Permanental Polynomials of Graphs

Russell Merris*
Department of Mathematics
California State University, Hayward
Hayward, California 94542

Kenneth R. Rebman
Department of Mathematics
California State University, Hayward
Hayward, California 94542

and

William Watkins
Department of Mathematics
California State University, Northridge
Northridge, California 91330

Submitted by George P. Barker

ABSTRACT

The first section surveys recent results on the permanental polynomial of a square matrix A, i.e., $\operatorname{per}(xI-A)$. The second section concerns the permanental polynomial of the adjacency matrix of a graph. The final section is an introduction to the permanental polynomial of the Laplacian matrix of a graph. An appendix lists some of these latter polynomials.

1. PERMANENTAL POLYNOMIALS

Let $A=(a_{ij})$ be an n by n matrix over a field F. The permanental polynomial afforded by A is the permanent of the characteristic matrix, i.e.,

$$\operatorname{per}(xI-A)=x^n-c_1x^{n-1}+c_2x^{n-2}-\cdots+(-1)^nc_n. \qquad (1)$$

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52 Vanderbilt Ave., New York, NY 10017 0024-3795/81/040273+16$02.50

It is easy to show that c_k is the sum of the k by k principal subpermanents of A. In particular, $c_1 = \text{trace} A$ and $c_n = \text{per} A$. Of course, the permanental polynomial is not a linear algebraic function. It is not preserved under similarity. However, it *is* preserved under permutation similarity:

$$\text{per}(xI - P^{-1}AP) = \text{per}\left[P^{-1}(xI - A)P\right] = \text{per}(xI - A) \tag{2}$$

for all n by n permutation matrices P. Following Engel, we denote the set of roots of the permanental polynomial of A (over an algebraic closure of F) by $S_{\text{per}}(A)$ and call the elements of this set the permanental roots of A. It is the purpose of this first section to describe some of the recent results concerning these polynomials and their roots. In our description, we have taken the liberty of specializing some very general results to the particular case of interest to us here. In no case should our chatty remarks be taken as a definitive summary of any of the articles in the references. Finally, in order not to be bothered by the various restrictions placed on the field in the following results, we will henceforth take F to be the field of complex numbers.

Chronologically first in our survey is a 1967 paper of J. L. Brenner and R. A. Brualdi. They proved the following: Suppose A is an n by n matrix with nonnegative entries and spectral radius ρ. Then $S_{\text{per}}(A) \subset \{z : |z| \leqslant \rho\}$. (It follows from this result that the permanental roots of an M-matrix have positive real parts.) In 1975, R. Merris proved that for a normal n by n matrix A,

$$S_{\text{per}}(A) \subset \{z : |z| \leqslant c\rho\}, \tag{3}$$

where $c = [1 + (2n/\pi)^2]^{1/2}$. In addition, he observed that if A is hermitian with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, then each *real* permanental root of A is in the interval $[\lambda_n, \lambda_1]$. This resulted in the speculation that, for hermitian matrices, the constant c in (3) could be replaced with 1. In 1979, Miroslav Fiedler (unpublished) found the following example which ended the speculation: Let $A = 4I_4 - J_4$, where I_4 is the 4 by 4 identity matrix and J_4 is the 4 by 4 matrix each of whose entries is 1. The characteristic polynomial of A is $x(x-4)^3$, but A has a permanental root of modulus approximately $4\frac{1}{2}$. (It may still be true that the real parts of the permanental roots of a hermitian matrix are bounded by the spectral radius.)

In 1972, G. N. de Oliveira showed that $S_{\text{per}}(A)$ is contained in the Geršgorin circles. In the same year, P. M. Gibson proved that $S_{\text{per}}(A)$ lies in the ovals of Casini. In 1973, G. M. Engel established significantly more general (but not so easily stated) results. A special case of Engel's theorem is this. If A is positive semidefinite hermitian, then any bound for the eigenvalues of A involving absolute values of the entries of A is also a bound for the permanental roots. (Using Engel's result, it can be shown that $\operatorname{per} M \geqslant \det M$ for any M-matrix M.)

In 1971, G. N. de Oliveira proved that there exists an n-square matrix A with diagonal elements $a_1, a_2, \dots, a_n$ and such that $\operatorname{per}(xI - A)$ is a prescribed monic polynomial of degree n, if and only if $-\Sigma a_i$ is the coefficient of x^{n-1} in the prescribed polynomial. In 1972, S. Friedland showed that there always exists an n by n matrix with prescribed off diagonal elements and prescribed (monic, degree n) permanental polynomial. In a 1975 paper, Friedland considered the following problem: Given an n by n complex matrix A, and a set S of n complex numbers, does there exist a diagonal matrix D such that $S_{\text{per}}(AD) = S$? In 1978, D. K. Baxter gave necessary and sufficient conditions for the existence of a matrix with prescribed characteristic and permanental polynomials.

In 1970, Oliveira conjectured that if A is irreducible and doubly stochastic, then $\operatorname{per}(xI - A)$ has no real roots if n is even and exactly one if n is odd. Almost immediately, B. N. Datta established Oliveira's conjecture in some special cases. However, in the next three years, counterexamples were given by J. Csima, D. J. Hartfiel, and R. B. Levow. Finally, in 1978, P. M. Gibson showed that for each $n \geqslant 7$, there exists an n by n irreducible, doubly stochastic matrix A such that $\operatorname{per}(xI - A)$ has n real roots.

If σ is a permutation of degree n, denote by $P_\sigma = (\delta_{i\sigma(j)})$ the corresponding permutation matrix. Write

$$\begin{aligned} g_\sigma(x) &= \operatorname{per}(xI - P_\sigma) \\ &= x^n - c_1(\sigma)x^{n-1} + c_2(\sigma)x^{n-2} - \cdots + (-1)^n c_n(\sigma). \end{aligned}$$

Then c_i is a (permutation) character of the symmetric group S_n. If χ is a class function of a subgroup G of S_n, define

$$g_\chi^G(x) = \frac{1}{o(G)} \sum_{\sigma \in G} \chi(\sigma) g_\sigma(x).$$

Finally, let $h_t(\sigma)$ be the number of cycles of length t in the disjoint cycle factorization of σ. Then the generalized cycle index polynomial of combinatorial analysis is

$$Z_\chi^G(y_1, y_2, \dots, y_n) = \frac{1}{o(G)} \sum_{\sigma \in G} \chi(\sigma) \prod_{t=1}^{n} y_t^{h_t(\sigma)}.$$

In 1979, K. Bogart and J. Gordon showed that

$$g_X^G(x) = Z_X^G\left(x-1, x^2+1, \dots, x^n+(-1)^n\right),$$

a result which supplies Pólya type enumeration theorems for certain hypergraphs.

2. THE ADJACENCY MATRIX OF A GRAPH

Let $G=(V, E)$ be a finite, nondirected graph without loops or multiple edges. Suppose the vertex set $V=\{v_1, v_2, \dots, v_n\}$. The adjacency matrix $A(G)=(a_{ij})$ of G is the n by n matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad \{v_i, v_j\} \in E, \text{ the set of edges,} \\ 0 & \text{otherwise.} \end{cases}$$

Of course, $A(G)$ depends not only on G but on the labeling of the vertices. It is easy to see, however, that adjacency matrices afforded by the same graph with different labelings are permutation similar. Indeed, one can make the stronger statement that graphs G_1 and G_2 are isomorphic if and only if $A(G_1)$ is permutation similar to $A(G_2)$. Thus, any function of $A(G)$ which is invariant under permutation similarity is, in fact, a function of the underlying graph. It follows from (2) that $\operatorname{per}[xI-A(G)]$ is such a function. In the prejudice of the present authors, the permanental polynomial seems a natural tool in the study of graphs. Thus, we were surprised when our literature search turned up only one article on the subject [41]. In the interest of initiating further study, we present some elementary results concerning these polynomials. We begin with a Sachs type result.

THEOREM 2.1. *Let G be a graph. Write*

$$\operatorname{per}\left[xI-A(G)\right] = x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots + (-1)^n c_n.$$

Then

$$c_i = \sum_H 2^{k(H)}, \qquad 1 \le i \le n, \tag{4}$$

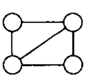
where the sum is over all subgraphs H on i vertices whose components are single edges or circuits, and $k(H)$ is the number of circuits.

Proof. Let $Z=(z_{st})$ be an i by i principal submatrix of $A(G)$. The rows (and columns) from which Z arises correspond to i vertices of G. Now

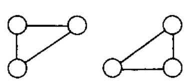
$$\operatorname{per} Z = \sum_{\sigma \in S_i} \prod_{j=1}^{i} z_{j\sigma(j)}. \tag{5}$$

Consider the disjoint cycle factorization, $\sigma=\sigma_1\sigma_2\cdots\sigma_r$, of a fixed $\sigma\in S_i$. This corresponds to a partition of the i vertices into r disjoint sets. The diagonal product corresponding to σ in (5) will be nonzero if and only if $A(G)$ contains a 1 in position $(j,\sigma_p(j))$ for each j in the orbit of σ_p, $1\leqslant p\leqslant r$ (in particular, only if σ has no fixed points). In other words, if σ corresponds to a nonzero diagonal product, then σ determines a unique subgraph of the required form. However, σ^{-1}, for example, determines the same subgraph. Indeed, it is not hard to see that $\pi\in S_i$ determines the same subgraph if and only if $\pi=\sigma_1^{q_1}\sigma_2^{q_2}\cdots\sigma_r^{q_r}$, where $q_j=\pm1$, $1\leqslant j\leqslant r$. Since $\sigma_p^{-1}=\sigma_p$ if and only if σ_p is a transposition (we are assuming σ has no fixed points), the number of permutations in S_i which correspond to a given subgraph H of the required form on the appropriate i vertices is $2^{k(H)}$. The result follows from the representation of c_i as the sum of all i by i principal subpermanents of $A(G)$. ∎

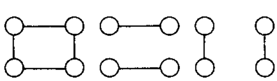
EXAMPLE 2.2. Let

$G=$  .

Then $c_1=0$, and $c_2=$(number of edges)$=5$.

c_3: 

$c_3=2^1+2^1=4$.

c_4: 

$c_4=2^1+2^0+2^0=4$. Thus, $\operatorname{per}[xI-A(G)]=x^4+5x^2-4x+4$.

It follows from Theorem 2.1 that $\operatorname{per} A(G)=c_n=0$ if and only if G does not contain a spanning subgraph the components of which are cycles and edges (e.g., a tree on an odd number of vertices). By König's theorem $\operatorname{per} A(G)=0$ if and only if $A(G)$ contains an s by t submatrix of zeros, where $s+t=n+1$. Putting these two conditions together, we may make the somewhat vague statement that the nonexistence of a spanning subgraph consisting of cycles and edges is equivalent to the presence in the complementary graph of some combination of a complete and a complete bipartite graph. (For example, if $s=t$ and the submatrix is principal, $G'\supset K_s$.)

COROLLARY 2.3. *Let T be a tree on n vertices. Let*

$$\det[xI-A(T)]=x^n-a_1x^{n-1}+a_2x^{n-2}-\cdots+(-1)^n a_n$$

and

$$\operatorname{per}[xI-A(T)]=x^n-c_1x^{n-1}+c_2x^{n-2}-\cdots+(-1)^n c_n.$$

Then

$$c_i=|a_i|, \qquad 1\leqslant i\leqslant n.$$

Proof. By Sachs's theorem,

$$a_i=(-1)^i\sum_H(-1)^{u(H)}2^{k(H)},$$

where the sum is over all subgraphs H on i vertices whose components are single edges or circuits, $u(H)$ is the number of components, and $k(H)$ is the number of circuits. Since T is a tree, it follows that $a_i=c_i=0$ for i odd, and $u(H)=i/2$ for all H on i vertices with i even. Thus, there is no cancellation in the expression for a_i, and comparing with (4),

$$c_i=\begin{cases} a_i & \text{if } i\equiv 0 \bmod 4\\ -a_i & \text{if } i\equiv 2 \bmod 4. \end{cases}$$ ∎

We say that two graphs, G_1 and G_2, are adjacency cospectral if $\det[xI-A(G_1)]=\det[xI-A(G_2)]$, and adjacency copermanental if $\operatorname{per}[xI-A(G_1)]=\operatorname{per}[xI-A(G_2)]$. Then isomorphic graphs are both adjacency cospectral and adjacency copermanental. A natural question is whether one or both of these polynomials can distinguish nonisomorphic graphs. Corollary 2.3 shows that, at least for trees, the permanental polynomial distinguishes nothing which was not already distinguished by the characteristic polynomial. Along these lines, A. J. Schwenk has proved that if t_p is the number of unlabeled trees on p vertices, and if s_p is the number of such trees which are adjacency cospectral with no other tree, then

$$\lim_{p\to\infty} \frac{s_p}{t_p} = 0.$$

It follows from Corollary 2.3 that Schwenk's theorem remains true if "adjacency cospectral" is replaced with "adjacency copermanental."

The permanental polynomial seems a little better than the characteristic polynomial when it comes to distinguishing graphs which are not trees. For example, the permanental polynomial distinguishes the five adjacency cospectral graphs of [20]. However, J. Turner has given two graphs on 9 vertices which are not trees, are nonisomorphic, and yet are adjacency copermanental (and adjacency cospectral). In a very recent article [23], C. R. Johnson and M. Newman consider a modified adjacency matrix, equivalent to replacing each zero in $A(G)$ with an indeterminate y. Calling the new matrix $A_y(G)$, they find that $\det[xI - A_y(G)]$ distinguishes many (but not all) adjacency cospectral graphs. It might be worthwhile to investigate this idea with determinant replaced by permanent.

3. THE LAPLACIAN MATRIX OF A GRAPH

Let $G=(V,E)$, with vertex set $V=\{v_1, v_2,\dots, v_n\}$ and edge set $E=\{e_1, e_2,\dots, e_m\}$, be a graph as in the previous section. Let $D(G)$ be the n by n diagonal matrix, the (i,i) entry of which is the valence (degree) of vertex v_i. Using the terminology of W. N. Anderson, we define the Laplacian of G to be the matrix $L(G)=D(G)-A(G)$. Then, as observed by Kirchhoff, for any connected labeled graph G, all cofactors of $L(G)$ are equal and their common value is the number of spanning trees of G.

Of course, $L(G)$ is symmetric. And, since all rows sum to zero, it is singular. Moreover, it follows from the Geršgorin circle theorem that $L(G)$ is positive semidefinite. There are two other (well-known) proofs of this last fact which we would like to mention. The first of these involves an explicit, graph theoretical interpretation of the quadratic form of $L(G)$, namely,

$$x^t L(G) x = \sum_{\{v_i, v_j\}\in E} (x_i - x_j)^2.$$

The second proof involves the concept of the $(0,1,-1)$ vertex-edge incidence matrix, which we proceed to develop.

Suppose G has n vertices and m edges. Convert G into a directed graph by arbitrarily directing each of the edges [i.e., by converting each unordered edge $e_r=\{v_s,v_t\}$ into an ordered edge (v_s,v_t)]. The corresponding n by m incidence matrix $Q(G)=(q_{ij})$ is defined by

$$q_{ij}=\begin{cases}+1 & \text{if } v_i \text{ is the "positive" end of } e_j,\\ -1 & \text{if } v_i \text{ is the "negative" end of } e_j,\\ 0 & \text{otherwise.}\end{cases}$$

Of course, $Q(G)$ depends not only on G, but on the way in which it was directed. It turns out, however, that $L(G)=Q(G)Q(G)^t$ no matter how G is directed. (Unless explicitly stated otherwise, we will return to thinking of G as an undirected graph.)

Suppose the eigenvalues of $L(G)$ are $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_{n-1}\geqslant\lambda_n=0$. It is not difficult to show that $\lambda_{n-1}\neq 0$ if and only if G is connected. (In a brilliant sequence of papers [11–13], M. Fiedler has obtained some remarkable results concerning the "algebraic connectivity" λ_{n-1} and its eigenspace.)

In keeping with the spirit of this article, we wish to consider the permanental polynomial of the Laplacian:

$$\operatorname{per}[xI-L(G)]=x^n-b_1x^{n-1}+b_2x^{n-2}-\cdots+(-1)^n b_n. \tag{6}$$

It is easy to see that $b_1=\operatorname{trace}L(G)=\operatorname{trace}D(G)=\Sigma(\text{vertex degrees})=2m$. Of course, $b_n=\operatorname{per}L(G)$. Since $L(G)$ is positive semidefinite symmetric, so are each of its principal submatrices. Since the permanent of a positive semidefinite symmetric matrix is nonnegative, it follows that each b_i, being a sum of permanents of principal submatrices, is nonnegative. In fact, we can say more.

THEOREM 3.1. *Let G be a connected graph with vertex valencies $d_1,d_2,\ldots,d_n$. If $r>1$, then*

$$r!e_r\geqslant b_r>e_r,$$

where e_r is the rth elementary symmetric function of $d_1,d_2,\ldots,d_n$.

Proof. The result is immediate from [27, Theorem 1] and the fact that $d_1,\ldots,d_n$ are the main diagonal elements of the positive semidefinite symmetric matrix $L(G)$. ■

If G is not connected, $\operatorname{per}[xI-L(G)]$ is the product of the permanental polynomials of the Laplacians of the connected components of G.

THEOREM 3.2. *Let G_1 be a proper subgraph of the connected graph G. Suppose the Laplacian permanental polynomial of G is given by* (6). *If*

$$\operatorname{per}[xI-L(G_1)]=x^k-c_1x^{k-1}+c_2x^{k-2}-\cdots+(-1)^k c_k,$$

then $b_i>c_i$, $1\leqslant i\leqslant k$, and $b_{n-j}>c_{k-j}$, $0\leqslant j<k$.

Proof. Let $G_1=(V_1,E_1)$, where $V_1\subseteq V$ and $E_1\subset E$. Since E_1 is a proper subset of E, $b_1>c_1$. Next, consider a t by t principal submatrix of $L(G_1)$, call it L_1. The rows (and columns) of L_1 correspond to a subset W of the vertices. Let G^W be the subgraph of G consisting of the vertices W and all edges of G which join vertices in W. Let G_1^W be the corresponding subgraph of G_1. Then $L_1=L(G_1^W)+D_1^W$, where D_1^W is a nonnegative (possibly zero) diagonal matrix. Each main diagonal entry of D_1^W counts the number of edges joining the corresponding vertex in G_1 to a vertex outside W. Now consider G_2^W, the complement of G_1^W in G^W, i.e., G_2^W is a graph on the vertices W whose edges are precisely those edges of G^W which are not edges of G_1^W. Then $L(G^W)=L(G_1^W)+L(G_2^W)$. Finally, the principal submatrix L of $L(G)$, corresponding to the vertices of W, is given by $L=L(G^W)+D^W$, where D^W is a nonnegative diagonal matrix. Each diagonal entry of D^W counts the number of edges joining the corresponding vertex in G to a vertex outside W. Since G_1 is a subgraph of G, $D^W-D_1^W$ is a diagonal matrix with nonnegative entries. It follows that

$$L=L_1+L\left(G_2^W\right)+\left(D^W-D_1^W\right). \tag{7}$$

Applying [31, Theorem 1], we obtain $\operatorname{per}L\geqslant\operatorname{per}L_1$, with equality if and only if $G_1^W=G_1$ and $D^W=D_1^W$. We have obtained a one to one correspondence between the t by t principal submatrices L_1 of $L(G_1)$ and some of the t by t principal submatrices L of $L(G)$ such that $\operatorname{per}L\geqslant\operatorname{per}L_1$. Moreover, since G_1 is a proper subgraph of G, there is at least one case of strict inequality. Thus, $b_t>c_t$, $1\leqslant t\leqslant k$.

To prove the second group of inequalities, we may take $k<n$. Without loss of generality, we may assume G_1 is a graph on the last k vertices. (In the notation above, $W=\{v_{n-k+1},v_{n-k+2},\dots,v_n\}$.) Let L_1 be a $k-t$ by $k-t$ principal submatrix of $L(G_1)$. Let L be the corresponding principal submatrix of $L(G)$. Then, as above, $\operatorname{per}L\geqslant\operatorname{per}L_1$ with strict inequality for an appropriate selection of the $k-t$ vertices. Let $\hat{L}$ be the $n-t$ by $n-t$ principal

submatrix of $L(G)$ lying in rows and columns $1,2,\dots,n-k$ and the $k-t$ rows and columns from which L is drawn. It follows from [27, Theorem 1] (also see [26]) that

$$\operatorname{per}\hat{L} \geqslant d_1 d_2 \cdots d_{n-k} \operatorname{per} L,$$

where $d_1, d_2, \dots, d_{n-k}$ are the valences of $v_1, v_2, \dots, v_{n-k}$. Since each of $d_1, \dots, d_{n-k} \geqslant 1$, we have found a one to one correspondence between the $k-t$ by $k-t$ principal submatrices L_1 of $L(G_1)$ and some of the $n-t$ by $n-t$ principal submatrices $\hat{L}$ of $L(G)$, such that $\operatorname{per}\hat{L} \geqslant \operatorname{per} L_1$, with at least one (and usually very many) strict inequalities. It follows that $b_{n-t} > c_{k-t}$, $0 \leqslant t < k$. ■

In spite of results like Theorems 3.1 and 3.2, a general Sachs type theorem for the permanental polynomial of the Laplacian seems far away.

In the previous section we observed that there are nonisomorphic graphs G_1 and G_2 with $\operatorname{per}[xI - A(G_1)] = \operatorname{per}[xI - A(G_2)]$. However, we do not know of a pair of nonisomorphic graphs for which $\operatorname{per}[xI - L(G_1)] = \operatorname{per}[xI - L(G_2)]$.

COROLLARY 3.3. *Let G be a graph on n vertices. Then*

$$\operatorname{per} L(G) \leqslant n! \sum_{r=0}^{n-2} \frac{(-1)^{n-r} n^r}{r!}, \tag{8}$$

with equality if and only if $G = K_n$, the complete graph on n vertices.

Proof. By the second set of inequalities in Theorem 3.2, $\operatorname{per} L(K_n) = b_n > c_n = \operatorname{per} L(G)$ unless $G = K_n$. It suffices to show that the right hand side of (8) is the permanent of $L(K_n)$. Notice that $L(K_n) = nI_n - J_n$, where I_n is the n by n identity matrix and J_n is the n by n matrix each of whose entries is 1. Let $p(x)$ be the permanental polynomial of J_n. Then $\operatorname{per} L(K_n) = p(n)$. If $p(x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^n a_n$, then a_i is the sum of all i by i principal subpermanents of J_i. There are $n!/i!(n-i)!$ of these, each with permanent $i!$. Since $a_1 = \operatorname{trace} J_n = n$, the result follows. ■

If G is the star graph on n vertices (the tree with one vertex of valence $n-1$ and the other $n-1$ vertices of valence 1), it can be shown that $\operatorname{per} L(G) = 2(n-1)$.

CONJECTURE 3.4. If G is a connected graph on n vertices, then $\operatorname{per} L(G) \geqslant 2(n-1)$ with equality if and only if G is the star graph.

APPENDIX

If G is a connected graph on $n=6$ vertices, let

$$\operatorname{per}[xI-L(G)]=x^6+\sum_{i=1}^{6}(-1)^i b_i x^{6-i}.$$

Then $b_1=2m$, where m is the number of edges of G. F. Harary [19, Appendix 1] lists all graphs on 6 vertices (using q where we use m). The list of Laplacian permanental polynomials in Table 1 follows Harary's numbering.

TABLE 1

m	Harary number	b_2	b_3	b_4	b_5	b_6
5	7	46	120	185	158	58
	8	45	112	160	122	38
	9	43	98	124	82	22
	10	45	112	161	126	42
	11	44	104	137	94	26
	13	40	80	85	46	10
6	7	66	208	393	420	200
	8	65	198	353	342	138
	9	65	198	356	356	156
	10	65	196	344	330	134
	11	64	188	317	284	104
	12	63	180	292	248	84
	13	64	186	308	274	102
	14	63	178	284	242	86
	15	60	154	213	150	42
	16	64	188	316	280	100
	18	63	176	273	222	74
	20	64	186	305	262	90
	21	62	168	249	190	58
7	5	88	312	657	778	410
	6	88	310	641	732	360
	7	88	312	654	760	380
	8	86	286	542	556	244
	9	87	296	578	612	276
	10	86	286	537	534	218

TABLE 1 (continued)

m	Harary number	b_2	b_3	b_4	b_5	b_6
7	11	85	274	489	452	168
	12	87	298	586	614	262
	13	87	300	604	672	324
	14	87	300	602	656	296
	16	87	298	589	630	282
	17	86	290	562	580	240
	18	83	256	430	368	124
	19	84	268	477	446	170
	20	86	288	550	558	230
	21	86	284	524	504	196
	22	84	264	454	400	140
	23	88	308	625	686	314
	24	85	274	486	440	156
8	1	111	416	886	1016	492
	2	110	402	817	862	362
	3	109	390	770	788	320
	4	109	386	736	700	256
	5	114	452	1041	1300	680
	6	114	454	1061	1370	762
	7	114	452	1046	1332	728
	8	112	426	918	1040	468
	9	113	442	1006	1252	660
	10	112	426	921	1056	488
	11	111	412	853	918	394
	12	113	440	994	1234	662
	13	113	438	972	1148	548
	14	113	438	974	1168	588
	15	112	432	968	1184	608
	16	113	440	989	1204	616
	17	112	422	898	1024	496
	20	112	424	906	1022	470
	21	112	428	940	1120	564
	22	110	396	774	764	296
	23	114	456	1081	1440	848
	24	111	412	858	944	424
9	1	138	560	1272	1536	768
	2	141	600	1457	1906	1050
	3	140	584	1365	1662	794

TABLE 1 (continued)

m	Harary number	b_2	b_3	b_4	b_5	b_6
9	4	139	568	1280	1480	668
	5	140	586	1388	1756	924
	6	141	598	1440	1860	1008
	7	144	644	1689	2442	1514
	8	143	628	1601	2236	1336
	9	143	628	1596	2200	1272
	10	143	630	1620	2298	1406
	11	142	612	1509	2002	1114
	12	141	594	1404	1740	868
	13	142	614	1528	2062	1174
	14	141	596	1416	1754	854
	16	142	612	1516	2048	1188
	17	144	648	1737	2646	1818
	18	141	598	1437	1836	968
	19	142	616	1544	2104	1208
	20	142	614	1533	2096	1232
	21	140	580	1333	1584	744
10	1	173	806	2121	2970	1722
	2	175	840	2325	3506	2250
	3	174	824	2233	3270	2018
	4	174	822	2212	3196	1932
	5	172	792	2052	2824	1608
	6	172	786	1985	2566	1274
	7	176	856	2417	3736	2464
	8	175	838	2308	3458	2206
	9	173	800	2056	2722	1398
	10	175	838	2301	3404	2104
	11	174	820	2193	3136	1872
	12	176	858	2445	3870	2682
	14	176	856	2412	3696	2384
	15	175	840	2324	3496	2228
11	1	209	1070	3098	4784	3068
	2	210	1088	3212	5096	3384
	3	211	1106	3329	5434	3754
	4	208	1044	2885	4046	2138
	5	210	1086	3189	5006	3266
	6	211	1108	3354	5544	3924
	7	212	1124	3449	5798	4186

TABLE 1 (continued)

m	Harary number	b_2	b_3	b_4	b_5	b_6
11	8	212	1124	3442	5736	4048
	9	211	1104	3308	5360	3668
12	1	250	1416	4578	7972	5828
	2	249	1398	4464	7668	5544
	3	249	1394	4410	7416	5148
	4	251	1436	4722	8432	6388
	5	252	1456	4860	8832	6800
13	1	293	1794	6264	11772	9288
	2	294	1816	6436	12360	10040
14		340	2248	8496	17328	14880
15		390	2760	11160	24336	22320

Note Added in Proof: In a forthcoming paper, the first author obtains a "Sachs type" theorem for the Laplacian permanental polynomial of a *tree*. As an application of his result, he confirms Conjecture 3.4.

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