MATHEMATICS

SINGULAR SUBMODULE AND INJECTIVE HULL

BY

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This note is in a broad sense a continuation of [7]. If R is a ring (with identity) and its singular ideal is zero then by results of Johnson and Wong any (left) injective hull $h(R)$ of R admits a ring structure which extends the original structure of (left) R-module and makes $h(R)$ a (left) self-injective ring. Here we are interested essentially in rings without zero divisors and we make the objects singular submodule and injective hull of R play the roles of the torsion submodule and the field of quotients of R respectively. We study related questions in the framework of Chapter VII of H. Cartan–S. Eilenberg's Homological Algebra. We refer to this book for the notation and terminology in homological algebra used in the text.

Throughout this paper ring means ring with identity $1 \neq 0$ and all modules are unitary. We use freely the well-known fact that any (left) module has a (left) injective hull unique up to canonical isomorphisms.

I. *The singular submodule and the injective hull*

Let M be a (left) R-module. If m is in M we denote with 0_m the left ideal of R of left annihilators of m. A submodule M' of M will be called antiprimitive in M if it has non-zero intersection with every non-zero submodule of M.

Definition: (R. E. JOHNSON [1]). The singular submodule of M is the submodule of M of all m in M such that 0_m is antiprimitive in R.

It will be denoted by $s(M)$. If R is a ring without zero divisors and with a left field of quotients then $s(M)=$ torsion submodule of M. If R is left semihereditary (that is, if every finitely generated left ideal is projective) then $s(R)=0$. In fact, if x is in R then the exact sequence $0 \to 0_x \to R \to R/0_x \to 0$ splits since $R/0_x \simeq R\cdot x$ is projective. In the case where R is commutative then it is enough to assume that every principal ideal is flat. To see this we use the isomorphism of [5], Ex. 19, page 126. We have $0=Tor_1{}^R(R/(x\cdot R),\ R\cdot x)=Tor_1{}^R(R/(x\cdot R),\ R/0_x)=$ $=(x\cdot R\cap 0_x)/(x\cdot R)\cdot 0_x=(x\cdot R\cap 0_x)/(Rx)\cdot 0_x=R\cdot x\cap 0_x$.

It is easily seen that on the category of left R-modules, the mapping $s: M \to s(M)$ defines a covariant left exact functor which commutes with arbitrary direct sums. Thus, $s(M)=0$ for any projective left module M

if and only if $s(R)=0$. $s=0$ if and only if R is a semisimple ring (d.c.c.) or more generally s is exact if and only if R is a semisimple ring (d.c.c.). In fact, let s be exact. It will be enough to prove that every maximal left ideal of R is a direct summand of R. Now a maximal left ideal I of R is a direct summand of R if and only if $s(R/I)=0$. In the case $s(R)=0$ the exactness of s and of $0 \to I \to R \to R/I \to 0$ give $s(R/I)=0$. So let us prove that $s(R)=0$ is actually the case. Referee's proof: Let I be a left ideal of R maximal respect to the property $I \cap s(R)=0$. The ideal $I \oplus s(R)=J$ is antiprimitive in R. Therefore $s(R/J)=R/J$ and the exactness of s implies that $s(R) \to R/J$ is an epimorphism which is possible only if $R/J=0$, that is $R=J=s(R) \oplus I$. Let $1=e \oplus f$, $e \in s(R)$ and $f \in I$. Since $I=R\cdot f=0_e$ is antiprimitive in R we must have $s(R)=0$ as wished.

With any module M we will consider an injective hull of M, that is an injective module $h(M)$ which contains M and such that M is antiprimitive in $h(M)$.

Proposition 1: Let $s(R)=0$. A submodule M' of M is contained in $s(M)$ if and only if $Hom_R(M', h(R))=0$. Therefore $s(M)= \cap \{Ker\, f;\ f \in Hom_R(M, h(R))\}$.

Proof: To see the necessity we observe that $0=s(R)=s(h(R)) \cap R$ implies $s(h(R))=0$; therefore if $f: M' \to h(R)$ then image $f \subset s(h(R))=0$ and so $f=0$. Conversely, let $m \neq 0$ be any element of M'. Let r be any non-zero element of R. The map

$$x\cdot(rm) \to x \in R$$

is not well defined (otherwise could be extended to a non-zero homomorphism of M' into $h(R)$), therefore there is an element $t \in R$ such that $(tr)\cdot m=0$ and $tr \neq 0$, which means precisely that $0_m \cap R\cdot r \neq 0$, that is 0_m is antiprimitive in R and hence $m \in s(M)$.

Proposition 2: Let K be a left R-module such that $s(K)=0$. Then if M' is antiprimitive in M, $Hom_R(M/M', K)=0$.

Proof: Let $f: M \to K$ be such that $f/M'=0$. Let $m \in M$ be such that $fm \neq 0$. Let $\mu \in R$ be a non-zero element of R satisfying $R\mu \cap 0_{fm}=0$. Let $\pi \in R$ satisfy $0 \neq \pi\mu\cdot m \in M'$. We have $(\pi\mu)fm=f(\pi\mu m)=0$, a contradiction.

Corollary: (Johnson [2] and also [3] Th. 5). If $s(R)=0$ then there is a canonical isomorphism $Hom_R(h(R), h(R)) \to h(R)$.

Proof: The exact sequence $0 \to R \to h(R) \to h(R)/R \to 0$ induces an exact sequence

$$0 \to Hom_R(h(R)/R, h(R)) \to Hom_R(h(R), h(R)) \to Hom_R(R, h(R)) \to 0$$

and by Proposition 2 $Hom_R(h(R)/R, h(R))=0$.

The isomorphism is given by $(f : h(R) \to h(R)) \to f(1)$, where 1 is the identity element of R. The mapping $h(R) \times h(R) \to h(R)$ defined by $(f, g) \to g(f(1))$ defines on $h(R)$ a structure of ring which satisfies the two following conditions: i) it extends the structure of left R-module of $h(R)$, ii) (up to canonical isomorphism) the structure satisfying i) is unique.

From now on, R will be a ring with $s(R)=0$ and $h(R)$ will be a fixed injective hull with the ring structure defined above.

Proposition 3 (Utumi): $h(R)$ is a von Neumann ring.

Proof: See [4] and [3].

Proposition 3′ (Wong–Johnson [3]): $h(R)$ is left $h(R)$-injective.

Proof: More generally from [5], Proposition 1.4, p. 107 we have that if C is a left injective R-module then $Hom_R(h(R), C)$ is left $h(R)$-injective.

Proposition 4: If $h(R)$ is right R-flat then every right $h(R)$-module in R-flat.

Proof: By Proposition 3 and from the fact that the von Neumann rings are those of weak global dimension 0 we have the $Tor_1^{h(R)}=0$. Since $h(R)$ is right flat there is a canonical isomorphism ([5], Prop. 4.1.2, p. 117)

$$(A_{h(R),R}\, C) \qquad Tor_1^R(A, C) \to Tor_1^{h(R)}(A, h(R) \otimes_R C)$$

and from here follows that $Tor_1^R(A, C)=0$ for every C, that is, A is right R-flat.

Proposition 5: If R is left noetherian then $h(R)$ is semisimple (d.c.c.).

Proof: It is enough (by Proposition 3′ and the fact that for any von Neumann ring all finitely generated (left or right) ideals are direct summands of the ring) to show that $h(R)$ is left noetherian. Since any finitely generated left ideal of $h(R)$ is R-injective and the direct limit of R-injectives is R-injective, follows that every ideal of $h(R)$ is an R-direct summand of $h(R)$. If I_j were a strictly increasing infinite sequence of left ideals of $h(R)$, then every member I_j would be direct summand of I_{j+1} and we would have in $h(R)$ an infinite direct sum of non-zero R-modules. This would give rise to an infinite direct sum in R of non-zero left ideals, a contradiction since R is left noetherian.

Proposition 6: (Utumi, see [4′], p. 5). Let R_n be the ring of all $n \times n$ matrices over R. Then $s(R_n)=0$. Let $h(R_n)$ be a R_n-left injective hull of R_n. Then there is a canonical ring isomorphism of $h(R_n)$ onto $(h(R))_n$.

Proof: The first part follows from the more general statement $s(R_n)=(s(R))_n$ whose proof is immediate. To prove the second part will be enough to prove that $(h(R))_n$ is, under the canonical inclusion $R_n \to (h(R))_n$, a R_n-injective hull of R_n. First, it is clear that R_n is antiprimitive in $(h(R))_n$. Next, to prove that $(h(R))_n$ is R_n-injective we need to make some previous considerations. Let A be a left R-module. Let A^n and A_n be the R-direct sum of n and n^2 copies of A respectively. We introduce in A^n and A_n structures of left R_n-module by defining:

$$(r_{ij})\cdot(a_1, \ldots, a_n) = (\sum_j r_{1j}\, a_j, \ldots, \sum_j r_{nj}\, a_j)$$
$$(r_{ij})\cdot(a_{ij}) = (\sum_k r_{ik}\, a_{kj}).$$

Let $\sum A^n$ denote the R_n-direct sum of n copies of A^n. Then the mapping: $(a_{ij}) \to (a_{11}, \ldots, a_{n1}) \oplus \ldots \oplus (a_{1n}, \ldots, a_{nn})$ defines a R_n-isomorphism of A_n onto $\sum A^n$.

Now, it is known (see [6], Cor. 1, p. 19) that A is R-injective if and only if A^n is R_n-injective. It follows then that A is R-injective if and only if A_n is R_n-injective.

To conclude the proof of Prop. 6 it will be enough to apply the above argument to the case $A = h(R)$.

II. *Rings without zero divisors*

Throughout this section, ring will mean ring without zero divisors and $h(R)$ will be any fixed left injective hull together with the ring structure already defined. If $h(R)$ is a division ring then it is a left field of quotients of R. It is clear that $h(R)$ is a division ring if and only if $h(R)$ is left torsion free (that is, for every x in $h(R)$, $0_x = 0$). In particular,

Proposition 1: Let $h(R)$ be left R-flat. Then $h(R)$ is a division ring.

Proof: We will prove that $h(R)$ is left torsion free. In fact, let $\mu \cdot x = 0$ with $\mu \neq 0$ in R. By [5] Ex. 6, p. 123 there exist $x_1, \ldots, x_n$ in $h(R)$ and $\mu_1, \ldots, \mu_n$ in R such that $x = \sum_i \mu_i x_i$ and $\mu\mu_i = 0$. Thus $\mu_1 = \ldots = \mu_n = 0$ and therefore $x = 0$.

In general if $h(R)$ is a division ring, it is not necessarily left R-flat. We have

Proposition 2: Let R have a left field of quotients. R has a right field of quotients (which coincides with $h(R)$) if and only if $h(R)$ is left R-flat.

Proof: If R has a right field of quotients $h(R)$ then it is the direct limit of the R-projective submodules $R\cdot x$ where $x = \mu^{-1}$, $0 \neq \mu \in R$ or $x = 0$, (see [7]).

Conversely, suppose $h(R)$ is left flat and R has no right field of quotients. Let (μ_i), $i \in [1, n]$, $n > 1$ be a set of right R-linearly independent elements of R. Let (q_i), $i \in [1, n]$ be a set of not all zero elements of $h(R)$ satisfying $\sum \mu_i q_i = 0$. Since $h(R)$ is left flat we can find elements b_j in $h(R)$ and α_{ij} in R (in a finite number) such that

$$q_i = \sum_j \alpha_{ij}\, b_j \text{ and } \sum_i \mu_i\, \alpha_{ij} = 0.$$

Therefore all the α_{ij} are zero which implies that the q_i are zero, a contradiction.

Remarks:

1) For any left R-module M over a ring R with a left field of quotients $h(R), h(R)\otimes_R M$ can be described as follows. Let M_* be the set of all elements of the form $\mu^{-1}\cdot m$, where $\mu\in R-(0)$, $m\in M$ and where $\mu^{-1}\cdot m$ is identified to $\mu'^{-1}\cdot m'$ if there exist α, α' not both zero elements of R satisfying $\alpha\mu=\alpha'\mu'$ and $\alpha\cdot m=\alpha'\cdot m'$. Define $\pi(\mu^{-1}\cdot m)=(\mu\,\pi^{-1})^{-1}\cdot m$ for $\pi\neq 0$ and 0 otherwise, and $\mu_1{}^{-1}\cdot m_1+\mu_2{}^{-1}\cdot m_2=(\alpha_1\,\mu_1)\,(\alpha_1\,m_1+\alpha_2\,m_2)$ where α_1, α_2 are not both zero elements of R satisfying $\alpha_1\,\mu_1=\alpha_2\,\mu_2$. These operations define on M_* a structure of left R-module. The mapping $\mu^{-1}\cdot\alpha\otimes_R m\to\mu^{-1}(y\cdot m)$ defines an isomorphism of $h(R)\otimes_R M$ onto M_*. If M is torsion free then the natural mapping $M\to M_*$ is a monomorphism and M_* is an injective hull of M.

2) If R can be imbedded in a division ring D and R has no left field of quotients then D cannot be left injective. In particular if R has a right field of quotients D but no left field of quotients, then D is not left injective.

3) It follows from what we have seen at the beginning of this section that a ring R has a left field of quotients if and only if for every left R-module A, the set of all torsion elements of A is a submodule of A. Actually, it is true that R has a left field of quotients if and only if for every module the sum of two torsion elements is a torsion element. In fact, let μ and μ' be two non-zero elements of R. Let $A=R/(\mu)\oplus R/(\mu')$ and let e, e' be the respective images of 1 by the canonical mappings $R\to R/(\mu)$ and $R\to R/(\mu')$. Then we have $\mu\cdot e=\mu'\cdot e'=0$ and therefore there exists a non-zero element $\alpha\in R$ such that $\alpha(e\oplus e')=0$ and this implies that $\alpha\cdot e=\alpha\cdot e'=0$, that is $0\neq\alpha\in(\mu)\cap(\mu')$.

III. $Tor_1{}^R(A, h(R))$

As in the previous section, ring will mean ring without zero divisors. Let A be any right R-module and let $h(R)$ be a right injective hull of R. From the exact sequence of left R-modules

$$0\to R\to h(R)\to h(R)/R\to 0$$

we get the exact sequence

$$0\to Tor_1{}^R(A, h(R))\to Tor_1{}^R(A, h(R)/R)\to A\to A\otimes_R h(R).$$

Proposition 1: *Ker* $(A\to A\otimes_R h(R))\subset s(A)$.

Proof: Let $a\in A$ be such that $a\otimes 1=0$ in $A\otimes h(R)$, $[\otimes=\otimes_R]$. If $a\notin s(A)$ then there is $0\neq\alpha\in R$ such that $0_a\cap\alpha\cdot R=0$. Since the mapping $A\to A\otimes h(R)$ is a R-homomorphism we have $a\cdot\alpha\otimes 1=0$ in $A\otimes h(R)$. Let R' be a left R-submodule of $h(R)$ generated by a finite number of elements $x_i\in h(R)$ such that $a\cdot\alpha\otimes 1=0$ in $A\otimes R'$. Let $\mu\in R$ satisfy $x_i\cdot\mu\in R$ and with at least one of them different from zero. So, a fortiori $\mu \neq 0$. The $x_i \cdot \mu$'s generate a left ideal I of R and the inclusion $I \to R$ induces a homomorphism $A \otimes I \to A$ which maps $0 = a \cdot \mu \otimes \mu$ into $a \cdot \alpha\mu = 0$ and therefore $0 \neq \alpha\mu \in 0_a$, a contradiction.

Corollary: a) If $h(R)$ is left R-flat then $Tor_1^R(A, h(R)/R) \subset s(A)$,
b) $A \otimes h(R) = 0$ implies $s(A) = A$.

Remarks:

1) There is equality, in corollary a) if, for instance, R has a right field of quotients.

2) We do not know whether $h(R)$ is always left R-flat and equality always holds in corollary a).

Proposition 2: If R is right semihereditary then $h(R)$ is left R-flat.

Proof: We first prove the following

Lemma: For any right R-module A

$$s[Tor_i^R(A, h(R))] \approx Tor_i^R(A, h(R)) \qquad i \geqslant 1.$$

Let $h(R)$ be left R-flat. Then the inclusion $s(A) \to A$ induces a monomorphism $s(A) \otimes h(R) \to A \otimes h(R)$ by which we identify $s(A) \otimes h(R)$ with its image. Then $s(A) \otimes h(R) \subset s(A \otimes h(R))$.

Proof: By using the canonical isomorphism

$$Ext_R(A, Hom_R(h(R), h(R))) \approx Hom_R(Tor_1^R(A, h(R)), h(R))$$

corresponding to the situation $(A_{R,R}\, h(R)_R, h(R)_R)$ and the fact that $Hom_R(h(R), h(R)) \approx h(R)$ is right R-injective, we get

$$Hom_R(Tor_i^R(A, h(R)), h(R)) = 0 \qquad i \geqslant 1$$

and the first part of the lemma follows from I Prop. 1. The second part follows from the isomorphism:

$$Hom_R(s(A) \otimes h(R), h(R)) \approx Hom_R(s(A), h(R))$$

and two applications of I. Prop. 1.

Returning to the proof of the Proposition 2, let I be any finitely generated right ideal of R. From the exact sequence $0 \to I \to R \to R/I \to 0$ we get the exact sequence $0 \to Tor_1^R(R/I, h(R)) \to I \otimes h(R) \to h(R)$. Since I is R-projective $s(I \otimes h(R)) = 0$ and by the Lemma we have $Tor_1^R(R/I, h(R)) = 0$ for every finitely generated right ideal I of R. To conclude that $h(R)$ is left flat is enough to make a direct limit argument.

Remark. It is known (see [5], page 133–4) that if R is a commutative semihereditary integral domain (that is, a Prüfer Ring) then $Tor_1^R(A, C)$ is always a torsion module. We will show here that in the case of a left (or right) semihereditary ring, with a left field of quotients Q and no right field of quotients, there can be R-modules ${}_RA'_{R,R}C$, such that $Tor_1^R(A', C)$ is different from zero and left R-torsion free (actually R-flat).

In fact, since R is semihereditary there is a natural isomorphism (see [5], page 115):

$$Tor_1^R(Q/R, Tor_1^R(h(R), A)) \approx Tor_1^R(Tor_1^R(Q/R, h(R)), A)$$

corresponding to the situation $((Q/R)_R, {}_R h(R)_R, {}_R A)$. Let us recall that for every left R-module M, $s(M) = Tor_1^R(Q/R, M)$ (see [7]). Furthermore since by Proposition 1 of II, $h(R)$ is not right flat, there is a left R-module A such that $0 \neq Tor_1^R(h(R), A)$. We have then,

$$\begin{aligned} s(Tor_1^R(h(R), A) &\approx Tor_1^R(Q/R, Tor_1^R(h(R), A)) \\ &\approx Tor_1^R(Tor_1^R(Q/R, h(R)), A) \\ &\approx Tor_1^R(s(h(R)), A) \\ &= 0 \end{aligned}$$

since $s(h(R)) = 0$. Therefore $0 \neq Tor_1^R(h(R), A)$ is left R-torsion free.

IV. *The singular submodule of $A \otimes h(R)$*

In the section we use the same hypothesis on R and $h(R)$ as in section III. Let A be a right R-module. If $s(A) = 0$ then by Proposition 1 of III the natural mapping $A \to A \otimes h(R)$ is a monomorphism, by which we identify A with its image. A natural question is to ask whether $A \otimes h(R)$ is an injective hull of A, that is, if A is antiprimitive in $A \otimes h(R)$ and if $A \otimes h(R)$ is right R-injective.

Proposition 1. A is antiprimitive in $A \otimes h(R)$ if and only if $s(A \otimes h(R)) = 0$.

Proof: If A is antiprimitive in $A \otimes h(R)$ then $0 = s(A) = s(A \otimes h(R)) \cap A$ which implies $s(A \otimes h(R)) = 0$. (Next, we exclude the trivial case $A = 0$). Conversely, let $z = \sum_i a_i \otimes x_i$ be a non-zero element of $A \otimes h(R)$. By hypothesis there is $0 \neq \alpha \in R$ such that $0_z \cap \alpha \cdot R = 0$. Let $\mu \in R$ satisfy $x_i \cdot \alpha\mu \in R$ and at least one of them be different from zero. A fortiori $0 \neq \mu$. Then we have

$$0 \neq (\textstyle\sum_i a_i \otimes x_i)\alpha\mu = \sum_i a_i \otimes x_i \cdot \alpha\mu = \sum_i a_i(x_i \alpha\mu) \in A.$$

Proposition 2: Let A be a finitely generated flat right R-module. Then $s(A \otimes h(R)) = 0$ and A is antiprimitive in $A \otimes h(R)$. If, in addition, R is right hereditary, then $A \otimes h(R)$ is an injective hull of A.

Proof: Let $[a_1, \ldots, a_n]$ be a minimal set of generators of A. We first prove that if $\sum_i(a_i \otimes r_i)$ belongs to $A \otimes h(R)$ then $\sum_i(a_i \otimes r_i) = 0$ if and only if all r_i are zero. In fact, let H be the left R-submodule of $h(R)$ generated by $r_1, \ldots, r_n$. The inclusion $H \to h(R)$ induces a monomorphism $A \otimes H \to A \otimes h(R)$ and therefore we have $\sum_i(a_i \otimes r_i) = 0$ in $A \otimes H$. Now, by known properties of the tensor product, there exist elements π_{ij} in R and a_i' in A (in a finite number) such that

$$a_j = \textstyle\sum_i a_i' \cdot \pi_{ij} \text{ and } \sum_j \pi_{ij} r_j = 0.$$

Since $[a_1, \ldots, a_n]$ generates A we have

$$a_i' = \sum_k a_k \cdot \mu_{ki},\ \mu_{ki} \in R$$

and thus

$$a_j = \sum_i (\sum_k a_k \cdot \mu_{ki})\, \pi_{ij} = \sum_k a_k (\sum_i \mu_{ki}\, \pi_{ij})$$

and by the minimality of $[a_1, \ldots, a_n]$ the following relations

$$\sum_i \mu_{ki}\, \pi_{ij} \in 0_{a_k} \qquad k \neq j$$

$$(\sum_i \mu_{ji}\, \pi_{ij} - 1) \in 0_{a_j}$$

hold. The fact that A is torsion free (flat implies torsion free, see the proof of Prop. II.1) makes all $0_{a_k} = 0$ and consequently we have

$$\sum_i \mu_{ki}\, \pi_{ij} = 0 \qquad k \neq j$$

$$\sum_i \mu_{ji}\, \pi_{ij} = 1.$$

Therefore $0 = \sum_j \pi_{ij}\, r_j$ implies $0 = \sum_i \mu_{ki}(\sum_j \pi_{ij}\, r_j) = \sum_j (\sum_i \mu_{ji}\, \pi_{ij}) r_j = (\sum_i \mu_{ji}\, \pi_{ij}) r_j = r_j$.

Let now $z = \sum_i (a_i \otimes r_i)$ be any element of $A \otimes h(R)$. If $\mu \in 0_z$ then $0 = z \cdot \mu = \sum_i a_i \otimes r_i \mu$ and by the preceding argument we have $r_i \mu = 0$, that is $0_z \subset 0_{r_i}$ for every i. Since $s(h(R)) = 0$, 0_z is antiprimitive in R if and only if $z = 0$. This proves that $s(A \otimes h(R)) = 0$. By Proposition 1 follows that A is antiprimitive in $A \otimes h(R)$. Finally in the case where R is right hereditary, $A \otimes h(R)$ is right injective and so an injective hull of A.

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REFERENCES

1. Johnson, R. E., The extended centralizer of a ring over a module, Proc. Amer. Math. Soc., 2, 894–895 (1951).
2. ———, Structure theory of faithful rings, Trans. Amer. Math. Soc., **84**, 508–544 (1957).
3. Wong, E. T. and R. E. Johnson, Self-injective rings, Can. Math. Bull., 2, 167–173 (1959).
4. Utumi, Y., On a theorem on modular lattices, Proc. of the Japan Academy, **35**, 1 (1959).
4′. ———, On quotient rings, Osaka Mathematical Journal, **8**, 1–18 (1956).
5. Cartan, H. and S. Eilenberg, Homological Algebra, Princeton University Press (1956).
6. Harada, M., Note on the dimension of modules and algebras, Journal of the Institute of Polytechnics, Osaka City University, 7, 17 (1957).
7. Gentile, E. R., On rings with one-sided field of quotients, Proc. of the Amer. Math. Soc., **11**, 380–384 (1960).