Applied Mathematics Letters 23 (2010) 68–72

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On the properties of Lucas numbers with binomial coefficients

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ARTICLE INFO

Article history:
Received 13 April 2009
Received in revised form 18 August 2009
Accepted 18 August 2009

Keywords:
Lucas numbers
Fibonacci numbers
Binomial coefficients

ABSTRACT

In this study, some new properties of Lucas numbers with binomial coefficients have been obtained to write Lucas sequences in a new direct way. In addition, some important consequences of these results related to the Fibonacci numbers have been given.

1. Introduction

Fibonacci and Lucas numbers have long interested mathematicians for their intrinsic theory and their applications. For rich applications of these numbers in science and nature, one can see the citations in [1–5]. For instance, the ratio of two consecutive of these numbers converges to the Golden section $\alpha = \frac{1+\sqrt{5}}{2}$. (The applications of Golden ratio appears in many research areas, particularly in Physics, Engineering, Architecture, Nature and Art. Physicists Naschie and Marek-Crnjac gave some examples of the Golden ratio in Theoretical Physics and Physics of High Energy Particles [6–9]). Therefore, in this paper, we are mainly interested in whether some new mathematical developments can be applied to these numbers. In this paper we obtain new results about Lucas numbers. As a reminder for the rest of this paper, for $n > 2$, the well-known Fibonacci $\{F_n\}$ and Lucas $\{L_n\}$ sequences are defined by $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$, where $F_1 = F_2 = 1$ and $L_1 = 2, L_2 = 1$, respectively. Moreover, for the first n Fibonacci numbers, it is well known that the sum of the squares is $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$. Also $\sum_{i=0}^{n} \binom{n-i}{i} = F_{n+1}$.

The sum of the squares formula is our motivation to look for combinatorial sums related to the square of Lucas numbers. Thus, again for the motivation of the paper, we should note that, in [10], Spivey presented a new approach for evaluating combinatorial sums by using finite differences. Also, he extended this new approach to handle binomial sums of the form $\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$, $\sum_k \binom{n}{2k} a_k$ and $\sum_k \binom{n}{2k+1} a_k$, as well as sums involving unsigned and signed Stirling numbers of the first kind $\sum_{k=0}^{n} \left[{n \atop k}\right] a_k$ and $\sum_{k=0}^{n} s(n, k) a_k$.

There is also interest for k-Fibonacci polynomials. Let $\left\{F_{k,n}\right\}_{n\in\mathbb{N}}$ be a k-Fibonacci sequence. Note that if k is a real variable x then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \begin{cases} 1 & \text{if } n = 0, \\ x & \text{if } n = 1, \\ xF_n(x) + F_{n-1}(x) & \text{if } n > 1, \end{cases}$$

(see [11]). Actually many relations for the derivatives of Fibonacci polynomials proved in that paper. As a final sentence of this section, we note that in the reference [12], some new properties of Fibonacci numbers with binomial coefficients have been investigated. Actually these new properties will be needed in the proof of one of the main results.

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doi:10.1016/j.aml.2009.08.007

2. Main results

In this study, we investigate the new properties of Lucas numbers in relation with Fibonacci numbers by using binomial coefficients. This strategy allows us to obtain in easy form a family of Lucas sequences in a new and direct way.

Theorem 1. *For $n > 2$ and $n \in \mathbb{Z}$, we have the relation*

$$L_{n+6} = 21L_{n-1} + 13L_{n-2}. \tag{1}$$

Proof. Let us use the principle of mathematical induction on n.

For $n = 3$, it is easy to see that

$$L_9 = 21L_2 + 13L_1 = 47.$$

Assume that it is true for all positive integers $n = k$, that is,

$$L_{k+6} = 21L_{k-1} + 13L_{k-2}. \tag{2}$$

Adding L_{k+5} to both sides of (2), we have

$$L_{k+6} + L_{k+5} = 21L_{k-1} + 13L_{k-2} + L_{k+5}.$$

Since $L_k = L_{k-1} + L_{k-2}$, we first obtain $L_{k+7} = L_{k+6} + L_{k+5}$ on the left hand side of the above equality, and for on the right hand side of the equality, we can write $L_{k+5} = L_{k+4} + L_{k+3}$. Hence, by iterating this procedure, we can write $L_{k+1} = L_k + L_{k-1}$. Therefore

$$L_{k+7} = 21L_k + 13L_{k-1},$$

as required. ∎

In the following theorem, for special values of $n \in \mathbb{Z}$, we will formulate special Lucas numbers in terms of their different indices.

Theorem 2. *For $n \geq 0$ and $\frac{n}{2} \in \mathbb{Z}$, we have the following relations:*

(a) $L_{3n+4} = \left[5\left(\sum_{i=0}^{n/2} 2^{2n+1-4i}\binom{n-i}{n-2i}\right)^2 - 4\right]^{1/2}$,

(b) $L_{2n+3} = 5\left(\sum_{i=0}^{n/2}\binom{n-i}{n-2i}\right)^2 - 2$.

Proof. (a) For $n \geq 0$ and $\frac{n}{2} \in \mathbb{Z}$, we know

$$F_{3(n+1)} = \sum_{i=0}^{n/2} 2^{2n+1-4i}\binom{n-i}{n-2i} \tag{3}$$

from [12]. Using the property in (3) and the equality $L_n^2 = 5F_{n-1}^2 - 4$ given in reference [4], in the following iteration, we have a generalization

$$L_4 = \left[5F_3^2 - 4\right]^{1/2} = \left[5\left(\binom{0}{0}2^1\right)^2 - 4\right]^{1/2} = 4$$

$$L_{10} = \left[5F_9^2 - 4\right]^{1/2} = \left[5\left(\binom{2}{2}2^5 + \binom{1}{0}2^1\right)^2 - 4\right]^{1/2} = 76$$

$$L_{16} = \left[5F_{15}^2 - 4\right]^{1/2} = \left[5\left(\binom{4}{4}2^9 + \binom{3}{2}2^5 + \binom{2}{0}2^1\right)^2 - 4\right]^{1/2} = 1364$$

$$\vdots$$

$$\begin{aligned} L_{3n+4} &= \left[5F_{3n+3}^2 - 4\right]^{1/2} \\ &= \left[5\left(\binom{n}{n}2^{2n+1} + \binom{n-1}{n-2}2^{2n-3} + \cdots + \binom{\frac{n}{2}}{0}2^1\right)^2 - 4\right]^{1/2}. \end{aligned}$$

or using the summation symbol, we write

$$L_{3n+4} = \left[5 \left(\sum_{i=0}^{n/2} 2^{2n+1-4i} \binom{n-i}{n-2i} \right)^2 - 4 \right]^{1/2}.$$

(b) Using the equality $L_{2n+3} = 5F_{n+1}^2 - 2$ given in reference [4], for $\frac{n+1}{2} \in \mathbb{Z}$ and $n \geq 1$, $F_{n+1} = \sum_{i=0}^{\frac{n-1}{2}} \binom{n-i}{n-2i}$ as given in [12], the proof can be seen easily. ■

By considering the proof of this above result, we can obtain the following theorem.

Theorem 3. *For $n \geq 1$ and $\frac{n-1}{2} \in \mathbb{Z}$, we have the following relations:*

(a) $L_{3n+4} = \left[5 \left(\sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i} \binom{n-i}{n-2i} \right)^2 + 4 \right]^{1/2}$,

(b) $L_{2n+3} = 5 \left(\sum_{i=0}^{\frac{n-1}{2}} \binom{n-i}{n-2i} \right)^2 + 2.$

Proof. Proof of this theorem can be seen easily in a similar manner with Theorem 2. ■

In addition to Theorem 2, we may also obtain more special Lucas numbers as in the following.

Theorem 4. *For $n \geq 0$ and $\frac{n}{2} \in \mathbb{Z}$, we have the following relations:*

(a) $L_{3n+2} = \frac{5}{4} \sum_{i=0}^{n/2} 2^{2n+1-4i} \binom{n-i}{n-2i} - \frac{3}{4} L_{3n+1}$,

(b) $L_{3n+3} = \sum_{i=0}^{n/2} 2^{2n+1-4i} \binom{n-i}{n-2i} + F_{3n+1}$.

Proof. (a) Let us use the principle of mathematical induction on n.

For $n = 0$, it is easy to see that

$$L_2 = \frac{5}{4} \left[2^1 \binom{0}{0} \right] - \frac{3}{4} L_1 = 1.$$

For $n = 2$, we write

$$L_8 = \frac{5}{4} \left[2^5 \binom{2}{2} + 2^1 \binom{1}{0} \right] - \frac{3}{4} L_7 = 29.$$

Assume that it is true for all positive integers $n = 2k$. That is,

$$L_{6k+2} = \frac{5}{4} \sum_{i=0}^{k} 2^{4k+1-4i} \binom{2k-i}{2k-2i} - \frac{3}{4} L_{6k+1}. \tag{4}$$

Therefore, we have to show that it is true for $n = 2k + 2$. In other words,

$$L_{6k+8} = \frac{5}{4} \sum_{i=0}^{k+1} 2^{4k+5-4i} \binom{2k+2-i}{2k+2-2i} - \frac{3}{4} L_{6k+7}.$$

Let us rewrite (4) by using (3),

$$L_{6k+2} = \frac{5}{4} F_{6k+3} - \frac{3}{4} L_{6k+1}. \tag{5}$$

Adding $\sum_{i=1}^{3} L_{6k+2i+1}$ to both sides of (5), we have

$$L_{6k+2} + \sum_{i=1}^{3} L_{6k+2i+1} = \frac{5}{4} F_{6k+3} - \frac{3}{4} L_{6k+1} + \sum_{i=1}^{3} L_{6k+2i+1}$$

$$L_{6k+8} = \frac{5}{4} F_{6k+3} - \frac{3}{4} L_{6k+1} + \sum_{i=1}^{3} L_{6k+2i+1}.$$

Since $F_n = F_{n+2} - F_{n+1}$, we first obtain $F_{6k+3} = F_{6k+5} - F_{6k+4}$ on the right hand side of above equality. Thus we can write

$$L_{6k+8} = \frac{5}{4}\left(F_{6k+5} - F_{6k+4}\right) - \frac{3}{4}L_{6k+1} + \sum_{i=1}^{3} L_{6k+2i+1}.$$

Hence, by iterating this procedure, we have

$$L_{6k+8} = \frac{5}{4}F_{6k+9} + 5F_{6k+9} - 10F_{6k+8} - \frac{3}{4}L_{6k+1} + \sum_{i=1}^{3} L_{6k+2i+1}. \tag{6}$$

Also, it is known from [4] that

$$F_n = \frac{1}{5}\left(L_n + L_{n+2}\right). \tag{7}$$

Using (7), we see $F_{6k+9} = \frac{1}{5}\left(L_{6k+9} + L_{6k+11}\right)$ and $F_{6k+8} = \frac{1}{5}\left(L_{6k+8} + L_{6k+10}\right)$. So one can easily rearrange (6) and have

$$L_{6k+8} = \frac{5}{4}F_{6k+9} + L_{6k+9} + L_{6k+11} - 2L_{6k+8} - 2L_{6k+10} - \frac{3}{4}L_{6k+1} + \sum_{i=1}^{3} L_{6k+2i+1}.$$

Since $L_k = L_{k-1} + L_{k-2}$, we can write $L_{6k+1} = L_{6k+3} - L_{6k+2}$ and $L_{6k+11} = L_{6k+10} + L_{6k+9}$. Hence, by iterating this procedure, we obtain

$$\frac{3}{4}L_{6k+7} = L_{6k+9} + L_{6k+11} - 2L_{6k+8} - 2L_{6k+10} - \frac{3}{4}L_{6k+1} + \sum_{i=1}^{3} L_{6k+2i+1}.$$

It is obvious that

$$L_{6k+8} = \frac{5}{4}F_{6k+9} - \frac{3}{4}L_{6k+7}.$$

After all, by using (3), we obtain

$$L_{6k+8} = \frac{5}{4}\sum_{i=0}^{k+1} 2^{4k+5-4i}\binom{2k+2-i}{2k+2-2i} - \frac{3}{4}L_{6k+7},$$

which ends up the induction. Therefore we have the required formulate on L_{3n+2}.

(b) The proof can be seen by using the principle of induction on n. ∎

By applying the same method as in the proof of Theorem 4, we have the following.

Theorem 5. *For $n \geq 1$ and $\frac{n-1}{2} \in \mathbb{Z}$, we have the following relations:*

(a) $L_{3n+2} = \frac{5}{4}\sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i} - \frac{3}{4}L_{3n+1}$,

(b) $L_{3n+3} = \sum_{i=0}^{\frac{n-1}{2}} 2^{2n+1-4i}\binom{n-i}{n-2i} + F_{3n+1}$.

Proof. The proof is similar to the proof of Theorem 4. ∎

In the last part of this paper, we would like to present the following two facts other than the above results about how to obtain some Lucas numbers with binomial coefficients. In fact we thought that this would be needed for the reader.

For $n \geq 0$ and $n \in \mathbb{Z}$, we have relations

$$L_{2n+2} = \left[5\left(\sum_{i=0}^{n}\binom{n+i}{2i}\right)^2 - 4\right]^{1/2}$$

and

$$L_{2n+3} = \left[5\left(\sum_{i=0}^{n}\binom{n+1+i}{1+2i}\right)^2 + 4\right]^{1/2}.$$

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