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Nonrepetitive colorings of trees

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Abstract

A coloring of the vertices of a graph G is *nonrepetitive* if no path in G forms a sequence consisting of two identical blocks. The minimum number of colors needed is the *Thue chromatic number*, denoted by $\pi(G)$. A famous theorem of Thue asserts that $\pi(P)=3$ for any path P with at least four vertices. In this paper we study the Thue chromatic number of trees. In view of the fact that $\pi(T)$ is bounded by 4 in this class we aim to describe the 4-chromatic trees. In particular, we study the 4-*critical trees* which are minimal with respect to this property. Though there are many trees T with $\pi(T)=4$ we show that any of them has a sufficiently large subdivision H such that $\pi(H)=3$. The proof relies on Thue sequences with additional properties involving *palindromic words*. We also investigate nonrepetitive edge colorings of trees. By a similar argument we prove that any tree has a subdivision which can be edge-colored by at most $\Delta+1$ colors without repetitions on paths.

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1. Introduction

Let A be a set of symbols and let $a=a_1a_2\ldots a_{2n}$ be a sequence, with $a_i\in A$, $n\geqslant 1$. A sequence a is called a *square* if $a_i=a_{i+n}$ for all $i=1,\ldots,n$. Let G be a simple graph and let f be a coloring of the vertices of G by symbols of A. We say that f is *nonrepetitive* if for any simple path $v_1v_2\ldots v_{2n}$ in G the associated sequence of colors $f(v_1)f(v_2)\ldots f(v_{2n})$ is not a square.

The minimum number of colors in a nonrepetitive coloring of G will be denoted by $\pi(G)$. We will call it the *Thue chromatic number* for reasons to be clear in a moment. For instance, if P_n is a path with n vertices then $\pi(P_3)=2$ while $\pi(P_4)=3$. Notice that a nonrepetitive coloring of G must be proper in the usual sense, but determining $\pi(G)$ is a nontrivial task even for paths or cycles. Indeed, the fact that $\pi(P_n)=3$ for all $n\geqslant 4$ follows from the famous result of Thue [22] asserting the existence of nonrepetitive ternary sequences of any length (see [1,6–10,17,18]). This implies that $\pi(C_n)\leqslant 4$ where C_n is a cycle with n vertices. In fact, $\pi(C_n)=3$ for all $n\geqslant 3$ except for $n=5, 7, 9, 10, 14, 17$, as proved by Currie [12].

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Let $\pi(d)$ denote the supremum of $\pi(G)$ where G ranges over graphs of maximum degree at most d. Thus, $\pi(2)=4$. In [2] it was proved that there are absolute positive constants c_1 and c_2 such that

$$c_1 \frac{d^2}{\log d} \leqslant \pi(d) \leqslant c_2 d^2.$$

The proof uses random graphs and the Lovász local lemma (see [3]).

In this paper we concentrate on trees as the first natural class of graphs (beyond paths) for a more detailed study. Indeed, it is not hard to show that $\pi(T)\leqslant 4$ for any tree T (Section 3). So, the main problem is to describe the class of trees for which $\pi(T)=4$. This naturally leads to investigating 4-*critical* trees, that is, trees satisfying $\pi(T)=4$ with $\pi(T')<4$ for any proper subgraph T' of T. The task appears, however, unexpectedly complex and leads to rather difficult questions about the structure of infinite nonrepetitive words (Sections 4, 6). In particular, the question whether there are infinitely many 4-critical trees is left open. One reason for this situation is perhaps a striking property (Theorem 3.5) which says that any tree has a subdivision S with $\pi(S)=3$. We consider also nonrepetitive edge colorings and the related *Thue chromatic index* $\pi'(G)$. As demonstrated in [2], $\pi'(T)\leqslant 4(\Delta(T)-1)$ for any tree T. We prove (Theorem 5.1) that any tree T has a subdivision S satisfying $\pi'(S)\leqslant \Delta(T)+1$. Proofs of both (vertex and edge) properties are constructive and use palindromic structures in nonrepetitive ternary words.

2. Squares and palindromes

In this section we provide some necessary preliminaries. Let A be a set of symbols and let A^+ denote the *free semigroup* generated by A, that is, the set of all finite sequences (*words*) over A with *concatenation* of words as a semigroup operation. A *substitution* over A is a map assigning to every symbol of A an element of A^+. Any substitution $h : A \to A^+$ may be extended to a homomorphism of A^+ in the natural way: if $w=w_1\ldots w_n$ is a word then $h(w)=h(w_1)\ldots h(w_n)$. For instance, if $A=\{0,1,2\}$ and $h(0)=1$, $h(1)=20$, $h(2)=210$, then for $w=210$ we have

$$h(w)=h(210)=h(2)h(1)h(0)=210201.$$

Now, we define two types of words that will be crucial for our further purposes. A *square* is a word w that can be written as $w=xx$ for some $x\in A^+$. A *factor* of a word w is any subsequence of consecutive terms of w. For instance, 01120112 is a square containing 120 as a factor. A word is *square-free* if none of its factors is a square. A *palindrome* is a word $w=w_1\ldots w_n$ which looks the same when written backward, that is $w=w_n\ldots w_1$. For instance, 0121021201210 is a square-free palindrome.

A substitution h is *square-free* if for any square-free word w its image $h(w)$ is also square-free. The first example of such peculiar object found by Thue is defined by $h(0)=01201$, $h(1)=020121$ and $h(2)=0212021$. Note that using h we may produce arbitrarily long square-free words by a sequence of iterations $h(0), h(h(0)), h(h(h(0))), \ldots$.

Let h be a substitution over $A=\{0,1\}$ defined by $h(0)=01$, $h(1)=10$. Define recursively a sequence of words t_n by $t_0=0$ and $t_n=h(t_{n-1})$ for $n\geqslant 1$. For instance,

$$t_0=0,$$

$$t_1=01,$$

$$t_2=0110,$$

$$t_3=01101001,$$

$$t_4=0110100110010110.$$

Notice that t_{2n} is a palindrome for any $n\geqslant 1$. Further, let q_n be a word obtained from t_{2n} by counting ones between consecutive zeros. For instance, $q_1=2$ and $q_2=2102012$.

Theorem 2.1 (*Thue [23]*). *The words t_n do not contain factors of the form $axaxa$, where $a\in A$ and $x\in A^+$. In consequence, the words q_n are square-free palindromes.*

In the sequel we will refer to t_n and q_n as the *Thue words*.

3. The Thue number of trees

We start with a proof of a general bound on $\pi(T)$ based on square-free sequences avoiding palindromes.

Theorem 3.1. *Any tree has a nonrepetitive* 4-*coloring*.

Proof. Let T be a tree with root r and $k \geqslant 1$ the maximum distance from r. Let L_i be the set of vertices at distance i from the root, $i = 0, \dots, k$. Construct a sequence $a = a_0 a_1 \dots a_k$ which is at the same time square-free and palindrome-free, that is, no factor of a is a square nor a palindrome. Such a sequence may be obtained from any ternary square-free word by inserting the fourth symbol between factors of length two. For instance, the word 0121021201210 gives

0132130231230132130.

Now, consider a coloring $f : V(T) \to \{0, 1, 2, 3\}$ defined by $f(v) = a_i$ whenever $v \in L_i$. We claim that this coloring is nonrepetitive. Indeed, suppose that there is a path $P = v_1 \dots v_{2n}$ in T such that the word $w = f(v_1) \dots f(v_{2n})$ is a square. Since a is square-free there must be a vertex in P, say v_h, whose neighbors v_{h-1}, v_{h+1} are on the same level L_i. Without loss of generality we may assume that $1 < h \leqslant n$ and that v_h is the root of T. Then the word w looks as follows:

$$w = a_{h-1} a_{h-2} \dots a_1 a_0 a_1 \dots a_{h-1} a_h \dots a_{2n-h}.$$

If $h < n$ then a palindrome $a_1 a_0 a_1$ lies entirely in the first half of w. Since w is a square this palindrome appears in the second half of w, and thus in a. If $h = n$ we get

$$w = (a_{n-1} a_{n-2} \dots a_1 a_0)(a_1 \dots a_{n-1} a_n).$$

Since w is a square we have $a_i = a_{n-i}$ for all $i = 0, \dots, n-1$. Hence the word $a_0 \dots a_n$ is a palindrome. In both cases we get a contradiction which completes the proof. □

By generalizing this argument Kündgen and Pelsmajer [15] proved that $\pi(G)$ is at most 4^k for every graph G of treewidth at most k.

Recall that the *eccentricity* of a vertex u is the maximum distance between u and any other vertex, and that the *radius* of a graph G, denoted $\mathrm{rad}(G)$, is the minimum eccentricity of its vertices. The *center* of a graph is the subgraph induced by the vertices of minimum eccentricity. It is well-known that the center of a tree T consists of a vertex or an edge and that it can be determined by deleting every leaf of T and continuing this procedure until the center is reached.

By a similar approach as in the proof of Theorem 3.1 we can show the following:

Lemma 3.2. *Let T be a tree of* $\mathrm{rad}(T) \leqslant 4$. *Then* $\pi(T) \leqslant 3$.

Proof. Let u be a vertex of T from its center and arrange the vertices into levels L_i, $0 \leqslant i \leqslant 4$, where L_i is the set of vertices x with $d_T(u, x) = i$. Let $a = a_0 a_1 a_2 a_3 a_4 = 21021$ be the beginning of q_2. Then the coloring $f : V(F) \to \{0, 1, 2\}$ defined with $f(v) = a_i$ for $v \in L_i$ is easily verified to be square-free. □

There are many trees with $\pi(T) = 4$. Perhaps the simplest way to convince oneself of this is to consider a 3-regular tree of height 5. However, this tree is not minimal with respect to having this property. Define a 4-*critical tree* as a tree T such that $\pi(T) = 4$, but $\pi(T') < 4$ for any proper subtree T' of T.

A tree is called *caterpillar* if it consists of a path P_k on vertices $v_1, \dots, v_k$ with some leaves added to each vertex v_i. The caterpillar with exactly one leaf in each vertex of the path P_k is called a *comb* H_k. Leafs of H_k will be denoted by $u_1, u_2, \dots, u_k$.

Now, consider a comb H_5 with vertices u_1, v_1, v_2 colored as shown below:

$$\begin{array}{ccccccccc} b & - & a & - & & - & & - & \\ | & & | & & | & & | & & |, \\ a & & & & & & & & \end{array}$$

where a, b are any different symbols from the set $\{0, 1, 2\}$. We denote this particular partial coloring of H_5 by F and call it a *flop*.

Claim 1. *A flop F cannot be extended to a nonrepetitive* 3*-coloring of* H_5.

Proof. Indeed, to avoid a square *abab* we must color the next two vertices by a new symbol $c \in \{0, 1, 2\}$:

$$\begin{array}{ccccccccc} b & - & a & - & c & - & & - & \\ | & & | & & | & & | & & |. \\ a & & c & & & & & & \end{array}$$

Now, to avoid *caca* we are forced to put b on the next two vertices:

$$\begin{array}{ccccccccc} b & - & a & - & c & - & b & - & \\ | & & | & & | & & | & & | \\ a & & c & & b & & & & \end{array}$$

and similarly in the next step:

$$\begin{array}{ccccccccc} b & - & a & - & c & - & b & - & a \\ | & & | & & | & & | & & |. \\ a & & c & & b & & a & & \end{array}$$

Finally, the only possibility for the last vertex is c:

$$\begin{array}{ccccccccc} \mathbf{b} & - & \mathbf{a} & - & \mathbf{c} & - & \mathbf{b} & - & \mathbf{a} \\ | & & | & & | & & | & & | \\ a & & c & & b & & a & & \mathbf{c} \end{array}$$

which produces a square *bacbac*. □

Proposition 3.3. H_8 *is a* 4*-critical tree*.

Proof. First we prove that H_8 is not 3-colorable (in the sense of Thue). So, assume on the contrary that there is a nonrepetitive coloring of H_8 with colors 0, 1, 2. We distinguish two cases with respect to a position of a palindrome in the v_2, v_7-path. In fact, any square-free ternary word of length six must contain a factor of the form *aba*. By symmetry, it suffices to consider only the following two cases: (1) *aba* appears on the triple $v_2v_3v_4$:

$$\begin{array}{ccccccccccccccc} & - & a & - & b & - & a & - & \circ & - & \circ & - & \circ & - & \\ | & & | & & | & & | & & | & & | & & | & & | \\ & & & & & & \circ & & \circ & & \circ & & \circ & & \end{array}$$

and (2) *aba* appears on the triple $v_3v_4v_5$:

$$\begin{array}{ccccccccccccccc} & - & & - & a & - & b & - & a & - & \circ & - & \circ & - & \circ \\ | & & | & & | & & | & & | & & | & & | & & |. \\ & & & & & & & & \circ & & \circ & & \circ & & \circ \end{array}$$

In both cases, however, we can find subgraphs isomorphic to a flop F (as depicted in diagrams above). This proves that $\pi(H_8) \geqslant 4$ by Claim 1. Equality follows from Theorem 3.1.

To see that H_8 is critical consider, for example, the graph $H' = H_8 - u_8$. We can color the path P by the word 12010210 and the sequence $u_1 \ldots u_7$ by 0120210 as is shown below:

$$\begin{array}{ccccccccccccccc} 1 & - & 2 & - & 0 & - & 1 & - & 0 & - & 2 & - & 1 & - & 0 \\ | & & | & & | & & | & & | & & | & & | & & . \\ 0 & & 1 & & 2 & & 0 & & 2 & & 1 & & 0 & & \end{array}$$

Other cases can be derived easily from this coloring and are left to the reader. □

The 4-critical trees will be further investigated in the next section. Using H_8 as a subgraph we next present a large class of trees with the Thue number equal to 4. In fact, all trees with no vertices of degree 2 have the Thue number 4 except the trees that are covered by Lemma 3.2.

Theorem 3.4. *Let T be a tree in which no vertex is of degree two. Then* $\pi(T)\leqslant 3$ *if and only if* $\mathrm{rad}(T)\leqslant 4$.

Proof. If $\mathrm{rad}(T)\leqslant 4$, $\pi(T)\leqslant 3$ by Lemma 3.2. Conversely, suppose that $\mathrm{rad}(T)\geqslant 5$ and let u be a vertex of T from its center. Then there exist vertices v and w such that $d_T(v, w) = 9$ (and such that u is on the v, w-shortest path P). Then the inner vertices of P induce the path on eight vertices and every vertex of it is of degree at least three. It follows that H_8 (the graph from Proposition 3.3) is a subgraph of T and consequently $\pi(T)\geqslant 4$. □

The following result complements the above theorem by presenting a large class of trees with the Thue number less than 4.

Theorem 3.5. *Any tree has a subdivision which has a nonrepetitive* 3*-coloring*.

Proof. Let T be a tree with root v and let $k\geqslant 2$ be the maximum distance from v to any vertex of T. Arrange the vertices of T into levels L_i, $i=1,\ldots,k+1$, so that the vertices of L_i are at distance $k+1-i$ from v. (Notice that this is different numbering than the one we used earlier.) Next, consider the Thue words q_i for $i = 2, \ldots, k + 1$ defined in Section 2. Clearly each of them is a square-free palindrome with symbol 2 in the middle, and may be written as $q_i = x_i 2\widetilde{x_i}$, where $\widetilde{x_i}$ is a reversal of x_i. Since q_{i-1} is an initial segment of q_i, the word q_{k+1} may be written as

$$q_{k+1} = 2y_1 2y_2 \ldots 2y_k 2y_{k+1},$$

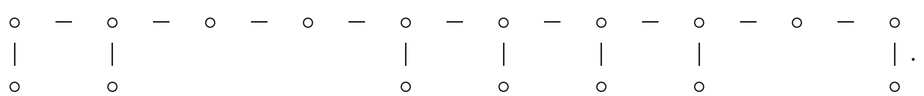
where $y_i \in A^+$ is a nonempty word of length n_i.

Now, subdivide each edge from L_i to L_{i+1} with n_i new vertices, $1\leqslant i\leqslant k$. We claim that this subdivision can be colored with symbols 0, 1, 2 without creating a square. To this end color the vertices of each level L_i by color 2, for $i = 1, \ldots, k + 1$. Finally, color the added vertices so that reading along any added path from L_i to L_{i+1} produces the word y_i, for $i = 1, \ldots, k$. It is not hard to see that the sequence of colors on any path of the subdivided tree must form a factor of q_{k+1}. Therefore, the coloring is nonrepetitive which completes the proof. □

4. 4-Critical trees

In this section we provide further examples of 4-critical trees to give more flavor of the problem. All are subgraphs of sufficiently large combs with one exception.

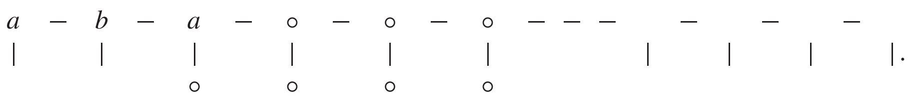
Let H_n be a comb. We call a tree T a *quasi-comb* if it can obtained from H_n by deleting some leafs of H_n. Similarly as in the previous section we will denote the vertices on a path by $v_1, v_2, \ldots, v_n$ and the corresponding leafs, if they exist, by $u_1, u_2, \ldots, u_n$. We will denote a quasi-comb by H_s, where s is a (finite) sequence that consists of integers and symbols "−" defined in the following way. For any sequence of consecutive vertices $v_{i+1}, \ldots, v_{i+k}$ in the path P_n such that all v_j have leafs, while each of v_i and v_{i+k+1} either does not have a leaf or its index is not within range $\{1, \ldots, n\}$ we put integer k in the corresponding place in s. For each vertex v_i which does not have a leaf we put the symbol "−" in the corresponding place of s. For instance, H_{2--4-1} denotes the following quasi-comb:



Note that this notation is consistent with our notation for combs.

Proposition 4.1. H_{6--4} *is a* 4*-critical quasi-comb*.

Proof. By the same reasoning as in the proof of Proposition 3.3 there is a palindrome 010 on the v_1, v_6-path. If this palindrome appears on $v_1v_2v_3$ (or on $v_4v_5v_6$) we have a flop F and we are done by Claim 1:



If the palindrome appears on $v_2v_3v_4$ again we have a flop F since H_5 is isomorphic to H_{--3--}:

```
    –  a  –  b  –  a  –  ○  –  ○  –  ○  –  ○  –     –     –     –
 |     |     |     |     |     |                 |     |     |     |.
                   ○     ○     ○
```

The only possibility that remains is when the palindrome aba, say 010, appears on vertices $v_3v_4v_5$. Then, arguing as in the proof of Claim 1, we obtain the unique partial coloring as below:

```
 1  –  2  –  0  –  1  –  0  –  2  –  1  –  0  –     –     –     –
 |     |     |     |     |     |                 |     |     |     |.
 0     1     2           2     1
```

Now, the path $v_4, \ldots, v_8$ is colored with 10210 and thus the color of v_9 must be 1. This gives the palindrome 101 on the vertices $v_7v_8v_9$. All other vertices must then be colored in unique following way:

```
 1  –  2  –  0  –  1  –  0  –  2  –  1  –  0  –  1  –  2  –  0  –  1
 |     |     |     |     |     |                 |     |     |     |.
 0     1     2           2     1                 2     0     1     2
```

However, in the above scheme we have a square on vertices $v_8v_9 \ldots v_{12}u_{12}$ and we have thus shown that $\pi(H_{6--4})=4$.

It remains to verify that H_{6--4} is 4-critical. If we delete u_{12} the coloring below is sufficient:

```
 1  –  2  –  0  –  1  –  0  –  2  –  1  –  0  –  1  –  2  –  0  –  1
 |     |     |     |     |     |                 |     |     |      .
 0     1     2     0     2     1                 2     0     1
```

If we delete u_{11}, we can again use the above coloring and color u_{12} with 0. If u_{10} is missing we change the color of v_{12} to 2 and color u_{12} with 1. When u_9 is removed we color u_{10} and v_{11} with 1, u_{11} and v_{12} with 0, and u_{12} with 2. When u_6 is deleted one can color v_8 with 0 and there is no problem to color the rest without squares. Similarly, v_7 can receive color 0 if u_5 is missing and the rest is easy.

If we remove u_1, we have the following square-free coloring:

```
 0– 1  –  2  –  0  –  1  –  0  –  2  –  1  –  0  –  1  –  2  –  0
    |     |     |     |     |                 |     |     |     |.
    0     1     2     0     2                 1     2     0     1
```

When u_2, u_3, or u_4 are missing, we just adapt this coloring in an analogous way as before when u_{11}, u_{10}, or u_9 were missing, respectively. This completes the proof. □

Note that $\pi(H_{6-5})=4$, since H_{6--4} is a subtree of H_{6-5}. However, H_{6-5} is not 4-critical, since $H_{6-5}-\{u_8\}=H_{6--4}$. Also the quasi-combs H_{i-5-5} are not 4-critical for any i, since they all contain H_{6-5} as a subgraph. After several 4-critical quasi-combs have been obtained, it appeared that some sort of systematic approach would be needed in their study. As a first step Table 1 has been computed containing all 4-critical quasi-combs with up to 20 base vertices. To make the table more transparent we used the following convention. A sequence of k symbols "–" is replaced by a small integer k and H is omitted. For instance, the quasi-combs H_8, H_{6--4} from Propositions 3.3 and 4.1 are denoted as 8, and $6\,{\scriptstyle 2}\,4$ (=$4\,{\scriptstyle 2}\,6$), respectively.

In the next proposition we give an example of a graph containing a 4-critical tree which is not a quasi-comb. Let $T_1 = H_{------5------}$ and $T_2 = H_{------3------}$. Let T be a tree obtained from disjoint copies of T_1, T_2 in the following way: remove the middle leaf u_8 from T_1 and u_7 from T_2 and join the two base vertices (v_8 in T_1 and v_7 in T_2) by an edge.

Proposition 4.2. *There exists a* 4*-critical tree which is not a quasi-comb.*

Proof. Let T be the tree constructed above. First we show that $\pi(T) = 4$. We claim that T_1 is uniquely 3-colorable (up to a permutation of colors). Indeed, by the flop property a palindrome, say 010, must occupy the middle of this comb.

Table 1
4-Critical quasi-combs of length $n \leqslant 20$

n	4-Critical quasi-combs with n base vertices
8	8
12	$4_{2}6$, $5_{2}5$
14	$1_{2}2_{2}5_{1}1$, $1_{2}3_{2}4_{1}1$, $1_{2}4_{2}3_{1}1$, $1_{2}5_{2}2_{1}1$, $2_{2}2_{2}6$, $2_{2}3_{2}5$, $2_{2}4_{2}4$, $2_{2}5_{2}3$
15	$1_{4}2_{2}6$, $1_{4}3_{2}5$, $1_{4}4_{2}4$, $1_{4}5_{2}3$, $3_{3}1_{2}6$, $4_{5}6$, $4_{3}1_{2}5$, $4_{2}3_{2}4$, $5_{5}5$
16	$1_{1}5_{5}1_{1}2$, $1_{1}5_{2}1_{4}2$, $5_{6}5$
17	$4_{3}3_{3}4$
18	$1_{1}5_{7}4$, $3_{1}1_{4}1_{2}6$, $3_{1}1_{1}1_{5}6$, $3_{1}1_{1}1_{2}2_{2}5$, $5_{8}5$
19	$4_{6}1_{2}6$, $4_{3}1_{5}6$, $4_{3}1_{2}2_{2}5$, $4_{3}1_{2}3_{2}4$, $4_{2}2_{3}1_{2}5$, $4_{2}3_{5}5$, $5_{5}2_{2}5$, $5_{3}1_{1}1_{3}5$, $5_{2}1_{3}1_{2}5$
20	$4_{2}3_{6}5$, $5_{6}2_{2}5$

This implies the following unique coloring of T_1:

$$\begin{array}{ccccccccccccccccc}
2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & \mathbf{0} & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 \\
 & & & & & & | & | & | & | & | & & & & & & \\
 & & & & & & 0 & 2 & \mathbf{1} & 2 & 0 & & & & & &
\end{array}.$$

Now observe that actually there are five different copies of T_1 in T. This fact together with the uniqueness of 3-coloring of T_1 forces the following configuration of colors:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & 0 & & 0 & & & & & & & \\
 & & & & & & & | & & | & & & & & & & \\
 & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 2 & 1 & 2 & \\
 & & & & & & & & | & & & & & & & & \\
2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 \\
 & & & & & & | & | & & | & | & & & & & & \\
 & & & & & & 0 & 2 & & 2 & 0 & & & & & &
\end{array}.$$

However, a square appears on the copy of T_2, which proves that $\pi(T) = 4$.

Hence T must contain a 4-critical tree. But, as one can check, all quasi-combs contained in T are 3-colorable, which completes the proof. □

Our inspection of quasi-combs suggests the following heuristic method of proving that there are arbitrarily long critical quasi-combs. Let F_k be an infinite quasi-comb built of an infinite number of copies of H_5 separated by paths of length k. If $\pi(F_k) = 4$ then by the compactness principle F_k must contain a finite subgraph G with $\pi(G) = 4$, which in turn must contain a 4-critical subgraph. The infinitude of critical quasi-combs would follow if one could show that $\pi(F_k) = 4$ for arbitrarily large k, and that $\pi(H_{\ldots\ldots 5\ldots\ldots}) = 3$ (which seems plausible). Actually, it would even suffice if one could demonstrate that the set of different 3-colorings of F_k is countable for every k. This, however, does not seem to follow from the known facts on nonrepetitive sequences. In particular, it is known that there are continuum many ternary nonrepetitive sequences that differ in their final segments (cf. [6]).

5. The Thue chromatic index

In this section we switch to edge colorings. The defining condition of a *nonrepetitive edge* coloring of a graph G is the same as in the vertex case: no path looks like a square. Note, however, that squares forming full cycles are allowed. The minimum number of colors in a nonrepetitive edge coloring of a graph G is denoted by $\pi'(G)$ and is called the *Thue chromatic index* of G. For instance, $\pi'(C_4) = 2$ while $\pi'(C_5) = 4$. Clearly, $\pi'(G)$ is at least $\chi'(G)$, the usual chromatic index of a graph G. This trivial lower bound is in [4] improved for certain trees. In general, $\pi'(G) \leqslant c\Delta^2$, for some absolute constant c, as proved in [2]. For trees we have a better estimate $\pi'(T) \leqslant 4(\Delta(T) - 1)$, the proof of which is also based on square-free words without palindromes.

In analogy to Theorem 3.5 we will prove that any tree can be subdivided so that the Thue index of the subdivision will be close to the optimal value.

Theorem 5.1. *Any tree T has a subdivision that has a nonrepetitive edge coloring with at most $\Delta(T)+1$ colors.*

Proof. Choose any vertex of degree less than $\Delta(T)$ as a root of T and order the vertices into $k+1$ levels so that the members of the ith level L_i, $i=0,\dots,k$, are at distance i from the root. Then take a square-free ternary word w that can be written as

$$w = 0x_1 0x_2 0 \dots 0x_k,$$

where none of the words $x_i 0 x_{i+1} \dots x_{j-1} 0 x_j$ is a palindrome (clearly, such a word exists for any k). Denote the length of x_i by ℓ_i. Next subdivide each edge $e=uv$, with $u \in L_i$, $v \in L_{i+1}$, by ℓ_i new vertices and color the uv-path along the pattern $0x_i$. Finally, recolor all stars centered at the old vertices using $\Delta - 1$ *shades* $0, 0', 0'', \dots, 0^{(\Delta-2)}$ of the color 0, so as to eliminate the situation in which incident edges have the same color.

We claim that this coloring is nonrepetitive. To prove it conveniently direct all edges towards the root. Clearly, any directed path is colored nonrepetitively. So, assume $p = e_1 \dots e_r e_{r+1} \dots e_{2n}$ is a path with a square coloring, where the edges e_r and e_{r+1} have the same out-neighbor on the level L_j. By construction the colors of e_r and e_{r+1} are different shades of 0. Since two shades of 0 may appear only once in the path p we infer that $r=n$ and e_1, e_{2n} are also colored by different shades of 0. It follows that the edges e_1, e_{2n} are incident with old vertices on the same level, say L_i. Hence the color pattern on the path $e_{r+1} \dots e_{2n-1}$ coincides with the word $x_i 0 x_{i+1} \dots x_{j-1} 0 x_j$. On the other hand, it coincides with the color pattern of the path $e_2 \dots e_{n-1}$. Thus it must be a palindrome by construction of the coloring. This contradiction completes the proof. □

The above result is in general optimal (for a path P_n for instance), however, for some trees we can do slightly better. In the proof of our next result we will apply the following substitution found by Leech [16]:

$$h(0) = 0121021201210,$$

$$h(1) = 1202102012021,$$

$$h(2) = 2010210120102.$$

Leech proved that all words of the form $h^{(n)}(0)$ are square-free, but we will need a stronger property, that h is a square-free substitution. This can be obtained easily by the following result of Crochemore characterizing square-free substitutions.

Lemma 5.2 (*Crochemore [11]*). *Let h be a substitution over $A=\{0,1,2\}$. Let m and M be the minimal and maximal length of a word $h(i)$, $i \in A$, respectively. Let $k=\max\{3, 1+\lfloor (M-3)/m \rfloor\}$. Then h is square-free provided $h(w)$ is a square-free word for any square-free word of length at most k.*

Lemma 5.3. *The substitution of Leech is square-free.*

Proof. By Lemma 5.2 it is enough to check that $h(w)$ is square-free for all square-free words w of length 3. It is readily seen that any such word is a factor of $h(i)$ for some $i=0,1,2$. Therefore, $h(w)$ is a factor of the word $h(h(h(0)))$ which is already known to be square-free. Hence the same must be true of $h(w)$. □

Proposition 5.4. *Let T be any tree with $\pi'(T) \leqslant 3$. Let H be a graph obtained from T by subdividing each edge of T with exactly 12 vertices. Then $\pi'(H) \leqslant 3$.*

Proof. Let f be a nonrepetitive coloring of the edges of T with colors 0, 1, 2. Let $e=uv$ be any edge of T and let P_e be the corresponding path with 13 edges joining u and v in the subdivided graph H. Color the edges of the path P_e consecutively by the symbols of the word $h(f(i))$, where h is the Leech's substitution. The assertion follows from the fact that h is square-free and each of the words $h(i)$ is a palindrome. □

Corollary 5.5. *Let H be any subdivision of a star with at least* 3 *rays. Then* $\pi'(H) = \Delta(H)$.

Proof. First notice that by the theorem any subdivision of a claw $K_{1,3}$ is nonrepetitively 3-colorable. Now, let H be a star subdivision with $\Delta > 3$ rays $R_1, R_2, \ldots, R_\Delta$. Color R_1, R_2, R_3 as in a claw subdivision and the rest of rays $R_4, \ldots, R_\Delta$ along the same pattern as R_3, say. Next, change the colors of the edges of $R_4, \ldots, R_\Delta$ incident with the center of S into symbols $4, \ldots, \Delta$. Clearly, this coloring is nonrepetitive. □

As for the vertex case we may ask for edge-critical quasi-combs. Indeed, it is easy to see that $\pi'(G) \leqslant 4$ for any quasi-comb (color the base path with three colors and the rest of the edges by the fourth color). An easy example of a 4-critical quasi-comb is H_{-1--1-}, but the question whether there are infinitely many of them also remains open.

6. Remarks and questions

It seems that eventual progress in studying nonrepetitive colorings will depend on our knowledge about distribution of palindromes in sequences without repetitions. Let $S = a_1a_2\ldots$ be an infinite square-free sequence of symbols $a_i \in \{0, 1, 2\}$. Consider the related sequence $D(S) = d_1d_2\ldots$ defined by $d_i = 2$ if $a_ia_{i+1}a_{i+2}$ is a palindrome and $d_i = 3$, otherwise. Notice that the sequence $D(S)$ determines S uniquely up to a permutation of symbols. So, there are continuum many of such sequences $D(S)$.

Problem 1. *Let k be a positive integer. Is there a set A of positive integers, with gaps of size at least k, such that for every square-free sequence S over* $\{0, 1, 2\}$, $d_i = 3$ *for at least one number* $i \in A$?

A positive answer would imply the existence of an infinitude of 4-critical quasi-combs in both versions of colorings we considered.

Another approach is to look for a directed tree $Q = (V, E)$ defined as follows. $V(Q)$ is the set of all finite square-free words with initial symbol 0, and $(u, v) \in E(Q)$ if there is a symbol $a \in \{0, 1, 2\}$ such that $ua = v$. A lot of deep results were derived so far about the structure of Q, but none of them seems to be sufficiently strong for the problem of critical trees. For instance, it is known that Q contains a subdivision of an infinite binary tree, which implies that the set of infinite square-free words is perfect (with a natural topology) (cf. [13,20,21]). On the other hand, any path starting from the root 0 can be extended to a path ending in a leaf of Q. If T is any 3-colorable tree then for any rooted copy of T there must exist a homomorphism to Q mapping the root of T to the root of Q, and preserving colors of all vertices.

Problem 2. *Let* B_k *be an infinite binary tree with each edge subdivided by k vertices. Is it true that there are infinitely many k such that* B_k *is not a subgraph of* Q?

It is not hard to demonstrate [14] that every graph has a subdivision which is nonrepetitively 5-colorable. In [5] this bound was improved to 4. However, the following question remains open.

Problem 3. *Is it true that every graph G has a subdivision S such that* $\pi(S) \leqslant 3$?

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