On some classes of inverse series relations and their applications

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Abstract

This paper gives a brief exposition of several recent results obtained by the authors concerning some classes of inverse relations, including the general binomial-type inversions and two kinds of general Stirling reciprocal transforms. Also described are some applications of the related inversion techniques to combinatorics (including new proofs of Rogers–Ramanujan identities and MacMahon's master theorem), interpolation methods and certain problems related to special polynomials and number sequences.

1. General binomial-type inversions (univariate case)

By the general binomial-type inversion (GBI), we mean the pair of inverse relations due to Gould and Hsu [14]: Let $\{a_i\}$ and $\{b_i\}$ be any two sequences of real or complex numbers such that

$$\psi(x,n) = \prod_{k=1}^{n} (a_k + xb_k) \neq 0$$

for nonnegative integers x and n with $\psi(x,0)=1$. Then the following inverse relations hold:

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(k,n) g(k), \tag{1.1}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a_{k+1} + kb_{k+1}) \psi^{-1}(n,k+1) f(k), \tag{1.2}$$

where either $\{f(n)\}$ or $\{g(n)\}$ is an arbitrary given sequence. Moreover, there is a rotated form for the inverse relations $(1.1) \Leftrightarrow (1.2)$.

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It is known that Carlitz [3] has found a useful q-analogue of the above result. Denote

$$(a,b)_n = \prod_{i=1}^{n} (1 - ab^{i-1}), \qquad (a,b)_0 = 1.$$

Accordingly, Gaussian q-coefficient may be written as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = (q,q)_n / (q,q)_k (q,q)_{n-k}, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1 \quad (0 \le k \le n).$$

Let the sequence of q-polynomials

$$\phi(x,n \mid q) = \prod_{k=1}^{n} (a_k + q^x b_k)$$

be different from zero for nonnegative integers x and n with $\Phi(x,0 \mid q) = 1$. Then there holds the q-analogue of GBI:

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(k,n \mid q) g(k), \tag{1.3}$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (a_{k+1} + q^k b_{k+1}) \phi^{-1}(n,k+1 \mid q) f(k). \tag{1.4}$$

Very recently, we have shown that the inverse relations (1.3) $\Leftrightarrow$ (1.4) may be used as a bridge that leads to a new proof of Rogers–Ramanujan identities. This will now be sketched.

Consider the sequence transform for a matrix sequence $\{c_{nk}\}$:

$$\sum_{k=0}^{n} c_{n,k} g(k) = \sum_{i=0}^{n} f(i)(-1)^i (a_{i+1} + q^i b_{i+1}) \sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix} \frac{c_{n,k}}{\phi(k,i+1 \mid q)}, \tag{1.5}$$

where $\{g(k)\}$ and $\{f(k)\}$ are related by (1.3) and (1.4). The important step is to choose a suitable matrix $C = [c_{nk}]$ such that the inner sum contained on the right-hand side of (1.5) can be evaluated as a closed form. This will be done if we choose

$$\phi(x,n \mid q) := (aq^x, q)_n = (aq^x)_n, \qquad c_{n,k} := \begin{bmatrix} n \\ k \end{bmatrix} \frac{a^k q^{k^2}}{(aq)_{k-1}}.$$

Using the known identity

$$\sum_{k=0}^{n} q^{k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} t^k (t)_k^{-1} = (t)_n^{-1},$$

we obtain

$$\sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix} \frac{c_{n,k}}{\phi(k,i+1 \mid q)} = \begin{bmatrix} n \\ i \end{bmatrix} \frac{a^i q^{i^2}}{(aq)_{n+i}}.$$

Thus, (1.5) gives

$$\sum_{k=0}^{n} c_{n,k}\, g(k) = \sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} \frac{(1-aq^{2i})a^i q^{i^2}}{(aq)_{n+i}} f(i). \tag{1.6}$$

Now choose $g(k)=(aq)_{k-1}$. Making use of (1.3) and (1.4) and the identity

$$x^n = \sum_{k=0}^{n} (-1)^k q^{\binom{n-k}{2}-\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix} (x)_k,$$

we find

$$f(n) = a^n q^{n^2 + \binom{n}{2}} (aq)_{n-1}.$$

Substituting $\{g(k)\}$ and $\{f(k)\}$ into (1.6), we get the finite form of Rogers–Ramanujan's identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^k q^{k^2} = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} \frac{1-aq^{2j}}{(aq^j)_{n+1}} a^{2j} q^{2j^2 + \binom{j}{2}}. \tag{1.7}$$

Finally, letting $n \to \infty$ and putting $a=1$ and q in (1.7), respectively, and using the well-known triple-product formula of Jacobi,

$$(z,q)_\infty (z^{-1}q,q)_\infty (q,q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k, \tag{1.8}$$

we obtain the pair of Rogers–Ramanujan identities after some simplifications:

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q)_k} = \frac{1}{(q,q^5)_\infty (q^4,q^5)_\infty}, \tag{1.9}$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q,q)_k} = \frac{1}{(q^2,q^5)_\infty (q^3,q^5)_\infty}. \tag{1.10}$$

A certain class of combinatorial identities may be defined via (1.1) and (1.2) or their rotated forms. More precisely, any identity is said to belong to class Σ if it can be embedded in either of the forms (1.1) and (1.2). Similarly, an identity is said to belong to class Σ^* if it may be embedded in either of the rotated forms of (1.1) and (1.2). Recently, Hsu and Hsu [19] have verified that nearly 30% of the total 500 identities displayed in Gould's formulary [13] can be embedded in (1.1) or (1.2) or one of the rotated forms, so that they belong to the union of set $\Sigma \cup \Sigma^*$. In particular, it has been shown that the classical identities due to Abel, Hagen–Rothe, Jensen, Rohatgi, Moriarty, Van Ebbenhorst–Tengbergen all belong to $\Sigma \cup \Sigma^*$, so that they can be proved rather straightforwardly via the GBI or its rotated form.

In solving enumeration problems for graphs with labelled vertices, it is known that Liskovets conjectured that there exist polynomials $\phi_\lambda(x)$ of degree λ with integer

coefficient such that

$$\sum_{k=1}^{n} \binom{n-1}{k-1} n^{n-k} \phi_\lambda(k) \, (k+\lambda)! = (2\lambda)! \, n^{n+\lambda}. \tag{1.11}$$

The solution found by Egorychev [12] is expressed in terms of Stirling numbers of the second kind, viz.

$$\phi_\lambda(k) = \Delta^k O^{k+\lambda} \frac{(2\lambda)!}{(k+\lambda)!} \quad (k \geqslant 1, \ \lambda \geqslant 0). \tag{1.12}$$

Note that (1.11) may be rewritten in the form

$$\sum_{k=0}^{n} \binom{n}{k} n^{-(k+1)} k \, \phi_\lambda(k) \, (k+\lambda)! = (2\lambda)! \, n^{\lambda}, \tag{1.13}$$

so that it can be embedded in (1.2) by defining $\psi(n, k+1) = n^{k+1}$ (with $a_i = 0$, $b_i = 1$) and letting $f(k) = \phi_\lambda(k) \, (k+\lambda)! (-1)^k$ and $g(k) = (2\lambda)! k^{\lambda}$. Consequently, (1.13) may be inverted by (1.1) to get

$$f(n) = (-1)^n (n+\lambda)! \, \phi_\lambda(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^n (2\lambda)! \, k^{\lambda}$$

and

$$\phi_\lambda(n) = \frac{(2\lambda)!}{(n+\lambda)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{n+\lambda} = \frac{(2\lambda)!}{(n+\lambda)!} \Delta^n O^{n+\lambda}.$$

Apparently, this procedure for solution is much easier and more immediate than that presented in [12].

2. General binomial-type inversions (bivariate and multivariate cases)

Note that (1.1) and (1.2) can be expressed more briefly using the difference operator Δ, namely

$$f(n) = \left(-\frac{\Delta}{x}\right)_0^n \{g(x) \psi(x, n)\}, \tag{2.1}$$

$$g(n) = \left(-\frac{\Delta}{x}\right)_0^n \{f(x) (a_{x+1} + x b_{x+1}) \psi^{-1}(n, x+1)\}, \tag{2.2}$$

where $\frac{\Delta}{x}$ is the difference operator with unit increment in x, and

$$\left(-\frac{\Delta}{x}\right)_t^n f(x) = (-1)^n \left\{\frac{\Delta^n}{x} f(x)\right\}_{x=t}.$$

Thus, (2.2) may also be regarded as an explicit solution of difference equations defined by (2.1).

Not long ago, we established a bivariate extension of the GBI (1.1) and (1.2). Let $\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, \{e_i\}$, and $\{f_i\}$ be six sequences of complex numbers such that the binary polynomials

$$\phi(x, y; m, n) = \prod_{i=1}^{m} (a_i + b_i x + c_i y) \prod_{j=1}^{n} (d_j + e_j x + f_j y) \tag{2.3}$$

differ from zero for nonnegative integers x, y, m, n with the convention $\prod_{i=1}^{m} * = 1$ for $m = 0$. Define an adjunct function (a linear function of x and y) of the form

$$\varepsilon(x, y; m, n) = (a_{m+1} + m b_{m+1} + y c_{m+1})(d_{n+1} + x e_{n+1} + n f_{n+1})$$

$$- c_{m+1} e_{n+1} (x - m)(y - n). \tag{2.4}$$

Our main result is the following theorem (cf. [11]).

Theorem 2.1. *With ϕ and ε being defined by* (2.3) *and* (2.4) *we have the inverse relations*

$$g(m, n) = \left(-\frac{\Delta}{x}\right)_0^m \left(-\frac{\Delta}{y}\right)_0^n \{f(x, y) \phi(x, y; m, n)\}, \tag{2.5}$$

$$f(m, n) = \left(-\frac{\Delta}{x}\right)_0^m \left(-\frac{\Delta}{y}\right)_0^n \left\{g(x, y) \frac{\varepsilon(m, n; x, y)}{\phi(m, n; x+1, y+1)}\right\}. \tag{2.6}$$

This result has a nice application to a problem of rational interpolation on rectangular lattices. More precisely, let M and N be any positive integers. Then a kind of rational interpolation formula can be constructed as follows:

$$S(f; x, y) := \sum_{i=0}^{M} \sum_{j=0}^{N} \psi(x, y; i, j) \left(\frac{\Delta}{x}\right)_0^i \left(\frac{\Delta}{y}\right)_0^j \{f(x, y) \phi(x, y; i, j)\}, \tag{2.7}$$

where the basic functions $\psi(x, y; i, j)$ are rational functions defined by

$$\psi(x, y; i, j) = \binom{x}{i} \binom{y}{j} \frac{\varepsilon(x, y; i, j)}{\phi(x, y; i+1, j+1)}.$$

It is clear that the interpolation conditions

$$S(f; m, n) = f(m, n), \quad 0 \leqslant m \leqslant M, \quad 0 \leqslant n \leqslant N$$

are automatically satisfied in accordance with the inversion (2.5) and (2.6).

Generally, such a type of interpolation formula (2.7) is applicable to those interpolated functions that have algebraic singularities near the lines $a_i + x b_i + y c_i = 0$ and $d_j + x e_j + y f_j = 0$ ($1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N$). Evidently, the most special case of (2.7) is the binary Newton interpolation formula that just corresponds to the case $a_i = d_j = 1$ and $b_i = c_i = e_j = f_j = 0$ so that $\phi(x, y; i, j) = \varepsilon(x, y; i, j) = 1$.

Very recently, Chu [9] has established a multidimensional generalization of the GBI and its rotated form which can be used to deduce Carlitz's convolution formulas [4, 5] and MacMahon's master theorem as consequences.

We shall need several usual notations. Denote by N_0 and C the sets of nonnegative integers and complex numbers, respectively. For $\bar{x} = (x_1, x_2, \ldots, x_k)$, $\bar{y} = (y_1, y_2, \ldots, y_k) \in C^k$ and $\bar{n} = (n_1, n_2, \ldots, n_k) \in N_0^k$, we denote

$$|\bar{x}| = \sum_{i=1}^{k} x_i, \qquad \langle \bar{x}, \bar{y} \rangle = \sum_{i=1}^{k} x_i y_i, \qquad \binom{\bar{x}}{\bar{n}} = \prod_{i=1}^{k} \binom{x_i}{n_i}.$$

Moreover, we write $\bar{n}! = \prod_{i=1}^{k} n_i!$ and denote

$$\binom{x}{\bar{n}} = \frac{x(x-1) \cdots (x - |\bar{n}| + 1)}{\bar{n}!}.$$

In addition, $\bar{0} \in N_0^k$ will denote the zero vector and $I \in N_0^k$, the vector with each coordinate being one, respectively.

Let $\{c_{ij}(n) : 1 \leqslant i \leqslant k, 0 \leqslant j \leqslant k\}_n$ be a matrix sequence of complex numbers such that the polynomials

$$\psi(\bar{x}; \bar{n}) = \prod_{i=1}^{k} \prod_{j=1}^{n_i} (c_{i,0}(j) + \langle \bar{c}_i(j), \bar{x} \rangle) \tag{2.8}$$

differ from zero for $\bar{x}, \bar{n} \in N_0^k$ with the convention that empty product equals one, where $\bar{c}_i(j) = (c_{i1}(j), c_{i2}(j), \ldots, c_{ik}(j))$. Similarly, generalizing (2.4), we introduce an adjunct function in a form using $k \times k$ determinant

$$\varepsilon(\bar{x}; \bar{n}) = \det_{k \times k} [(c_{i0}(n_i+1) + \langle \bar{c}_i(n_i+1), \bar{x} \rangle) \delta_{ij} + (n_j - x_j) c_{ij}(n_i+1)], \tag{2.9}$$

where δ_{ij} is the Kronecker delta. Then by some technical calculation, one may obtain a multivariate extension of the GBI, namely the following theorem.

Theorem 2.2. *With ψ and ε being defined by (2.8) and (2.9), respectively, there hold the inverse relations*

$$F(\bar{n}) = \sum_{\bar{0} \leqslant \bar{r} \leqslant \bar{n}} (-1)^{|\bar{r}|} \binom{\bar{n}}{\bar{r}} \psi(\bar{r}; \bar{n}) G(\bar{r}), \tag{2.10}$$

$$G(\bar{n}) = \sum_{\bar{0} \leqslant \bar{r} \leqslant \bar{n}} (-1)^{|\bar{r}|} \binom{\bar{n}}{\bar{r}} \frac{\varepsilon(\bar{n}; \bar{r})}{\psi(\bar{n}; \bar{r} + I)} F(\bar{r}), \tag{2.11}$$

In particular, if

$$D_a(\bar{n}) = \det_{k \times k} [(a_i + \langle \bar{c}_i, \bar{n} \rangle) \delta_{ij} - n_j c_{ij}]$$

is denoted by $D_a(\bar{n})$, then the embedding technique implied by (2.10) and (2.11) will yield some convolution identities due to Carlitz [4, 5].

Moreover, using the rotated form of Theorem 2.2, one can prove the following expansions.

Theorem 2.3. *Let*

$$u_j = x_j \exp\left\{ -\sum_{i=1}^{k} c_{ij} x_i \right\}, \quad (j=1,2,\ldots,k), \tag{2.12}$$

$$v_j = y_j \prod_{i=1}^{k} (1+y_i) - c_{ij} - \delta_{ij}, \quad (j=1,2,\ldots,k). \tag{2.13}$$

Then the following expansions hold:

$$\det_{k \times k}{}^{-1} [\delta_{ij} - c_{ij} x_j] = \sum_{\bar{n} \geqslant 0} \prod_{i=1}^{k} \frac{\langle \bar{c}_i, \bar{n} \rangle^{n_i}}{n_i!} \prod_{i=1}^{k} u_i^{n_i} \tag{2.14}$$

$$\det_{k \times k}{}^{-1} [\delta_{ij} - c_{ij} y_j] = \sum_{\bar{n} \geqslant 0} \prod_{i=1}^{k} \binom{n_i + \langle \bar{c}_i, \bar{n} \rangle - 1}{n_i} \prod_{i=1}^{k} v_i^{n_i}. \tag{2.15}$$

It may easily be observed that the first expansion given above implies the famous master theorem of MacMahon. Certainly, the rational interpolation formula (2.7) can be generalized to the multivariate case via the inverse relations (2.10) and (2.11). For details, refer to [9].

3. General Stirling reciprocal transforms of the first kind

Let $\Gamma = (\Gamma, +, \cdot)$ denote the commutative ring of formal power series with real or complex coefficients, in which the ordinary addition and Cauchy multiplication are defined. Substitution of formal power series is defined as usual. Any two elements ϕ and ψ of Γ are said to be reciprocal (compositional inverse) of each other if $\phi(\psi(t)) = \psi(\phi(t)) = t$ with $\phi(0) = \psi(0) = 0$.

It is known that the Stirling numbers $S_1(n,k)$ and $S_2(n,k)$ of both kinds may be defined by the exponential generating functions $(f(t))^k/k!$ and $(g(t))^k/k!$, respectively, where $f(t) = \log(1+t)$ and $g(t) = e^t - 1$ just form a pair of reciprocal elements in Γ. Also, we have the pair of Stirling reciprocal transforms (inverse relations):

$$a_n = \sum_{k=0}^{n} S_1(n,k) b_k, \qquad b_n = \sum_{k=0}^{n} S_2(n,k) a_k.$$

The natural connection of the above inverse relations with the Stirling number pair defined by the pair of generating functions involving reciprocal elements has been investigated and generalized in [16, 17]. Indeed, it has been shown that the concept of a generalized Stirling number pair (GSN pair) can be completely characterized by a pair of inverse relations.

Recently, we have found two classes of extended GSN pairs that imply two kinds of Stirling reciprocal transforms. In what follows, we will need several basic concepts.

A sequence of polynomials $\{p_n(t)\}$ is said to be normal if $p_n(t)$ is of degree n with $p_0(t) = 1$ and $p_n(0) = 0$ $(n \geqslant 1)$. A normal sequence $\sigma = \{p_n(t)\}$ is called a basic set if every polynomial $p(t)$ has a unique representation as a linear sum $p(t) = \sum c_n p_n(t)$, c_n being real or complex coefficients.

Given a normal basic set $\sigma = \{p_n(t)\}$, a linear operator $\mathscr{P}$ is called a basic operator associated with σ if

$$\mathscr{P} p_0(t) = 0, \qquad \mathscr{P} p_n(t) = n p_{n-1}(t) \quad (n \geqslant 1)$$

and if for every formal series $\sum_0^\infty c_k p_k(t)$, we have

$$\mathscr{P} \sum_0^\infty c_k p_k(t) = \sum_1^\infty k c_k p_{k-1}(t).$$

The concept of a GSN pair described in [16] may now be extended by means of the following definition: Suppose that $\sigma = \{p_n(t)\}$ and $\tau = \{q_n(t)\}$ are two normal basic sets of polynomials. Let $\phi(t)$ and $\psi(t)$ be elements of Γ with $\phi(0) = \psi(0) = 0$, and let the following formal expansions hold:

$$p_k(\phi(t)) = \sum_{n \geqslant 0} A_1(n,k) q_n(t), \qquad q_k(\psi(t)) = \sum_{n \geqslant 0} A_2(n,k) p_n(t). \tag{3.1}$$

Then $\langle A_1(n,k), A_2(n,k) \rangle$ is called an extended GSN pair generated by $\{\sigma, \tau, \phi, \psi\}$ if and only if ϕ and ψ are reciprocal elements of Γ, viz. $\phi(\psi(t)) = \psi(\phi(t)) = t$. Making use of some algebraic manipulations with formal power series, it is not difficult to establish the following results.

Theorem 3.1. *Any pair* $\langle A_1(n,k), A_2(n,k) \rangle$ *defined by* (3.1) *is an extended GSN pair if and only if the following formal inverse series relations hold:*

$$y_n = \sum_{k \geqslant 0} A_1(n,k) x_k, \qquad x_n = \sum_{k \geqslant 0} A_2(n,k) y_k, \tag{3.2}$$

where either $\{x_k\}$ *or* $\{y_k\}$ *is any finite sequence.*

Corollary 3.2. *The infinite matrices* $[A_1(n,k)]$ *and* $[A_2(n,k)]$ *are reciprocal (inverse) of each other if and only if* ϕ *and* ψ *are reciprocal elements of* Γ.

Theorem 3.3. *Let* $\{p_n(t)\}$ *and* $\{q_n(t)\}$ *be two normal basic sets of polynomials associated with the basic operators* $\mathscr{P}$ *and* $\mathscr{Q}$, *respectively. Then the pair of general Stirling reciprocal transforms,*

$$y_n = \sum_{k \geqslant 0} \frac{1}{k!} [\mathscr{Q}^k p_n(\phi(t))]_{t=0} x_k, \tag{3.3}$$

$$x_n = \sum_{k \geqslant 0} \frac{1}{k!} [\mathscr{P}^k q_n(\psi(t))]_{t=0} y_k, \tag{3.4}$$

hold if and only if ϕ and ψ are reciprocal elements of Γ, where either $\{x_n\}$ or $\{y_n\}$ is an arbitrary finite sequence.

Remark. Both the inverse relations contained in Theorems 3.1 and 3.3 can be replaced by their rotated forms.

Evidently, for $p_n(t) = q_n(t) = t^n/n!$, both (3.1) and (3.2) will reduce to the ordinary case that has been treated in some detail in Hsu [16]. Also, there are various special pairs of inverse relations implied by (3.3) and (3.4). For instance, the simplest case is given by $\phi(t) = \psi(t) = t$, which implies the classical Stirling transforms when $p_n(t) = t^n$, $q_n(t) = (t)_n$ (falling factorial), $\mathscr{P} = D = d/dt$ and $\mathscr{Q} = \Delta$ (difference operator with unit increment). For other typical examples and more details about (3.2)–(3.4), refer to Hsu [16–18].

4. General Stirling reciprocal transforms of the second kind

Let $A(t), B(t), f(t)$ and $g(t)$ be elements of Γ such that $A(0) = B(0) = 1$ and $f(0) = g(0) = 0$. We will consider the numbers $G_1(n, k)$ and $G_2(n, k)$ generated by the formal expansions

$$\frac{A(t)}{B(f(t))} \frac{(f(t))^k}{k!} = \sum_{n \geqslant k} G_1(n, k) \frac{t^n}{n!}, \tag{4.1}$$

$$\frac{B(t)}{A(g(t))} \frac{(g(t))^k}{k!} = \sum_{n \geqslant k} G_1(n, k) \frac{t^n}{n!}. \tag{4.2}$$

These expansions are similar to those used by Barrucand [1] in his investigation on generalized Appell polynomials. In our recent paper [18], we have defined $\langle G_1(n, k), G_2(n, k) \rangle$ as an extended GSN pair of the second kind under the condition that $f(t)$ and $g(t)$ are reciprocal elements of Γ, and we have proved the following result.

Theorem 4.1. *Any pair $\langle G_1(n, k), G_2(n, k) \rangle$ defined by (4.1) and (4.2) is an extended GSN pair if and only if the following inverse relations hold:*

$$y_n = \sum_{k \geqslant 0}^{n} G_1(n, k) x_k, \qquad x_n = \sum_{k \geqslant 0}^{n} G_2(n, k) y_k. \tag{4.3}$$

Moreover, the above relations can be replaced by their rotated forms.

Corollary 4.2. *The matrices $[G_1(n, k)]$ and $[G_2(n, k)]$ of infinite order with elements generated by (4.1) and (4.2) are reciprocal (inverse) of each other if and only if $f(g(t)) = g(f(t)) = t$.*

The proof of the above theorem again depends on a suitable manipulation with some formal power series, in which a certain simple lemma is needed, namely, the identity $g(f(t))/A(g(f(t)))=t/A(t)$ implies $g(f(t))=f(g(t))=t$. For details, refer to [18].

The extended GSN pair $\langle G_1, G_2 \rangle = \langle G_1(n,k), G_2(n,k) \rangle$ may be expressed by formal derivatives. In fact, using $D_0^n=(d/dt)_0^n$ we have

$$G_1 = \frac{1}{k!} D_0^n \left(\frac{A(t)(f(t))^k}{B(f(t))} \right), \qquad G_2 = \frac{1}{k!} D_0^n \left(\frac{B(t)(g(t))^k}{A(g(t))} \right). \qquad (4.4)$$

On replacing $A(\cdot)$ and $B(\cdot)$ by $A(f(\cdot))$ and $B(g(\cdot))$, respectively, and soon, one may obtain some variations as follows:

$$G_1 = \frac{1}{k!} D_0^n (A(f(t))(f(t))^k/B(t)), \qquad G_2 = \frac{1}{k!} D_0^n (B(g(t))(g(t))^k/A(t)), \quad (4.5)$$

$$G_1 = \frac{1}{k!} D_0^n (A(f(t))(f(t))^k/B(f(t))), \qquad G_2 = \frac{1}{k!} D_0^n (B(t)(g(t))^k/A(t)), \quad (4.6)$$

$$G_1 = \frac{1}{k!} D_0^n (A(f(t))(f(t))^k), \qquad G_2 = \frac{1}{k!} D_0^n ((g(t))^k/A(t)). \qquad (4.7)$$

In particular, for the pair given by (4.7), the first number $G_1 = G_1(n,k)$ can be expressed in terms of $g(t)$ only, namely

$$G_1 = \binom{n-1}{k-1} D_0^{n-k} \{ A(t)(t/g(t))^n \} + \binom{n-1}{k} D_0^{n-k-1} \{ A'(t)(t/g(t))^n \} \quad (4.8)$$

This expression is easily verified by means of the Lagrange inversion formula involving an inverse function. Clearly, (4.8) is useful when compositional inverse $f(t)$ of $g(t)$ is not so easily determined.

Surely, Theorem 4.1 may be used to obtain a good many of special inverse relations. Moreover, it may be worth mentioning that (4.1)–(4.3) for the case $A(t) = B(t) = 1$ can be extended to the multivariate case. Actually, this has been done by Chu [6].

For a given GSN pair $\langle G_1, G_2 \rangle$ one may say that the numbers $G_1 = G_1(n,k)$ and $G_2 = G_2(n,k)$ are associated with each other. Accordingly, G_2 is called an associated number with respect to G_1, and vice versa.

We will show that certain associated numbers with respect to the generalized Bernoulli numbers $B_n^{(k)}$ and Euler numbers $E_n^{(k)}$ can be determined explicitly by means of (4.8). Similarly, we will obtain two kinds of special polynomials that are associated with the generalized Bernoulli polynomials $B_n^{(k)}(x)$ and Euler polynomials $E_n^{(k)}(x)$, respectively.

Recalling the generating functions

$$(t/(e^t-1))^k = \sum_{n \geq 0} B_n^{(k)} t^n/n!, \qquad (2e^t/(e^{2t}+1))^k = \sum_{n \geq 0} E_n^{(k)} t^n/n!,$$

it is clear that we have to take functions

$$f_1(t) = t^2/(e^t-1), \qquad f_2(t) = 2te^t/(e^{2t}+1),$$

so that $f_1(0) = f_2(0) = 0$ and both f_1 and f_2 have reciprocal elements in Γ.

Accordingly, generalized Bernoulli numbers $B_1(n,k)$ and Euler numbers $E_1(n,k)$ of the first kind may be defined by the following equations:

$$(f_1(t))^k = \sum_{m=k}^{\infty} B_1(m,k) \frac{k! \, t^m}{m!} = \sum_{m=k}^{\infty} B_{m-k}^{(k)} \frac{t^m}{(m-k)!},$$

$$(f_2(t))^k = \sum_{m=k}^{\infty} E_1(m,k) \frac{k! \, t^m}{m!} = \sum_{m=k}^{\infty} E_{m-k}^{(k)} \frac{t^m}{(m-k)!}.$$

Comparing both sides, we have

$$B_1(m,k) = \binom{m}{k} B_{m-k}^{(k)}, \qquad E_1(m,k) = \binom{m}{k} E_{m-k}^{(k)}.$$

Now in order to find the numbers $B_2(m,k)$ and $E_2(m,k)$ that are associated with $B_1(m,k)$ and $E_1(m,k)$, respectively, one may make use of (4.8) with $A(t)=1$ (so that $A'(t)=0$). More precisely, we have

$$B_2(m,k) = \binom{m-1}{k-1} D_0^{m-k} \left(\frac{e^t-1}{t}\right)^m = \frac{m!(m-1)!}{(2m-k)!(k-1)!} S_2(2m-k,m),$$

$$E_2(m,k) = \binom{m-1}{k-1} D_0^{m-k} \left(\frac{e^{2t}+1}{2e^t}\right)^m = 2^{-m} \binom{m-1}{k-1} \sum_{j=0}^{m} \binom{m}{j} (m-2j)^{m-k},$$

where $S_2(\cdot,\cdot)$ is the ordinary Stirling number of the second kind. Thus, we obtain two special GSN pairs $\langle B_1(m,k), B_2(m,k) \rangle$ and $\langle E_1(m,k), E_2(m,k) \rangle$, and consequently, we have two matrix inverses:

$$[B_1(n,k)]^{-1} = [B_2(n,k)], \qquad [E_1(n,k)]^{-1} = [E_2(n,k)].$$

A similar procedure may be applied to obtaining associated polynomials with respect to $B_n^{(k)}(x)$ and $E_n^{(k)}(x)$, respectively. As in the foregoing case, we are naturally led to consider polynomials of the form

$$B_1(m,k;x) = \binom{m}{k} B_{m-k}^{(k)}(x), \qquad E_1(m,k;x) = \binom{m}{k} E_{m-k}^{(k)}(x).$$

These polynomials are generated by the exponential generating functions

$$\frac{1}{k!} A^{-1}(t) (g_1(t))^k, \qquad \frac{1}{k!} A^{-1}(t) (g_2(t))^k,$$

respectively, where $A(t) = e^{-xt}$, $g_1(t) = t^2/(e^t - 1)$ and $g_2(t) = 2t/(e^t + 1)$. Thus, in accordance with (4.7) and by the aid of (4.8), one can obtain the following associated polynomials:

$$B_2(m, k; x) = \left\{ \binom{m-1}{k-1} D_0^{m-k} - x \binom{m-1}{k} D_0^{m-k-1} \right\} \left(e^{-xt} \left(\frac{e^t - 1}{t} \right)^m \right)$$

$$= \frac{(m!)^2}{m \cdot k!} \sum_{j=0}^{m-k} \frac{(m-j) S_2}{(m-k-j)!} \frac{(m+j, m)}{(m+j)!} (-x)^{m-k-j}$$

and

$$E_2(m, k; x) = 2^{-m} \left\{ \binom{m-1}{k-1} D_0^{m-k} - x \binom{m-1}{k} D_0^{m-k-1} \right\} (e^{-xt}(e^t + 1)^m)$$

$$= 2^{-m} \binom{m}{k} \sum_{j=0}^{m} \binom{m}{j} \left(\frac{k}{m} j - x \right) (j - x)^{m-k-1}.$$

Consequently, Corollary 4.2 implies the following proposition for matrix inverses:

$$[B_1(n, k; x)]^{-1} = [B_2(n, k; x)], \qquad [E_1(n, k; x)]^{-1} = [E_2(n, k; x)].$$

Finally, it may be noted that the above procedure may also be used to treat special polynomials of the similar kind. For instance, one can find special polynomials associated with Humbert polynomials, etc.

References

[1] P. Barrucand, Sur les polynômes d'Appell généralisés, C.R. Acad. Sci. Paris 274 (1972) 1109–1111.

[2] R.P. Boas and R.C. Buck, Polynomial Expansions of Analytic Functions (Springer, Berlin, 1958).

[3] L. Carlitz, Some inverse relations, Duke Math. J. 40 (1973) 893–901.

[4] L. Carlitz, An application of MacMahon's master theorem, SIAM J. Appl. Math. 26 (1974) 431–436.

[5] L. Carlitz, Some expansions and convolution formulas related to MacMahon's master theorem, SIAM J. Math. Anal. 8 (1977) 320–336.

[6] W.C. Chu, Reciprocal formula and its multifold analogue (Chinese, English summary), J. Dalian Inst. Tech. Math. (1986) 14–18.

[7] W.C. Chu, A class of bivariate reciprocal relations and combinatorial identities (Chinese, English summary), Math. Appl. 1 (1988) 63–66.

[8] W.C. Chu, A multifold form of Gould–Hsu's inversion (Chinese), Acta Math. Sinica 31 (1988) 837–844.

[9] W.C. Chu, On the multidimensional generalizations of Gould–Hsu reciprocal transforms, preprint, 1988.

[10] W.C. Chu, Some multiple inversions and applications to power series expansions, Acta Math. Appl. Sinica (English series), to appear.

[11] W.C. Chu and L.C. Hsu, A class of bivariate inverse relations with an application to interpolation process, preprint, 1987.

[12] G.P. Egorychev, Intergal Representation and Computation of Combinatorial Sums, translation of Math. Monographs, Vol. 59 (Amer. Math. Soc., Providence, RI, 1984).

[13] H.W. Gould, Combinatorial Identities (Morgantown, Printing Company, 1972).

[14] H.W. Gould and L.C. Hsu, Some new inverse series relations, Duke Math. J. 40 (1973) 885–891.

[15] D.H. Greene and D.E. Knuth, Mathematics for the Analysis of Algorithms (Birkhauser, Basel, 1981).

[16] L.C. Hsu, Generalized Stirling number pairs associated with inverse relations, Fibonacci Quart. 25 (1987) 346–351.

[17] L.C. Hsu, A class of generalized Stirling number pairs, J. Math. Res. Exposition 8 (1988) 358.

[18] L.C. Hsu, Theory and application of generalized Stirling number pairs, J. Math. Res. Exposition 9 (1989) 211–220.

[19] F. Hsu and L.C. Hsu, A unified treatment of a class of combinatorial sums, preprint, 1987.

[20] J. Riordan, Cominatorial Identities (Wiley, New York, 1968).