A^R_{\Gamma,\mathrm{loc}} = (\mathbb{Q}[\exp(H^1(\Gamma,\mathbb{Z}))]_{\mathrm{loc}})^\Gamma$$

is the Γ-invariant subring of the (Cohn) localization of the group-ring $\mathbb{Q}[\exp(H^1(\Gamma,\mathbb{Z}))]$ with respect to the ideal of elements that augment to ± 1. We will think of $A^R_{\Gamma,\mathrm{loc}}$ as the coefficients by which a graph Γ is multiplied.

Note that $\mathbb{Q}[\exp(H^1(\Gamma,\mathbb{Z}))]$ can be identified with the ring of Laurent polynomials in $b_1(\Gamma)$ variables, where $b_1(\Gamma)$ is the first betti number of Γ. Thus $\Lambda_{\Gamma,\mathrm{loc}}$ can be identified with the ring of rational functions $p(s)/q(s)$ in $b_1(\Gamma)$ variables $\{s\}$ for polynomials p and q such that $q(1)=\pm 1$. Let $\Lambda^{(p)}_{\Gamma,\mathrm{loc}}$ denote the subring of $\Lambda_{\Gamma,\mathrm{loc}}$ that consists of functions $p(s)/q(s)$ as above such that q, evaluated at any complex pth roots of unity is nonzero. In [13] (see also [21]) the authors considered a map:

$$\mathrm{Res}_p : A^{R,(p)}_{\Gamma,\mathrm{loc}} \to \mathbb{C}$$

defined by

$$\mathrm{Res}_p\left(\frac{f(s)}{g(s)}\right) = p^{\chi(\Gamma)} \sum_{\omega^p=1} \frac{f(\omega)}{g(\omega)},$$

where the sum is over all $b_1(\Gamma)$-tuples $(\omega_1,\ldots,w_{b_1(\Gamma)})$ of complex pth root of unity and where $\chi(\Gamma)$ is the Euler characteristic of Γ. This gives rise to a map $\mathrm{Res}_p : \mathscr{A}^R \to \mathscr{A}(\phi)$.

Similarly, we have that

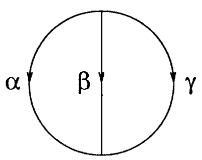
$$\mathscr{A}(\Lambda_{\mathrm{loc}}) = (\oplus_\Gamma A_\Gamma(\Lambda_{\mathrm{loc}})\cdot\Gamma)/(\mathrm{Relations}),$$

where $A_\Gamma(\Lambda_{\mathrm{loc}})$ is the Γ-invariant subspace of the vector space spanned by α: $\mathrm{Edge}(\Gamma) \to \Lambda_{\mathrm{loc}}$ modulo the Relations of Garoufalidis and Kricker [12, Figs. 2,3] which include the AS,IHX relations, multilinearity on the beads of the edges and the Vertex Invariance Relation. An important difference between $\mathscr{A}^R$ and $\mathscr{A}(\Lambda_{\mathrm{loc}})$ is the fact that $A^R_{\Gamma,\mathrm{loc}}$ is an *algebra* whereas $A_\Gamma(\Lambda_{\mathrm{loc}})$ is only a *vector space*. Nevertheless, there is a map $\phi_{R,\Gamma} : A_\Gamma(\Lambda_{\mathrm{loc}}) \to A^R_{\Gamma,\mathrm{loc}}$ defined by

$$\phi_{R,\Gamma}(\alpha) = \frac{1}{\mathrm{Aut}(\Gamma)} \sum_{\sigma\in\mathrm{Aut}(\Gamma)} \prod_{e\in\mathrm{Edge}(\Gamma)} \alpha_e(t_{\sigma(e)})$$

where $\alpha=(\alpha_e(t))$: $\mathrm{Edge}(\Gamma) \to \Lambda_{\mathrm{loc}}$. The maps $\phi_{R,\Gamma}$ assemble together to define a map $\phi_R : \mathscr{A}(\Lambda_{\mathrm{loc}}) \to \mathscr{A}^R$.

For example, consider the trivalent graph Θ whose edges are labeled by $\alpha,\beta,\gamma \in \Lambda_{\mathrm{loc}}$ as shown below



with automorphism group $\mathrm{Aut}(\Theta) = \mathrm{Sym}_2 \times \mathrm{Sym}_3$ that acts on the algebra of rational functions in three variables by permuting the variables and by inverting all variables simultaneously. Then, we have

$$\phi_{R,\Theta}(\alpha,\beta,\gamma)=\frac{1}{12}\sum_{\sigma\in \mathrm{Aut}(\Theta)}\alpha(t_{\sigma(1)})\beta(t_{\sigma(2)})\gamma(t_{\sigma(3)})\in\mathbb{Q}(t_1,t_2,t_3).$$

We finish by giving a promised relation between Lift_p and Res_p for p-regular rational functions:

Theorem 7. *The following diagram commutes*:

$$\begin{array}{ccc} \mathscr{A}(\Lambda^{(p)}_{\mathrm{loc}}) & \xrightarrow{\phi_R} & \mathscr{A}^R \\ \mathrm{Lift}_p \searrow & & \swarrow \mathrm{Res}_p \\ & \mathscr{A}(\phi). & \end{array}$$

Proof. Using the properties of Lift_p and Res_p it suffices to consider only trivalent graphs Γ with edges decorated by elements in Λ, and in fact only those graphs whose edges are decorated by powers of t. Moreover, since both Lift_p and Res_p satisfy the push relations, it suffices to consider graphs whose edges along any forest are labeled by 1.

Fix a trivalent graph Γ with ordered edges e_i decorated by $\alpha=(t^{m_1},\dots,t^{m_{3n}})$. We begin by giving a description of the algebra $A_{R,\mathrm{loc}}$ in terms of local coordinates as follows. Choose a maximal forest T and assume, without loss of generality, that the edges of $\Gamma\setminus T$ are $e_1,\dots,e_b$ where $b=b_1(\Gamma)$. Each edge e_i corresponds to a 1-cocycle $x_i\in C^1(\Gamma,\mathbb{Q})$. Since $H^1(\Gamma,\mathbb{Z})=\mathrm{Ker}(C^1(\Gamma,\mathbb{Z})\to C^0(\Gamma,\mathbb{Z}))$, it follows that $H^1(\Gamma,\mathbb{Z})$ is a (free) abelian group with generators $x_1,\dots,x_{3n}$ and relations $\sum_{j:\,v\in\partial e_j}\varepsilon_{j,v}x_j=0$ for all vertices v of Γ and for appropriate local orientation signs $\varepsilon_{j,v}=\pm1$. It follows that

$$\mathbb{Q}[H^1(\Gamma,\mathbb{Q})]=\frac{\mathbb{Q}[t_1^{\pm1},\dots,t_{3n}^{\pm1}]}{\left(\prod_{j:\,v\in\partial e_j}t_j^{\varepsilon_{j,v}}=1\ \text{for}\ v\in\mathrm{vertex}(\Gamma)\right)}=\mathbb{Q}[t_1^{\pm1},\dots,t_b^{\pm1}],$$

where $t_i=\mathrm{e}^{x_i}$. This implies that $A^R_{\Gamma,\mathrm{loc}}$ is a Γ-invariant subalgebra of $\mathbb{Q}(t_1,\dots,t_b)$.

Now, $\phi_{R,\Gamma}(\alpha)$ is obtained by symmetrizing over Γ-automorphisms of the monomial $t_1^{m_1}\dots t_b^{m_b}$. We may assume that $m_i\in\{0,\dots,p-1\}$ for all i. Thus,

$$\mathrm{Res}_p(t_1^{m_1}\dots t_b^{m_b})=\frac{p^{b_0(\Gamma)}}{p^b}\left(\sum_{\omega_1^p=1}\omega_1^{m_1}\right)\dots\left(\sum_{\omega_b^p=1}\omega_b^{m_b}\right)=p^{b_0(\Gamma)}\delta_{m_1,0}\dots\delta_{m_b,0}.$$

On the other hand, an admissible p-coloring of (Γ,α) necessarily assigns the same color to each connected component of Γ and then the consistency relations along the edges e_i for $i=1,\dots,b$ show that an admissible coloring exists only if $m_i=0$, for $i=1,\dots,b$, and in that case there are p admissible colorings for each connected component of Γ. Thus, the number of admissible p-colorings is $p_0^b(\Gamma)$. After symmetrization over Γ, the result follows. □

The reader is encouraged to compare the above proof with [21, Lemmas 3.4.1, 3.4.2].

5.3. The degree 2 part of Z^{rat}

In this section, we prove Corollary 1.2. The following lemma reformulates where $Q = Z_2^{\text{rat}}$ takes values. Consider the vector space

$$\Lambda_\Theta = \otimes^3 \Lambda_{\text{loc}} / ((f, g, h) = (tf, tg, th), \text{ Aut}(\Theta))$$

$\text{Aut}(\Theta) = \text{Sym}_3 \times \text{Sym}_2$ acts on $\otimes^3 \Lambda_{\text{loc}}$ by permuting the three factors and by applying the involution of Λ_{loc} simultaneously to all three factors.

Lemma 5.9. *Q takes values in $\Lambda_\Theta \cdot \Theta$.*

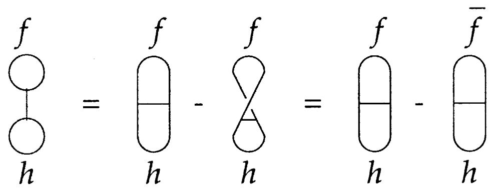
Proof. There are two trivalent graphs of degree 2, namely Θ and $\bigcirc\!\!-\!\!\bigcirc$. Label the three oriented edges of $\bigcirc\!\!-\!\!\bigcirc$ e_i for $i = 1, 2, 3$ where e_2 is the label in the middle (nonloop) edge of $\bigcirc\!\!-\!\!\bigcirc$. For $f, g, h \in \Lambda_{\text{loc}}$, let $\alpha_{\bigcirc\!-\!\bigcirc}(f, g, h) \in \alpha_{\bigcirc\!-\!\bigcirc}(\Lambda_{\text{loc}}) \cdot \bigcirc\!\!-\!\!\bigcirc$ denote the corresponding element.

For $p, q \in \Lambda$, $f, h \in \Lambda_{\text{loc}}$, we write $q = \sum_k a_k t^k$ and compute

$$\begin{aligned} \alpha_{\bigcirc\!-\!\bigcirc}(f, p, h) &= \alpha(f, (p/q).q, h) \\ &= \sum_k \alpha(f, (p/q) a_k t^k, h) \quad \text{by multilinearity} \\ &= \sum_k \alpha_{(f,\, p/q,\, a_k t^k h t^{-k})} \quad \text{by the vertex invariance relation} \\ &= \alpha(f, p/q, q(1)h). \end{aligned}$$

Thus, $\alpha_{\bigcirc\!-\!\bigcirc}(\Lambda_{\text{loc}})$ is spanned by $\alpha(f, p, h)$ for f, p, h as above. Applying the above reasoning once again, it follows that $\alpha_{\bigcirc\!-\!\bigcirc}(\Lambda_{\text{loc}})$ is spanned by $\alpha(f, 1, h)$ for f, h as above.

Applying the IHX relation



it follows that the natural map $\Lambda_\Theta \to \mathscr{A}_2(\Lambda_{\text{loc}})$ is onto. It is easy to see that it is also 1–1, thus a vector space isomorphism. □

Remark 5.10. In fact, one can show that Q takes values in the abelian subgroup $\Lambda_{\Theta,\mathbb{Z}}$ of Λ_Θ generated by $\otimes^3 \Lambda_{\mathbb{Z}}$.

Proof of Corollary 1.2. Consider the degree 2 part in the Equation of Theorem 1. On the one hand, we have $Z_2 = 1/2\lambda \cdot \Theta$ (see [22, Section 5.2]) and on the other hand, it follows by definition and Lemma 5.9 that $Z_2^{\text{rat}} = 1/6Q \cdot \Theta$. Theorem 7 which compares liftings and residues concludes first part of the corollary.

For the second part, observe that Q is a rational function on $S^1 \times S^1$, which is regular when evaluated at complex roots of unity. Furthermore, by definition of Res_p, it follows that

$$\frac{1}{p}\,\mathrm{Res}_p^{t_1,t_2,t_3} Q(M,K) = \frac{1}{p^2} \sum_{\omega_1^p=\omega_2^p=1} Q(M,K)(\omega_1,\omega_2,(\omega_1\omega_2)^{-1})$$

is the average of $Q(M,K)$ on $S^1 \times S^1$ (evaluated at pairs of complex pth roots of unity) and converges to $\int_{S^1\times S^1} Q(M,K)(s)\,\mathrm{d}\mu(s)$. This concludes the proof of Corollary 1.2. □

6. Remembering the knot

In this section, we will briefly discuss an extension of Theorem 1 for invariants of cyclic branched covers in the presence of the lift of the branch locus.

We begin by noting that the rational invariant Z^{rat} can be extended to an invariant of pairs (M,K) of *null homologous knots K in rational homology 3-spheres M*, [12]. The extended invariant (which we will denote by the same name), takes values in $\mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}) = \mathscr{B}(\Lambda_{\mathbb{Z}} \to \mathbb{Q}) \times \mathscr{A}^{\mathrm{gp}}(\Lambda_{\mathrm{loc}})$. In this section, we will work in this generality.

Consider a pair (M,K) of a null homologous knot K in a rational homology 3-sphere M, and the corresponding cyclic branched covers $\Sigma^p_{(M,K)}$. The preimage of K in $\Sigma^p_{(M,K)}$ is a knot K_{br}, which we claim is null homologous. Indeed, we can construct the branched coverings by cutting $M - K$ along a Seifert surface of K and gluing several copies side by side. This implies that a Seifert surface of K in M lifts to a Seifert surface of K_{br} in $\Sigma^p_{(M,K)}$.

If we wish, we may think of K_{br} as a 0-framed knot in $\Sigma^p_{(M,K)}$ (where a 0-framing is obtained by a parallel of K_{br} along a Seifert surface, and is independent of the Seifert surface chosen).

We now consider the rational invariant $Z^{\mathrm{rat}}(\Sigma^p_{(M,K)}, K_{\mathrm{br}})$ of a *p-regular pair* (M,K), that is a pair such that M and $\Sigma^p_{(M,K)}$ are rational homology 3-spheres and K is null homologous in M. For the rational version of the lift map

$$\mathrm{Lift}_p^{\mathrm{rat}} : \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}).$$

defined below, we have the following improved version of Theorem 1.

Theorem 8. *For all p and p-regular pairs (M,K), we have*

$$Z^{\mathrm{rat}}(\Sigma^p_{(M,K)}, K_{\mathrm{br}}) = \mathrm{e}^{\sigma_p(M,K)\Theta/16}\mathrm{Lift}_p^{\mathrm{rat}} \circ \tau^{\mathrm{rat}}_{\alpha_p} \circ Z^{\mathrm{rat}}(M,K) \in \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}),$$

where $\alpha_p = v^{-(p-1)/p}$, $v = Z(S^3, \mathrm{unknot})$.

The meaning of multiplying elements $(M,s) \in \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}})$ by elements $a \in \mathscr{A}(\phi)$ is as follows: $a \cdot (M,s) = (M, a \sqcup s)$.

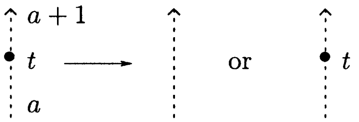
Remark 6.1. Evaluating $\mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}) \to \mathscr{A}^{\mathrm{gp}}(\phi)$ at $t=1$ corresponds to forgetting the knot K_{br}, thus the above theorem is an improved version of Theorem 1.

The proof of Theorem 8, which is left as an exercise, follows the same lines as the proof of Theorem 1 using properties of the $\mathrm{Lift}_p^{\mathrm{rat}}$ map rather than properties of the Lift_p map.

In the remaining section, we introduce the map $\mathrm{Lift}_p^{\mathrm{rat}}$ which is an enhancement of the map Lift_p of Section 5. We start by defining a map

$$\mathrm{Lift}_p^{\mathrm{rat}} : \mathscr{A}(\uparrow_X, \Lambda) \to \mathscr{A}(\uparrow_{X^{(p)}}, \Lambda).$$

This map is defined in exactly the same way as the map Lift_p of Lemma 5.3, except that instead of forgetting all labels as the last step, we do the following replacement:



depending on $a \neq p-1$ or $a = p-1$. As in Section 5.1, this leads to a well-defined map

$$\mathrm{Lift}_p^{\mathrm{rat}} : \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda) \to \mathscr{A}^{\mathrm{gp}}(\circledast_{X^{(p)}}, \Lambda).$$

The next step is to extend this to a map of diagrams with rational beads in $\Lambda_{\mathrm{loc}}^{(p)}$. The following lemma considers elements of the ring $\Lambda_{\mathrm{loc}}^{(p)}$.

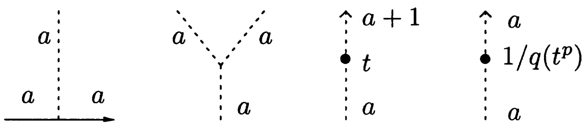
Lemma 6.2. *Every $r(t) \in \Lambda_{\mathrm{loc}}^{(p)}$ can be written in the form $r(t) = p(t)/q(t^p)$ where $p(t), q(t) \in \Lambda \otimes \mathbb{C}$.*

Proof. Using a partial fraction expansion of the denominator of $r(t)$, it suffices to assume that $r(t) = 1/(t-a)^k$ for some $k \geqslant 1$. In that case, we have

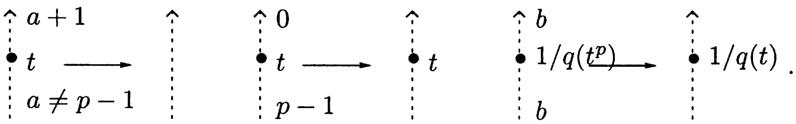
$$\frac{1}{t-a} = \frac{\prod_{i=1}^{p-1}(t - a\omega^i)}{t^p - a^p}$$

where $\omega = \exp(2\pi \mathrm{i}/p)$. □

Now, we can introduce the definition of $\mathrm{Lift}_p^{\mathrm{rat}}$ for diagrams with labels in $\Lambda_{\mathrm{loc}}^{(p)}$. Consider such a diagram D, and replace each bead $r(t)$ by a product of beads $p(t)\,1/q(t^p)$ using Lemma 6.2. Now, consider the diagrams obtained by p-admissible colorings of the lift $\pi_p^{-1}(D)$, that is colorings of the lift that satisfy the following conditions:



Finally forget the beads of the edges, as follows:



$\mathrm{Lift}_p^{\mathrm{rat}}(D)$ is defined to be the resulting combination of diagrams. We leave as an exercise to show that this is well-defined, independent of the quotient used in Lemma 6.2 above.

Remark 6.3. The map $\mathrm{Lift}_p^{\mathrm{rat}} : \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda_{\mathrm{loc}}^{(p)}) \to \mathscr{A}^{\mathrm{gp}}(\circledast_X, \Lambda_{\mathrm{loc}})$ is an algebra map, using the disjoint union multiplication.

Finally, we define

$$\mathrm{Lift}_p^{\mathrm{rat}} : \mathscr{A}^{\mathrm{gp},0}(\circledast_X, \Lambda_{\mathrm{loc}}^{(p)}) \to \mathscr{A}^{\mathrm{gp},0}(\circledast_{X^{(p)}}, \Lambda_{\mathrm{loc}})$$

by

$$\mathrm{Lift}_p^{\mathrm{rat}}(M(t), s) = (M(t \to T_t^{(p)}), \mathrm{Lift}_p^{\mathrm{rat}}(s)),$$

where $T_t^{(p)}$ is the p by p matrix (given by example for $p = 4$)

$$T_t^{(p)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ t & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

The substitution of Eq. (8) is motivated from the combinatorics of lifting struts (the analogue of Proposition 5.8 for the $\mathrm{Lift}_p^{\mathrm{rat}}$ map), but also from the following lemma from algebraic topology, that was communicated to us by Levine, and improved our understanding.

Lemma 6.4. *Consider a null homotopic link L in a standard solid torus ST, with equivariant linking matrix $A(t)$ and its lift $L^{(p)}$ in ST under the p-fold covering map $\pi_p : ST \to ST$. Then, $L^{(p)}$ is null homotopic in ST with equivariant linking matrix given by $A(t \to T_t^{(p)})$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{ST} & \overset{\pi'}{\to} & ST \\ & \searrow_{\pi} & \downarrow_{\pi_p} \\ & & ST \end{array}$$

where π and π' are universal covering maps. Since π_p is 1–1 on fundamental groups, it follows that $L^{(p)}$ is null homotopic in ST. Choose representatives L_i' of the components L_i of L in the universal cover $\widetilde{ST}$, for $i = 1, \ldots, l$ where l is the number of components of L. Then,

$$A_{ij}(t) = \sum_{k=0}^{\infty} \mathrm{lk}(L_i', t^k L_j') t^k.$$

On the other hand, $\{t^r L_i'\}$ is a choice of representatives of the lifts of $L^{(p)}$ to $\widetilde{ST}$, for $r = 0, \ldots, p-1$, and $l = 1, \ldots, l$. Furthermore, if $(B_{ij,rs}(t))$ is the equivariant linking matrix of $L^{(p)}$, we have

$$B(t)_{ij,rs}(t) = \sum_{k=0}^{\infty} \mathrm{lk}(t^r L_i', t^{k+j} L_j') t^k.$$

It follows that if we collect all powers of t modulo p in Laurent polynomials a_{ijk} such that

$$t^{r-s}A_{ij}(t) = \sum_{k=0}^{p-1} a_{ij,rs,k}(t^p)$$

(for $r,s = 0,\dots,p-1$), then

$$B_{ij,rs}(t) = a_{ij,rs,0}.$$

Writing this in matrix form, gives the result. □

We end this section with a comment regarding the commutativity of τ^{rat} and $\mathrm{Lift}_p^{\mathrm{rat}}$ as endomorphisms of $\mathscr{A}^{\mathrm{gp}}(\Lambda_{\mathrm{loc}}^{(p)})$:

Lemma 6.5. *For $\alpha \in \mathscr{A}^{\mathrm{gp}}(\star)$, we have*

$$\mathrm{Lift}_p^{\mathrm{rat}} \circ \tau_\alpha^{\mathrm{rat}} = \tau_{\alpha^p}^{\mathrm{rat}} \circ \mathrm{Lift}_p^{\mathrm{rat}}.$$

7. The wheeled invariants

The goal of this independent section is to discuss the relation between twisting and wheeling of diagrams and, as an application, to give an alternative version of Theorem 1 in terms of the wheeled rational invariant $Z^{\mathrm{rat},\text{¤}}$ introduced below.

Recall the wheeling and unwheeling maps from Section 4.2.

Lemma 7.1. *For $x \in \mathscr{A}(\star)$, we have*

$$\tau_\Omega(x) = \langle \Omega, \Omega \rangle^{-1} x^{\text{¤}^{-1}},$$

$$\tau_{\Omega_\#^{-1}}(x) = \langle \Omega, \Omega \rangle x^{\text{¤}},$$

where the notation $\Omega_\#^{-1}$ means the inverse of $\Omega \in \mathscr{A}(\star)$ using the # multiplication (rather than the disjoint union multiplication).

Note that $\chi(\Omega_\#^r) = v^r$, for all $r \in \mathbb{Q}$, by notation.

Proof. The first identity follows from Lemma 4.4(a) using the identity

$$\hat{\Omega}^{-1}(\Omega) = \langle \Omega, \Omega \rangle^{-1} \Omega$$

of [3, Proposition 3.3, Corollary 3.5].

The second identity follows from the first, after inverting the operators involved. Specifically, Lemma 4.4(b) implies that

$$\begin{aligned} y &= \tau_1(y) \\ &= \tau_{\Omega_\#^{-1}}(\tau_\Omega(y)) \end{aligned}$$

$$= \tau_{\Omega_{\#}^{-1}}(\langle \Omega, \Omega \rangle^{-1}\, y^{¤^{-1}})$$

$$= \langle \Omega, \Omega \rangle^{-1} \tau_{\Omega_{\#}^{-1}}(y^{¤^{-1}}).$$

Setting $x = y^{¤^{-1}}$, we have that $y = x^{¤}$ and the above implies that

$$x^{¤} = \langle \Omega, \Omega \rangle^{-1}\, \tau_{\Omega_{\#}^{-1}}(x). \qquad \square$$

The *wheeled invariant* $Z^{¤}$ is defined by wheeling the Z invariant of each of the component of a link. Although $Z^{¤}$ is an invariant of links equivalent to the Z invariant, in many cases the $Z^{¤}$ invariant behaves in a more natural way, as was explained in [3]. Similarly, we define the *wheeled rational invariant* $Z^{\mathrm{rat},¤}$ by

$$Z^{\mathrm{rat},¤}(M,K) = \tau^{\mathrm{rat}}_{\Omega_{\#}^{-1}} \circ Z^{\mathrm{rat}}(M,K) \in \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}).$$

The naming of $Z^{\mathrm{rat},¤}$ is justified by the following equation

$$\mathrm{Hair}^{\Omega} \circ Z^{\mathrm{rat},¤}(M,K) = \langle \Omega, \Omega \rangle Z^{¤}(M,K) \in \mathscr{A}(\star),$$

which follows from Corollary 4.10 (with $\alpha = \Omega_{\#}^{-1}$) and Lemma 7.1.

The rational wheeled invariant $Z^{\mathrm{rat},¤}$ behaves in some ways more naturally than the Z^{rat} invariant. A support of this belief is the following version of Theorem 8:

Theorem 9. *For all p and p-regular pairs (M,K) we have*

$$Z^{\mathrm{rat},¤}(\Sigma^{p}_{(M,K)}, K_{\mathrm{br}}) = \mathrm{e}^{\sigma_p(M,K)\Theta/16}\, \mathrm{Lift}_p \circ Z^{\mathrm{rat},¤}(M,K) \in \mathscr{A}^{\mathrm{gp},0}(\Lambda_{\mathrm{loc}}).$$

The proof uses the same formal calculation that proves Theorem 1, together with the following version of Theorem 4.

Theorem 10. *With the notation of Theorem* 4, *we have*

$$\mathrm{Lift}_p\left(\int^{\mathrm{rat}} \mathrm{d}X\ \check{Z}^{\mathrm{rat},¤}(L) \right) = \int \mathrm{d}X^{(p)} \check{Z}^{¤}(L^{(p)}).$$

The proof of Theorem 10 follows from the proof of Proposition 5.6.

Acknowledgements

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Appendix A. Diagrammatic calculus

In this section, we finish the proof of Theorem 4.11 using the identities and using the notation of [12, Appendices A–E]. The rest of the proof uses the function-theory properties of the $\overset{\mathrm{rat}}{\int}$-integration [12, Appendix A–E]. These properties are expressed in terms of the combinatorics of gluings of legs of diagrams. $\overset{\mathrm{rat}}{\int}$-integration is a diagrammatic formal Gaussian integration that mimics closely the Feynman diagram expansion of perturbative quantum field theory. Keep this in mind particularly with manipulations below called the "δ-function trick", "integration by parts lemma" and "completing the square". The uninitiated reader may consult [2, Part I,II] for examples and motivation of the combinatorial calculus and also [3,4,1]. We will follow the notation of [2,12] here.

We focus on the term $\overset{\mathrm{rat}}{\int} \mathrm{d}X(\varphi(\check{Z}^{\mathrm{rat}}(L)))$. Let us assume that the canonical decomposition of $\check{Z}^{\mathrm{rat}}(L)$ is

$$\check{Z}^{\mathrm{rat}}(L) = \exp\left(\frac{1}{2}\sum \overset{x_i}{\underset{x_j}{\updownarrow}} W_{ij}\right) \sqcup R,$$

suppressing summation indices. We perform a standard move (the "δ-function trick") to write this as

$$\left\langle R(y), \exp\left(\frac{1}{2}\sum \overset{x_i}{\underset{x_j}{\updownarrow}} W_{ij} + \overset{y_j}{\underset{x_i}{\updownarrow}}\right)\right\rangle_Y,$$

where Y is a set in 1–1 correspondence with X. Continuing,

$$\overset{\mathrm{rat}}{\int} \mathrm{d}X(\varphi(\check{Z}^{\mathrm{rat}}(L))) = \left\langle \varphi(R(y)), \overset{\mathrm{rat}}{\int} \mathrm{d}X \left(\exp\left(\frac{1}{2}\sum \varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow}} W_{ij}\right) + \overset{y_i}{\underset{x_i}{\updownarrow}}\right)\right)\right\rangle_Y.$$

The "integration by parts lemma" [12] implies that

$$\overset{\mathrm{rat}}{\int} \mathrm{d}X \left(\exp\left(\frac{1}{2}\sum \varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow}} W_{ij}\right) + \overset{y_j}{\underset{x_i}{\updownarrow}}\right)\right)$$

$$= \overset{\mathrm{rat}}{\int} \mathrm{d}X \left(\left(\exp\left(\sum \varphi\left(-\overset{x_j}{\underset{y_i}{\updownarrow}} W_{ij}^{-1}\right)\right)\right) \flat_X \exp\left(\frac{1}{2}\sum \varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow}} W_{ij}\right) + \overset{y_i}{\underset{x_i}{\updownarrow}}\right)\right).$$

"Completing the square" implies that the above equals to:

$$\exp\left(-\frac{1}{2}\sum \varphi\left(\overset{y_j}{\underset{y_i}{\updownarrow}} W_{ij}^{-1}\right)\right) \overset{\mathrm{rat}}{\int} \mathrm{d}X \left(\exp\left(\frac{1}{2}\sum \varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow}} W_{ij}\right)\right)\right).$$

Returning to the expression in question:

$$\int^{\mathrm{rat}} \mathrm{d}X(\varphi(\check{Z}^{\mathrm{rat}}(L))) = \int^{\mathrm{rat}} \mathrm{d}X\left(\exp\left(\frac{1}{2}\sum \varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij}\right)\right)\right)$$

$$\sqcup \left\langle \varphi\left(R(y)\right), \exp\left(-\frac{1}{2}\sum \varphi\left(\overset{y_j}{\underset{y_i}{\updownarrow\!\bullet}} W_{ij}^{-1}\right)\right)\right\rangle_Y .$$

The second factor equals to $\varphi(Z^{\mathrm{rat}}(M,K))$. The first factor contains only sums of disjoint union of wheels. We can repeat the arguments which lead to the proof of the of the Wheels identity in this case, [12, Appendix E].

$$\int^{\mathrm{rat}} \mathrm{d}X\left(\exp\left(\frac{1}{2}\sum \varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij}\right)\right)\right)$$

$$= \int^{\mathrm{rat}} \mathrm{d}X\left(\exp\left(\frac{1}{2}\sum\left(\overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij} + \left(\varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij}\right) - \overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij}\right)\right)\right)\right)$$

$$= \left\langle \exp\left(-\frac{1}{2}\sum \overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij}^{-1}\right), \exp\left(\frac{1}{2}\left(\varphi\left(\overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij}\right) - \overset{x_j}{\underset{x_i}{\updownarrow\!\bullet}} W_{ij}\right)\right)\right\rangle_X$$

$$= \exp\left(-\frac{1}{2}\operatorname{tr}\log(W^{-1}\varphi(W))\right). \qquad \square$$

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